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EXISTENCE AND REGULARITY RESULTS FOR THE PRIMITIVE EQUATIONS IN TWO SPACE DIMENSIONS

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Abstract. Our aim in this article is to present some existence, uniqueness and regularity results for the Primitive Equations of the ocean in space dimension two with periodic boundary conditions. We prove the existence of weak solutions for the PEs, the existence and uniqueness of strong solutions and the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity.

Contents

1. Introduction

The objective of this article is to derive various results of existence and regularity of solutions for the Primitive Equations of the ocean (PEs) in two space dimensions. These results, besides their intrinsic interest, are needed in [9] which is another motivation of this work.

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We consider the PEs in their nondimensional form (see Section 5) :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\,\Delta u + S_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u = \nu_{\boldsymbol{v}}\,\Delta v + S_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon}w = \nu_\rho\,\Delta\rho + S_\rho. \tag{1.1e}$$

All the independent variables (t, x, z) and the dependent variables (u, v, w, ρ, p) are dimensionless, as are the forcing and source terms (S_u, S_v, S_ρ). Here $(u,\ v,\ w)$ are the three components of the velocity vector and, as usual, we denote by p and ρ the pressure and density deviations, respectively, from prescribed background states. The (dimensionless) parameters are the Rossby number ε, the Burger number N, and the inverse (eddy) Reynolds numbers $\nu_{\boldsymbol{v}}$ and ν_ρ.

Some motivations on the physical background and the derivation of these equations are given in the Appendix (Section 5). The two spatial directions are $0x$ and $0z$, corresponding to the west–east and vertical directions in the so-called f-plane approximation for geophysical flows (for details, see the Appendix); $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$.

The article is organized as follows: We start in Section 2 by recalling the variational formulation of problem (1.1) under suitable assumptions and we prove the existence of weak solutions for the PEs. We continue in Section 3 by proving the existence and uniqueness of strong solutions. Finally in Section 4 we prove the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity. We thought that it is useful to end the article with an Appendix (Section 5) containing some physical explanations regarding the PEs and the derivation of (1.1).

We mention here the similar works of Bresch, Kazhikhov and Lemoine [2] and of Ziane [13], who consider different boundary conditions and do not consider the higher regularity results needed in [9]; see also [11]. For the non-dimensional form of the PEs, we refer here for example to [4], [8], and [12] but a substantial amount of literature is available on this subject.

2. Existence of the Weak Solutions for the PEs

We work in a limited domain

$$\mathcal{M} = (0,\ L_1) \times (-L_3/2,\ L_3/2), \tag{2.1}$$

and, since this is needed in [9], we assume space periodicity with period $\mathcal{M}$, that is, all functions are taken to satisfy $f(x + L_1, z, t) = f(x, z, t) = f(x, z + L_3, t)$ when extended to $\mathbb{R}^2$. Moreover, we assume that the following symmetries hold:

$$\begin{aligned} u(x,z,t) &= u(x,-z,t), & S_u(x,z,t) &= S_u(x,-z,t),\\ v(x,z,t) &= v(x,-z,t), & S_v(x,z,t) &= S_v(x,-z,t),\\ \rho(x,z,t) &= -\rho(x,-z,t), & S_\rho(x,z,t) &= -S_\rho(x,-z,t),\\ w(x,z,t) &= -w(x,-z,t), & p(x,z,t) &= p(x,-z,t). \end{aligned}$$

(Here u, v and p are said to be even in z, and w and ρ odd in z.)

We note that these conditions are often used in numerical studies of rotating stratified turbulence (see e.g., [1]).

Our aim is to solve the problem (1.1) with initial data

$$u = u_0,\ v = v_0,\ \rho = \rho_0 \quad \text{at } t = 0. \tag{2.2}$$

Hence the natural function spaces for this problem are as follows:

$$\begin{aligned} V = \{(u,\, v,\, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3,&\\ u,\, v \text{ even in } z,\ \rho \text{ odd in } z,\ &\textstyle\int_{-L_3/2}^{L_3/2} u(x,z')\,\mathrm{d}z' = 0\}, \end{aligned} \tag{2.3}$$

$$H = \text{closure of } V \text{ in } (\dot{L}^2(\mathcal{M}))^3. \tag{2.4}$$

Here the dot above $\dot{H}^1_{\text{per}}$ or $\dot{L}^2$ denotes the functions with average in $\mathcal{M}$ equal to zero. These spaces are endowed with Hilbert scalar products; in H the scalar product is

$$(U,\, \tilde{U})_H = (u,\, \tilde{u})_{L^2} + (v,\, \tilde{v})_{L^2} + \kappa(\rho,\, \tilde{\rho})_{L^2}, \tag{2.5}$$

and in $\dot{H}^1_{\text{per}}$ and V the scalar product is (using the same notation when there is no ambiguity):

$$((U,\, \tilde{U})) = ((u,\, \tilde{u})) + ((v,\, \tilde{v})) + \kappa((\rho,\, \tilde{\rho})); \tag{2.6}$$

where we have written $\mathrm{d}\mathcal{M}$ for $\mathrm{d}x\,\mathrm{d}z$, and

$$((\phi,\, \tilde{\phi})) = \int_{\mathcal{M}} \Big(\frac{\partial\phi}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial\tilde{\phi}}{\partial z} \Big)\,\mathrm{d}\mathcal{M}. \tag{2.7}$$

The positive constant κ is defined below. We have

$$|U|_H \le c_0\|U\|,\ \forall\, U \in V. \tag{2.8}$$

where $c_0 > 0$ is a positive constant related to κ and the Poincaré constant in $\dot{H}^1_{\text{per}}(\mathcal{M})$. More generally, the c_i, c_i', c_i'' will denote various positive constants. Inequality (2.8) implies that $\|U\| = ((U,\, U))^{1/2}$ is indeed a norm on V.

We first show how we can express the diagnostic variables w and p in terms of the prognostic variables u, v and ρ. For each $U = (u,\, v,\, \rho) \in V$ we can determine uniquely $w = w(U)$ from (1.1d),

$$w(U) = w(x,z,t) = -\int_0^z u_x(x,z',t)\,\mathrm{d}z', \tag{2.9}$$

since $w(x,\,0) = 0$, w being odd in z. Furthermore, writing that $w(x,-L_3/2,t) = w(x,L_3/2,t)$, we also have

$$\int_{-L_3/2}^{L_3/2} u_x(x,z',t)\,\mathrm{d}z' = 0. \tag{2.10}$$

As for the pressure, we obtain from (1.1c),

$$p(x,z,t) = p_{\mathrm{s}}(x,t) - \int_0^z \rho(x,z',t)\,\mathrm{d}z', \tag{2.11}$$

where $p_{\mathrm{s}} = p(x,0,t)$ is the surface pressure. Thus, we can uniquely determine the pressure p in terms of ρ up to p_{s}.

It is appropriate to use Fourier series and we write, e.g., for u,

$$u(x,z,t) = \sum_{(k_1,k_3)\in\mathbb{Z}} u_{k_1,k_3}(t)\mathrm{e}^{i(k_1'x+k_3'z)}, \tag{2.12}$$

where for notational conciseness we set $k_1' = 2\pi k_1/L_1$ and $k_3' = 2\pi k_3/L_3$. Since u is real and even in z, we have $u_{-k_1,-k_3} = \bar{u}_{k_1,k_3} = \bar{u}_{k_1,-k_3}$, where $\bar{u}$ denotes the complex conjugate of u. Regarding the pressure, we obtain from (1.1c):

$$\begin{aligned}
p(x,z,t) &= p(x,0,t) - \int_0^z \sum_{(k_1,\,k_3)} \rho_{k_1,\,k_3} e^{i(k_1'x+k_3'z')}\mathrm{d}z' \\
&= \sum_{k_1} p_{\mathrm{s}k_1} e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{ik_1'x}(e^{ik_3'z}-1) \\
&\qquad \text{[using the fact that } \rho_{k_1,\,0} = 0,\ \rho \text{ being odd in } z\text{]} \\
&= \sum_{k_1} \big(p_{\mathrm{s}k_1} + \sum_{k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'}\big) e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{i(k_1'x+k_3'z)} \\
&= \sum_{k_1} p_{\star k_1} e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{i(k_1'x+k_3'z)},
\end{aligned}$$

where we denoted by p_{s} the surface pressure and $p_\star = \sum_{k_1\in\mathbb{Z}} p_{\star k_1} e^{ik_1'x}$, which is the average of p in the vertical direction, is defined by

$$p_{\star,k_1} = p_{\mathrm{s}k_1} + \sum_{k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'}.$$

Note that p is fully determined by ρ, up to one of the terms p_{s} or $p_\star$ which are connected by the relation above.

We now obtain the variational formulation of problem (1.1). For that purpose we consider a test function $\tilde{U} = (\tilde{u},\, \tilde{v},\, \tilde{\rho}) \in V$ and we multiply (1.1a), (1.1b) and (1.1e), respectively by $\tilde{u}$, $\tilde{v}$ and $\kappa\,\tilde{\rho}$, where the constant κ (which was already introduced in (2.5) and (2.6)) will be chosen later. We add the resulting equations and integrate over $\mathcal{M}$. We find:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_H + b(U,U,\tilde{U}) + a(U,\tilde{U}) + \frac{1}{\varepsilon} e(U,\tilde{U}) = (S,\tilde{U})_H, \qquad \forall \tilde{U} \in V. \tag{2.13}$$

Here we set

$$\begin{aligned}
a(U,\tilde{U}) &= \nu_{\boldsymbol{v}}((u,\tilde{u})) + \nu_{\boldsymbol{v}}((v,\tilde{v})) + \kappa\nu_\rho((\rho,\tilde{\rho})),\\
e(U,\tilde{U}) &= \int_{\mathcal{M}} (u\tilde{v} - v\tilde{u})\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (\rho\tilde{w} - \kappa N^2\, w\tilde{\rho})\,\mathrm{d}\mathcal{M},\\
b(U,U^\sharp,\tilde{U}) &= \int_{\mathcal{M}} \Big(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\Big)\tilde{u}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} \Big(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\Big)\tilde{v}\,\mathrm{d}\mathcal{M}\\
&\qquad + \kappa\int_{\mathcal{M}} \Big(u\frac{\partial \rho^\sharp}{\partial x} + w(U)\frac{\partial \rho^\sharp}{\partial z}\Big)\tilde{\rho}\,\mathrm{d}\mathcal{M}.
\end{aligned}$$

We now choose $\kappa = 1/N^2$ and this way we find $e(U,\, U) = 0$. Also it can be easily seen that:

$$\begin{aligned}
&a : V\times V \to \mathbb{R} \text{ is bilinear, continuous, coercive, } a(U,U) \geq c_1\|U\|^2,\\
&e : V\times V \to \mathbb{R} \text{ is bilinear, continuous, } e(U,U) = 0,\\
&b \text{ is trilinear, continuous from } V\times V_2\times V \text{ into } \mathbb{R},\\
&\qquad\qquad \text{and from } V\times V\times V_2 \text{ into } \mathbb{R},
\end{aligned} \tag{2.14}$$

where V_2 is the closure of $V \cap (H^2_{\mathrm{per}}(\mathcal{M}))^3$ in $(H^2_{\mathrm{per}}(\mathcal{M}))^3$. Furthermore,

$$\begin{aligned}
&b(U,\tilde{U},U^\sharp) = -b(U,U^\sharp,\tilde{U}),\\
&b(U,\tilde{U},\tilde{U}) = 0,
\end{aligned} \tag{2.15}$$

when U, $\tilde{U}$, $U^\sharp \in V$ with $\tilde{U}$ or $U^\sharp$ in V_2. We also have the following:

Lemma 2.1. *There exists a constant $c_2 > 0$ such that, for all $U \in V$, $\tilde{U} \in V_2$ and $U^\sharp \in V$:*

$$\begin{aligned}
|b(U,U^\sharp,\tilde{U})| \leq{}& c_2|U|^{1/2}_{L^2}\|U\|^{1/2}\|U^\sharp\||\tilde{U}|^{1/2}_{L^2}\|\tilde{U}\|^{1/2}\\
&+ c_2\|U\|\|U^\sharp\|^{1/2}|U^\sharp|^{1/2}_{V_2}|\tilde{U}|^{1/2}_{L^2}\|\tilde{U}\|^{1/2}.
\end{aligned} \tag{2.16}$$

Proof. We only estimate two typical terms; the other terms are estimated exactly in the same way. Using the Hölder, Sobolev and interpolation inequalities, we write:

$$\begin{aligned}\Big|\int_{\mathcal{M}} u\frac{\partial u^\sharp}{\partial x}\tilde u\,\mathrm{d}\mathcal{M}\Big| &\le |u|_{L^4}\Big|\frac{\partial u^\sharp}{\partial x}\Big|_{L^2}|\tilde u|_{L^4}\\ &\le c_1'|u|_{L^2}^{1/2}\|u\|^{1/2}\Big|\frac{\partial u^\sharp}{\partial x}\Big|_{L^2}|\tilde u|_{L^2}^{1/2}\|\tilde u\|^{1/2},\end{aligned}$$

$$\begin{aligned}\Big|\int_{\mathcal{M}} w(U)\frac{\partial u^\sharp}{\partial z}\tilde u\,\mathrm{d}\mathcal{M}\Big| &\le |w(U)|_{L^2}\Big|\frac{\partial u^\sharp}{\partial z}\Big|_{L^4}|\tilde u|_{L^4}\\ &\le c_2'\|u\|\Big|\frac{\partial u^\sharp}{\partial z}\Big|_{L^2}^{1/2}\Big\|\frac{\partial u^\sharp}{\partial z}\Big\|^{1/2}|\tilde u|^{1/2}\|\tilde u\|^{1/2};\end{aligned}$$

(2.16) follows from these estimates and the analogous estimates for the other terms. □

We now recall the result regarding the existence of weak solutions for the PEs of the ocean; see [7]. In [7] the existence of the weak solutions is established in three space dimensions with different boundary conditions, but the proof applies as well to two dimensions with our boundary conditions.

Theorem 2.1. *Given $U_0 \in H$ and $S \in L^\infty(\mathbb{R}_+;\, H)$, there exists at least one solution U of (2.13), $U \in L^\infty(\mathbb{R}_+;\, H) \cap L^2(0, t_\star;\, V)$, $\forall\, t_\star > 0$, with $U(0) = U_0$.*

The proof of this theorem is based on the a priori estimates given below, which gives, as in [7], that $U \in L^\infty(0,\, t_\star;\, H)$, $\forall\, t_\star > 0$; however, as shown below, we have in fact,

$$U \in L^\infty(\mathbb{R}_+;\, H).$$

Taking $\tilde U = U$ in equation (2.13), after some simple computations and using (2.14), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_1\|U\|^2 \le c_1'|S|_\infty^2\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_0c_1|U|_H^2 \le c_1'|S|_\infty^2\,, \tag{2.17}$$

where $|S|_\infty$ is the norm of S in $L^\infty(\mathbb{R}_+;\, H)$. Using the Gronwall inequality, we infer from (2.17) that:

$$|U(t)|_H^2 \le |U(0)|_H^2\,\mathrm{e}^{-c_1c_0t} + \frac{c_1'}{c_1c_0}(1-\mathrm{e}^{-c_1c_0t})\,|S|_\infty^2\,, \quad \forall\, t > 0. \tag{2.18}$$

Hence

$$\limsup_{t\to\infty}|U(t)|_H^2 \le \frac{c_1'}{c_1c_0}|S|_\infty^2 =: r_0^2\,,$$

and any ball $B(0,\, r_0')$ in H with $r_0' > r_0$ is an absorbing ball; that is, for all U_0, there exists $t_0 = t_0(|U_0|_H)$ depending increasingly on $|U_0|_H$ (and depending also on

r_0', $|S|_\infty$ and other data), such that $|U(t)|_H \le r_0'$, $\forall\, t \ge t_0(|U_0|_H)$. Furthermore, integrating equation (2.17) from t to $t+r$, with $r > 0$ arbitrarily chosen, we find:

$$\int_t^{t+r} \|U(t')\|^2 \,\mathrm{d}t' \le K_1, \qquad \text{for all } t \ge t_0(|U_0|_H), \tag{2.19}$$

where K_1 denotes a constant depending on the data but not on U_0. As mentioned before, (2.18) implies also that

$$U \in L^\infty(\mathbb{R}_+;\, H), \qquad |U(t)|_H \le \max(|U_0|_H,\, r_0).$$

Remark 2.1. We notice that, in the inviscid case ($\nu_{\boldsymbol{v}} = \nu_\rho = 0$ with $S = 0$), taking $\tilde{U} = U$ in (2.13), we find, at least formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(|u|_{L^2}^2 + |v|_{L^2}^2 + \frac{1}{N^2}|\rho|_{L^2}^2\Big) = 0. \tag{2.20}$$

The physical meaning of (2.20) is that the sum of the kinetic energy (given by $\frac{1}{2}(|u|_{L^2}^2 + |v|_{L^2}^2)$) and the available potential energy (given by $\frac{1}{2N^2}|\rho|_{L^2}^2$) is conserved in time. This is the physical justification of the introduction of the constant $\kappa = N^{-2}$ in (2.5).

3. Existence and Uniqueness of Strong Solutions for the PEs

The solutions given by Theorem 2.1 are usually called weak solutions. We are now interested in strong solutions (and even more regular solutions in Section 4). We use here the same terminology as in fluid mechanics (incompressible Navier–Stokes equations): weak solutions are those in $L^\infty(L^2)$ and $L^2(H^1)$, strong solutions are those in $L^\infty(H^1)$ and $L^2(H^2)$. We notice that we cannot obtain directly the global existence of strong solutions for the PEs as, e.g., for the Navier–Stokes equations using a single a priori estimate (obtained by replacing $\tilde{U}$ by ΔU in (2.13)). Instead, to derive the necessary a priori estimates we proceed by steps: we successively derive estimates in $L^\infty(L^2)$ and $L^2(H^1)$ for u_z, u_x, v_z, v_x, ρ_z and ρ_x (here the subscripts t, x, z denote differentiation). Notice that the order in which we obtain these estimates cannot be changed in the calculations below.

Firstly, using (2.11) we rewrite (1.1a) as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p_{\mathrm{s}}}{\partial x} - \frac{1}{\varepsilon}\int_0^z \rho_x(x,\, z',\, t)\,\mathrm{d}z' = \nu_{\boldsymbol{v}}\Delta u + S_u. \tag{3.1}$$

We differentiate (3.1) with respect to z and we find, with $w_z = -u_x$:

$$u_{tz} + uu_{xz} + wu_{zz} - \frac{1}{\varepsilon}v_z - \frac{1}{\varepsilon}\rho_x - \nu_{\boldsymbol{v}}u_{xxz} - \nu_{\boldsymbol{v}}u_{zzz} = S_{u,\,z}\,,$$

where $S_{u,\,z} = \partial_z S_u = \partial S_u/\partial z$. After multiplying this equation by u_z and integrating over $\mathcal{M}$, we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\|u_z\|^2 + \int_{\mathcal{M}} u u_z u_{xz}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w u_z u_{zz}\,\mathrm{d}\mathcal{M}\\ - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_z u_z\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} \rho_x u_z\,\mathrm{d}\mathcal{M} = \int_{\mathcal{M}} u_z S_{u,\,z}\,\mathrm{d}\mathcal{M}.\end{aligned}$$

Integrating by parts and taking into account the periodicity and the conservation of mass equation (1.1d) we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\|u_z\|^2 - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_z u_z\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} \rho_x u_z\,\mathrm{d}\mathcal{M} = \int_{\mathcal{M}} u_z S_{u,\,z}\,\mathrm{d}\mathcal{M}. \tag{3.2}$$

In all that follows $K(\varepsilon)$, $K'(\varepsilon)$, $K''(\varepsilon)$, ..., denote constants depending on ε and other data but not on U_0; we use the same symbol for different constants. We easily obtain the following estimates:

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} v_z u_z\,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} v u_{zz}\,\mathrm{d}\mathcal{M}\Big| \le K(\varepsilon)|v|^2_{L^2} + \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2,\\ &\frac{1}{\varepsilon}\,\Big|\int_{\mathcal{M}} \rho_x u_z\,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\,\Big|\int_{\mathcal{M}} \rho\, u_{xz}\,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2 + K(\varepsilon)|\rho|^2_{L^2},\\ &\Big|\int_{\mathcal{M}} S_{u,z} u_z\,\mathrm{d}\mathcal{M}\Big| = \Big|\int_{\mathcal{M}} S_u u_{zz}\,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2 + c_1'|S_u|^2_{L^2}\,;\end{aligned}$$

applied to (3.2), these give:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\,\|u_z\|^2 \le K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.3}$$

We apply Poincaré's inequality (2.8) and we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + c_0\nu_{\boldsymbol{v}}|u_z|^2_{L^2} \le K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.4}$$

Using Gronwall's lemma, we infer from (3.4) that:

(3.5)

$$\begin{aligned}|u_z(t)|^2_{L^2} &\le |u_z(0)|^2_{L^2} e^{-c_0\nu_{\boldsymbol{v}} t} + K(\varepsilon)e^{-c_0\nu_{\boldsymbol{v}} t}\int_0^t (|v(t')|^2_{L^2} + |\rho(t')|^2_{L^2})e^{c_0\nu_{\boldsymbol{v}} t'}\,\mathrm{d}t' + c_2'|S_u|^2_\infty\\ &\le |u_z(0)|^2_{L^2} e^{-c_0\nu_{\boldsymbol{v}} t} + K'(\varepsilon)(1 - e^{-c_0\nu_{\boldsymbol{v}} t})(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty\\ &\le |u_z(0)|^2_{L^2} e^{-c_0\nu_{\boldsymbol{v}} t} + K'(\varepsilon)(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty,\end{aligned}$$

where $|v|_\infty = |v|_{L^\infty(\mathbb{R}_+;\,L^2(\mathcal{M}))}$, and similarly for ρ and S_u. We obtain an explicit bound for the norm of u_z in $L^\infty(\mathbb{R}_+;\,H)$:

$$|u_z(t)|^2_{L^2} \le |u_z(0)|^2_{L^2} + K'(\varepsilon)(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty. \tag{3.6}$$

For what follows, we recall here the uniform Gronwall lemma (see e.g., [10]):

If ξ, η and y are three positive locally integrable functions on $(t_1,\,\infty)$ such that y' is locally integrable on $(t_1,\,\infty)$ and which satisfy

$$\begin{aligned}&y' \le \xi y + \eta,\\ &\int_t^{t+r} \xi(s)\,\mathrm{d}s \le a_1,\ \int_t^{t+r} \eta(s)\,\mathrm{d}s \le a_2,\ \int_t^{t+r} y(s)\,\mathrm{d}s \le a_3, \quad \forall\, t \ge t_1,\end{aligned} \tag{3.7}$$

where r, a_1, a_2, a_3 are positive constants, then

$$y(t+r) \le \Big(\frac{a_3}{r} + a_2\Big)\,\mathrm{e}^{a_1}, \quad t \ge t_1. \tag{3.8}$$

The bound (3.6) depends on the initial data U_0. In order to obtain a bound independent of U_0 we apply the uniform Gronwall lemma to the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} \le K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.9}$$

to obtain

$$|u_z(t)| \le K'(\varepsilon, r, r_0'), \quad \forall\, t \ge t_1', \tag{3.10}$$

where $t_1' = t_0(|U_0|_{L^2}) + r$ and $r > 0$ is fixed. Integrating equation (3.3) from t to $t + r$ with $r > 0$ as before, we also find:

$$\int_t^{t+r} \|u_z(s)\|^2\,\mathrm{d}s \le K''(\varepsilon, r, r_0'), \qquad \forall\, t \ge t_1'. \tag{3.11}$$

We now derive the same kind of estimates for u_x: We differentiate (3.1) with respect to x and we obtain

$$\begin{aligned} u_{tx} + u_x^2 + u u_{xx} + w u_{xz} + w_x u_z - \frac{1}{\varepsilon} v_x + \frac{1}{\varepsilon} p_{s,xx} + \int_z^0 \rho_{xx}(z')\,\mathrm{d}z'&\\ - \nu_{\boldsymbol{v}}\, u_{xxx} - \nu_{\boldsymbol{v}}\, u_{zzx} = S_{u,\,x};& \end{aligned} \tag{3.12}$$

multiplying this equation by u_x and integrating over $\mathcal{M}$ we find, using (1.1d):

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_x|^2_{L^2} + \int_{\mathcal{M}} u_x^3\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w_x u_z u_x\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_x u_x\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} p_{s,\,x\,x} u_x\,\mathrm{d}\mathcal{M}&\\ + \int_{\mathcal{M}}\Big(\int_z^0 \rho_{x\,x}(z')\,\mathrm{d}z'\Big)\, u_x\,\mathrm{d}\mathcal{M} + \nu_{\boldsymbol{v}}\|u_x\|^2 = \int_{\mathcal{M}} u_x S_{u,\,x}\,\mathrm{d}\mathcal{M}.& \end{aligned} \tag{3.13}$$

Based on the Hölder, Sobolev and interpolation inequalities, we derive the following estimates:

$$\begin{aligned} \Big|\int_{\mathcal{M}} u_x^3\,\mathrm{d}\mathcal{M}\Big| \le |u_x|^3_{L^3(\mathcal{M})} \le c_4'|u_x|^3_{H^{1/3}(\mathcal{M})} \le c_5'|u_x|^2_{L^2}\|u_x\|&\\ \le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + c_6'|u_x|^4_{L^2},& \end{aligned}$$

$$\Big|\int_{\mathcal{M}} w_x u_z u_x \,\mathrm{d}\mathcal{M}\Big| \le c_7'|w_x|_{L^2}|u_z|_{L^2}^{1/2}\|u_z\|^{1/2}|u_x|_{L^2}^{1/2}\|u_x\|^{1/2}$$
$$\le c_8'|u_{xx}|_{L^2}|u_z|_{L^2}^{1/2}\|u_z\|^{1/2}|u_x|_{L^2}^{1/2}\|u_x\|^{1/2}$$
$$\le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + c_9'|u_z|_{L^2}^2\|u_z\|^2|u_x|_{L^2}^2,$$

By the definition of V, and since p_{s} is independent of z, we find:

$$\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} p_{\mathrm{s},\,x\,x} u_x \,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_0^L p_{\mathrm{s},\,xx}\int_{-L_3/2}^{L_3/2} u_x \,\mathrm{d}z\,\mathrm{d}x\Big| = 0.$$

We can also prove the following estimates:

$$\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} v_x u_x \,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + K'(\varepsilon)|v|_{L^2}^2,$$

$$\frac{1}{\varepsilon}\,\Big|\int_{\mathcal{M}}\Big(\int_z^0 \rho_{x\,x}(z')\,\mathrm{d}z'\Big)\,u_x\,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}}\Big(\int_z^0 \rho_x(z')\,\mathrm{d}z'\Big)\,u_{xx}\,\mathrm{d}\mathcal{M}\Big|$$
$$\le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + K''(\varepsilon)|\rho_x|_{L^2}^2,$$

$$\Big|\int_{\mathcal{M}} u_x S_{u,\,x}\,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + c_{10}'|S_u|_\infty^2.$$

With these relations (3.13) implies:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_x|_{L^2}^2 + \nu_{\boldsymbol{v}}\|u_x\|^2 \le \xi\,|u_x|_{L^2}^2 + \eta, \tag{3.14}$$

where we denoted

$$\xi = \xi(t) = 2c_6'|u_x|_{L^2}^2 + 2c_9'|u_z|_{L^2}^2\|u_z\|^2,$$

and

$$\eta = \eta(t) = 2K'(\varepsilon)|v|_{L^2}^2 + 2K''(\varepsilon)|\rho_x|_{L^2}^2 + 2c_{10}'|S_u|_\infty^2.$$

We easily conclude from (3.14) that

$$u_x \in L^\infty(0,\, t_\star;\, L^2)\cap L^2(0,\, t_\star;\, H^1), \quad \forall\, t_\star > 0. \tag{3.15}$$

However, for later purposes, (3.15) is not sufficient, and we need estimates uniform in time.

We will apply the uniform Gronwall lemma to (3.14) with $t_1 = t_1'$ as in (3.10). Noting that

$$\begin{aligned}\int_t^{t+r}\xi(t')\,\mathrm{d}t' &= \int_t^{t+r}[2c_6'|u_x|_{L^2}^2 + 2c_9'|u_z(t')|_{L^2}^2\|u_z(t')\|^2]\,\mathrm{d}t'\\ &\le 2c_6'\int_t^{t+r}|u_x(t')|_{L^2}^2\,\mathrm{d}t' + 2c_9'|u_z|_\infty^2\int_t^{t+r}\|u_z(t')\|^2\,\mathrm{d}t'\\ &\le a_1, \qquad \forall\, t \ge t_1',\end{aligned} \tag{3.16}$$

$$\begin{aligned}\int_t^{t+r}\eta(t')\,\mathrm{d}t' &= \int_t^{t+r}[2K'(\varepsilon)|v|_{L^2}^2+2K''(\varepsilon)|\rho_x|_{L^2}^2+2c'_{10}|S_u|_\infty^2]\,\mathrm{d}t'\\ &\le K(\varepsilon)+2c'_{10}r|S_u|_\infty^2\\ &= a_2, \qquad \forall\, t\ge t'_1,\end{aligned}\tag{3.17}$$

$$\int_t^{t+r}|u_x(t')|_{L^2}^2\,\mathrm{d}t'\le a_3, \qquad \forall\, t\ge t'_1, \tag{3.18}$$

(3.8) then yields:

$$|u_x(t)|_{L^2}^2\le\Big(\frac{a_3}{r}+a_2\Big)e^{a_1}, \quad \forall\, t\ge t'_1+r, \tag{3.19}$$

and thus

$$|u_x|_{L^2}\in L^\infty(\mathbb{R}_+). \tag{3.20}$$

Note that in (3.16)–(3.18) we can use bounds on $|u_z|_\infty$ (and other similar terms) independent of U_0, since $t\ge t_0(|U_0|_{L^2})+r$. Integrating equation (3.14) from 0 to t'_1+r where $t'_1=t'_1(|U_0|_{L^2})$, we obtain a bound for u_x in $L^2(0,\,t'_1+r;\,H^1)$ which depends on $\|U_0\|$. A bound independent of U_0 is obtained if we work with $t\ge t'_1+r=t''_1=t''_1(|U_0|_{L^2})$: Integrating equation (3.14) from t to $t+r$ with r as before, we find:

$$\int_t^{t+r}\|u_x(s)\|^2\,\mathrm{d}s\le K(\varepsilon), \qquad \forall\, t\ge t''_1. \tag{3.21}$$

We perform similar computations for v_z: We differentiate (1.1b) with respect to z, multiply the resulting equation by v_z and integrate over $\mathcal{M}$. Using again the conservation of mass relation, we arrive at:

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}|v_z|_{L^2}^2+\int_{\mathcal{M}}u_zv_xv_z\,\mathrm{d}\mathcal{M}+\int_{\mathcal{M}}w_zv_z^2\,\mathrm{d}\mathcal{M}+\frac1\varepsilon\int_{\mathcal{M}}u_zv_z\,\mathrm{d}\mathcal{M}+\nu_{\boldsymbol{v}}\|v_z\|^2&\\ =\int_{\mathcal{M}}v_zS_{u,\,z}\;\mathrm{d}\mathcal{M}.&\end{aligned}\tag{3.22}$$

We notice the following estimate:

$$\begin{aligned}\Big|\int_{\mathcal{M}}u_zv_xv_z\,\mathrm{d}\mathcal{M}\Big| &\le c'_{11}|u_z|_{L^2}^{1/2}\|u_z\|^{1/2}|v_x|_{L^2}|v_z|_{L^2}^{1/2}\|v_z\|^{1/2}\\ &\le\frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2+c'_{12}|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3}|v_z|_{L^2}^{2/3}\\ &\le\frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2+c'_{12}|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3}(1+|v_z|_{L^2}^2).\end{aligned}$$

We also see that

$$\begin{aligned}
\Big|\int_{\mathcal{M}} w_z v_z v_z \,\mathrm{d}\mathcal{M}\Big| &= \Big|\int_{\mathcal{M}} u_x v_z v_z \,\mathrm{d}\mathcal{M}\Big| \le c'_{13}|u_x|_{L^2}^{1/2}\|u_x\|^{1/2}|v_z|_{L^2}^{3/2}\|v_z\|^{1/2}\\
&\le \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c'_{14}|u_x|_{L^2}^{2/3}\|u_x\|^{2/3}|v_z|_{L^2}^2,\\
\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} u_z v_z \,\mathrm{d}\mathcal{M}\Big| &= \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} u\, v_{zz} \,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + K(\varepsilon)|u|_{L^2}^2,\\
\Big|\int_{\mathcal{M}} S_{v,z} v_z \,\mathrm{d}\mathcal{M}\Big| &= \Big|\int_{\mathcal{M}} S_v v_{zz} \,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c'_{15}|S_v|_\infty^2,
\end{aligned}$$

which gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}|v_z|_{L^2}^2 + \nu_{\boldsymbol{v}}\|v_z\|^2 \le \xi|v_z|^2 + \eta, \tag{3.23}$$

where we denoted

$$\eta = \eta(t) = 2c'_{12}|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3} + 2K(\varepsilon)|u|^2 + 2c'_{15}|S_v|_\infty^2,$$

and

$$\xi = \xi(t) = 2c'_{12}|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3} + 2c'_{14}|u_x|_{L^2}^{2/3}\|u_x\|^{2/3}.$$

From (3.23), using the estimates obtained before and applying the classical Gronwall lemma we obtain bounds depending on the initial data for v_z in $L^\infty_{\mathrm{loc}}(0,t_\star;L^2)$ and $L^2_{\mathrm{loc}}(0,t_\star;H^1)$, valid for any finite interval of time $(0,t_\star)$.

To obtain estimates valid for all time, we apply the uniform Gronwall lemma observing that:

$$\begin{aligned}
\int_t^{t+r}\xi(t')\,\mathrm{d}t' &\le 2c'_{12}|u_z|_\infty^{2/3}\Big(\int_t^{t+r}\|u_z(t')\|\,\mathrm{d}t'\Big)^{1/3}\Big(\int_t^{t+r}|v_x(t')|_{L^2}^2\,\mathrm{d}t'\Big)^{2/3}\\
&\qquad + 2c'_{14}|u_x|_\infty^{2/3}\int_t^{t+r}\|u_x(t')\|^{2/3}\,\mathrm{d}t'\\
&\le a_1, \quad \forall\, t \ge t''_1,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\int_t^{t+r}\eta(t')\,\mathrm{d}t' &\le 2c'_{12}|u_z|_\infty^{2/3}\Big(\int_t^{t+r}\|u_z(t')\|\,\mathrm{d}t'\Big)^{1/3}\Big(\int_t^{t+r}|v_x(t')|_{L^2}^2\,\mathrm{d}t'\Big)^{2/3}\\
&\qquad + 2K(\varepsilon)|u|_\infty^2 r + 2c'_{15}r|S_v|_\infty^2\\
&\le a_2, \quad \forall\, t \ge t''_1,
\end{aligned} \tag{3.25}$$

$$\int_t^{t+r}|v_z(t')|^2\,\mathrm{d}t' \le a_3, \quad \forall\, t \ge t''_1. \tag{3.26}$$

Then the uniform Gronwall lemma gives:

$$|v_z(t)|_{L^2}^2 \le \Big(\frac{a_3}{r} + a_2\Big)e^{a_1}, \quad \forall\, t \ge t''_1 + r, \tag{3.27}$$

with a_1, a_2, a_3 as in (3.25), (3.24) and (3.26). Integrating equation (3.23) from t to $t+r$ with $r>0$ as above and $t \geq t_1'' + r$, we find:

$$\int_t^{t+r} \|v_z(s)\|^2 \,\mathrm{d}s \leq K(\varepsilon), \quad \forall\, t \geq t_1'' + r. \tag{3.28}$$

The same methods apply to v_x, ρ_z and ρ_x, noticing that at each step we precisely use the estimates from the previous steps, so the order can not be changed in this calculations.

With these estimates, the Galerkin method as used for the proof of Theorem 2.1 gives the existence of strong solutions:

Theorem 3.1. *Given $U_0 \in V$ and $S \in L^\infty(\mathbb{R}_+;\, H)$, there is a unique solution U of equation (2.13) with $U(0) = U_0$ such that*

$$U \in L^\infty(\mathbb{R}_+;\, V) \cap L^2(0,\, t_\star;\, (\dot{H}^2(\mathcal{M}))^3),\ \forall\, t_\star > 0. \tag{3.29}$$

Proof. As we said, the existence of strong solutions follows from the previous estimates. It remains to prove the uniqueness.

Assume U_1 and U_2 are two solutions of problem (2.13) satisfying (3.29), and let $U = U_1 - U_2$. We write (2.13) for U_1 and U_2 with $\tilde{U} = U$; combining the resulting equations, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + a(U,U) + b(U_1,U_1,U) - b(U_2,U_2,U) = 0. \tag{3.30}$$

Using (2.15b) we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_1\|U\|^2 + b(U,U_2,U) \leq 0. \tag{3.31}$$

From Lemma (2.1) and using Young's inequality we find that:

$$\begin{aligned} b(U,U_2,U) &\leq c_1'|U|_{L^2}\|U\|\|U_2\| + c_2'|U|_{L^2}^{1/2}\|U\|^{3/2}|U_2|_{V_2}^{1/2} \\ &\leq \frac{c_1}{2}\|U\|^2 + c_3'|U|_{L^2}^2\|U_2\|^2 + c_4'|U|_{L^2}^2|U_2|_{V_2}^2. \end{aligned} \tag{3.32}$$

Going back to (3.31) we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 \leq c_5'|U|_H^2(\|U_2\|^2 + |U_2|_{V_2}^2). \tag{3.33}$$

Since U_2 satisfies (3.29) the function

$$t \to \|U_2(t)\|^2 + |U_2(t)|_{V_2}^2 \text{ is integrable,}$$

and we can apply the Gronwall lemma which yields, since $U_1(0) = U_2(0)$,

$$|U(t)|_H^2 \leq 0, \quad \forall\, t \in [0, t_\star]. \tag{3.34}$$

From (3.34) we conclude that $U_1 = U_2$. ☐

4. More Regular Solutions for the PEs

In this section we show how to obtain estimates on the higher order derivatives from which one can derive the existence of solutions of the PEs in $(\dot{H}^m(\mathcal{M}))^3$ for all $m \in \mathbb{N}$, $m \geq 2$ (hence up to $\mathcal{C}^\infty$ regularity). In all that follows we work with U_0 in $(\dot{H}^m_{\text{per}}(\mathcal{M}))^3$.

We set $|U|_m = \big(\sum_{[\alpha]=m} |D^\alpha U|^2_{L^2}\big)^{1/2}$. We fix $m \geq 2$ and, proceeding by induction, we assume that for all $0 \leq l \leq m-1$, we have shown that

$$U \in L^\infty(\mathbb{R}_+;\, (\dot{H}^l(\mathcal{M}))^3) \cap L^2(0,\, t_\star;\, (\dot{H}^{l+1}(\mathcal{M}))^3), \quad \forall\, t_\star > 0, \tag{4.1}$$

with

$$\int_t^{t+r} |U(t')|^2_{l+1}\, \mathrm{d}t' \leq a_l, \quad \forall\, t \geq t_l(U_0), \tag{4.2}$$

where a_l is a constant depending on the data (and l) but not on U_0, and $r > 0$ is fixed (the same as before). We then want to establish the same results for $l = m$.

In equation (2.13) we take $\tilde{U} = \Delta^m U(t)$ with $m \geq 2$ and t arbitrarily fixed, and we obtain:

$$\begin{aligned}\Big(\frac{\mathrm{d}U}{\mathrm{d}t},\, \Delta^m U\Big)_{L^2} + a(U, \Delta^m U) + b(U, U, \Delta^m U) + \frac{1}{\varepsilon} e(U,\, \Delta^m U)& \\ = (S,\, \Delta^m U)_{L^2}&.\end{aligned} \tag{4.3}$$

Integrating by parts, using periodicity and the coercivity of a and the fact that $e(U, U) = 0$, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U(t)|^2_m + c_1 |U|^2_{m+1} \leq |b(U, U, \Delta^m U)| + |(S,\, \Delta^m U)_{L^2}|. \tag{4.4}$$

We need to estimate the terms on the right hand side of (4.4). We first notice that

$$|(S,\, \Delta^m U)_{L^2}| \leq c|S|^2_{m-1} + \frac{c_1}{2(m+3)} |U|^2_{m+1}, \tag{4.5}$$

and it remains to estimate $|b(U, U, \Delta^m U)|$.

By the definition of b we have:

$$\begin{aligned} b(U, U, \Delta^m U) = \int_{\mathcal{M}} (u u_x + w(U) u_z) \Delta^m u \, \mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (u v_x + w(U) v_z) \Delta^m v \, \mathrm{d}\mathcal{M} \\ + \kappa \int_{\mathcal{M}} (u \rho_x + w(U) \rho_z) \Delta^m \rho \, \mathrm{d}\mathcal{M}. \end{aligned} \tag{4.6}$$

The computations are similar for all the terms, and, for simplicity, we shall only estimate the first integral on the right hand side of (4.6).

We notice that $b(U, U, \Delta^m U)$ is a sum of integrals of the type

$$\int_{\mathcal{M}} u \frac{\partial u}{\partial x} D_1^{2\alpha_1} D_3^{2\alpha_3} u \, \mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} w(U) \frac{\partial u}{\partial z} D_1^{2\alpha_1} D_3^{2\alpha_3} u \, \mathrm{d}\mathcal{M},$$

where $\alpha_i \in \mathbb{N}$ with $\alpha_1 + \alpha_3 = m$. By D_i we denoted the differential operator $\partial/\partial x_i$. Integrating by parts and using periodicity, the integrals take the form

$$\int_{\mathcal{M}} D^\alpha\Big(u\frac{\partial u}{\partial x}\Big) D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} D^\alpha\Big(w(U)\frac{\partial u}{\partial z}\Big) D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.7}$$

where $D^\alpha = D_1^{\alpha_1} D_3^{\alpha_3}$. Using Leibniz' formula, we see that the integrals are sums of integrals of the form

$$\int_{\mathcal{M}} u\, D^\alpha \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} w(U)\, D^\alpha \frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.8}$$

and of integrals of the form

$$\int_{\mathcal{M}} \delta^k u\, \delta^{m-k} \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} \delta^k w(U)\, \delta^{m-k} \frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.9}$$

with $k = 1, ..., m$, where δ^k is some differential operator D^α with $[\alpha] = \alpha_1 + \alpha_3 = k$. For each α, after integration by parts we see that the sum of the two integrals in (4.8) is zero because of the mass conservation equation (1.1d). It remains to estimate the integrals of type (4.9). We use here the Sobolev and interpolation inequalities. For the first term in (4.9) we write:

$$\begin{aligned}\Big| \int_{\mathcal{M}} \delta^k u\, \delta^{m-k} \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M} \Big| &\le |\delta^k u|_{L^4} \Big| \delta^{m-k} \frac{\partial u}{\partial x} \Big|_{L^4} |D^\alpha u|_{L^2} \\ &\le c_1' |\delta^k u|_{L^2}^{1/2} |\delta^k u|_{H^1}^{1/2} \Big| \delta^{m-k} \frac{\partial u}{\partial x} \Big|_{L^2}^{1/2} \Big| \delta^{m-k} \frac{\partial u}{\partial x} \Big|_{H^1}^{1/2} |D^\alpha u|_{L^2} \\ &\le c_1' |U|_k^{1/2} |U|_{k+1}^{1/2} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_m,\end{aligned} \tag{4.10}$$

where $k = 1, ..., m$.

The second term from (4.9) is estimated as follows:

$$\begin{aligned}\Big| \int_{\mathcal{M}} \delta^k w(U)\, \delta^{m-k} \frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M} \Big| &\le |\delta^k w(U)|_{L^2} \Big| \delta^{m-k} \frac{\partial u}{\partial z} \Big|_{L^4} |D^\alpha u|_{L^4} \\ &\le c_2' |\delta^k w(U)|_{L^2} \Big| \delta^{m-k} \frac{\partial u}{\partial z} \Big|_{L^2}^{1/2} \Big| \delta^{m-k} \frac{\partial u}{\partial z} \Big|_{H^1}^{1/2} |D^\alpha u|_{L^2}^{1/2} |D^\alpha u|_{H^1}^{1/2} \\ &\le c_3' |U|_{k+1} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_m^{1/2} |U|_{m+1}^{1/2},\end{aligned} \tag{4.11}$$

where $k = 1, ..., m$.

From (4.10) and (4.11) we obtain that:

$$
\begin{aligned}
|b(U,U,\Delta^m U)| \le c_3 \sum_{k=1}^{m} |U|_k^{1/2} |U|_{k+1}^{1/2} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_m \\
+ c_3 \sum_{k=1}^{m} |U|_{k+1} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_m^{1/2} |U|_{m+1}^{1/2}.
\end{aligned} \tag{4.12}
$$

We now need to bound the terms on the right hand side of (4.12). The terms corresponding to $k = 2, ..., m-1$ in the first sum do not contain $|U|_{m+1}$ and we leave them as they are. For $k = 1$ and $k = m$, we apply Young's inequality and we obtain:

$$
c_3 |U|_1^{1/2} |U|_2^{1/2} |U|_m^{3/2} |U|_{m+1}^{1/2} \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_4' |U|_1^{2/3} |U|_2^{2/3} |U|_m^2. \tag{4.13}
$$

For the terms in the second sum in (4.12) we distinguish between $k = 1$, $k = m$ and $k = 2, ..., m-1$. The term corresponding to $k = 1$ is bounded by:

$$
c_3 |U|_2 |U|_m |U|_{m+1} \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_5' |U|_2^2 |U|_m^2. \tag{4.14}
$$

For $k = m$ we find:

$$
c_3 |U|_1^{1/2} |U|_2^{1/2} |U|_m^{1/2} |U|_{m+1}^{3/2} \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_6' |U|_1^2 |U|_2^2 |U|_m^2. \tag{4.15}
$$

For the terms corresponding to $k = 2, ..., m-1$ we apply Young's inequality in the following way:

$$
\begin{aligned}
&c_3 |U|_{k+1} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_m^{1/2} |U|_{m+1}^{1/2} \\
&\qquad \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_7' |U|_{k+1}^{4/3} |U|_{m-k+1}^{2/3} |U|_{m-k+2}^{2/3} |U|_m^{2/3}.
\end{aligned} \tag{4.16}
$$

Gathering all the estimates above we find:

$$
\frac{\mathrm{d}}{\mathrm{d}t} |U|_m^2 + c_1 |U|_{m+1}^2 \le \xi + \eta |U|_m^2,
$$

where the expressions of ξ and η are easily derived from (4.4), (4.13), (4.14), (4.15) and (4.16). Using the Gronwall lemma and the induction hypotheses (4.1)–(4.2) we obtain a bound for U in $L^\infty(0, t_\star; H^m)$ and $L^2(0, t_\star; H^{m+1})$, for all fixed $t_\star > 0$, this bound depending also on $|U_0|_m$. We also see that, because of the induction hypotheses (4.1)–(4.2), we can apply the uniform Gronwall lemma and we obtain U bounded in $L^\infty(\mathbb{R}_+; H^m)$ with a bound independent of $|U_0|_m$ when $t \ge t_m(U_0)$; we also obtain an analogue of (4.2). The details regarding the way we apply the uniform Gronwall lemma and derive these bounds are similar to the developments in Section 3.

In summary we have proven the following result:

Theorem 4.1. *Given $m \in \mathbb{N}$, $m \geq 1$, $U_0 \in V \cap (\dot{H}^m_{\text{per}}(\mathcal{M}))^3$ and $S \in L^\infty(\mathbb{R}_+; H \cap (\dot{H}^{m-1}_{\text{per}}(\mathcal{M}))^3)$, equation (2.13) has a unique solution U such that*

$$U \in L^\infty(\mathbb{R}_+; (\dot{H}^m_{\text{per}}(\mathcal{M}))^3) \cap L^2(0, t_\star; (\dot{H}^{m+1}_{\text{per}}(\mathcal{M}))^3), \quad \forall\, t_\star > 0. \tag{4.17}$$

Remark 4.1. Since $\cap_{m\geq 0} \dot{H}^m_{\text{per}}(\mathcal{M}) = \dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M})$, given $U_0 \in (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3$ and $S \in L^\infty(\mathbb{R}_+; (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3)$, equation (2.13) has a unique solution U belonging to $L^\infty(\mathbb{R}_+; (\dot{H}^m_{\text{per}}(\mathcal{M}))^3)$ for all $m \in \mathbb{N}$; that is, U is in $L^\infty(\mathbb{R}_+; (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3)$. Regularity (differentiability) in time can be also derived if S is also $\mathcal{C}^\infty$ in time. However the arguments above do not provide the existence of an absorbing set in $(\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3$.

5. Appendix: Physical Background

The large-scale ocean equations considered in this article, also called the Primitive Equations (PEs), are derived from the general conservation laws of physics using the Boussinesq and hydrostatic approximations. They comprise: the conservation of horizontal momentum equation, the hydrostatic equation, the continuity equation, the equation for the temperature (conservation of energy), the equation of diffusion for the salinity and the equation of state (see, e.g., [7], [8] or [12]):

$$\frac{\partial \boldsymbol{v}^*}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)\boldsymbol{v}^* + w^* \frac{\partial \boldsymbol{v}^*}{\partial z^*} + fk \times \boldsymbol{v}^* + \frac{1}{\rho_{\text{ref}}} \nabla p^* = \mu^*_{\boldsymbol{v}}\, \Delta^*_{\text{h}} \boldsymbol{v}^* + \nu^*_{\boldsymbol{v}} \frac{\partial^2 \boldsymbol{v}^*}{\partial z^{*2}}, \tag{5.1a}$$

$$\frac{\partial p^*_{\text{full}}}{\partial z^*} = -\rho^*_{\text{full}}\, g, \tag{5.1b}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0, \tag{5.1c}$$

$$\frac{\partial T}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)T + w^* \frac{\partial T}{\partial z^*} = \mu_T\, \Delta^*_{\text{h}} T + \nu_T \frac{\partial^2 T}{\partial z^{*2}}, \tag{5.1d}$$

$$\frac{\partial S}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)S + w^* \frac{\partial S}{\partial z^*} = \mu_S\, \Delta^*_{\text{h}} S + \nu_S \frac{\partial^2 S}{\partial z^{*2}}, \tag{5.1e}$$

$$\rho^*_{\text{full}} = \rho_{\text{ref}} \big[1 - \beta_T(T - T_{\text{ref}}) - \beta_S(S - S_{\text{ref}})\big]. \tag{5.1f}$$

Here $\boldsymbol{v}^* = (u^*, v^*)$ is the horizontal velocity, w^* the vertical velocity, p^*_{full} the (full) pressure, ρ^*_{full} the (full) density, T the temperature and S the salinity. Asterisks denote dimensional quantities, a notation which will be useful below when we non-dimensionalise. The constants ρ_{ref}, T_{ref}, S_{ref} denote reference (average) values respectively for the density, temperature and salinity; g is the gravitational acceleration and f the Coriolis parameter. The horizontal gradient and Laplacian operators are denoted by ∇^* and Δ^*_{h}, respectively.

We recall that in the hydrostatic approximation of the Boussinesq equation, the conservation of the vertical momentum equation is replaced by the hydrostatic equation corresponding to its leading terms (5.1b). We chose a linear equation of state (5.1f), but this is not essential; appropriate nonlinear equations could be handled similarly; also $\rho_{\rm ref}$, $T_{\rm ref}$ and $S_{\rm ref}$ could be nonconstant with suitable changes in the following. Equations (5.1) correspond to the f-plane approximation of equations on the sphere, hence $f = \text{constant} = 2\Omega$, Ω being the angular velocity of the Earth in its rotation around the poles' axes.

A simplification of this system can be obtained if we assume that $\beta_T\nu_T = \beta_S\nu_S$ and $\beta_T\mu_T = \beta_S\mu_S$ so that (5.1d)–(5.1f) can be combined into a single equation for ρ, namely:

$$\frac{\partial\rho^*_{\rm full}}{\partial t^*} + u^*\frac{\partial\rho^*_{\rm full}}{\partial x^*} + v^*\frac{\partial\rho^*_{\rm full}}{\partial y^*} + w^*\frac{\partial\rho^*_{\rm full}}{\partial z^*} = \mu^*_\rho\,\Delta^*_{\rm h}\rho^*_{\rm full} + \nu^*_\rho\,\frac{\partial^2\rho^*_{\rm full}}{\partial z^{*2}}\,. \tag{5.2}$$

We are interested in the case where the density $\rho^*_{\rm full}$ is of the form

$$\rho^*_{\rm full}(x,\,y,\,z,\,t) = \rho_{\rm ref} + \bar\rho(z) + \rho^*(x,\,y,\,z,\,t), \tag{5.3}$$

where $\bar\rho = \bar\rho(z)$ is a stratification profile of the density. Similarly, we write the pressure as,

$$p^*_{\rm full}(x,\,y,\,z,\,t) = p_{\rm ref} + \bar p(z) + p^*(x,\,y,\,z,\,t), \tag{5.4}$$

where $\partial p_{\rm ref}/\partial z = -g\rho_{\rm ref}$ and $\partial\bar p/\partial z^* = -g\bar\rho$. With this, (5.1b) reduces to

$$\frac{\partial p^*}{\partial z^*} = -g\rho^*. \tag{5.5}$$

We shall be interested in the physical regimes where $|\bar\rho| \ll |\rho_{\rm ref}|$ and $|\rho^*| \ll |\bar\rho|$, the first inequality meaning that the density profile $\bar\rho$ does not depart too much from a mean reference value $\rho_{\rm ref}$ and the second one meaning that the horizontal and temporal variations of the density surfaces are very small compared to the vertical stratification. Furthermore, we consider a part of the ocean where $\bar\rho(z)$ is a linear function of z and introduce the (constant) Brunt–Väisälä frequency N^*, defined by

$$(N^*)^2 = -\frac{g}{\rho_{\rm ref}}\,\frac{{\rm d}\bar\rho}{{\rm d}z}\,. \tag{5.6}$$

With this, the evolution equation for density (5.2) can be written as,

$$\frac{\partial\rho^*}{\partial t^*} + u^*\frac{\partial\rho^*}{\partial x^*} + v^*\frac{\partial\rho^*}{\partial y^*} + w^*\frac{\partial\rho^*}{\partial z^*} - \frac{\rho_{\rm ref}}{g}(N^*)^2\,w^* = \mu_\rho\,\Delta^*_{\rm h}\rho^* + \nu_\rho\,\frac{\partial^2\rho^*}{\partial z^{*2}}\,. \tag{5.7}$$

At this point, we have reduced the PEs to (5.1a), (5.5), (5.1c), and (5.7), with the dependent variables being $(\boldsymbol{v}^*, w^*, \rho^*, p^*)$. We now non-dimensionalise this set of equations by means of the following typical scales: For length and velocity, we write

$$\begin{aligned} x^* &= Lx, & y^* &= Ly, & z^* &= Hz, \\ u^* &= Uu, & v^* &= Uv, & w^* &= Ww, \end{aligned}$$

where (x, y, z) and (u, v, w) are dimensionless variables. We also define the aspect ratio

$$\delta := H/L. \tag{5.8}$$

Since we are interested in the advective timescale, we write $t^* = Tt$ with t dimensionless, where

$$T = L/U, \tag{5.9}$$

and define the Rossby number as

$$\varepsilon = U/fL. \tag{5.10}$$

The (perturbation) pressure p^* is non-dimensionalised by

$$p^* = (U^2\rho_{\mathrm{ref}}/\varepsilon)\, p, \tag{5.11}$$

and the (perturbation) density ρ^* by

$$\rho^* = (U^2\rho_{\mathrm{ref}}/\varepsilon gH)\, \rho, \tag{5.12}$$

where again $p(x, y, z, t)$ and $\rho(x, y, z, t)$ are dimensionless. We define the Burger number as

$$N = N^* H/fL. \tag{5.13}$$

Finally, we define the non-dimensional eddy viscosity coefficients (inverse Reynolds numbers) by

$$\begin{aligned} \mu_{\boldsymbol{v}} &= \mu_{\boldsymbol{v}}^*/UL, & \nu_{\boldsymbol{v}} &= \nu_{\boldsymbol{v}}^* L/UH^2, \\ \mu_\rho &= \mu_\rho^*/UL, & \nu_\rho &= \nu_\rho^* L/UH^2\,. \end{aligned}$$

We shall choose $\mu_{\boldsymbol{v}} = \nu_{\boldsymbol{v}}$ and $\mu_\rho = \nu_\rho$ for the sake of simplicity.

With these, we can write the PEs in the completely non-dimensional form,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\Delta_3 u + S_u\,, \tag{5.14a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u + \frac{1}{\varepsilon}\frac{\partial p}{\partial y} = \nu_{\boldsymbol{v}}\Delta_3 v + S_v\,, \tag{5.14b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{5.14c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5.14d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon}w = \nu_\rho\Delta_3\rho + S_\rho\,. \tag{5.14e}$$

Here the forcing and source terms $S_u,\ S_v,\ S_\rho$ have been added to the right-hand sides for mathematical generality.

In this paper, we shall consider the case of two spatial dimensions by assuming that all functions are independent of y, but we allow v to be non-zero. We intend to study the three-dimensional case in a similar paper. The system (5.14) becomes now (1.1). We notice easily that if $u,\ v,\ \rho,\ w,\ p$ are solutions of (1.1) for $S = (S_u,\ S_v,\ S_\rho)$, then $\tilde{u},\ \tilde{v},\ \tilde{\rho},\ \tilde{w},\ \tilde{p}$ are solutions of (1.1) for $\tilde{S}_u,\ \tilde{S}_v,\ \tilde{S}_\rho$ where:

$$\begin{aligned}
\tilde{u}(x,\, z,\, t) &= u(x,\, -z,\, t), & \tilde{v}(x,\, z,\, t) &= v(x,\, -z,\, t),\\
\tilde{w}(x,\, z,\, t) &= -\, w(x,\, -z,\, t), & \tilde{p}(x,\, z,\, t) &= p(x,\, -z,\, t),\\
\tilde{\rho}(x,\, z,\, t) &= -\, \rho(x,\, -z,\, t), & &\\
\tilde{S}_u(x,\, z,\, t) &= S_u(x,\, -z,\, t), & \tilde{S}_v(x,\, z,\, t) &= S_v(x,\, -z,\, t),\\
\tilde{S}_\rho(x,\, z,\, t) &= -\, S_\rho(x,\, -z,\, t). & &
\end{aligned}$$

Therefore if we assume that $S_u,\ S_v$ are even in z and S_ρ is odd in z, then we can anticipate the existence of a solution of (1.1) such that:

$$u,\ v,\ w,\ p,\ \rho \text{ are periodic in } x \text{ and } z \text{ with periods } L_1 \text{ and } L_3,$$

and

$$u,\ v \text{ and } p \text{ are even in } z;\ w \text{ and } \rho \text{ are odd in } z,$$

provided the initial conditions satisfy the same symmetry properties. Our aim is to solve the problem (1.1) with the periodicity and symmetry properties above and with initial data

$$u = u_0,\ v = v_0,\ \rho = \rho_0 \text{ at } t = 0.$$

Hence the natural function spaces for this problem are as follows:

$$V = \{(u,\, v,\, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3,\, u,\, v \text{ even in } z \,,\, \rho \text{ odd in } z,\, u_{(k_1,\,0)} = 0\},$$
$$H = \text{closure of } V \text{ in } L^2(\mathcal{M})^3.$$

The motivations for considering periodic boundary conditions is that there are needed in studies on homogeneous turbulence of the atmosphere and also for the study of the renormalized equations considered in [9].

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