FICHE DE TRAVAUX DIRIGÉS N° 1

Il a été suggéré qu'une partie des enseignements de master s'effectue en anglais. Sans sombrer dans le ridicule, on peut introduire un certain multilinguisme à travers la documentation ou le matériel pédagogique. À titre d'essai, certains énoncés d'exercices, peut-être tous, de ces fiches de travaux dirigés seront écrits en anglais. Un bénéfice immédiat et indiscutable sera l'acquisition des termes usuels de la discipline dans la langue de Shakespeare.

Otherwise stated, $\mathbf{R}$, $\mathbf{R}^n$, and more generally topological spaces, are equipped with their Borel measurable structure. So, the Borel σ-algebras $\mathcal{B}(\mathbf{R})$, $\mathcal{B}(\mathbf{R}^n)$, $\mathcal{B}(E)$, will be usually omitted in notations. General states spaces will be denoted by $(E, \mathcal{E})$, $(F, \mathcal{F})$, etc.

Exercice 1. — Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Suppose that $X : \Omega \to \mathbf{R}$ is a function which assumes only countably many values $a_1, a_2, \ldots \in \mathbf{R}$.

(i) Show that X is a random variable if and only if

$$X^{-1}\{a_k\} \in \mathcal{A} \qquad \text{for all } k = 1, 2, \ldots \tag{$*$}$$

(ii) Suppose that the former property $(*)$ holds. Show that

$$\mathbf{E}[|X|] = \sum_{k=1}^{\infty} |a_k| \, \mathbf{P}\{X = a_k\}.$$

(iii) If $(*)$ holds and $\mathbf{E}[|X|] < \infty$, show that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} a_k \, \mathbf{P}\{X = a_k\}.$$

(iv) If $(*)$ holds and $f : \mathbf{R} \to \mathbf{R}$ is a bounded Borel measurable function, show that

$$\mathbf{E}[f(X)] = \sum_{k=1}^{\infty} f(a_k) \, \mathbf{P}\{X = a_k\}.$$

Answer. — (i) The condition $(*)$ is necessary: since $X^{-1}(B) \in \mathcal{A}$ for every Borel measurable set $B \in \mathcal{B}(\mathbf{R})$, it is also the case for every $B = \{a_k\}$. Conversely, if $(*)$ holds, then for every $B \subset \mathbf{R}$, especially $B \in \mathcal{B}(\mathbf{R})$, one has $X^{-1}(B) = X^{-1}\{a_k : a_k \in B\} = X^{-1}(\bigcup_{a_k \in B}\{a_k\}) = \bigcup_{a_k \in B} X^{-1}\{a_k\} \in \mathcal{A}$ since it is a countable union of $\mathcal{A}$-measurable sets.

(ii) The sets $(\{X = a_k\})_{k=1}^{\infty}$ form a countable measurable partition of Ω, thus $1 = \mathbf{1}_\Omega = \mathbf{1}_{\cup_{k=1}^{\infty}\{X=a_k\}} = \sum_k \mathbf{1}_{\{X=a_k\}}$. Then it is a simple use of Fubini–Tonelli's theorem to the non-negative measurable function $(\omega, k) \in \Omega \times \mathbf{N}^* \mapsto \mathbf{1}_{\{X=a_k\}}(\omega)|X(\omega)|$ integrated with respect to the σ-finite product measure $\mathbf{P} \otimes (\sum_{k=1}^{\infty} \delta_{\{k\}})$:

$$\begin{aligned}\mathbf{E}[|X|] &= \int_\Omega |X(\omega)| \, \mathbf{P}(\mathrm{d}\omega) = \int_\Omega 1 \times |X(\omega)| \, \mathbf{P}(\mathrm{d}\omega) = \sum_{k=1}^{\infty} \int_\Omega \mathbf{1}_{\{X=a_k\}} \times |X(\omega)| \, \mathbf{P}(\mathrm{d}\omega) \\ &= \sum_{k=1}^{\infty} \int_\Omega \mathbf{1}_{\{X=a_k\}} \times |a_k| \, \mathbf{P}(\mathrm{d}\omega) = \sum_k |a_k| \int_\Omega \mathbf{1}_{\{X=a_k\}} \, \mathbf{P}(\mathrm{d}\omega) = \sum_{k=1}^{\infty} |a_k| \, \mathbf{P}\{X = a_k\}.\end{aligned}$$

This result is a such usual corollary of Fubini–Tonelli's theorem, it is invoked by saying that expectaction is additive with respect to countable sums of non-negative random variables.

(iii) When one of the three (double or multiple) integrals is finite, all of them are so and one can remove safely absolute values: that's Fubini's theorem. Here, $\mathbf{E}[|X|] < \infty$ if and only if $\sum_{k=1}^{\infty} |a_k| \, \mathbf{P}\{X = a_k\} < \infty$ and then

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} a_k \, \mathbf{P}\{X = a_k\}.$$

(iv) Consider $f : \mathbf{R} \to \mathbf{R}$ a bounded Borel measurable function, or more generally any bounded function (since we consider in fact $f|_{\{a_k : 1 \le k < \infty\}} : \{a_k : 1 \le k < \infty\} \to \mathbf{R}$ there is no real measubility condition). Since f is bounded $\mathbf{E}[|f(X)|]$ is finite, then Once again, by Fubini's theorem applied to the non-negative measurable function $(\omega, k) \in \Omega \times \mathbf{N}^* \mapsto \mathbf{1}_{\{X=a_k\}}(\omega) f(X(\omega))$ integrated with respect to the σ-finite product measure $\mathbf{P} \otimes (\sum_{k=1}^{\infty} \delta_{\{k\}})$, we have

$$\mathbf{E}[f(X)] = \sum_{k=1}^{n} f(a_k) \, \mathbf{P}\{X = a_k\}.$$

Remark. — The results of the previous exercice are obvious when X takes only finitely many values.

Exercice 2. — Let $X : (\Omega, \mathcal{A}, \mathbf{P}) \to \mathbf{R}$ be a random variable. The *distribution function* F_X of the law or distribution of X is defined by

$$F_X(x) = \mathbf{P}\{X \le x\}, \qquad x \in \mathbf{R}.$$

(i) Prove that F_X has the following properties:

a) $0 \le F_X \le 1$, $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to +\infty} F_X(x) = 1$;

b) F_X is increasing, i.e. non decreasing;

c) F_X is right continuous, i.e. $F_X(x) = \lim_{h \downarrow 0} F_X(x+h)$.

(ii) Let $g : \mathbf{R} \to \mathbf{R}$ be a Borel mesurable function such that $\mathbf{E}[|g(X)|] < \infty$. Prove that

$$\mathbf{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) \, \mathrm{d}F_X(x),$$

where the integral on the right is interpreted in the Lebesgue–Stieljes sense.

(iii) Let $p : \mathbf{R} \to \mathbf{R}_+$ be a non negative measurable function on $\mathbf{R}$. We say that the law of X admits the *density* p if

$$F_X(x) = \int_{-\infty}^{x} p(y) \, \mathrm{d}y \qquad \text{for all } x \in \mathbf{R}.$$

We know that the law at time t of a 1-dimensional Brownian motion started from 0, B_t, has the density

$$p(x) = \frac{1}{\sqrt{2\pi t}} \, \mathrm{e}^{-x^2/2t}, \qquad x \in \mathbf{R}.$$

Find the density of B_t^2.

Answer. — (i) *a)* Obvious, use continuity properties of probablity measures along monotone sequences of measurable sets.

b) Obvious, probability measures are monotone.

c) Once again, this is the continuity property of probability measures along decreasing measurable sets.

(ii) By definition

$$\int_{-\infty}^{+\infty} g(x)\,\mathrm{d}F_X(x) = \int_{-\infty}^{+\infty} g(x)\mu_X(\mathrm{d}x)$$

which is equal to $\mathbf{E}[g(X)]$ by transfert theorem. More precisely, $\mathrm{d}F_X$ must be seen as the unique probability measure whose F_X is the distribution function, that is to say μ_X.

(iii) The random variable B_t^2 is non-negative, it distribution function is 0 over $\mathbf{R}_-^*$ and for $x \geq 0$, $F_{B_t^2}(x) = \mathbf{P}\{B_t^2 \leq x\} = \mathbf{P}^0\{B_t \in [-\sqrt{x}, \sqrt{x}]\} = 2\,\mathbf{P}^0\{B_t \leq \sqrt{x}\} - 1 = 2F_{B_t}(\sqrt{x}) - 1$. This shows that $F_{B_t^2}$ is quite regular over $\mathbf{R}_+^*$ with derivative

$$p(x) = \frac{1}{\sqrt{x}}\,F'_{B_t}(\sqrt{x}) = \frac{1}{\sqrt{2\pi t x}}\,\mathrm{e}^{-x/2t}$$

Thus,

$$p(x) = \frac{\mathbf{1}_{\mathbf{R}_+}(x)}{\sqrt{2\pi t x}}\,\mathrm{e}^{-x/2t}, \qquad x \in \mathbf{R}$$

is a density of the law of B_t^2 when the initial value is 0 (it is a bit more complicated when $B_0 \neq 0$ since the symmetry argument used in computation is no longer true).

Exercice 3. — Let $(\mathcal{H}_i)_{i\in I}$ be a family of σ-algebras on Ω. Prove that

$$\mathcal{H} = \bigcap_{i\in I} \mathcal{H}_i = \big\{A \subset \Omega : A \in \mathcal{H}_i \text{ for all } i \in I\big\}$$

is again a σ-algebra on Ω.

Answer. — This is absolutely obvious. It is always funny to look at the case where I is an empty set. What should be $\mathcal{H}$ in this case? Since the infimum of the empty set in an ordered set should the the natural supremum, if any, of this set, one should take $\mathcal{H} = \mathcal{P}(\Omega)$.

Exercice 4. — (i) Let $X : (\Omega, \mathcal{A}, \mathbf{P}) \to \mathbf{R}$ be a random variable such that

$$\mathbf{E}\big[|X|^p\big] < \infty \qquad \text{for some } 0 < p < \infty.$$

Prove the Markov(–Chebychev) inequality

$$\mathbf{P}\{|X| \geq \lambda\} \leq \frac{1}{\lambda^p}\,\mathbf{E}\big[|X|^p\big] \qquad \text{for all } \lambda \geq 0.$$

(*Hint.* — We have $\int_\Omega |X|^p\,\mathrm{d}\mathbf{P} \geq \int_A |X|^p\,\mathrm{d}\mathbf{P}$, where $A = \{|X| \geq \lambda\}$.)

(ii) Suppose that there exists $k > 0$ such that

$$M = \mathbf{E}[\exp(k|X|)] < \infty.$$

Prove that $\mathbf{P}\{|X| \geq \lambda\} \leq M\,\mathrm{e}^{-k\lambda}$ for all $\lambda \geq 0$.

Answer. — (i) For $\lambda \geq 0$, we have

$$\begin{aligned}\mathbf{P}\{|X| > \lambda\} = \mathbf{P}\{|X|^p > \lambda^p\} &= \int_{\{|X|^p > \lambda^p\}} 1\,\mathbf{P}(\mathrm{d}\omega) \\ &\leq \int_{\{|X|^p > \lambda^p\}} \frac{|X|^p}{\lambda^p}\,\mathbf{P}(\mathrm{d}\omega) \leq \int_\Omega \frac{|X|^p}{\lambda^p}\,\mathbf{P}(\mathrm{d}\omega) = \frac{1}{\lambda^p}\,\mathbf{E}\big[|X|^p\big].\end{aligned}$$

Note that we allow λ to be 0 since there is no $0/0$ case in this computation.

(ii) One has

$$\mathbf{P}\{|X| > \lambda\} = \mathbf{P}\{\exp(k|X|) > \exp(k\lambda)\} \leq \frac{1}{\exp(k\lambda)}\,\mathbf{E}[\exp(k|X|)] \leq M\exp(-k\lambda),$$

bry the former inequality.

Exercice 5. — Let $X, Y : (\Omega, \mathcal{A}, \mathbf{P}) \to \mathbf{R}$ be two independent random variables and assume for simplicity that X and Y are bounded. Prove that

$$\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y].$$

(*Hint.* — Assume that $|X| \le M$, $|Y| \le N$. Approximate X and Y by simple random variables $X_m(\omega) = \sum_{i=1}^m x_i \mathbf{1}_{A_i}(\omega)$, $Y_m(\omega) = \sum_{j=1}^n y_j \mathbf{1}_{B_j}(\omega)$ respectively, where $A_i = X^{-1}([x_i, x_{i+1}[)$, $B_j = Y^{-1}([y_j, y_{j+1}[)$, $-M = x_0 < x_1 < \cdots < x_{m-1} < x_m = M$, $-N = y_0 < y_1 < \cdots < y_{n-1} < y_n = N$. Then

$$\mathbf{E}[X] \approx \mathbf{E}[X_m] = \sum_{0\le i\le m} x_i\,\mathbf{P}(A_i), \qquad \mathbf{E}[Y] \approx \mathbf{E}[Y_n] = \sum_{0\le j\le n} y_i\,\mathbf{P}(B_i)$$

and

$$\mathbf{E}[XY] \approx \mathbf{E}[X_m Y_n] = \sum_{\substack{0\le i\le m\\ 0\le j\le n}} x_i y_j\,\mathbf{P}(A_i \cap B_j)$$

and so on...)

Answer. — We skip the reduction of the from integrable variables to bounded ones: replace X by $\mathbf{1}_{\{|X|\le M\}}X$ and Y by $\mathbf{1}_{\{|Y|\le N\}}Y$ which are still independent variables and whose expectations are close to X and Y ones for M and N sufficiently large. The case of the products may be a little more delicate, so, forget about it.

Discretization allows that $\|X - X_m\|_\infty \le \varepsilon$ and $\|Y - Y_m\|_\infty \le \varepsilon$. Then $|\,\mathbf{E}[X] - \mathbf{E}[X_m]| \le \varepsilon$, $|\,\mathbf{E}[Y] - \mathbf{E}[Y_n]| \le \varepsilon$, and

$$\begin{aligned}|\,\mathbf{E}[X]\,\mathbf{E}[Y] - \mathbf{E}[X_m]\,\mathbf{E}[Y_n]| &= |\,\mathbf{E}[X]\,\mathbf{E}[Y] - \mathbf{E}[X]\,\mathbf{E}[Y_n] + \mathbf{E}[X]\,\mathbf{E}[Y_n] - \mathbf{E}[X_m]\,\mathbf{E}[Y_n]|\\ &\le |\,\mathbf{E}[X]\,\mathbf{E}[Y] - \mathbf{E}[X]\,\mathbf{E}[Y_n]| + |\,\mathbf{E}[X]\,\mathbf{E}[Y_n] - \mathbf{E}[X_m]\,\mathbf{E}[Y_n]|\\ &\le M\varepsilon + N\varepsilon = (M+N)\varepsilon.\end{aligned}$$

Moreover,

$$\begin{aligned}\|XY - X_mY_m\|_\infty &= \|XY - XY_m + XY_m - X_mY_m|_\infty\\ &\le \|XY - XY_m\|_\infty + \|XY_m - X_mY_m|_\infty \le N\varepsilon + M\varepsilon = (M+N)\varepsilon\end{aligned}$$

and $\mathbf{E}[XY] - \mathbf{E}[X_mY_n]| \le (M+N)\varepsilon$. Finally, since $\mathbf{E}[X_mY_n] = \mathbf{E}[X_m]\,\mathbf{E}[Y_n]$,

$$\begin{aligned}|\,\mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y]| &= |\,\mathbf{E}[XY] - \mathbf{E}[X_mY_n] + \mathbf{E}[X_mY_n] - \mathbf{E}[X]\,\mathbf{E}[Y]|\\ &\le |\,\mathbf{E}[XY] - \mathbf{E}[X_mY_n]| + |\,\mathbf{E}[X_mY_n] - \mathbf{E}[X]\,\mathbf{E}[Y]|\\ &\le |\,\mathbf{E}[XY] - \mathbf{E}[X_mY_n]| + |\,\mathbf{E}[X_m]\,\mathbf{E}[Y_n] - \mathbf{E}[X]\,\mathbf{E}[Y]|\\ &\le (M+N)\varepsilon + (M+N)\varepsilon = 2(M+N)\varepsilon.\end{aligned}$$

Thus $|\,\mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y]|$ is arbitrarily close too zero, so $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.

Exercice 6. — Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let $A_1, A_2, \ldots$ be sets in $\mathcal{A}$ such that

$$\sum_{k=1}^{\infty} \mathbf{P}(A_n) < \infty.$$

Prove the weak sense of the Borel–Cantelli lemma:

$$\mathbf{P}\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big), = 0,$$

that is to say: the set of all $\omega \in \Omega$ such that ω belongs to infinitely many $A_k's$ has probability zero.

Answer. — This is obvious: for every $N \geq 1$, on has

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \subset \bigcup_{k=N}^{\infty} A_k \quad \text{and} \quad \mathbf{P}\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big) \leq \mathbf{P}\Big(\bigcup_{k=N}^{\infty} A_k\Big) \leq \sum_{k=N}^{\infty} \mathbf{P}(A_k).$$

The term on the right being the remainder of a convergent series, it tends to zero as $N \to \infty$. This also proves that if $\mathbf{P}\big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\big) > 0$, then $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$.

Let us see the strong sens of the Borel–Cantelli lemma. Suppose moreover that $(A_n)_{n\geq 1}$ are independent events. Then, if $\sum_n \mathbf{P}(A_n) = \infty$, $\mathbf{P}\big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\big) = 1$ (it is also a strictly positive probability).

For $1 \leq n \leq N$,

$$\mathbf{P}\Big(\bigcup_{n\leq m\leq N} A_m\Big) = 1 - \mathbf{P}\Big(\bigcap_{n\leq m\leq N} A_m^c\Big) = 1 - \prod_{n\leq m\leq N} \mathbf{P}(A_m^c) = 1 - \prod_{n\leq m\leq N} \big(1 - \mathbf{P}(A_m)\big).$$

Since $1 - x \leq \mathrm{e}^{-x}$ for every $x \geq 0$,

$$\mathbf{P}\Big(\bigcup_{n\leq m\leq N} A_m\Big) \geq 1 - \exp\Big(-\sum_{m=n}^{N} \mathbf{P}(A_m)\Big).$$

By hypothesis, when N goes to infinity, $\sum_{m=n}^{N} \mathbf{P}(A_m)$ tends, for every n, to infinity and then

$$\mathbf{P}\Big(\bigcup_{n\leq m} A_m\Big) = 1$$

for every n, then the conclusion follows by decreasing limit.

Exercice 7. — Let Ω be non empty set.

(i) Suppose that $A_1, \ldots, A_m$ are disjoints subset of Ω such that $\Omega = \bigcup_{k=1}^{m} A_k$. Prove that the family $\mathcal{A}$ consisting of all unions of some, none, or all $A_1, \ldots, A_m$ is a σ-algebra on Ω.

(ii) Prove that any finite σ-algebra $\mathcal{A}$ on Ω is of the type described in the former question.

(iii) Let $\mathcal{A}$ be a finite σ-algebra on Ω and let $X : \Omega \to \mathbf{R}$ be a $\mathcal{A}$-measurable function. Prove that X assumes only finitely many possible values. More precisely, there exists a disjoint family of subsets $B_1, \ldots, B_n \in \mathcal{A}$ and real numbers $x_1, \ldots, x_m$ such that

$$X = \sum_{i=1}^{m} x_i \mathbf{1}_{B_i}.$$

Answer. — (i) Let $(A_k)_{k=1}^{m}$ be a finite partition of the set Ω and let $\mathcal{A} = \{\bigcup_{k\in K} A_k : K \subset \{1, \ldots, m\}$. We shall prove that $\mathcal{A}$ is a σ-algebra on Ω :

a) taking $K = \{1, \ldots, m\}$, one has $\Omega \in \mathcal{A}$ (and taking $K = \emptyset$, one has $\emptyset \in \Omega$);

b) if $A = \bigcup_{k\in K} A_k \in \mathcal{A}$, then

$$A^c = \Omega \setminus A = \Big(\bigcup_{k\in\{1,\ldots,m\}} A_k\Big) \setminus \Big(\bigcup_{k\in K} A_k\Big) = \bigcup_{k\in\{1,\ldots,m\}\setminus K} A_k = \bigcup_{k\in K^c} A_k$$

which is in $\mathcal{A}$;

c) if $B_\ell = \bigcup_{k\in K_\ell} A_k \in \mathcal{A}$, $\ell = 1, 2, \ldots$, then

$$\bigcup_{\ell} B_\ell = \bigcup_{\ell}\Big(\bigcup_{k\in K_\ell} A_k\Big) = \bigcup_{k\in K} A_k, \qquad \text{with } K = \textstyle\bigcup_\ell K_\ell,$$

which is in $\mathcal{A}$.

(ii) Let $\mathcal{A}$ a finite σ-algebra on Ω. For $\omega, \omega' \in \Omega$ we set $\omega \ \mathcal{R} \ \omega'$ if and only if, for every $A \in \mathcal{A}$, $\omega \in A$ implies $\omega' \in A$. It is easy to prove that $\mathcal{R} : \Omega \times \Omega \to \{\text{true}, \text{false}\}$ is an equivalence relation (reflexivity, symmetry, transitivity). Its equivalence classes $A_1, \ldots, A_m$ form a partition of Ω and, moreover, each A_k is in $\mathcal{A}$: for any $\omega \in A_k$, $\dot{\omega} = A_k = \bigcap_{A \in \mathcal{A}, \omega \in A} A$. Furthermore, if $A \in \mathcal{A}$ and $\omega \in A$ then $\dot{\omega} = A_k \subset A$, thus A is a (finite) union of A_k.

(iii) Let $\mathcal{P}_X = \{X^{-1}\{x\} : x \in \mathbf{R}\}$. Since X is $\mathcal{A}/\mathcal{B}(\mathbf{R})$-measurable, $\mathcal{P}_X \subset \mathcal{A}$ and is finite. This shows that there is only finitely many $x \in \mathbf{R}$ such that $X^{-1}\{x\} \neq \emptyset$, i.e. X assumes only finitely many possible values. Let $x_1, \ldots, x_n$ be these values, $B_i = X^{-1}\{x_i\}$, and then

$$X = \sum_{i=1}^{n} x_i \mathbf{1}_{B_i}.$$

One should notice that, since X is constant over each A_k,

$$X = \sum_{k=1}^{m} x'_k \mathbf{1}_{A_k}$$

where x'_k is the value taken by X on A_k.

Remark. — The last question can be setted with $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ where $(E, \mathcal{E})$ is a measurable space such that all singletons belong to $\mathcal{E}$, but one would have to write anew the last formula since the sum may have no meaning if E is not a vector space. Moreover, one can take $B_i = A_i$ if there is no need for $x_1, \ldots, x_n$ to be distincts.

Exercice 8. — Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on $\mathbf{R}$ with $B_0 = 0$, and put $\mathbf{E} = \mathbf{E}^0$.

(i) Show that

$$\mathbf{E}\big[\mathrm{e}^{\mathrm{i}\theta X_t}\big] = \exp(-\theta^2 t/2) \qquad \text{for all } \theta \in \mathbf{R}.$$

(*Hint.* — Compute the characteristic function of the standard normal distribution $\mathcal{N}(0,1)$ by solving an ordinary differential equation and deduce the result from this computation.)

(ii) Use the power series expansion of the exponential function on both sides, compare the terms with the same power of θ and deduce that

$$\mathbf{E}\big[B_t^4\big] = 3t^2$$

and more generally that

$$\mathbf{E}\big[B_t^{2k}\big] = \frac{(2k)!}{2^k k!}\, t^k, \qquad k \in \mathbf{N}.$$

(iii) If you feel uneasy about the lack of rigour in the method in (ii), you can proceed as follows: we have

$$\mathbf{E}^0[f(B_t)] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} f(x)\, \mathrm{e}^{-x^2/2t}\, \mathrm{d}x$$

for all function f such that the integral on the right converges ; apply this to $f(x) = x^{2k}$ and use integration by parts and induction on k.

(iv) Suppose that B is a n-dimensional Brownian motion. Prove that

$$\mathbf{E}^x\left[\|B_t - B_s\|^4\right] = n(n+2)|t-s|^2$$

by using (ii) and induction on n. *Answer.* — (i) Let $Z : (\Omega, \mathcal{A}, \mathbf{P}) \to \mathbf{R}$ a random variable with standard normal distribution. Its characteristic function is defined by

$$\phi(\theta) = \mathbf{E}\big[\mathrm{e}^{\mathrm{i}\theta Z}\big] = \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}\theta x}\, \mathrm{e}^{-x^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}}, \qquad \theta \in \mathbf{R}.$$

We can differentiate ϕ to obtain

$$\phi'(\theta) = \int_{\mathbf{R}} \mathrm{i}x\,\mathrm{e}^{\mathrm{i}\theta x}\,\mathrm{e}^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbf{R}} \mathrm{i}x\,\mathrm{e}^{\mathrm{i}\theta x}\,\mathrm{e}^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} = -\mathrm{i}\int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}\theta x}\big(-x\,\mathrm{e}^{-x^2/2}\big)\frac{\mathrm{d}x}{\sqrt{2\pi}}$$
$$= -\mathrm{i}\int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}\theta x}\frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{e}^{-x^2/2}\big)\frac{\mathrm{d}x}{\sqrt{2\pi}} = -\mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}\theta x}\,\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty} - \theta\int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}\theta x}\,\mathrm{e}^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} = -\theta\phi(\theta)$$

The differential equation $\phi'(\theta) = -\theta\phi(\theta)$, $\phi(0) = 1$ admits a unique solution on $\mathbf{R}$. We check that $\theta \mapsto \exp(-\theta^2/2)$ is a solution, hence the solution: $\phi(\theta) = \exp(-\theta^2/2)$ for all $\theta \in \mathbf{R}$.

Let's consider B_t. We know that $B_t/\sqrt{t}$ has standard normal distribution, then

$$\varphi_{B_t}(\theta) = \mathbf{E}\big[\mathrm{e}^{\mathrm{i}\theta B_t}\big] = \mathbf{E}\big[\mathrm{e}^{\mathrm{i}(\theta\sqrt{t})(B_t/\sqrt{t})}\big] = \phi\big(\theta\sqrt{t}\big) = \exp(-\theta^2 t/2), \qquad \theta \in \mathbf{R}.$$

(ii) On one side

$$\mathbf{E}\big[\mathrm{e}^{\mathrm{i}\theta B_t}\big] = \mathbf{E}\left[\sum_{n=0}^{\infty}\frac{(\mathrm{i}\theta B_t)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{(\mathrm{i}\theta)^n}{n!}\,\mathbf{E}[B_t^n],$$

on the other side,

$$\exp(-\theta^2 t/2) = \sum_{n=0}^{\infty}\frac{(-\theta^2 t)^n}{2^n n!}.$$

By identifying the different powers, one has for every $n \geq 0$

$$\mathbf{E}[B_t^{2n+1}] = 0 \quad \text{and} \quad \mathbf{E}[B_t^{2n}] = \frac{(2n)!}{2^n n!}\,t^n,$$

in particular, for $n = 2$, $\mathbf{E}[B_t^4] = 24/4/2 \times t^2 = 3t^2$.

(iii) Note that B_t has $\mathcal{N}(0,t)$ distribution when the initial value is 0. Then call $M_n(t) = \mathbf{E}[B_t^n]$. One has, for $n \geq 1$,

$$M_n(t) = \int_{\mathbf{R}} x^n\,\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi}} = -t\int_{\mathbf{R}} x^{n-1}\big(-x/t\,\mathrm{e}^{-x^2/2t}\big)\frac{\mathrm{d}x}{\sqrt{2\pi}}$$
$$= -t\left[\frac{x^{n-1}\,\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi}}\right]_{-\infty}^{\infty} + (n-1)t\int_{\mathbf{R}} x^{n-2}\,\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi}}$$
$$= 0 + (n-1)tM_{n-2}(t).$$

Then,

$$M_{2n}(t) = (2n-1)(2n-3)\times\cdots\times 3\times 1\times t^n M_0(t)$$
$$= \frac{2n(2n-1)(2n-2)(2n-3)\times\cdots\times 4\times 3\times 2\times 1}{2^n\times n(n-1)\times\cdots\times 2\times 1}\,t^n M_0(t) = \frac{(2n)!}{2^n n!}\,t^n$$

and

$$M_{2n+1}(t) = (2n)(2n-2)\times\cdots\times 4\times 2\times t^n M_1(t) = 2^n n! t^n M_1(t) = 0$$

since $M_0(t) = 1$ and $M_1(t) = 0$.

(iv) Set $\tau = |t-s|$. Whatever is the initial value, we know that $Z = B_t - B_s = (Z_1,\ldots,Z_n)$ has $\mathcal{N}(0,\tau\,\mathrm{Id}_n)$ n-dimensional normal distribution: Z_i are independent and each Z_i has $\mathcal{N}(0,\tau)$ normal distribution. Then $\|Z\|^2 = Z_1^2 + \cdots + Z_n^2$ and

$$\|Z\|^4 = \sum_{i=1}^{n} Z_i^4 + 2\sum_{1\leq i<j\leq n} Z_i^2 Z_j^2,$$

then, by linearity and independance,

$$\begin{aligned}\mathbf{E}[\|Z\|^4] &= \sum_{i=1}^{n} \mathbf{E}\big[Z_i^4\big] + 2 \sum_{1\le i<j\le n} \mathbf{E}\big[Z_i^2\big] \times \mathbf{E}\big[Z_j^2\big] \\ &= n\,\mathbf{E}\big[Z_1^4\big] + 2\frac{n(n-1)}{2}\,\mathbf{E}\big[Z_1^2\big]^2 = 3n\tau^2 + n(n-1)\tau^2 = n(n+2)\tau^2.\end{aligned}$$

EXERCICE 9. — To illustrate that the (finite-dimensional) distributions alone do not give all the information regarding the continuity properties of a process, consider the following example:

Let $(\Omega, \mathcal{A}, \mathbf{P}) = (\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+), \mu)$ where μ is a probability measure on $\mathbf{R}_+$ with no mass on single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_t(\omega) = 0 \qquad \text{for all } (t,\omega) \in \mathbf{R}_+ \times \mathbf{R}_+.$$

Prove that $x = (X_t)_{t\ge 0}$ and $y = (Y_t)_{t\ge 0}$ have the same (finite-dimensional) distributions and that X is a version of Y. And yet we have that $t \mapsto Y_t(\omega)$ is continuous for all $\omega \in \Omega$, while $t \mapsto X_t(\omega)$ is discontinuous for all $\omega \in \Omega$.

Answer. — For every $t \le 0$, $\mathbf{P}\{X_t = 1\} = \mathbf{P}\{\omega \in \Omega : X_t(\omega) = 1\} = \mathbf{P}\{\omega \in \Omega : \omega = t\} = \mu\{t\} = 0$. Thus, since all possible values of X are $\{0,1\}$, for every $0 \le t_1 < t_2 < \cdots < t_n$, $\mathbf{P}\{X_{t_1} = 0, \ldots, X_{t_n} = 0\} = 1$ which is the same as Y. This shows that X and Y have the same finite dimensional distributions. We recall that X and Y are versions of each others if and only if $\mathbf{P}\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = 1$ for all $t \ge 0$. Here for a given $t \ge 0$, we have $\mathbf{P}\{\omega \in \Omega : X_t(\omega) \ne Y_t(\omega)\} = \mu\{t\} = 0$. Thus X and Y are also versions of each others.

EXERCICE 10. — A stochastic process $X = (X_t)_{t\ge 0}$ is called *stationary* if X has the same distribution as $(X_{t+h})_{t\ge 0}$ for all $h \ge 0$. Prove that the Brownian motion $B = (B_t)_{t\ge 0}$ has stationary increments, i.e. that the process $(B_{t+h} - B_t)_{h\ge 0}$ has the same distribution for all $t \ge 0$.

Answer. — Immediate. Use $(**)$ for instance. More generally, if X is a homogeneous Markov process with values in $\mathbf{R}$ whose transition function P satisfies $P_t(x, B) = \mu_t(B - x)$ where μ_t is a probability measure on $\mathbf{R}$, then X as stationary increments.

EXERCICE 11. — Prove that, if $B = (B^{(1)}, \ldots, B^{(n)})$ is a n-dimensional Brownian motion with independent initial coordinates $(B_0^{(1)}, \ldots, B_0^{(n)})$, then the 1-dimensional processes $B^{(i)} = (B_t^{(i)})_{t\ge 0}$, $1 \le i \le n$, are independent 1-dimensional Brownian motions.

Answer. — Il suffit de regarder la caractérisation en terme de processus à accroissements indépendants gaussiens et de voir que toutes les variables réelles obtenues sont indépendantes et ont la loi qui convient. (C'est trop lourd pour etre ecrit. Passer par les fonctions caractéristiques ? Ça laisse les étudiants très sceptiques.

EXERCICE 12. — Let $B = (B_t)_{t\ge 0}$ be a Brownian motion and fix $t_0 \ge 0$. Prove that

$$\widetilde{B}_t = B_{t_0+t} - B_{t_0}, \qquad t \ge 0,$$

is a Brownian motion.

Answer. — Il suffit de regarder la caractérisation en terme de processus à accroissements indépendants gaussiens.

EXERCICE 13. — Let $B = (B_t)_{t\ge 0}$ be a 2-dimensional Brownian motion and put

$$D_\varrho = \{x \in \mathbf{R}^2 : |x| < \varrho\} \qquad \text{for } \varrho > 0.$$

Compute $\mathbf{P}^0\{B_t \in D_\varrho\}$.

Answer. — We have

$$\mathbf{P}^0\{B_t \in D_\varrho\} = \int_{D_\varrho} \mathrm{e}^{-((x^2+y^2)}/2t \,\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi t} = \int_0^\varrho \int_0^{2\pi} \frac{r}{t}\,\mathrm{e}^{-r^2/2t}\,\mathrm{d}r = -\big[\mathrm{e}^{-r^2/2t}\big]_0^\varrho = 1-\mathrm{e}^{-\varrho^2/2t}.$$

Exercice 14. — Let $B = (B_t)_{t\geq 0}$ be a n-dimensional Brownian motion and let $K \subset \mathbf{R}^n$ have zero n-dimensional Lebesgue measure. Prove that the expected total time spent in K by B is zero. (*Potential theoritical remark.* — This implies that the *Green measure* associated with B is absolutely continuous with respect to the Lebesgue measure.)

Answer. — By Fubini's theorem, we have

$$\begin{aligned}\mathbf{E}^x\Big[\int_0^\infty \mathbf{1}_{\{B_t\in K\}}\,\mathrm{d}t\Big] &= \int_0^\infty \mathbf{E}^x\big[\mathbf{1}_{\{B_t\in K\}}\big]\,\mathrm{d}t \\ &= \int_0^\infty \int_{\mathbf{R}^n} \mathbf{1}_{\{y\in K\}}\,\mathrm{e}^{-\|y-x\|^2/2t}\,\frac{\mathrm{d}y}{(2\pi t)^{n/2}}\,\mathrm{d}t = \int_0^\infty 0\,\mathrm{d}t = 0.\end{aligned}$$

This result simply use the fact fact the law of each B_t, $t>0$, is absolutely continuous with respect to the Lebesgue measure. Many other processes share the same property (uniform motion on $\mathbf{R}$, compensated Poisson process, ...).

Exercice 15. — Let $B = (B_t)_{t\geq 0}$ be a n-dimensional Brownian motion started from 0 and let $O \in \mathbf{R}^{n\times n}$ be a (constant) orthogonal matrix, i.e. $OO^t = \mathrm{Id}_n$. Prove that

$$\widetilde{B}_t = OB_t, \qquad t \geq 0,$$

is also a Brownian motion.

Answer. — C'est clairement un processus à accroissements indépendants. Ces accroissements sont des images par une application linéaire de vecteurs gaussiens, ce sont donc des vecteurs gaussiens. Leur vecteur moyen est nul comme il se doit et leur matrice de covariance est inchangée (il suffit de l'écrire).

Exercice 16 (Brownian scaling property). — Let $B = (B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion and let $c>0$ be a constant. Prove that

$$\widehat{B}_t = \frac{1}{c}B_{c^2t}, \qquad t\geq 0,$$

is also a Brownian motion.

Answer. — C'est encore un processus à accroissements indépendants. Il suffit de déterminer la loi des accroissements, ce qui est presque immédiat.

Exercice 17 (À revoir !). — If $X_t : (\Omega, \mathcal{A}, \mathbf{P}) \to \mathbf{R}$, $t \geq 0$, is a continuous stochastic process, then, for $p>0$, the *p'th variation process* of X, $\langle X, X\rangle^{(p)}$ is defined by

$$\langle X, X\rangle_t^{(p)} = \lim_{\Delta t_k \to 0} \sum_{0=t_1<t_2<\cdots<t_n=t} \big|X_{t_{k+1}} - X_{t_k}\big|^p,$$

where $\Delta t_k = t_{k+1} - t_k$, as a limit for the convergence in probability taken over all subdivisions of $[0,t]$.

In particular, if $p=1$ this process is called the *total variation process* of X, and if $p=2$ this is called the *quadratic variation process*.

For 1-dimensional brownian motion B we now prove that the quadratic variation process is simply

$$\langle B, B\rangle_t(\omega) = \langle B, B\rangle_t^{(2)}(\omega) = t \qquad \text{almost surely.}$$

Proceed as follows:

(i) Define $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and put

$$Y_t(\omega) = \sum_{0=t_1<t_2<\cdots<t_n=t} \big(\Delta B_k(\omega)\big)^2.$$

Show that

$$\mathbf{E}\big[(Y_t - t)^2\big] = 2 \sum_{0=t_1<t_2<\cdots<t_n=t} (\Delta t_k)^2$$

and deduce that $Y_t \to t$ in $L^2(\Omega, \mathcal{A}, \mathbf{P})$ as $\Delta t_k \to 0$.

(ii) Use (i) to prove that almost surely paths of Brownian motions do not have a bounded variation on $[0, t]$, i.e. the total variation of Brownian motion is infinite almost surely.

Exercice 18. — Let $\Omega = \{1, 2, 3, 4, 5, \}$ and $\mathcal{U}$ be the collection

$$\mathcal{U} = \big\{\{1, 2, 3, \},\ \{3, 4, 5\}\big\}$$

of subsets of Ω. Find the smallest σ-algebra containing $\mathcal{U}$ (i.e. the σ-algebra $\sigma(U)$ generated by $\mathcal{U}$).

(i) Define $X : \Omega \to \mathbf{R}$ by

$$X(1) = X(2) = 0, \quad X(3) = 10, \quad X(4) = X(5) = 1.$$

Is X measurable with respect to $\sigma(\mathcal{U})$?

(ii) Define $Y : \Omega \to \mathbf{R}$ by

$$Y(1) = 1, \quad Y(2) = Y(3)Y(4) = Y(5) = 1.$$

Find the σ-algebra $\sigma(Y)$ generated by Y.

Exercice 19. — Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let $p \in [1, \infty]$. A sequence $(f_n)_{n=1}^{\infty}$ of functions in $L^p(\mathbf{P}) = L^p(\Omega, \mathcal{A}, \mathbf{P}; \mathbf{R})$ is called a *Cauchy sequence* if

$$\|f_n - f_m\|_p \qquad \text{as } m, n \to \infty.$$

The sequence is called *convergent* if there exists $f \in L^p(\mathbf{P})$ such that $f_n \to f$ in $L^p(\mathbf{P})$.

Prove that every convergent sequence is a Cauchy sequence.

A fundamental theorem in measure theory states that the converse is also true: *every Cauchy sequence in* $L^p(\mathbf{P})$ *is convergent.* A normed linear space with this property is called *complete*. Thus, the $L^p(\mathbf{P})$ spaces are complete.

Exercice 20. — Le B be a 1 dimensional Brownian motion, $\sigma \in \mathbf{R}$ a constant and $0 \leq s \leq t$. Use $(**)$ to prove that

$$\mathbf{E}\big[\exp\big(\sigma(B_s - B_t)\big)\big] = \exp\big(\tfrac{1}{2}\sigma^2(s - t)\big).$$