Advanced in Differential Equations

ON THE δ-PRIMITIVE AND BOUSSINESQ TYPE EQUATIONS

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Abstract. In this article we consider the Primitive Equations without horizontal viscosity but with a mild vertical viscosity added in the hydrostatic equation, as in [13] and [16], which are the so-called $\delta-$Primitive Equations. We prove that the problem is well posed in the sense of Hadamard in a certain type of spaces. This means that we prove the finite in time existence, uniqueness and continuous dependence on data for appropriate solutions. The results given in the 3D periodic space, easily extend to dimension 2.

We also consider a Boussinesq type of equations, meaning that the mild vertical viscosity present in the hydrostatic equation, is replaced by the time derivative of the vertical velocity. We prove the same type of results as for the $\delta-$Primitive Equations; periodic boundary conditions are similarly considered.

Contents

1. Introduction

One of the major challenges in the mathematical and physical sciences is to study and improve the long-term weather prediction and to understand the climate changes. This consists in studying the mathematical equations and the models governing the motion of the atmosphere and the oceans, and advancing the techniques for their numerical simulations. The general equations describing these motions are derived from the basic conservation laws. The resulting equations are very complex and unfortunately too complicated to be analyzed but using some scale analysis methods and meteorological observations, these equations are well approximated by a somehow simplier system called the Primitive Equations.

In this article we are interested in deriving various results regarding the so-called δ-Primitive Equations and a Boussinesq type equation. By δ-Primitive Equations we understand the equations governing the movement of the geophysical fluids (atmosphere, oceans) as follows; we consider the laws of conservation of horizontal momentum with some minor geometrical approximations and we add a dissipation term in the hydrostatic equation (namely the term δw in (1.1c) as in [13] and [16]). The Boussinesq type equations considered in this article are the same as the δ-Primitive Equations, the only difference being the hydrostatic equation, where the dissipative term δw is substituted by the time derivative of the vertical velocity multiplied by $\delta > 0$, $\delta\partial w/\partial t$, see e.g. [8].

We prove in this article that the equations obtained in this manner lead to the well-posedness of the problem, meaning that in a certain class of functions and in limited time, the equations have solutions which are unique and depend continuously in the initial data (for a similar result in the context of the Euler equation, see e.g. Kato [5], [4] or Temam [15]).

The δ-Primitive Equations for the ocean read[1]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = F_v, \tag{1.1b}$$

$$\delta w + \frac{\partial p}{\partial z} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = F_T. \tag{1.1e}$$

In the system above, $(u,\, v,\, w)$ are the three components of the velocity vector, p, ρ and T are respectively the perturbations of the pressure, of the density and of the temperature from a reference (average) state p_0, ρ_0 and T_0. The relation between the temperature and the density is given by the equation of state and we consider here a version of this equation

[1]Some slight modifications are necessary for the atmosphere, we refer the interested reader to Salmon [13], where it is shown that working with the potential temperature instead of the temperature and changing the coordinates, the equations for the lower atmosphere will have the same form as the equations (1.1) for the ocean.

linearized around the reference state ρ_0, T_0:

$$\rho = \rho_0(-\beta_T(T - T_0)). \tag{1.2}$$

In the system (1.1), f is the Coriolis parameter, (F_u, F_v) represent the body forces per unit of mass and F_T represents a heating source. In applications, F_u, F_v and F_T vanish for the ocean (but we consider here the nonzero forcing for mathematical generality).

The Boussinesq type equations are the same as (1.1), with the only difference that the hydrostatic equation (1.1c) is substituted by:

$$\delta \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -\rho g, \quad \delta > 0. \tag{1.3}$$

We recall here that the usual Primitive Equations correspond to $\delta = 0$ in (1.1c). For $\delta > 0$, the term δw is a friction (vertical viscosity) term. The term δw, called a drag term, is on one hand a mathematical remedy to ensure the well-posedness of the problem. On the other hand, as shown in [16], it has a smoothing effect, numerically filtering some undesirable oscillations. An interesting problem raised by these equations is the asymptotic study when $\delta \to 0$ (see e.g. [12], [11]). For more details regarding the motivations of this term see Temam and Tribbia [16] and also Salmon [13]. A coherent model can be also obtained substituting the term δw with $\delta\, \partial w/\partial t$; as we announced we also intend to study this model from the mathematical point of view.

The article is organized as follows: in Section 2 we recall the δ-Primitive Equations, then we prove the existence of solutions in some class of functions, and we finally derive some a priori estimates, showing the continuous dependence on the data and the uniqueness of the regular solutions and a regularity in time. In Section 3 we consider the Boussinesq type of equations and we prove the existence and uniqueness of regular solutions and also the regularity in time of the solutions.

For the interested reader, we mention that much work is available on the mathematical theory of the Primitive Equations in different contexts: the well-posedness of the Primitive Equations in the presence of viscosity has been established by Lions, Temam and Wang (see [6], [7]) for both the ocean and the atmosphere; improved results based on an anisotropic treatment of the vertical direction can be found in Ziane [19] and in Petcu, Temam and Wirosoetisno see [10]. The same problem, of the well-posedness of the Primitive Equations, has been considered in a thin domain by Hu, Temam and Ziane [3]. High regularity results for the Primitive Equations in 2D periodic space were derived in [10]. A review of numerous results available in the mathematical theory of geophysical fluid dynamics (as far as existence, uniqueness and regularity of solutions are concerned) can be found in [17].

For details regarding the derivation of these models (PEs, δ-PEs, Boussinesq type model) from the physical laws, we refer the reader to classical references in Geophysical Fluid Dynamics, e.g. Haltiner and Williams [2], Gill [1], Pedlosky [9], Washington and

Parkinson [18], and the references therein, as well as the references already quoted of Salmon, and Temam and Tribbia.

2. δ-Primitive Equations

In this section we consider the δ-Primitive Equations as described in the Introduction and we prove that the problem is well-posed in the sense of Hadamard in a certain type of spaces.

2.1. **The main result: existence and uniqueness of solutions.** In this article we work in a limited domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ and we assume space periodicity with period Ω, meaning that all functions are taken to satisfy:

$$f(x_1 + L_1, x_2, x_3, t) = f(x_1, x_2, x_3, t) = f(x_1, x_2 + L_2, x_3, t) = f(x_1, x_2, x_3 + L_3, t),$$

when extended to $\mathbb{R}^3$. All the functions being periodic, they admit Fourier series expansions, hence we can write:

$$f = \sum_{k \in \mathbb{Z}^3} f_k e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)}, \tag{2.1}$$

where $k_j' = 2\pi k_j / L_j$ for $j = 1, 2, 3$.

Our aim is to study the existence and regularity of the solutions of problem (1.1) with some initial data. In system (1.1), the prognostic variables are u, v and T, whereas ρ, w and p are the diagnostic variables. Indeed, the density is already expressed in terms of the temperature T by the state equation (1.2), hence taking into account that in (1.1) ρ and T are respectively the perturbations of the density and of the temperature from an average value, we have:

$$\rho = -\rho_0 \beta_T T. \tag{2.2}$$

In order to determine the vertical velocity in terms of the prognostic variables, we write the equations (1.1c) and (1.1d) in Fourier modes and we obtain:

$$\delta w_k + i k_3' p_k = -g \rho_k, \tag{2.3}$$

and

$$k_1' u_k + k_2' v_k + k_3' w_k = 0. \tag{2.4}$$

From equation (2.3) we find:

$$w_k = -\frac{g \rho_k}{\delta}, \quad \text{for} \quad k_3' = 0, \tag{2.5}$$

and from equation (2.4) we find:

$$w_k = -\frac{k_1' u_k + k_2' v_k}{k_3'}, \quad \text{for} \quad k_3' \neq 0. \tag{2.6}$$

So, for each $U = (u,\, v,\, T)$ we can define $w = w(U)$ by its Fourier series, namely:

$$w(U)_k = \begin{cases} -\dfrac{k_1' u_k + k_2' v_k}{k_3'}, & \text{for } k_3' \neq 0, \\ \dfrac{g}{\delta}\rho_0 \beta_T T_k, & \text{for } k_3 = 0. \end{cases} \tag{2.7}$$

From (2.3) we then determine the pressure p in terms of the diagnostic variables, up to its vertical average. This means that we can fully determine the Fourier coefficients p_k of the pressure p for $k_3 \neq 0$ but not for $k_3 = 0$. The part of the pressure which can not be expressed in terms of the prognostic variables is the average of the pressure in the vertical direction:

$$\frac{1}{L_3}\int_0^{L_3} p(x_1,\, x_2,\, x_3)\, \mathrm{d}x_3 = \sum_{k,\, k_3=0} p_k(t) e^{i(k_1' x_1 + k_2' x_2)}. \tag{2.8}$$

Some natural function spaces for this problem are as follows:

$$\mathbf{V} = \{(u,\, v,\, T) \in (\dot{H}^1_{\mathrm{per}}(\Omega))^3;\, \int_0^{L_3} (u_x + v_y)\, \mathrm{d}z = 0\}, \tag{2.9}$$

and

$$\mathbf{H} = \text{ the closure of } \mathbf{V} \text{ in } (\dot{L}^2(\Omega))^3. \tag{2.10}$$

Here the dot above $\dot{H}^1_{\mathrm{per}}$ and $\dot{L}^2$ denotes the functions with average in Ω equal to zero. These spaces are endowed with the usual scalar products, meaning that on $\mathbf{H}$ we take the scalar product from $L^2(\Omega)$ and on $\mathbf{V}$ we work with the following scalar product:

$$((\phi,\, \tilde{\phi}))_{\mathbf{V}} = \int_\Omega \Big(\frac{\partial \phi}{\partial x}\frac{\partial \tilde{\phi}}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \tilde{\phi}}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \tilde{\phi}}{\partial z} \Big)\, \mathrm{d}\Omega. \tag{2.11}$$

Note that because of the assumption that all the functions have zero average, the Poincaré inequality holds, meaning:

$$|U|_{L^2} \leq c_0 \|U\|, \quad \forall\, U \in \mathbf{V}, \tag{2.12}$$

which indeed guarantees that $\|\cdot\|$ is a norm on $\mathbf{V}$ equivalent to the usual norm on H^1.

In order to obtain the variational formulation of this problem, we consider a test function $\tilde{U} = (\tilde{u},\, \tilde{v},\, \tilde{T}) \in \mathbf{V}$, multiply (1.1a) by $\tilde{u}$, (1.1b) by $\tilde{v}$, and (1.1e) by $\tilde{T}$, and integrate over Ω. Using the integration by parts and the space periodicity we find that system (1.1) is formally equivalent to the following problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U, \tilde{U})_{L^2} + b(U,\, U,\, \tilde{U}) + e(U,\, \tilde{U}) = (F, \tilde{U})_{L^2}, \quad \forall\, \tilde{U} \in \mathbf{V}, \\ &U(0) = U_0. \end{aligned} \tag{2.13}$$

In (2.13) we have defined the bilinear form e as being:

$$e(U,\tilde{U}) = f\int_\Omega u\tilde{v}\,\mathrm{d}\Omega - f\int_\Omega v\tilde{u}\,\mathrm{d}\Omega - \beta_T g\int_\Omega T\tilde{w}\,\mathrm{d}\Omega + \frac{\delta}{\rho_0}\int_\Omega w\tilde{w}\,\mathrm{d}\Omega, \tag{2.14}$$

and the trilinear form b as:

$$\begin{aligned} b(U,U^\sharp,\tilde{U}) = &\int_\Omega (u\frac{\partial u^\sharp}{\partial x}\tilde{u} + v\frac{\partial u^\sharp}{\partial y}\tilde{u} + w(U)\frac{\partial u^\sharp}{\partial z}\tilde{u})\,\mathrm{d}\Omega \\ &+ \int_\Omega (u\frac{\partial v^\sharp}{\partial x}\tilde{v} + v\frac{\partial v^\sharp}{\partial y}\tilde{v} + w(U)\frac{\partial v^\sharp}{\partial z}\tilde{v})\,\mathrm{d}\Omega \\ &+ \int_\Omega (u\frac{\partial T^\sharp}{\partial x}\tilde{T} + v\frac{\partial T^\sharp}{\partial y}\tilde{T} + w(U)\frac{\partial T^\sharp}{\partial z}\tilde{T})\,\mathrm{d}\Omega. \end{aligned} \tag{2.15}$$

We also introduce the following notation: we denote by $(f,g)_m$ and $|f|_m$ the scalar product and the norm in $\dot{H}^m_{\mathrm{per}}(\Omega)$,

$$(f,g)_m = \sum_{|\alpha|=m} (D^\alpha f, D^\alpha g)_{L^2}, \tag{2.16}$$

where D^α is a multi-index derivation; $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}$, $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

In all that follows in this article, we are interested in proving the existence of solutions for this problem on a certain interval of time, the uniqueness and the continuous dependence on the data for a certain class of solutions. The main result is the existence and uniqueness theorem, stated here below:

Theorem 2.1. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$ as above. Then for each U_0 given in $\mathbf{V}\cap(\dot{H}^m_{\mathrm{per}}(\Omega))^3$ and F given in $L^\infty(0,t_1;\, (\dot{H}^m_{\mathrm{per}}(\Omega))^3)$, there exists a $t_\star \leq t_1$, depending on the data $(L_1,\, L_2,\, L_3,\, U_0,\, F)$ but independent of m, and a unique solution U of problem (2.13) defined on the interval $(0,t_\star)$, with*

$$U \in L^\infty(0,t_\star;\mathbf{V}\cap(\dot{H}^m_{\mathrm{per}}(\Omega))^3).$$

Proof. The proof of the existence of solutions is based on the Galerkin-Fourier method, using the a priori estimates obtained in the subsection below (for more details see e.g., in the context of the Euler equations, [4], [14], [15]). The uniqueness of solution will be proved in the next subsection. □

Remark 2.1. The same result of existence can be obtained in any dimension d, the proof is identical; because of the dimension of the space in the Sobolov imbedding theorems, we then require $m > 1 + d/2$.

2.2. **Existence of regular solutions.** In this section we are interested in obtaining some estimates on the high order derivatives, from which we will then derive the existence of solutions for the δ-Primitive Equations in $(\dot{H}^m_{\mathrm{per}}(\Omega))^3$, for m specified later on, and sufficiently large.

In all that follows, we assume that $m > 5/2$ so that $H^{m-1}(\Omega)$ is a multiplicative algebra.

We start by deriving the a priori estimates necessary to prove the existence results. Let α be a multi-index, $|\alpha| = m$. We apply the operator D^α to equations (1.1a), (1.1b) and (1.1e), then multiply the equations respectively by $D^\alpha u$, $D^\alpha v$ and $D^\alpha T$, integrate over Ω and add all these equations for $|\alpha| = m$. In this way we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_m+&(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w(U)\frac{\partial u}{\partial z},u)_m+(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w(U)\frac{\partial v}{\partial z},v)_m\\&+(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w(U)\frac{\partial T}{\partial z},T)_m+\frac{1}{\rho_0}(\frac{\partial p}{\partial x},u)_m+\frac{1}{\rho_0}(\frac{\partial p}{\partial y},v)_m\\&=(F,U)_m.\end{aligned}\tag{2.17}$$

Using periodicity and integrating by parts, we obtain:

$$\frac{1}{\rho_0}(\frac{\partial p}{\partial x},u)_m+\frac{1}{\rho_0}(\frac{\partial p}{\partial y},v)_m=-\frac{1}{\rho_0}(p,u_x+v_y)_m.\tag{2.18}$$

Using (1.1c), (1.1d) and integrating by parts we find:

$$-\frac{1}{\rho_0}(p,u_x+v_y)_m=-\frac{1}{\rho_0}(p_z,w)_m=\frac{\delta}{\rho_0}|w|^2_m-\beta_T\frac{g}{\rho_0}(T,w)_m.\tag{2.19}$$

It now remains to estimate the nonlinear terms:

$$\begin{aligned}I_1&=(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w(U)\frac{\partial u}{\partial z},u)_m,\\I_2&=(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w(U)\frac{\partial v}{\partial z},v)_m,\\I_3&=(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w(U)\frac{\partial T}{\partial z},T)_m.\end{aligned}$$

Since the three terms I_1, I_2 and I_3 have a similar structure, it suffices to estimate I_1 which we write in the form:

$$I_1=\sum_{|\alpha|=m}(D^\alpha\psi,D^\alpha u)_{L^2},\tag{2.20}$$

where $\psi = u\,\partial u/\partial x + v\,\partial u/\partial y + w(U)\,\partial u/\partial z$.

Using the Leibnitz rule we find:

$$
\begin{aligned}
D^\alpha \psi =& u\frac{\partial D^\alpha u}{\partial x} + v\frac{\partial D^\alpha u}{\partial y} + w(U)\frac{\partial D^\alpha u}{\partial z} \\
&+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta}(D^\beta u\frac{\partial D^{\alpha-\beta} u}{\partial x} + D^\beta v\frac{\partial D^{\alpha-\beta} u}{\partial y} + D^\beta w(U)\frac{\partial D^{\alpha-\beta} u}{\partial z}),
\end{aligned}
\tag{2.21}
$$

where $c_{\alpha,\beta}$ are some suitable coefficients.

The contribution in (2.20) of the first three terms from (2.21) is zero, because of the conservation of mass law (1.1d). In this way, I_1 becomes:

$$
I_1 = \sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} c_{\alpha,\beta}(D^\beta u\frac{\partial D^{\alpha-\beta} u}{\partial x} + D^\beta v\frac{\partial D^{\alpha-\beta} u}{\partial y} + D^\beta w(U)\frac{\partial D^{\alpha-\beta} u}{\partial z},\, D^\alpha u)_{L^2}. \tag{2.22}
$$

We bound I_1 as follows, c denoting an absolute constant which may be different at different places:

$$
\begin{aligned}
|I_1| \leq& c \sum_{\substack{|\alpha|=m\\0<\beta\leq\alpha}} \big[\big|D^\beta u\frac{\partial D^{\alpha-\beta} u}{\partial x}\big|_{L^2} + \big|D^\beta v\frac{\partial D^{\alpha-\beta} u}{\partial y}\big|_{L^2} \\
&+ \big|D^\beta w(U)\frac{\partial D^{\alpha-\beta} u}{\partial z}\big|_{L^2}\big]|D^\alpha u|_{L^2}.
\end{aligned}
\tag{2.23}
$$

We know that:

$$
|D^\alpha u|_{L^2} \leq |u|_m \leq |U|_m, \quad \forall\, \alpha \,\text{with}\, |\alpha| = m. \tag{2.24}
$$

The problem now reduces to finding a good way to estimate the terms:

$$
\big|D^\beta u\frac{\partial D^{\alpha-\beta} u}{\partial x}\big|_{L^2(\Omega)}, \tag{2.25}
$$

and

$$
\big|D^\beta w(U)\frac{\partial D^{\alpha-\beta} u}{\partial z}\big|_{L^2(\Omega)}, \tag{2.26}
$$

for α with $|\alpha| = m$ and $0 < \beta \leq \alpha$.

In order to bound these terms, we use the following inequalities:

$$
|\xi\eta|_{L^2(\Omega)} \leq c_1'|\xi|_{H^2(\Omega)}|\eta|_{L^2(\Omega)}, \tag{2.27}
$$

and

$$
|\xi\eta|_{L^2(\Omega)} \leq c_2'|\xi|_{H^1(\Omega)}|\eta|_{H^1(\Omega)}, \tag{2.28}
$$

where c_1' and c_2' are constants depending only on Ω.

We obtain:

$$\sum_{\substack{|\alpha|=m\\0<\beta\le\alpha}} \big|D^\beta u \frac{\partial D^{\alpha-\beta}u}{\partial x}\big|_{L^2} \le c_1(|U|_m|U|_3 + |U|^2_{m-1}). \tag{2.29}$$

For the sum from I_1 containing the terms of the form (2.26), we obtain:

$$\begin{aligned}\sum_{\substack{|\alpha|=m\\0<\beta\le\alpha}} \big|D^\beta w(U) \frac{\partial D^{\alpha-\beta}u}{\partial z}\big|_{L^2} \le & c_2(|w(U)|_m|U|_3 \\ & + |w(U)|_{m-1}|U|_{m-1} + |w(U)|_3|U|_m).\end{aligned} \tag{2.30}$$

Taking into account these estimates and using Young's inequality, we find the following energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_m + \frac{\delta}{\rho_0}|w(U)|^2_m \le \eta(t)|U|^2_m + \xi(t)|U|_m, \tag{2.31}$$

where

$$\eta(t) = c_1 + c_2|U|^2_3 + |w(U)|_3,$$

and

$$\xi(t) = 2|F|_m + c_3(|U|^2_{m-1} + |w(U)|_{m-1}|U|_{m-1}).$$

For $m = 3$ the differential inequality writes as:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_3 + \frac{\delta}{\rho_0}|w(U)|^2_3 \le (c_1 + c_2|U|^2_3)|U|^2_3 + 2|U|_3|F|_3. \tag{2.32}$$

We find that there exists a t_1 depending only on the initial data, such that:

$$|U(t)|_3 \le 1 + 2|U_0|_3,\ \forall\, 0 \le t \le t_1, \tag{2.33}$$

which leads us to:

$$U \in L^\infty(0, t_1; (\dot{H}^3_{\mathrm{per}}(\Omega))^3),\ w(U) \in L^2(0, t_1; \dot{H}^3_{\mathrm{per}}(\Omega)). \tag{2.34}$$

Recursively we find that, for $m \ge 3$, $|U(t)|_m$ remains bounded on $(0, t_1)$, where t_1 is exactly the time determined for $m = 3$.

Gathering all these estimates and using classical methods (the Galerkin-Fourier method), we obtain the existence of the solutions as enounced above. The Galerkin-Fourier method consists in constructing approximate solutions by the Galerkin approximation. The approximate solutions are the solutions of a finite-dimensional equation with bilinear nonlinearity. The a priori estimates work for each approximate solution. Since the bound is independent of the solution, we can pass to the limit finding the solution of the problem.

2.3. **Uniqueness and continuous dependence on the data.** In this section we prove the continuous dependence on data of the solutions. Hence, with the existence result proved above, we obtain that problem (2.13) is well posed in the sense of Hadamard, in suitable spaces.

Let us consider two solutions for the problem (2.13), namely $U' = (u',\, v',\, T')$ and $U'' = (u'',\, v'',\, T'')$, which respectively correspond to the initial data $U_0' = (u_0',\, v_0',\, T_0')$ and $U_0'' = (u_0'',\, v_0'',\, T_0'')$ and to the forcing $F' = (F_u',\, F_v',\, F_T')$ and $F'' = (F_u'',\, F_v'',\, F_T'')$. We set:

$$\begin{aligned} &u = u' - u'', && v = v' - v'',\\ &w(U) = w'(U') - w''(U''), && T = T' - T'',\\ &u_0 = u_0' - u_0'', && v_0 = v_0' - v_0'',\\ &T_0 = T_0' - T_0'', && F = F' - F''. \end{aligned}$$

Then, $U = (u,\, v,\, T)$ obeys the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde U)_{L^2} + b(U',U,\tilde U) + b(U,U'',\tilde U) + e(U,\tilde U) = (F,\tilde U)_{L^2}, \quad \forall \tilde U \in \mathbf{V}, \tag{2.35a}$$

$$U(0) = U_0. \tag{2.35b}$$

We set in all that follows $\mathbf{v} = (u,\, v)$.

In equation (2.35) we take formally $\tilde U = U(t)$ for t fixed but arbitrary. We notice that:

$$\int_\Omega [(\mathbf{v}'\cdot\nabla)u\,u + w'\frac{\partial u}{\partial z}u]\,\mathrm{d}\Omega = 0, \tag{2.36}$$

because $(\mathbf{v}',\, w')$ obeys the conservation of mass equation, namely (1.1d) (the same argument is used for the terms similar with (2.36)).

We obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[|u|_{L^2}^2 + |v|_{L^2}^2 + |T|_{L^2}^2] + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial x}u\,\mathrm{d}\Omega + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial y}v\,\mathrm{d}\Omega = (F,U)_{L^2} + \eta_1 + \eta_2 + \eta_3, \tag{2.37}$$

where:

$$\begin{aligned} \eta_1 &= \int_\Omega [(\mathbf{v}\cdot\nabla)u'' + w\frac{\partial u''}{\partial z}]u\,\mathrm{d}\Omega,\\ \eta_2 &= \int_\Omega [(\mathbf{v}\cdot\nabla)v'' + w\frac{\partial v''}{\partial z}]v\,\mathrm{d}\Omega,\\ \eta_3 &= \int_\Omega [(\mathbf{v}\cdot\nabla)T'' + w\frac{\partial T''}{\partial z}]T\,\mathrm{d}\Omega. \end{aligned}$$

Using integration by parts we notice that:

$$\begin{aligned}\frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial x} u \,\mathrm{d}\Omega + \frac{1}{\rho_0}\int_\Omega \frac{\partial p}{\partial y} v \,\mathrm{d}\Omega &= \frac{1}{\rho_0}\int_\Omega p w_z \,\mathrm{d}\Omega = -\frac{1}{\rho_0}\int_\Omega p w_z \,\mathrm{d}\Omega \\ &= \frac{\delta}{\rho_0}|w|_{L^2}^2 + \frac{g}{\rho_0}\int_\Omega \rho w \,\mathrm{d}\Omega.\end{aligned} \tag{2.38}$$

We need to estimate η_1, η_2 and η_3. Supposing that the solutions are functions having their gradient in L^∞, i.e. their first order spatial derivatives are bounded, we find:

$$\begin{aligned}|\eta_j| &\leq c_1' |\nabla U''|_{L^\infty} |U|_{L^2}^2 + c_2' |w(U)|_{L^2} |U|_{L^2} \Big|\frac{\partial U''}{\partial z}\Big|_{L^\infty} \\ &\leq k_1 |U|_{L^2}^2 + k_2, \text{ for } j = 1, 2, 3,\end{aligned} \tag{2.39}$$

where k_1 and k_2 are constants depending on $U'' = (u'', v'', T'')$ through the L^∞ norm of its spatial gradient.

Going back to (2.37), we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|_{L^2}^2 + \frac{\delta}{\rho_0}|w(U)|_{L^2}^2 \leq &\Big|\frac{g}{\rho_0}\int_\Omega \rho w(U) \,\mathrm{d}\Omega\Big| + |F|_{L^2}|U|_{L^2} + k_1|U|_{L^2}^2 \\ &+ k_2 |w(U)|_{L^2}|U|_{L^2}.\end{aligned} \tag{2.40}$$

Using Young's inequality, (2.40) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_{L^2}^2 + \frac{\delta}{\rho_0}|w(U)|_{L^2}^2 \leq k_1 |U|_{L^2}^2 + |F|_{L^2}^2. \tag{2.41}$$

By the Gronwall lemma, we find:

$$|U(t)|_{L^2}^2 \leq |U_0|_{L^2}^2 e^{k_1 t} + e^t \int_0^t |F(s)|_{L^2}^2 \mathrm{d}s. \tag{2.42}$$

From the estimate (2.42) we deduce immediately that the solutions having their first derivatives uniformly bounded, depend continuously on the data in the root-mean-sense.

Uniqueness of the solutions belonging to the class mentioned above can be deduced from (2.42), taking the same initial data $U_0' = U_0''$ and the same forcing $F' = F''$, which leads to $U_0 = 0$ and $F = 0$, so

$$|U(t)|_{L^2}^2 \leq 0, \quad \forall\, t > 0.$$

We now define the following function spaces:

$$\begin{aligned}&Y = \{U \in L^\infty(0, t_\star; (\dot{H}^1_{\text{per}}(\Omega))^3); D_j U \in (L^\infty(\Omega \times (0, t_1)))^3,\ j = 1,\ 2,\ 3\},\\ &X = [(\dot{H}^m_{\text{per}}(\Omega))^3 \cap \mathbf{V}] \times [L^\infty(0, t_1; (\dot{H}^m_{\text{per}}(\Omega))^3)],\end{aligned}$$

where $D_j = \partial/\partial x_j$, $m \geq 3$, $t_1 > 0$ arbitrarily chosen and $t_\star$ defined in Theorem 2.1. Both spaces are equipped with their natural norms which make them Banach spaces. For Theorem 2.2 we also consider these spaces equipped with the L^2 norm for X and with the norm $|U| = |\nabla U|_{L^\infty(\Omega\times(0,t_1))}$ and then call $\tilde{X}$ and $\tilde{Y}$ these spaces (which are normed, non-complete spaces).

We conclude this section by the following theorem:

Theorem 2.2. *For $(U_0, F) \in X$ and $m \geq 3$, the system (2.13) has a unique solution U in $L^\infty(0, t_\star; (\dot{H}^m_{\text{per}}(\Omega))^3 \cap \mathbf{V}) \subset Y$. Furthermore, the mapping $(U_0, F) \to U$ is bounded from X into Y and continuous from the bounded sets of X into Y for the norms of $\tilde{X}$ and $\tilde{Y}$.*

2.4. **Time regularity.** In this subsection we are interested in deriving some regularity results in time. We first derive the necessary a priori estimates. We differentiate the equations (1.1) l times in t and then take the scalar product in H^m of the equations resulting from (1.1a), (1.1b) and (1.1e) respectively with $u^{(l)}$, $v^{(l)}$ and $T^{(l)}$. Here and in all that follows we set $u^{(l)} = \partial^l u/\partial t^l$. We find:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U^{(l)}|^2_m &+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m \\
&+ \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m \\
&= (F^{(l)}, U^{(l)})_m.
\end{aligned} \tag{2.43}$$

Using periodicity, integrating by parts and taking into account (1.1c), (1.1d), we obtain:

$$\begin{aligned}
\frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m &= -\frac{1}{\rho_0}(p^{(l)}, u_x^{(l)} + v_y^{(l)})_m \\
&= \frac{\delta}{\rho_0}|w(U)^{(l)}|^2_m - \beta_T \frac{g}{\rho_0}(T^{(l)}, w(U)^{(l)})_m.
\end{aligned} \tag{2.44}$$

We now need to estimate the terms:

$$\begin{aligned}
J_1 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\
J_2 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\
J_3 &= \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m,
\end{aligned} \tag{2.45}$$

where the C_l^k are the binomial coefficients.

Since the terms J_1, J_2 and J_3 are similar, we concentrate our attention only on J_1. We notice that:

$$J_1 = \sum_{|\alpha|=m} (D^\alpha \eta, D^\alpha u^{(l)})_{L^2}, \tag{2.46}$$

where

$$\eta = \sum_{k=0}^{l} C_l^k \big(u^{(k)} \frac{\partial u^{(l-k)}}{\partial x} + v^{(k)} \frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)} \frac{\partial u^{(l-k)}}{\partial z} \big). \tag{2.47}$$

Computing $D^\alpha \eta$ we find:

$$\begin{aligned} D^\alpha \eta =& \sum_{k=0}^{l} C_l^k \big(u^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial x} + v^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial y} + w(U)^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial z} \big) \\ &+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} \sum_{k=0}^{l} C_l^k \big(D^\beta u^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial x} + D^\beta v^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial y} \\ &\quad + D^\beta w(U)^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial z} \big). \end{aligned} \tag{2.48}$$

For $k = 0$, the corresponding terms from the first sum will have the scalar product with $D^\alpha u^{(l)}$ equal to zero, because of the conservation of mass law (1.1d). Taking into account this simplification, J_1 writes as:

$$\begin{aligned} J_1 =& \sum_{k=0}^{l} C_l^k \big(u^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial x} + v^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial y} + w(U)^{(k)} \frac{\partial D^\alpha u^{(l-k)}}{\partial z},\ D^\alpha u^{(l)} \big)_{L^2} \\ &+ \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} \sum_{k=0}^{l} C_l^k \big(D^\beta u^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial x} + D^\beta v^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial y} \\ &\quad + D^\beta w(U)^{(k)} \frac{\partial D^{\alpha-\beta} u^{(l-k)}}{\partial z},\ D^\alpha u^{(l)} \big)_{L^2}. \end{aligned} \tag{2.49}$$

Using the inequalities (2.27) and (2.28), we estimate J_1 as follows:

$$\begin{aligned} |J_1| \leq& c_1 |U|_{m+1} |U^{(l)}|_m^2 + c_2 |w(U)^{(l)}|_m |U|_{m+1} |U^{(l)}|_m \\ &+ c_3 \sum_{k=1}^{l-1} |U^{(l-k)}|_{m+1} (|U^{(k)}|_m + |w(U)^{(k)}|_m) |U^{(l)}|_m. \end{aligned} \tag{2.50}$$

Gathering the estimates obtained above and using also Young's inequality, we arrive at the following differential inequality:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|U^{(l)}|_m^2 + \frac{\delta}{\rho_0}|w(U)^{(l)}|_m^2 \leq (c_1 + c_2|U|_{m+1}^2)|U^{(l)}|_m^2 + |F^{(l)}|_m|U^{(l)}|_m \\ + c_3|U^{(l)}|_m \sum_{k=1}^{l-1}|U^{(l-k)}|_{m+1}(|U^{(k)}|_m + |w(U)^{(k)}|_m).\end{aligned} \tag{2.51}$$

By classical methods (meaning we use the Fourier-Galerkin method, constructing approximating solutions for which all the a priori estimates deduced above hold, and then passing to the limit) and using an induction argument, we find that exists a time $t_\star$ depending only on the initial data, such that:

$$\frac{\partial^l U}{\partial t^l} \in L^\infty(0, t_\star, H^m_{\text{per}}(\Omega)^3), \tag{2.52}$$

for all $l \geq 1$ and all $m \geq 3$. This way we prove the following result:

Theorem 2.3. *Let there be given $m \geq 3$, $l \geq 1$ and $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ as above. Then for each U_0 given in $\mathbf{V} \cap (\dot{H}^m_{\text{per}}(\Omega))^3$ and F given such that $F^{(l)}$ is in $L^\infty(0, t_1, \dot{H}^m_{\text{per}}(\Omega)^3)$, there exists a time $t_\star$ depending on the initial data and not on m nor on l, and a unique solution U of problem (2.13) defined on the interval $(0, t_\star)$, with*

$$U^{(l)} \in L^\infty(0, t_\star, \dot{H}^m_{\text{per}}(\Omega)^3).$$

We conclude this section by the following theorem:

Theorem 2.4. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ as above. Then for each U_0 given in $\mathbf{V} \cap (\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^3$ and F given in $\mathcal{C}^\infty(0, t_1; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^3$, there exists a unique solution U of problem (2.13) defined on the interval $(0, t_\star)$, with*

$$U \in \mathcal{C}^\infty(0, t_\star; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^3),$$

where $t_\star = \min(t_1, t_2)$, t_2 given by (2.33).

Proof. To prove Theorem 2.4 we apply Theorem 2.3 for each $m \geq 3$, remembering that:

$$\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}) = \cap_{m \geq 3} H^m_{\text{per}}(\Omega);$$

of importance here is the fact that $t_\star$ in Theorem 2.3 is independent of m. We denote by $\mathcal{C}^\infty_{\text{per}}(\bar{\Omega})$ the set of functions in $\mathcal{C}^\infty(\bar{\Omega})$ whose periodic extension beyond Ω is $\mathcal{C}^\infty$ on $\mathbb{R}^3$ (smooth matching at the boundary of Ω). $\square$

3. A Boussinesq type equation

As we announced in the Introduction of this article, we are also interested in considering a Boussinesq type of equations, given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = F_u, \tag{3.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = F_v, \tag{3.1b}$$

$$\delta\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -\rho g, \tag{3.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = F_T, \tag{3.1e}$$

where $\delta > 0$ is given.

3.1. **Existence and uniqueness of regular solutions.** Our aim is to study the existence and regularity of solutions of problem (3.1) on a periodic domain with suitable initial data.

The natural function spaces for this problem are:

$$\tilde{\mathbf{V}} = \{(u,\, v,\, w,\, T) \in (\dot{H}^1_{\text{per}}(\Omega))^4;\ u_x + v_y + w_z = 0\}, \tag{3.2}$$

and

$$\tilde{\mathbf{H}} = \text{ the closure of } \tilde{\mathbf{V}} \text{ in } (\dot{L}^2(\Omega))^4. \tag{3.3}$$

As before, the dot above the spaces H^1_{per} and L^2 denotes the functions with average zero; for these functions the Poincaré inequality holds. The spaces are endowed with the usual scalar products.

We first derive the variational formulation of the problem. We consider a test function $\tilde{U} = (u,\, v,\, w,\, T) \in \tilde{\mathbf{V}}$, multiply (3.1a) by $\tilde{u}$, (3.1b) by $\tilde{v}$, (3.1c) by $\tilde{w}$ and (3.1e) by $\tilde{T}$ and integrate over Ω. Using the integration by parts and the space periodicity, we find that system (3.1) is formally equivalent to the variational problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{L^2} + b(U,U,\tilde{U}) + e(U,\tilde{U}) = (F,(\tilde{u},\tilde{v},\tilde{T}))_{L^2}, \quad \tilde{U} \in \tilde{\mathbf{V}}, \\ &U(0) = U_0. \end{aligned} \tag{3.4}$$

In relation (3.4) we defined the bilinear form e as:

$$e(U,\tilde{U}) = f\int_\Omega u\tilde{v}\,\mathrm{d}\Omega - f\int_\Omega v\tilde{u}\,\mathrm{d}\Omega - g\beta_T\int_\Omega T\tilde{w}\,\mathrm{d}\Omega, \tag{3.5}$$

and the trilinear form b as:

$$
\begin{aligned}
b(U, U^\sharp, \tilde{U}) = &\int_\Omega (u\frac{\partial u^\sharp}{\partial x}\tilde{u} + v\frac{\partial u^\sharp}{\partial y}\tilde{u} + w(U)\frac{\partial u^\sharp}{\partial z}\tilde{u})\,\mathrm{d}\Omega \\
&+ \int_\Omega (u\frac{\partial v^\sharp}{\partial x}\tilde{v} + v\frac{\partial v^\sharp}{\partial y}\tilde{v} + w(U)\frac{\partial v^\sharp}{\partial z}\tilde{v})\,\mathrm{d}\Omega \\
&+ \int_\Omega (u\frac{\partial T^\sharp}{\partial x}\tilde{T} + v\frac{\partial T^\sharp}{\partial y}\tilde{T} + w(U)\frac{\partial T^\sharp}{\partial z}\tilde{T})\,\mathrm{d}\Omega;
\end{aligned} \tag{3.6}
$$

by F we understand $F = (F_u,\, F_v,\, F_T)$.

In this section we prove the existence and uniqueness of solutions locally in time for this problem. We state here the main result for these equations:

Theorem 3.1. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1)\times(0, L_2)\times(0, L_3)$ as above. Then for each U_0 given in $\tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4$ and F given in $L^\infty(0, t_1; (\dot{H}^m_{\mathrm{per}}(\Omega))^3)$, there exists a $0 < t_\star \leq t_1$, independent of m, and a unique solution U of problem (2.13) defined on the interval $(0, t_\star)$, with*

$$
U \in L^\infty(0, t_\star; \tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4).
$$

Proof. The proof of the existence of solutions is based on the Galerkin-Fourier method. The method is based on the priori estimates obtained below.

The estimates we deduce here are on the high order derivatives; these estimates lead us to conclude the existence of solutions for the Boussinesq type equations in $(\dot{H}^m_{\mathrm{per}}(\Omega))^4$, with $m > 5/2$.

Let α be a multi-index, $|\alpha| = m$. We take the operator D^α and apply it to equations (3.1a), (3.1b), (3.1d) and (3.1e), then we multiply these equations respectively by $D^\alpha u$, $D^\alpha v$, $D^\alpha w/\rho_0$ and $D^\alpha T$, integrate over Ω and add all these equations for $|\alpha| = m$.

The terms containing the Coriolis parameter obviously disappear. Integrating by parts and using the periodicity and the conservation of mass law we also have:

$$
(p_x, u)_m + (p_y, v)_m + (p_z, w)_m = -(p, u_x + v_y + w_z)_m = 0. \tag{3.7}
$$

We then have:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{|u|_m^2 + |v|_m^2 + \frac{\delta}{\rho_0}|w|_m^2 + |T|_m^2\} - g\beta_T(T, w)_m \\
&\quad + ((([(u,v,w)\cdot \mathrm{grad}]u, u))_m + ((([(u,v,w)\cdot \mathrm{grad}]v, v))_m \\
&\quad + ((([(u,v,w)\cdot \mathrm{grad}]T, T))_m = (F, (u,v,T))_m.
\end{aligned} \tag{3.8}
$$

We easily estimate:

$$
|g\beta_T(T, w)_m| \leq g\beta_T |T|_m |w|_m, \tag{3.9}
$$

and

$$
|(F, (u,v,T))_m| \leq |F|_m(|u|_m + |v|_m + |T|_m). \tag{3.10}
$$

It remains to estimate the terms:

$$\begin{aligned} I_1 &= ([(u,v,w)\cdot \mathrm{grad}]u,u)_m, \quad I_2 = ([(u,v,w)\cdot \mathrm{grad}]v,v)_m, \\ I_3 &= ([(u,v,w)\cdot \mathrm{grad}]T,T)_m. \end{aligned} \tag{3.11}$$

Since the terms from (3.11) are the same as the terms considered in the previous section for the Primitive Equations, we can use the inequalities (2.27) and (2.28) and apply the same kind of reasonings. We do not repeat here the details of the computations.

We also introduce the following definition:

$$\|U\|_m^2 = |u|_m^2 + |v|_m^2 + \frac{\delta}{\rho_0}|w|_m^2 + |T|_m^2; \tag{3.12}$$

$\|\cdot\|_m$ is a norm on $(\dot{H}^m_{\mathrm{per}}(\Omega))^4$ equivalent to the usual norm.

Returning to (3.8), we find the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_m^2 \le g\beta_T\rho_0\|U\|_m^2 + |F|_m\|U\|_m + c_1\|U\|_m^2\|U\|_3 + c_2\|U\|_{m-1}^2\|U\|_m, \tag{3.13}$$

where c_1 and c_2 are some constants independent of the initial data, which may vary at different appearances.

For $m = 3$ we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_3^2 \le 2g\beta_T\rho_0\|U\|_3^2 + 2|F|_3\|U\|_3 + c_1\|U\|_3^3 + c_2\|U\|_2^2\|U\|_3. \tag{3.14}$$

Applying the Gronwall lemma to the estimate (3.14), we find that there exists a time $t_\star$ depending on the initial data such that the following estimate in $L^\infty(0,t_\star;(\dot{H}^3(\Omega))^4)$ holds:

$$\|U(t)\|_3 \le 2\|U_0\|_3, \quad \forall\, 0 \le t \le t_\star. \tag{3.15}$$

Recursively we find that, for $m > 3$, $|U(t)|_m$ remains bounded in $(0,t_\star)$, where $t_\star$ is the time determined for $m = 3$. Gathering these estimates and using the Galerkin–Fourier method, we obtain the existence of the solutions.

In order to prove the uniqueness of the solutions, we consider two solutions of the problem (3.4), namely $U' = (u', v', w', T')$ and $U'' = (u'', v'', w'', T'')$. We set $U = U' - U''$. Substituting the corresponding equation (3.4) for U'' from the equation for U' we find that U satisfies the following equation:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{\tilde{\mathbf{H}}} + b(U',U,\tilde{U}) + b(U,U'',\tilde{U}) + e(U,\tilde{U}) = 0, \quad \forall\, \tilde{U} \in \mathbf{V}, \\ &U(0) = 0, \end{aligned} \tag{3.16}$$

where the scalar product on $\tilde{\mathbf{H}}$ is:

$$(U,\tilde{U})_{\tilde{\mathbf{H}}} = (u,\tilde{u})_{L^2} + (v,\tilde{v})_{L^2} + \frac{\delta}{\rho_0}(w,\tilde{w})_{L^2} + (T,\tilde{T})_{L^2}. \tag{3.17}$$

In equation (3.16), we take $\tilde{U} = U(t)$ for an arbitrary but fixed instant of time t. Applying the conservation of mass equation (1.1d) we find $b(U', U, U) = 0$. From the definition of the form e we also find $e(U, U) = 0$. We then obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_{\tilde{\mathbf{H}}} + b(U, U'', U) = 0, \tag{3.18}$$

which leads us to the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_{\tilde{\mathbf{H}}} \leq c|DU'|_{L^\infty}|U|^2_{\tilde{\mathbf{H}}}. \tag{3.19}$$

Since the solutions are in $(\dot{H}^3_{\text{per}}(\Omega))^4$, we find $U = 0$, so the solution is unique. □

Similar to the case of δ–Primitive Equations, we can prove the continuous dependence on the data of the solutions. We define the following spaces:

$$X_1 = [(\dot{H}^m_{\text{per}}(\Omega))^4 \cap \tilde{\mathbf{V}}] \times [L^\infty(0, t_1; (\dot{H}^m_{\text{per}}(\Omega))^4)],$$
$$Y_1 = \{U \in L^\infty(0, t_\star; (\dot{H}^1_{\text{per}}(\Omega))^4);\ D_j U \in (L^\infty(\Omega \times (0, t_\star)))^4,\ j = 1, 2, 3, 4.\},$$

where $t_\star$ is defined in Theorem 3.1.

We equip these spaces with their natural norms, which make them Banach spaces. We also consider X_1 equipped with the L^2–norm and Y_1 equipped with the L^∞–norm of the spatial gradient ($|U| = |\nabla U|_{L^\infty(\Omega\times(0,t_\star))}$) and we call these spaces respectively $\tilde{X}_1$ and $\tilde{Y}_1$. Then we can prove that:

Remark 3.1. For the spaces defined above, the analogue of Theorem 2.2 holds.

3.2. **Time regularity.** In this section we prove that a time regularity result, similar to the result obtained for the δ-Primitive Equations, is available for the Boussinesq type of equations considered in this section. As before, we are interested in obtaining some a priori estimates. In order to derive the necessary a priori estimates, we differentiate l times in time the equations (3.1a), (3.1b), (3.1c) and (3.1e), then we take the scalar product in H^m of the resulting equations respectively with $u^{(l)}$, $v^{(l)}$, $w^{(l)}/\rho_0$ and $T^{(l)}$. We find:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(l)}\|_m^2 + &\sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial u^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial u^{(l-k)}}{\partial y} + w(U)^{(k)}\frac{\partial u^{(l-k)}}{\partial z}, u^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial v^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial v^{(l-k)}}{\partial y} + w^{(k)}\frac{\partial v^{(l-k)}}{\partial z}, v^{(l)})_m \\
&+ \sum_{k=0}^{l} C_l^k (u^{(k)}\frac{\partial T^{(l-k)}}{\partial x} + v^{(k)}\frac{\partial T^{(l-k)}}{\partial y} + w^{(k)}\frac{\partial T^{(l-k)}}{\partial z}, T^{(l)})_m \\
&+ \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, w^{(l)})_m \\
= &(F^{(l)}, (u,v,T)^{(l)})_m.
\end{aligned} \tag{3.20}
$$

Using periodicity, the conservation of mass (3.1d) and integrating by parts, we have:

$$
\frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial x}, u^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial y}, v^{(l)})_m + \frac{1}{\rho_0}(\frac{\partial p^{(l)}}{\partial z}, w^{(l)})_m = 0. \tag{3.21}
$$

The terms that remain to be estimated are the same as the terms obtained for the δ-Primitive Equations, so the computations are identical and we skip them here. We obtain the following a priori estimate:

(3.22)

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(l)}\|_m^2 \le c_1\|U\|_{m+1}\|U^{(l)}\|_m^2 + c_2\sum_{k=1}^{l-1}\|U^{(l-k)}\|_{m+1}\|U^{(k)}\|_m\|U^{(l)}\|_m + 2|F^{(l)}|_m\|U^{(l)}\|_m.
$$

Using the same arguments as before we find that there exists a time $t_\star$, depending on the initial data, such that:

$$
\frac{\partial^l U}{\partial t^l} \in L^\infty(0, t_\star, (\dot{H}^m_{\mathrm{per}}(\Omega))^4), \tag{3.23}
$$

for all $l \ge 0$ and all $m \ge 3$.

We can now state the following results, similar to the results obtained for the δ-Primitive Equations:

Theorem 3.2. *Let there be given $m \ge 3$, $l \ge 0$ and $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ as above. Then for each U_0 given in $\tilde{\mathbf{V}} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^4$ and F given such that $F^{(l)}$ is in $L^\infty(0, t_1, \dot{H}^m_{\mathrm{per}}(\Omega)^3)$, there exists a time $t_\star$ depending only on the initial data and a unique solution U of problem (2.13) defined on the interval $(0, t_\star)$, with*

$$
U^{(l)} \in L^\infty(0, t_\star, \dot{H}^m_{\mathrm{per}}(\Omega)^4).
$$

Using the a priori estimates above, we can also find:

Theorem 3.3. *Let there be given $m \geq 3$, and $L_1, L_2, L_3 > 0$, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ as above. Then for each U_0 given in $\tilde{\mathbf{V}} \cap (\mathcal{C}^\infty_{\text{per}}(\bar{\Omega}))^4$ and F given in $\mathcal{C}^\infty(0, t_0; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^3)$, there exists a time $t_\star$, $0 < t_\star \leq t_0$ and a unique solution U of problem (2.13) defined on the interval $(0, t_\star)$, with*

$$U \in \mathcal{C}^\infty(0, t_\star; \mathcal{C}^\infty_{\text{per}}(\bar{\Omega})^4).$$

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