Asymptotic Analysis

GEVREY CLASS REGULARITY FOR THE PRIMITIVE EQUATIONS IN SPACE DIMENSION 2

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Abstract. The aim of this article is to prove results of space and time regularity of solutions for the Primitive Equations of the ocean in space dimension two with periodic boundary conditions. It is shown that these solutions belong to a certain Gevrey class of functions which is a subset of real analytic functions.

Contents

1. Introduction

In this article we consider the Primitive Equations for the ocean or for the atmosphere in space dimension two with periodic boundary conditions (for details regarding the form of the primitive equations see e.g. [8], [6] or [7]). The form of the equations used is in this article is close to that considered in [9], so for more details regarding the existence of the solutions for the primitive equations the reader is referred to [9]. In this article it is proved that, considering a forcing term which is an analytical function in time with values in some Gevrey space, the solutions of the Primitive Equations starting with initial data in the Sobolev space H^1 become, for some positive time, elements of a certain Gevrey class and the solutions are thus real analytic functions. One can show that the unique solution is restriction to the real time axis $t \geq 0$, of a complex function analytic in the temporal variable t in some complex neighborhood of the real time axis.

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This article was inspired by the article by Foias and Temam [5] who proved similar results for the Navier-Stokes equations in space dimension two and three with periodical boundary conditions (see also [3]). We also mention here the works of Ferrari and Titi [2] who proved that the solutions of a certain class of nonlinear parabolic equations belong to a certain Gevrey class; also that of Cao, Rammaha and Titi [1] who established the Gevrey regularity for a certain class of analytic nonlinear parabolic equations on the two-dimensional sphere.

1.1. **Preliminaries.** We consider the PEs in their usual (dimensional) form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = \nu\,\Delta u + F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + fu = \nu\,\Delta v + F_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \mu\,\Delta T + F_T. \tag{1.1e}$$

Here $(u,\, v,\, w)$ are the three components of the velocity vector and, as usual, we denote respectively by p, ρ and T, the pressure, density and temperature deviations from a prescribed main value corresponding to the natural stratification. The relationship between ρ and T is $\rho = -\alpha\rho_0 T$. In general the temperature and the density are related by the equation of state $\rho = \rho_0(1 - \alpha(T - T_0))$ where ρ_0 and T_0 are the reference values for the density and the temperature, but in our case we already subtracted the average values from the actual values. The constant g is the gravitational acceleration and f the Coriolis parameter; ν and μ are the eddy diffusivity coefficients. This form of the PEs corresponds to the ocean, although the salinity has been omitted which does not raise any new mathematical difficulty; some minor changes, not done here, are necessary for the atmosphere.

We consider the following domain:

$$\Omega = (0,\, L_1) \times (-L_3/2,\, L_3/2), \tag{1.2}$$

and we assume space periodicity with period Ω, that is, all functions are taken to satisfy $f(x + L_1, z, t) = f(x, z, t) = f(x, z + L_3, t)$ when extended to $\mathbb{R}^2$. Moreover, we assume, as in [9], that the following symmetries hold:

$$
\begin{aligned}
u(x,z,t) &= u(x,-z,t), & F_u(x,z,t) &= F_u(x,-z,t),\\
v(x,z,t) &= v(x,-z,t), & F_v(x,z,t) &= F_v(x,-z,t),\\
T(x,z,t) &= -T(x,-z,t), & F_T(x,z,t) &= -F_T(x,-z,t),\\
w(x,z,t) &= -w(x,-z,t), & p(x,z,t) &= p(x,-z,t),
\end{aligned}
\tag{1.3}
$$

that is to say that we search for u, v, p even and w, T odd; the motivations for considering such solutions are described in [9]. Note that without the symmetry properties (1.3), space periodicity is not consistent with the equations (1.1).

The natural function spaces for this problem are as follows:

$$
\begin{aligned}
V = \{(u,\, v,\, T) \in (\dot{H}^1_{\mathrm{per}}(\Omega))^3,\\
u,\, v \text{ even in } z,\, T \text{ odd in } z,\, \textstyle\int_{-L_3/2}^{L_3/2} u(x,z')\,\mathrm{d}z' = 0\},
\end{aligned}
\tag{1.4}
$$

$$
H = \text{closure of } V \text{ in } (\dot{L}^2(\Omega))^3. \tag{1.5}
$$

Here the dots above $\dot{H}^1_{\mathrm{per}}$ or $\dot{L}^2$ denote the functions with average in Ω equal to zero. These spaces are endowed with Hilbert scalar products; in H the scalar product is

$$
(U,\, \tilde{U})_H = (u,\, \tilde{u})_{L^2} + (v,\, \tilde{v})_{L^2} + \kappa(T,\, \tilde{T})_{L^2}, \tag{1.6}
$$

and in $\dot{H}^1_{\mathrm{per}}$ and V the scalar product is (using the same notation when there is no ambiguity):

$$
((U,\, \tilde{U}))_V = ((u,\, \tilde{u})) + ((v,\, \tilde{v})) + \kappa((T,\, \tilde{T})); \tag{1.7}
$$

here we have written $\mathrm{d}\Omega$ for $\mathrm{d}x\,\mathrm{d}z$, and

$$
((\phi,\, \tilde{\phi})) = \int_\Omega \Big(\frac{\partial \phi}{\partial x}\frac{\partial \tilde{\phi}}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \tilde{\phi}}{\partial z} \Big)\,\mathrm{d}\Omega. \tag{1.8}
$$

The relations above define the norms $|\cdot|_H$ and $\|\cdot\|_V$. The positive constant κ is chosen below. We have

$$
|U|_H \le c_0 \|U\|_V,\ \forall\, U \in V, \tag{1.9}
$$

where $c_0 > 0$ is a positive constant related to κ and the Poincaré constant in $\dot{H}^1_{\mathrm{per}}(\Omega)$.

The prognostic variables of the system are u, v and T and the diagnostic variables are w and p. We can express the diagnostic variables w and p in terms of the prognostic variables u, v, and T. For each $U = (u, v, T) \in V$ we can determine uniquely

$$
w = w(U) = -\int_0^z u_x(x, z', t)\,\mathrm{d}z'. \tag{1.10}
$$

Note that $w = 0$ at $z = 0$ and $L_3/2$ by the requirements on w (periodicity and anti-symmetry); see more details in [9]. By (1.10), the fact that $w = 0$ at $z = L_3/2$ gives the constraint on u:

$$\int_{-L_3/2}^{L_3/2} u_x \,\mathrm{d}z = 0. \tag{1.11}$$

As for the pressure, it can be determined uniquely in terms of T up to its value at $z = 0$, p_{s}, namely,

$$p(x,z,t) = p_{\mathrm{s}}(x,t) + \alpha\rho_0 \int_0^z T(x,z',t)\,\mathrm{d}z'.$$

Considering a test function $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{T})$ in V, we multiply equation (1.1a) by $\tilde{u}$, (1.1b) by $\tilde{v}$ and (1.1e) by $\kappa\tilde{T}$. We obtain the variational formulation of the problem as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_H + a(U,\tilde{U}) + b(U,U,\tilde{U}) + e(U,\tilde{U}) = (F,\tilde{U})_H, \qquad \forall \tilde{U} \in V, \tag{1.12}$$

and we supplement this equation with the initial condition $U = U_0$.

Here we set

$$\begin{aligned}
a(U,\tilde{U}) &= \nu((u,\tilde{u})) + \nu((v,\tilde{v})) + \kappa\mu((T,\tilde{T})),\\
e(U,\tilde{U}) &= f\int_\Omega (u\tilde{v} - v\tilde{u})\,\mathrm{d}\Omega - \alpha g \int_\Omega T\tilde{w}\,\mathrm{d}\Omega,\\
b(U,U^\sharp,\tilde{U}) &= \int_\Omega \Big(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\Big)\tilde{u}\,\mathrm{d}\Omega + \int_\Omega \Big(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\Big)\tilde{v}\,\mathrm{d}\Omega\\
&\qquad + \kappa\int_\Omega \Big(u\frac{\partial T^\sharp}{\partial x} + w(U)\frac{\partial T^\sharp}{\partial z}\Big)\tilde{T}\,\mathrm{d}\Omega.
\end{aligned}$$

We notice that:

$$a(U,U) + e(U,U) = \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - \alpha g\int_\Omega T w(U)\,\mathrm{d}\Omega,$$

and since

$$\Big|\alpha g \int_\Omega T w(U)\,\mathrm{d}\Omega\Big| \le \alpha g |T|_{L^2} |w(U)|_{L^2} \le c\alpha g \|T\|\|u\|,$$

we find that

$$a(U,U) + e(U,U) \ge \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - c\alpha g\|T\|\|u\|. \tag{1.13}$$

From equation (1.13) we see that for κ large enough, more specifically for $\kappa \ge (g^2\alpha^2c^2)/(\nu\mu)$ the bilinear, continuous form $a + e$ is coercive on V, and

$$a(U,U) + e(U,U) \ge \frac{\nu}{2}\|u\|^2 + \nu\|v\|^2 + \kappa\frac{\mu}{2}\|T\|^2 \ge c_1\|U\|_V^2. \tag{1.14}$$

We also mention that the form b is trilinear continuous from $V \times V \times V_2$ where V_2 is defined as the closure of V in $(\dot{H}^2_{\mathrm{per}}(\Omega))^3$; for more details regarding the way we obtain these results, see e.g. [9].

Equation (1.12) is equivalent to an evolution equation of the form:

$$\begin{aligned}&\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U,U) + E(U) = F,\\ &U(0) = U_0,\end{aligned} \tag{1.15}$$

in the space V_2', which is the dual of V_2. For more details regarding the derivation of the variational and evolutional form for the Primitive Equations and also for the derivation of the properties of the forms a and b the reader is referred to [9]. In that article existence and uniqueness of solutions and regularity results in all Sobolev spaces H^m are derived for the non-dimensionalised Primitive Equations for the ocean in periodic space dimension two; though the equations are not absolutely identical to those considered here, one can, with minimal changes, derive similar results for the equations considered here.

All the functions being periodic, they admit Fourier series expansions. Hence, for instance, for U we write

$$U = \sum_{(k_1,\, k_3)\in\mathbb{Z}^2} U_{(k_1,k_3)} e^{i(k_1'x + k_3'z)},$$

where $k_j' = 2\pi k_j/L_j$. We also introduce the following notation:

$$[U_k]_\kappa^2 = |u_k|^2 + |v_k|^2 + \kappa|T_k|^2.$$

Considering the Laplacian $-\Delta$, we define the Gevrey class $D(e^{\tau(-\Delta)^s})$ as the set of functions U in H satisfying

$$|\Omega| \sum_{k\in\mathbb{Z}^2} e^{2\tau|k'|^{2s}} [U_k]^2 = |e^{\tau(-\Delta)^s} U|_H^2 < \infty. \tag{1.16}$$

The norm of the Hilbert space $D(e^{\tau(-\Delta)^s})$ is given by

$$|U|_{D(e^{\tau(-\Delta)^s})} = |e^{\tau(-\Delta)^s} U|_H, \text{ for } U \in D(e^{\tau(-\Delta)^s}), \tag{1.17}$$

and the associated scalar product is

$$(U,V)_{D(e^{\tau(-\Delta)^s})} = (e^{\tau(-\Delta)^s} U, e^{\tau(-\Delta)^s} V)_H, \text{ for } U, V \in D(e^{\tau(-\Delta)^s}). \tag{1.18}$$

Another Gevrey type space that we will use is $D((-\Delta)^{1/2} e^{\tau(-\Delta)^s})$, which is a Hilbert space when endowed with the inner product:

$$\begin{aligned}(U,V)_{D((-\Delta)^{1/2}e^{(-\Delta)^s})} &= ((-\Delta)^{1/2} e^{\tau(-\Delta)^s} U, (-\Delta)^{1/2} e^{\tau(-\Delta)^s} V)_H\\ &= ((e^{\tau(-\Delta)^s} U, e^{\tau(-\Delta)^s} V))_V,\end{aligned} \tag{1.19}$$

for U, V in $D((-\Delta)^{1/2} e^{\tau(-\Delta)^s})$; the norm of the space is given by

$$\begin{aligned}|U|^2_{D((-\Delta)^{1/2}e^{\tau(-\Delta)^s})} = |(-\Delta)^{1/2} e^{\tau(-\Delta)^s} U|_H^2 &= \|e^{\tau(-\Delta)^s} U\|_V^2\\ &= |\Omega| \sum_{k\in\mathbb{Z}^2} |k'|^2 e^{2\tau|k'|^{2s}} [U_k]_\kappa^2.\end{aligned} \tag{1.20}$$

2. A Priori Estimates for the Real Case

As we already mentioned in the introduction, our aim is to prove that the solutions of the PEs are real functions analytic in time with values in Gevrey spaces and the restriction of some complex analytic functions in time in the neighborhood of a real positive interval. We start in this section by deriving some a priori estimates in the real case and then we consider the complex case.

We begin with the following technical result:

Lemma 2.1. *Let U, $U^\sharp$ and $\tilde{U}$ be given in $D(\Delta e^{\tau(-\Delta)^s})$ for $\tau \geq 0$. Then the following inequality holds:*

$$\begin{aligned}
|(e^{\tau(-\Delta)^{1/2}}B(U,U^\sharp), e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U})_H| \leq c_2 |e^{\tau(-\Delta)^{1/2}}(-\Delta)^{1/2}U|^{1/2} \\
|e^{\tau(-\Delta)^{1/2}}\Delta U|^{1/2} |e^{\tau(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\sharp|^{1/2} |e^{\tau(-\Delta)^{1/2}}\Delta U^\sharp|^{1/2} |e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U}|.
\end{aligned} \tag{2.1}$$

Proof. We start by writing the trilinear form b in Fourier modes. For that purpose we define, for each $j \in \mathbb{Z}^2$, δ_j as j_1'/j_3' when $j_3' \neq 0$ and as 0 when $j_3' = 0$. We obtain:

$$\begin{aligned}
b(U,U^\sharp,\tilde{U}) = \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)u_l u_j^\sharp \tilde{u}_k + \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)u_l v_j^\sharp \tilde{v}_k \\
+ \kappa \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)u_l T_j^\sharp \tilde{T}_k.
\end{aligned} \tag{2.2}$$

We then compute:

$$\begin{aligned}
(e^{\tau(-\Delta)^{1/2}}B(U,U^\sharp), e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U})_H = \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l u_j^\sharp \tilde{u}_k \\
+ \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l v_j^\sharp \tilde{v}_k + \kappa \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l T_j^\sharp \tilde{T}_k.
\end{aligned} \tag{2.3}$$

We now associate to each u the function $\check{u}$ defined by:

$$\check{u} = \sum_{j\in\mathbb{Z}^2} \check{u}_j e^{ij'\cdot x}, \text{ where } \check{u}_j = e^{\tau|j'|}|u_j|, \tag{2.4}$$

and we also use similar notations for the other functions.

Using the notation above and the fact that $|k| - |l| - |j| \leq 0$ since $j + l + k = 0$, we continue to bound the right-hand side of relation (2.3) and we obtain:

$$\begin{aligned}
|(e^{\tau(-\Delta)^{1/2}}B(U,U^\sharp), e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U})_H| \leq c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{u}_j^\sharp||\check{\tilde{u}}_k| \\
+ c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{v}_j^\sharp||\check{\tilde{v}}_k| + \kappa c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{T}_j^\sharp||\check{\tilde{T}}_k|,
\end{aligned} \tag{2.5}$$

where we also used the estimate $|j_1' - j_3'\delta_l| \leq c|j'||l'|$. Here and in the sequel c denotes a constant which may be different at different places.

We estimate the first term from the right-hand side of (2.5), the rest of the estimates being identical. For that purpose, we define the following functions:

$$\xi(x) = \sum_{j\in\mathbb{Z}^2} |j'|\check{u}_j e^{ij'\cdot x}, \quad \psi(x) = \sum_{j\in\mathbb{Z}^2} |j'|\check{u}^\sharp_j e^{ij'\cdot x}, \quad \theta(x) = \sum_{j\in\mathbb{Z}^2} |j'|^2 \check{\tilde{u}}_j e^{ij'\cdot x},$$

and we write:

$$\begin{aligned}
\sum_{j+l+k=0} |j'||l'||k'|^2|\check{u}_l||\check{u}^\sharp_j||\check{\tilde{u}}_k| &= \frac{1}{|\Omega|}\int_\Omega \xi(x)\psi(x)\theta(x)\,\mathrm{d}\Omega \le c|\xi|_{L^4}|\psi|_{L^4}|\theta|_{L^2} \\
&\le c|\xi|^{1/2}_{L^2}\|\xi\|^{1/2}|\psi|^{1/2}_{L^2}\|\psi\|^{1/2}|\theta|_{L^2} \\
&\le c\|e^{t(-\Delta)^{1/2}}U\|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}U|^{1/2}\|e^{t(-\Delta)^{1/2}}U^\sharp\|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}U^\sharp|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}\tilde{U}|.
\end{aligned}$$

Using the same kind of arguments for the other terms we find the relation (2.1). □

Decomposition of the solution

We want to derive the Gevrey regularity of the problem:

$$\begin{aligned}
&U' + AU + B(U,U) + EU = F, \text{ in } V_2', \\
&U(0) = U_0.
\end{aligned} \tag{2.6}$$

In all that follows we assume that the forcing F is an analytic function in time with values in the Gevrey space $D(e^{\sigma_1(-\Delta)^{1/2}})$ for some $\sigma_1 > 0$. To obtain the desired a priori estimates, we can suppose that the natural way would be to apply the operator $e^{t(-\Delta)^{1/2}}$ to equation (2.6) and to take the scalar product with $-\Delta e^{t(-\Delta)^{1/2}}$ in H. But taking into account the inequality (2.1), we see that, unlike in [5] for the Navier-Stokes equations, we would obtain a weak estimate for the nonlinear term which would force us to work with small initial data. In order to avoid imposing such a restriction, we split the solution U into $U = U^\star + \tilde{U}$, where $U^\star$ is the solution of the linear problem:

$$\begin{aligned}
&\frac{\mathrm{d}U^\star}{\mathrm{d}t} + AU^\star + EU^\star = F, \\
&U^\star(0) = U_0,
\end{aligned} \tag{2.7}$$

and $\tilde{U}$ is the solution of the nonlinear problem:

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star), \\
&\tilde{U}(0) = 0.
\end{aligned} \tag{2.8}$$

We will derive estimates and existence results for the linear problem (2.7) and then for the nonlinear problem (2.8) which is equivalent to (1.15), taking (2.7) into account. We start treating the linear problem:

The linear problem

We suppose that U_0 is in $D((-\Delta)^{1/2})$ and F is a function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}})$, for some $\sigma_1 > 0$. Setting $\varphi(t) = \min(t, \sigma_1)$, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}$ to equation (2.7) and then take the scalar product with $-\Delta e^{\varphi(t)(-\Delta)^{1/2}} U^\star$ in H.

With the same κ as in (1.13) we have:

$$\begin{aligned}(e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H \\ + (e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H \geq c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2.\end{aligned} \tag{2.9}$$

The relation above holds because:

$$\begin{aligned}&(e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H + (e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H \\ &= ((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star)_H \\ &\quad + ((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star)_H \\ &= a(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star) \\ &\quad + e(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star),\end{aligned}$$

where we used that A and E commute with $-\Delta$ and the fact that for the κ chosen before, $a+e$ is coercive. The commutativity of the operators A and E can be easily established using, for example, the Fourier series expansions.

We also have:

$$\begin{aligned}&(e^{\varphi(t)(-\Delta)^{1/2}}(U^\star)'(t), e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star(t))_H \\ &\quad = \Big(\frac{\mathrm{d}}{\mathrm{d}t}((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star), (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star\Big)_H \\ &\quad\quad - \varphi'(t)((-\Delta)e^{\varphi(t)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star)_H \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \varphi'(t)(\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star)_H \\ &\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - |\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H\|e^{\varphi(t)(-\Delta)^{1/2}}U^\star\|_V \\ &\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \frac{c_1}{4}|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \frac{1}{c_1}\|e^{\varphi(t)(-\Delta)^{1/2}}U^\star\|_V^2.\end{aligned}$$

The term containing the force F is estimated using the Schwarz inequality:

$$\begin{aligned}(e^{\varphi(t)(-\Delta)^{1/2}}F, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star)_H &\leq |e^{\varphi(t)(-\Delta)^{1/2}}F|_H|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star|_H \\ &\leq \frac{1}{c_1}|e^{\varphi(t)(-\Delta)^{1/2}}F|_H^2 + \frac{c_1}{4}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star|_H^2.\end{aligned} \tag{2.10}$$

Taking into account all the estimates above, we obtain:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 + c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 \\ &\quad\leq \frac{2}{c_1}|e^{\varphi(t)(-\Delta)^{1/2}}F|_H^2 + \frac{2}{c_1}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2.\end{aligned} \tag{2.11}$$

Applying the Gronwall lemma to (2.11), it follows that:

$$|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 \leq |(-\Delta)^{1/2}U_0|_H^2 e^{\frac{2}{c_1}t} + \sup_{0\leq s\leq t}|e^{\sigma_1(-\Delta)^{1/2}}F(s)|_H^2 e^{\frac{2}{c_1}t}, \tag{2.12}$$

which gives a bound of $(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star$ in $L^\infty(0,t_\star;H)$ for all $t_\star > 0$. Returning to (2.11) and integrating, we find a bound of $\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star$ in $L^2(0,t_\star;H)$ for all $t_\star > 0$.

The nonlinear problem

We now need to study the Gevrey regularity for the following nonlinear problem:

$$\begin{aligned}&\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star), \\ &\tilde{U}(0) = 0,\end{aligned} \tag{2.13}$$

where $U^\star$ is the solution of the linear problem presented above.

As for the linear case, at a time t, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}$ to each side of equation (2.13) and then we take the scalar product in H of the resulting equation with $e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)\tilde{U}$. The difference between this case and the linear case appears in the terms containing the operator B and, to estimate these terms, we use Lemma 2.1. Note that since the norm on H is equivalent to the usual norm on L^2, in the right hand side of (2.1) we can change the norm on L^2 with the norm on H, changing only the preceding constant. Thus, we obtain:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 \\ &\quad\leq f(t)|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(t) \\ &\qquad + c_1|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2,\end{aligned} \tag{2.14}$$

where

$$f(t) = c_1' + c_2' |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} U^\star(t)|_H^2 |\Delta e^{\varphi(t)(-\Delta)^{1/2}} U^\star(t)|_H^2,$$
$$g(t) = c_3' |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} U^\star(t)|_H^2 |\Delta e^{\varphi(t)(-\Delta)^{1/2}} U^\star(t)|_H^2.$$

We rewrite (2.14) as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 &+ (c_0 - c_1 |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H) |\Delta e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 \\ &\leq f(t) |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 + g(t).\end{aligned} \tag{2.15}$$

Since $\tilde{U}(0) = 0$, we may assume that:

$$|(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H \leq \frac{c_1}{2c_2}, \text{ on some finite interval of time } (0, t_0). \tag{2.16}$$

On that interval the following estimate holds:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 &+ \frac{c_1}{2} |\Delta e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 \\ &\leq f(t) |(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 + g(t).\end{aligned} \tag{2.17}$$

Taking into account the a priori estimates obtained for $U^\star$, we find that f and g are functions locally integrable. So, we can apply the Gronwall lemma and deduce the following estimate on $(0, t_0)$:

$$|(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}|_H^2 \leq \int_0^{t^\star} g(s) \exp\left(\int_s^{t^\star} f(\tau)\, \mathrm{d}\tau\right) \mathrm{d}s. \tag{2.18}$$

Since f and g are locally integrable, we can define $t^\star = t(F, U_0, \sigma_1)$ as the first time for which:

$$\int_0^{t^\star} g(s) \exp\left(\int_s^{t^\star} f(\tau)\, \mathrm{d}\tau\right) \mathrm{d}s = \frac{c_1}{2c_2}. \tag{2.19}$$

Then, on the interval $(0, t^\star)$ we find:

$$|(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}(t)|_H \leq c_0/2c_1.$$

Hence, on $(0, t^\star)$, with $t^\star$ defined by (2.19), the solution $\tilde{U}$ satisfies both (2.15) and (2.17).

3. Time Analyticity in Gevrey Spaces

As mentioned in the Introduction, the task of this article is to prove that the solutions of the Primitive Equations are analytic in time with values in some Gevrey spaces. In fact we show that the solution is the restriction to $\mathbb{R}_+$ of a complex analytic function in the temporal variable in a complex domain containing an interval $(0, t_1)$. In order to derive such a result, we use an already classical method (see e.g. [4] or [5]), the idea being to pass from the Primitive Equations written in real time to an extended equation in the complex time. To avoid too complicated notations and because there is no risk of confusion, for the extended spaces and operators we use the same notations as in the real case. In this way, equation (2.6) is rewritten as:

$$\frac{\mathrm{d}U}{\mathrm{d}\zeta} + AU + B(U,U) + EU = F, \tag{3.1}$$

where $\zeta \in \mathbb{C}$ is the complex time.

In all what follows, $\zeta = se^{i\theta}$, where $s > 0$ and $\cos\theta > 0$ so that the real part of ζ is positive. As for the real case, we need to split the solution of the equation (3.1) into $U^\star$ and $\tilde{U}$, where $U^\star$ is the solution of the linear equation:

$$\begin{aligned} &\frac{\mathrm{d}U^\star}{\mathrm{d}\zeta} + AU^\star + EU^\star = F, \\ &U^\star(0) = U_0, \end{aligned} \tag{3.2}$$

and $\tilde{U}$ is the solution of the nonlinear problem:

$$\begin{aligned} &\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\zeta} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star), \\ &\tilde{U}(0) = 0. \end{aligned} \tag{3.3}$$

We start by deriving the a priori estimates for $U^\star$. For that purpose, we apply the operator $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}$ to equation (3.2) and then take the scalar product in H with $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)U^\star$, multiply by $e^{i\theta}$ and take the real part.

We notice that:

$$\begin{aligned} &\operatorname{Re} e^{i\theta}\Big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\frac{\mathrm{d}U^\star}{\mathrm{d}\zeta}, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star\Big)_H \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 \\ &\quad - \varphi'(s\cos\theta)\cos\theta \operatorname{Re} e^{i\theta}(\Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star)_H. \end{aligned} \tag{3.4}$$

Using the same constant κ as in (1.13), we find:

$$\begin{aligned} &\operatorname{Re} e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}AU^\star, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star)_H \\ &\quad + \operatorname{Re} e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}EU^\star, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star)_H \\ &\quad \geq c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2. \end{aligned} \tag{3.5}$$

From all the computations above we conclude that:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}&|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + c_1\cos\theta\,|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2\\ &\le \cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H\\ &\quad + |e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H.\end{aligned}$$

Restricting θ so that $\cos\theta \ge \sqrt{2}/2$ and making use of the Cauchy-Schwarz inequality, we find:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}&|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + c_1\cos\theta\,|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2\\ &\le \frac{2\cos\theta}{c_1}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + \frac{2}{c_1\cos\theta}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H^2.\end{aligned} \tag{3.6}$$

We can now apply the Gronwall lemma to (3.6) and obtain:

$$\begin{aligned}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 \le &|(-\Delta)^{1/2}U_0|_H^2\exp\big(\frac{1}{c_1}s\big)\\ &+ 2|e^{\sigma_1(-\Delta)^{1/2}}F|_H^2\exp\big(\frac{1}{c_1}s\big).\end{aligned} \tag{3.7}$$

Since $U_0 \in D((-\Delta)^{1/2})$, we deduce from (3.7) a bound on $U^\star(se^{i\theta})$ in $D(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2})$ for θ such that $\sqrt{2}/2 \le \cos\theta \le 1$ and for $s \le t$, for all $t \ge 0$.

Integrating equation (3.7), one can see that

$$\int_0^s |e^{\varphi(s'\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2\,\mathrm{d}s' \le C(s,F,U_0,\sigma_1),\ \text{for all } s \ge 0. \tag{3.8}$$

Having in mind these estimates, we start deriving estimates for the solution $\tilde{U}$ of equation (3.3).

The calculations for obtaining the a priori estimates are the same as for the linear case: we apply $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}$ to equation (3.3), take the scalar product in H with $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)U^\star$ and then multiply the resulting equation by $e^{i\theta}$ and take the real part. Using Lemma 2.1 in order to estimate the terms containing the B operator, we find:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}&|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2\\ &\le f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(s)\\ &\quad + c_3|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2,\end{aligned} \tag{3.9}$$

where

$$f(s) = \frac{1}{c_1} + c_1'|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star|_H^2|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star|_H^2,$$

$$g(s) = \frac{\sqrt{2}}{c_1}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H^2.$$

We obtained the form of the functions f and g using the Cauchy-Schwarz inequality and restricting θ to $\sqrt{2}/2 \le \cos\theta \le 1$.

We can also write inequality (3.10) as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + (c_1\frac{\sqrt{2}}{2} - c_3|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H)\\ \cdot |e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 \le f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(s).\end{aligned} \tag{3.10}$$

Since $\tilde{U}(0) = 0$, we may assume that:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \le \frac{c_1\sqrt{2}}{4c_3},$$

on some finite interval $(0, t_0)$ and, on this interval, $\tilde{U}$ satisfies the inequality:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U|_H^2 + c_1\frac{\sqrt{2}}{4}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U|_H^2\\ \le f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U|_H^2 + g(s).\end{aligned} \tag{3.11}$$

Since f and g depend on the solution $U^\star$ of the linear problem and we already obtained a priori estimates on $U^\star$, we see that for all $\theta \in [-\pi/4, \pi/4]$, f and g are locally integrable functions. Thus we can apply the Gronwall lemma to (3.11) and we find the following estimate on $(0, t_0)$:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \le \int_0^t g(s)\exp(\int_s^t f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s. \tag{3.12}$$

Since f and g are locally integrable functions, we can define $t_1 = t(F, U_0, \sigma_1)$ as the time for which we have:

$$\int_0^{t_1} g(s)\exp(\int_s^{t_1} f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s = \frac{c_1\sqrt{2}}{4c_3}. \tag{3.13}$$

So on the interval $(0, t_1)$ we find:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \le \frac{c_1\sqrt{2}}{4c_3}. \tag{3.14}$$

We define the region:

$$D(U_0, F, \sigma_1) = \{\zeta = se^{i\theta},\ |\theta| \le \pi/4,\ 0 < s < t_1(U_0, F, \sigma_1)\}, \tag{3.15}$$

and from the previous estimates we obtain a bound on $U(\zeta)$ in $D((-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}})$:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \leq \frac{c_1\sqrt{2}}{4c_3}, \text{ for } \zeta \in \bar{D}(U_0, F, \sigma_1). \tag{3.16}$$

We can now state the main result of this article:

Theorem 3.1. *Let U_0 be given in $D((-\Delta)^{1/2})$ and let F be a function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}})$ for some $\sigma_1 > 0$. Then there exists t_1 depending on the initial data such that the function*

$$t \to (-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U(t),$$

is analytic on $(0, t_1)$, where $\varphi(t) = \min(t, \sigma_1)$ and t_1 is defined by relation (3.13).

Proof. In order to prove the existence of an analytic solution, we use the Galerkin approximation method based on the Fourier series, and the energy estimates obtained above. For the solutions of the Galerkin approximation the a priori estimates which are formally derived above hold rigorously and the bounds are independent of the order m of the Galerkin approximation. With these estimates we can pass to the limit $m \to \infty$ using classical theorems concerning convergence of analytic functions. From here follows Theorem 3.1. □

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References

[1] C. Cao, M. A. Rammaha, and E. S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations on the sphere*, J. Dynam. Differential Equations, 12 (2000), pp. 411–433.
[2] A. B. Ferrari and E. S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Differential Equations, 23 (1998), pp. 1–16.
[3] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes equations and turbulence*, vol. 83 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2001.
[4] C. Foiaş and R. Temam, *Some analytic and geometric properties of the solutions of the evolution Navier-Stokes equations*, J. Math. Pures Appl. (9), 58 (1979), pp. 339–368.
[5] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal., 87 (1989), pp. 359–369.
[6] J.-L. Lions, R. Temam, and S. Wang, *New formulations of the primitive equations of the atmosphere and applications*, Nonlinearity, 5 (1992), pp. 237–288.
[7] ———, *On the equations of the large-scale ocean*, Nonlinearity, 5 (1992), pp. 1007–1053.
[8] J. Pedlosky, *Geophysical fluid dynamics, 2nd Edition*, Springer, 1987.
[9] M. Petcu, R. Temam, and D. Wirosoetisno, *Existence and regularity results for the primitive equations*, Comm. Pure Appl. Analysis, 3 (2004), pp. 115–131.