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EULER EQUATION IN A CHANNEL IN SPACE DIMENSION 2 AND 3

M. PETCU[*♭♯]

[*]Laboratoire d'Analyse Numérique, Université de Paris–Sud, Orsay, France
[♭]The Institute of Mathematics of the Romanian Academy, Bucharest, Romania
[♯]The Institute for Scientific Computing and Applied Mathematics,
Indiana University, Bloomington, IN, USA

Abstract. In this article we consider the Euler equations of an ideal incompressible fluid in a $2D$ and $3D$ channel and we prove the existence and uniqueness of classical solutions for all time for the $2D$ case and the local in time existence for the $3D$ case. For the $2D$ case, the proof makes use of the Schauder fixed point, and specific properties of the Green function in a channel are derived. For the $3D$ case, we use a priori estimates on some appropriate Sobolev spaces and the existence of solution follows by the Galerkin method.

Contents

1. Introduction

In this article we consider the Euler equations in a 2D and $3D$ channel with the non-penetration boundary condition and we are interested in proving the existence and uniqueness of the solutions.

The case of the 3D channel with nonhomogeneous boundary conditions will be considered in a companion paper [10].

We mention here that the study of the Euler equations in various spaces with various boundary conditions, is a problem of significant interest in mathematical physics. Let us recall the pioneering work in this field of Kato [7], of Lichtenstein [8], of Wolibner [16], and among more recent references the works of Beale, Kato and Majda [1], of Ebin and Marsden [3], [4], of Temam [12], [11] and Vishik [14].

In this article we prove first the existence and uniqueness of classical solutions for all time in a 2D channel corresponding to the nonpenetration condition in the direction normal to the wall and space periodicity in the other direction. The proof extends that of Kato [7] who considered the case of a bounded domain with regular boundary and homogeneous boundary conditions. The form and the properties of the Green function for a two dimensional channel with homogeneous Dirichlet boundary conditions are provided in an Appendix.

We also prove the local existence of the solutions for the Euler equations in a $3D$ channel. The idea is to obtain a priori estimates in certain Sobolev spaces and then, by standard methods, we prove the existence and uniqueness of solution, locally in time.

2. Euler equation in a $2D$ channel

The Euler equation in a 2D channel $\Omega_\infty = \mathbb{R} \times (0, L_2)$ considered here reads:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \operatorname{grad} p = \mathbf{f}, \tag{2.1a}$$

$$\operatorname{div}\ \mathbf{u} = 0, \tag{2.1b}$$

where $\mathbf{u} = (u_1, u_2)$ is the two dimensional velocity of the fluid and p is the pressure. We also consider $\Omega = (0, L_1) \times (0, L_2)$, where L_1 is the period in the $0x_1$ direction. For a function given in Ω, we denote by $\tilde{u}$ its periodic extension to Ω_∞.

We supplement the system (2.1) with the initial condition:

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \forall\, x \in \Omega. \tag{2.2}$$

The boundary conditions for this system are:

$$\begin{aligned} &\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_2 = \{x_2 = 0,\ x_2 = L_2\}, \\ &\text{and the functions are periodic in } x_1. \end{aligned} \tag{2.3}$$

Remark 2.1. Because of the periodicity in the x_1 direction, it is convenient to introduce the average of u_1 over Ω,

$$m_{u_1}(t) = \frac{1}{L_1 L_2} \int_\Omega u_1(x, t)\, \mathrm{d}\Omega,$$

and we set

$$\bar{u}_1 = u_1 - m_{u_1}.$$

The average m_{u_1} is determined explicitly in terms of the data. We write equation (2.1a) for u_1 and integrate over Ω. Integrating by parts and using (2.3), and the divergence-free condition, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{u_1}(t) = m_{f_1}(t), \tag{2.4}$$

which leads to:

$$m_{u_1}(t) = m_{u_{1,0}} + \int_0^t m_{f_1}(s)\,\mathrm{d}s.$$

By substitution to (2.1a), we find:

$$\frac{\partial \bar{u}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x_1} + u_2 \frac{\partial \bar{u}_1}{\partial x_2} + m_{u_1} \frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial p}{\partial x_1} = \bar{f}_1. \tag{2.5}$$

Since the quantity m_{u_1} is known, equation (2.5) has a behavior very similar to the initial equation (2.1), so we can suppose from the beginning that we work with flows having $m_{u_1} = 0$.

The main result of the paper is the following:

Theorem 2.1. *Let L_1, L_2, T be given positive and arbitrary, and let $\Omega = (0, L_1) \times (0, L_2)$. Let $\mathbf{u_0}$ be given such that its periodic extension $\tilde{\mathbf{u}}_\mathbf{0}$ belongs to $\mathcal{C}^{1+\alpha}(\Omega_\infty)$ and satisfies* div $\tilde{\mathbf{u}}_\mathbf{0} = 0$ *and $\tilde{\mathbf{u}}_\mathbf{0} \cdot \mathbf{n} = 0$ at $x_2 = 0$ and $x_2 = L_2$. Let $\mathbf{f}$ be a given function on $Q_T = \bar{\Omega} \times [0, T]$ such that its periodic extension $\tilde{\mathbf{f}}$ on $Q_{\infty,T} = \Omega_\infty \times [0, T]$ belongs to $\mathcal{C}^{1+\alpha,0}(Q_{\infty,T})$, with $0 < \alpha < 1$. Then there exists a unique solution $(\mathbf{u}, p)$ of problem (2.1)–(2.3) such that:*

$$\mathbf{u} \in \mathcal{C}^1(Q_T),\ \nabla p \in \mathcal{C}(Q_T).$$

We recall here the definition of the Hölder spaces and of their norms. The norm is defined as:

$$|f|_{\alpha,Q_T} = |f|_{0,Q_T} + H_x^\alpha(f) + H_t^\alpha(f), \tag{2.6}$$

where $|\cdot|_{0,Q_T}$ is the norm of $\mathcal{C}(Q_T)$ and

$$H_x^\alpha(f) = \sup_{\substack{x_1, x_2 \in \Omega \\ t \in (0,T)}} \{|f(x_1,t) - f(x_2,t)| \cdot |x_1 - x_2|^{-\alpha}\},$$

$$H_t^\alpha(f) = \sup_{\substack{x \in \Omega \\ t_1, t_2 \in (0,T)}} \{|f(x,t_1) - f(x,t_2)| \cdot |t_1 - t_2|^{-\alpha}\}.$$

3. Existence and uniqueness of the solution for all time

In this section we prove Theorem 2.1 using an auxiliary independent result regarding the properties of the Green function, proven in Section 3. The proof of existence of solutions is based on Schauder's fixed point theorem.

We start by explaining the general plan of the proof, and give then the rigorous arguments. We apply the curl operator to equation (2.1a) and we obtain the following problems; for the vorticity the equation reads as:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \text{grad})\omega = \text{curl } \mathbf{f}, \tag{3.1a}$$

$$\omega \text{ is periodic in } x_1, \tag{3.1b}$$

$$\omega(x, 0) = \omega_0(x) = \text{curl } \mathbf{u_0}; \tag{3.1c}$$

and, for the velocity:

$$\text{curl } \mathbf{u} = \omega, \tag{3.2a}$$

$$\text{div } \mathbf{u} = 0, \tag{3.2b}$$

$$\mathbf{u} \text{ is periodic in } x_1,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ at } x_2 = 0 \text{ and } x_2 = L_2, \tag{3.2c}$$

$$m_{u_1} = 0. \tag{3.2d}$$

Note here that without the condition $m_{u_1} = 0$, the flow $\mathbf{u}$ would be determined up to a constant.

3.1. **The general approach and preliminary results.** In this section we will show that problem (3.1)–(3.2) is equivalent to (2.1)–(2.3) and we will find the solution $\mathbf{u}$ of (3.1)–(3.2) as the fixed point of the mapping Λ defined below.

In order to define the mapping Λ, we start with a given function ω^0 and determine a function $\mathbf{u^1}$ from (3.2); then, setting $\mathbf{u} = \mathbf{u^1}$ in (3.1), we find the vorticity ω^1 from (3.1). We thus define the function Λ by:

$$\Lambda : \bigcup_{0<\epsilon<1} \mathcal{C}^{\epsilon,0}(Q_T) \to \mathcal{C}(Q_T),\ \Lambda(\omega^0) = \omega^1, \tag{3.3}$$

and, as mentioned before, we want to prove that Λ has a fixed point ω which is the solution of our problem (2.1).

Equation (3.1) is a first-order partial differential equation for ω, which is equivalent to solving the ordinary differential equation:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = \mathbf{u}(x, t), \\ &x(s) = y, \end{aligned} \tag{3.4}$$

where $0 \le s \le t$, $x = x(t; y, s)$. The solutions of (3.4) are the streamlines (or the trajectories) of the flow $\mathbf{u}$.

Then the solution of (3.1) is determined from the formula:

$$\omega(x(t; y, s), t) = \omega_0(x(0; y, s)) + \int_0^t \text{curl } \mathbf{f}(x(t'; y, s), t')\, \mathrm{d}t'. \tag{3.5}$$

For problem (3.2), we look for solutions of the form $\mathbf{u} = \text{curl } \varphi$ since div $\mathbf{u} = 0$. We thus need to determine φ as the solution of the equation $-\Delta\varphi = \text{curl } \mathbf{u} = \omega$, with φ periodic in x_1 and equal to zero on the boundary $x_2 = 0$ and $x_2 = L_2$ [1]. We find:

$$\mathbf{u}(x,t) = \text{curl }_x \int_\Omega G(x,y)\omega(y,t)\,\mathrm{d}y, \tag{3.6}$$

where G is the Green function of the Dirichlet problem for the channel. We recall here some properties of the Green function we need to use and we give the proof in the Appendix (see Theorem 5.1):

(a) The Green function G is continuous on $\bar{\Omega} \times \bar{\Omega}$, except at $x = y$.

(b) G has continuous first-order partial derivatives with respect to x, except at $x = y$, in the neighborhood of which the following estimate holds:

$$|D_x G(x,y)| \leq K|x-y|^{-1}. \tag{3.7}$$

(c) G is symmetrical in x and y.

We now study the behavior of the function $\mathbf{u}$ given by (3.6). The quasi-Lipschitz condition satisfied by the velocity reads:

Lemma 3.1. *If $\omega \in L^\infty(\Omega)$, then the two following inequalities hold for every x, x' in Ω:*

$$|\mathbf{u}(x)| \leq K\|\omega\|_\infty, \tag{3.8a}$$

$$|\mathbf{u}(x) - \mathbf{u}(x')| \leq K\|\omega\|_\infty \chi(|x-x'|), \tag{3.8b}$$

where $\|\cdot\|_\infty$ is the $L^\infty(\Omega)$-norm, K is a constant depending only on Ω, not the same at each occurrence, and the function χ is:

$$\chi(r) = \begin{cases} r(1-\ln r), & \text{if } r < 1, \\ 1, & \text{if } r \geq 1. \end{cases}$$

Proof. The proof for (3.8a) is easy since:

$$\begin{aligned} |\mathbf{u}(x)| = &\left| \int_\Omega \text{curl } G(x,y)\omega(y)\,\mathrm{d}y \right| \leq \|\omega\|_\infty \int_\Omega |\text{curl }_x G(x,y)|\,\mathrm{d}y \\ &\leq \|\omega\|_\infty \int_\Omega \frac{K}{|x-y|}\,\mathrm{d}y < \infty; \end{aligned}$$

here we used the property (3.7) of the Green function G.

[1] Conditions (3.2c) imply that $\varphi = 0$ at $x_2 = 0$ and φ is constant at $x_2 = L_2$. Condition (3.2d) implies that this constant vanishes.

In order to prove (3.8b), we set $r = |x - x'|$. If $r \geq 1$, then:

$$
\begin{aligned}
|\mathbf{u}(x) - \mathbf{u}(x')| &\leq \|\omega\|_\infty \int_\Omega |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y \\
&\leq K\|\omega\|_\infty \Big\{ \int_\Omega \frac{1}{|x-y|}\,\mathrm{d}y + \int_\Omega \frac{1}{|x'-y|}\,\mathrm{d}y \Big\} \leq K\|\omega\|_\infty.
\end{aligned} \tag{3.9}
$$

If $r < 1$, we consider $B = \mathcal{B}(x, 2r)$ the ball centered at x of radius $2r$ and write:

$$
\begin{aligned}
\int_\Omega |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y &= \int_{\Omega\cap B} |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y \\
&+ \int_{\Omega\setminus B} |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y.
\end{aligned} \tag{3.10}
$$

For the first integral from (3.10), we use (3.7):

$$
\begin{aligned}
\int_{\Omega\cap B} |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y &\leq K \int_{\Omega\cap B} \Big\{ \frac{1}{|x-y|} + \frac{1}{|x'-y|} \Big\}\,\mathrm{d}y \\
&\leq K \int_{|x-y|\leq 2r} \frac{1}{|x-y|}\,\mathrm{d}y + K \int_{|x'-y|\leq 3r} \frac{1}{|x'-y|}\,\mathrm{d}y \\
&\leq K_2 r.
\end{aligned}
$$

In order to estimate the second integral from (3.10), we choose a particular point $\bar{x} = \bar{x}(xx')$ on the segment xx', such that:

$$
\begin{aligned}
\int_{\Omega\setminus B} |\mathrm{curl}\ _x G(x,y) - \mathrm{curl}\ _x G(x',y)|\,\mathrm{d}y &= \int_{\Omega\setminus B} |x-x'||\nabla_{(xx')}\mathrm{curl}\ _x G(\bar{x},y)|\,\mathrm{d}y \\
&\leq rK \int_{\Omega\setminus B} \frac{1}{|\bar{x}-y|^2}\,\mathrm{d}y,
\end{aligned}
$$

where $\nabla_{(xx')}$ is the derivative in the direction (x, x'). Since $\bar{x}$ belongs to the segment xx' we see that for y outside the ball B, $|\bar{x} - y| \geq |x - y|/2$ so the integral is bounded by:

$$
4rK \int_{\Omega\setminus B} \frac{1}{|x-y|^2}\,\mathrm{d}y \leq 4rK \int_{2r\leq|x-y|\leq R} \frac{1}{|x-y|^2}\,\mathrm{d}y \leq 8\pi rK \log(R/2r),
$$

where R is the diameter of the domain Ω.

Gathering the above relations we find (3.8b). □

We need also to establish some regularity results for the function $\mathbf{u}$ given by (3.6). We have:

Lemma 3.2. *Let $0 < \beta < 1$. If $\omega \in \mathcal{C}^{\beta,0}(Q_T)$, then $\mathbf{u} \in \mathcal{C}^{1+\beta',0}(Q_T)$ for any $\beta' < \beta$. If $\omega \in \mathcal{C}^{\beta,\epsilon}(Q_T)$, then $\mathbf{u} \in \mathcal{C}^{1+\beta',\epsilon'}(Q_T)$ for any $\beta' < \beta$, $\epsilon' < \epsilon$.*

Proof. Since this result is a classical one, we do not give the proof here. For details, we refer the interested reader to [7], [9] and the references herein. □

We now need to study the existence of the fluid trajectories and their properties. Firstly, we prove the following lemma:

Lemma 3.3. *Let us consider the Cauchy problem in* $\mathbb{R}^n$*:*

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= b(x,t),\\ x(0) &= x_0,\end{aligned} \tag{3.11}$$

with $b \in \mathcal{C}(\mathbb{R}^n \times [0,T])$*, uniformly bounded and satisfying the condition*

$$|b(x,t) - b(y,t)| \leq K_0 \chi(|x-y|), \tag{3.12}$$

where K_0 *is a positive constant independent of* t *and* χ *is the function defined above.*
Then problem (3.11) has a unique solution $x = x(t)$*.*

Proof. The proof makes use of the usual Picard iterative method. We define:

$$x_n(t) = x_0 + \int_0^t b(x_{n-1}(s), s)\,\mathrm{d}s,\ x_0(t) = x_0. \tag{3.13}$$

One can check that, for all $r > 0$ and $0 < \epsilon < 1$,

$$\chi(r) \leq (-\ln \epsilon) r + \epsilon; \tag{3.14}$$

we then compute:

$$\begin{aligned}|x_n(t) - x_{n-1}(t)| &= |\int_0^t b(x_{n-1}(s), s) - b(x_{n-2}(s), s)\,\mathrm{d}s|\\ &\leq K_0 \int_0^t \chi(|x_{n-1}(s) - x_{n-2}(s)|)\,\mathrm{d}s\\ &\leq K_0 \int_0^t L_\epsilon |x_{n-1}(s) - x_{n-2}(s)|\,\mathrm{d}s + K_0 \epsilon t,\end{aligned} \tag{3.15}$$

where $L_\epsilon = -\ln \epsilon$, for $\epsilon < 1$.

We iterate relation (3.15) and we obtain, for any $0 \leq t \leq T$ and $n \geq 2$ that:

$$|x_n(t) - x_{n-1}(t)| \leq CT\epsilon \sum_{k=0}^{n-2} \frac{K_0^k L_\epsilon^k t^k}{k!} + \frac{t^{n-1} K_0^{n-1} L_\epsilon^{n-1}}{(n-1)!} \sup_{t \leq T} |x_1(t) - x_0|. \tag{3.16}$$

Since b is bounded, we have:

$$|x_1(t) - x_0| \leq CT;$$

here C is the bound of b, and in the sequel C is a constant depending on the domain but not on the initial data, which can vary at different occurrences. Choosing $\epsilon = \exp(-n)$

and a time T sufficiently small so that $1 - K_0T > 1/2$, ($T < 1/2K_0$), we obtain (with $K_0 \leq C$):

$$|x_n(t) - x_{n-1}(t)| \leq CT\exp(-n/2) + C^n \frac{T^n n^{n-1}}{(n-1)!}. \tag{3.17}$$

Using the Stirling formula $n^n \leq n! \exp(Cn)$, we find:

$$|x_n(t) - x_{n-1}(t)| \leq CT\exp(-n/2) + \{CTe^C\}^n; \tag{3.18}$$

finally, choosing T small enough (more exactly such that $CTe^C < 1$, and taking into account the assumption above, we find $T < \min(1/2K_0, 1/Ce^C)$) we see that $|x_n(t) - x_{n-1}(t)|$ is bounded by the terms of a convergent geometrical series. This means that $\{x_n(t)\}_n$ for $0 < t < T$, is a uniformly convergent Cauchy sequence. The limit $x = x(t)$ of the sequence is a solution of equation (3.11). In the proof above the time T was imposed small but it does not depend on the initial data x_0, so the procedure can be iterated until arriving at arbitrary large times.

The uniqueness of solutions is proven in a similar way. We consider two different solutions x, y of equation (3.11) with the same initial data. We write the solutions x, y in the integral form:

$$y(t) = x_0 + \int_0^t b(y(s), s)\,\mathrm{d}s,\ x(t) = x_0 + \int_0^t b(x(s), s)\,\mathrm{d}s; \tag{3.19}$$

then the difference $y(t) - x(t)$ satisfies:

$$\begin{aligned} |y(t) - x(t)| &\leq \int_0^t |b(y(s), s) - b(x(s), s)|\,\mathrm{d}s \leq K_0 \int_0^t \chi(|y(s) - x(s)|)\,\mathrm{d}s \\ &\leq K_0L_\epsilon \int_0^t |y(s) - x(s)|\,\mathrm{d}s + K_0t\epsilon. \end{aligned} \tag{3.20}$$

Applying the Gronwall lemma to the integral inequality (3.20), we find:

$$|y(t) - x(t)| \leq K_0t\epsilon + \frac{\epsilon}{K_0L_\epsilon^2} e^{K_0L_\epsilon t}. \tag{3.21}$$

We take $\epsilon = e^{-n}$ and find:

$$|y(t) - x(t)| \leq K_0te^{-n} + \frac{1}{n} e^{-n+K_0nt}, \tag{3.22}$$

which implies that, for T chosen such that $1 - K_0T > 0$, $|x(t) - y(t)|$ is arbitrary small so $|x(t) - y(t)| = 0$ on the interval $(0, T)$. Since T chosen before does not depend on the initial data, we can repeat the argument for arbitrary large times. This implies that the solution is unique. □

Remark 3.1. Note that we can change t to $-t$ and solve problem (3.11) on the interval $[-T, 0]$ if b is defined on the interval $[-T, 0]$. In fact this means that we can also solve problem (3.11) backward in time.

With the above lemma, we can now prove that equation (3.4) is satisfied:

Lemma 3.4. *The differential equation (3.4) has a unique global solution $x(t; y, s)$, existing for $0 \le t \le T$, for any initial condition $x(s) = y$, where $0 \le s \le T$ and $y \in \Omega$.*

Proof. The proof of this result is based on the previous lemma. Because of (3.8), the property (3.12) is satisfied. The velocity $\mathbf{u}$, given by (3.6), is a continuous function, jointly in x and in t, by Lemma 3.2.

In order to conclude that equation (3.4) is satisfied, we need to prove that the trajectories do not leave the domain because the velocity $\mathbf{u}$ is only acting inside the domain Ω. Since $\mathbf{u}$ is periodic in the $0x_1$-direction, the problem appears only on the walls $x_2 = 0$ and $x_2 = L_2$.

Let us prove that if a trajectory touches the boundary, then it remains there.

We take a point x_0 of the boundary, with $x_{0,2} = 0$. We consider the equation:

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = u_1(x_1(t), 0, t), \\ &x_1(s) = x_{0,1}, \end{aligned} \tag{3.23}$$

and we see that the pair $(x_1(t), 0)$ solves equation (3.4). Since the function $\mathbf{u}$ is in $\mathcal{C}^{1,0}(Q_T)$, the equation has a unique solution, which does not leave the domain; if the trajectory touches the boundary, it stays there. □

Lemma 3.5. *Let $x = x(t; y, s)$ be the solution of (3.4). Then x is continuously differentiable as a function of the three variables. For s and t fixed, the function x is a one-to-one, measure preserving map of the domain into itself, with:*

$$\begin{aligned} &x(t; x(s; y, t), s) = x(t; y, t) = y, \\ &x(s; x(t; y, r), t) = x(s; y, r). \end{aligned} \tag{3.24}$$

Proof. The properties follows from the theory of ordinary differential equations and from the fact that div $\mathbf{u} = 0$. The result is a classical one (also known as the Liouville theorem).

Note here that by $x(t; y, s)$ with $t < s$ we understand the trajectory backward in time, which exists as we mentioned in Remark 3.1. □

Remark 3.2. The trajectories $x(\cdot)$ are periodic in x_1, i.e. they satisfy the property:

$$x(t; y_1 + L_1, y_2, s) = x(t; y_1, y_2, s) + (L_1, 0).$$

Proof. Using the fact that $\mathbf{u}$ is periodic in x_1, we see that $x(t; y_1, y_2, s) + (L_1, 0)$ verifies equation (3.4) with initial condition $x(s) = (y_1 + L_1, x_2)$. The result follows from the uniqueness of the solution. □

We also need to see in what sense the vorticity given by formula (3.5) is a solution of equation (3.1). We prove the following result:

Lemma 3.6. *If $\omega_0 \in \mathcal{C}(\bar{\Omega})$, $\operatorname{curl} \mathbf{f} \in \mathcal{C}(Q_T)$, then ω is a weak solution of equation (3.1), meaning that for every $\psi \in \mathcal{C}^1(\bar{\Omega})$ we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega, \psi) = (\operatorname{curl} \mathbf{f}, \psi) + (\omega, \mathbf{u} \cdot \operatorname{grad} \psi). \tag{3.25}$$

Proof. From formula (3.5) we see that ω can be written as:

$$\omega(y,t) = \omega_0(x(0;y,t)) + \int_0^t \operatorname{curl} \mathbf{f}(x(t';y,t),t')\,\mathrm{d}t'. \tag{3.26}$$

We write $\omega = \omega_1 + \omega_2$ where ω_1 and ω_2 are respectively the first and the second term in the right-hand side of (3.26). We have, using Lemma 3.5:

$$(\omega_1, \psi) = \int_\Omega \omega_0(x(0;y',t))\psi(y')\,\mathrm{d}y' = \int_\Omega \omega_0(y)\psi(x(t;y,0))\,\mathrm{d}y, \tag{3.27}$$

where we performed the change of variable $y' = x(t;y,0)$. Taking the derivative in time of (3.27), which is legitimate, we find:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1, \psi) &= \int_\Omega \omega_0(y)\frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t;y,0))\,\mathrm{d}y \\
&= \int_\Omega \omega_0(y)\mathrm{grad}\psi(x(t;y,0)) \cdot \mathbf{u}(x(t;y,0),t)\,\mathrm{d}y \\
&= \int_\Omega \omega_0(x(0;y',t))\mathrm{grad}\psi(y') \cdot \mathbf{u}(y',t)\,\mathrm{d}y' = (\omega_1,\ \mathbf{u} \cdot \operatorname{grad}\psi),
\end{aligned} \tag{3.28}$$

where on the last line we used the change of variable $y = x(0;y',t)$.

For the term in ω_2 we write:

$$(\omega_2, \psi) = \int_0^t \int_\Omega \operatorname{curl} \mathbf{f}(x(t',y',t),t')\psi(y')\,\mathrm{d}y'\,\mathrm{d}t' = \int_0^t \int_\Omega \operatorname{curl} \mathbf{f}(y,t')\psi(x(t;y,t'))\,\mathrm{d}y\,\mathrm{d}t'. \tag{3.29}$$

Taking the derivative in time of (3.29), we find in the same way as before, that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\omega_2, \psi) &= \int_\Omega \operatorname{curl} \mathbf{f}(y,t)\psi(y)\,\mathrm{d}y \\
&\quad + \int_0^t \int_\Omega \operatorname{curl} \mathbf{f}(y,t')\operatorname{grad}\psi(x(t;y,t')) \cdot \mathbf{u}(x(t;y,t'),t')\,\mathrm{d}y\,\mathrm{d}t' \\
&= (\operatorname{curl} \mathbf{f}, \psi) + (\omega_2, \mathbf{u} \cdot \operatorname{grad}\psi).
\end{aligned} \tag{3.30}$$

By gathering the results above, we find that ω is indeed a weak solution of equation (3.1). $\square$

3.2. **A priori estimates.** In this section we are interested in deducing some a priori estimates for the velocity, the trajectories and the vorticity, in order to prove that we can apply the fixed point theorem to the function Λ defined by (3.3).

The estimates for the velocity were already given in (3.8).

For the trajectories, we have the following estimates:

Lemma 3.7. *Let $x = x(t; y, s)$ be the solution of equation (3.4). Then the following estimate holds for $|y_1 - y_2| \le 1$, $|t_1 - t_2| \le 1$, $|s_1 - s_2| \le 1$:*

$$|x(t_1; y_1, s_1) - x(t_2; y_2, s_2)| \le C_5(\omega^0)(|t_1 - t_2|^\delta + |y_1 - y_2|^\delta + |s_1 - s_2|^\delta), \tag{3.31}$$

for a constant δ less than 1, which depends on the $L^\infty(Q_T)$ norm of ω^0.

Proof. (i) We start by deriving the Hölder estimates in space. Let us consider two arbitrary points y_1 and y_2 and the corresponding trajectories $x(t; y_1, s)$ and $x(t; y_2, s)$. Setting $z(t) = x(t; y_1, s) - x(t; y_2, s)$, z satisfies the equation:

$$\begin{aligned} &\frac{\mathrm{d}z}{\mathrm{d}t} = \mathbf{u}(x(t; y_1, s), t) - \mathbf{u}(x(t; y_2, s), t), \\ &z(s) = y_1 - y_2. \end{aligned} \tag{3.32}$$

We assume that $|y_1 - y_2| < 1$, which means $|z(s)| < 1$. Then, there exists a maximal subinterval I in $[0, T]$ containing s such that $|z(t)| < 1$ for all $t \in I$. Since z is a continuous function, the interval I is open.

From (3.32) one can deduce, using (3.8b):

$$\begin{aligned} \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| &= |\mathbf{u}(x(t; y_1, s), t) - \mathbf{u}(x(t; y_2, s), t)| \\ &\le K\|\omega^0\|_\infty \chi(|x(t; y_1, s) - x(t; y_2, s)|) \\ &= C_1\chi(|z(t)|), \end{aligned} \tag{3.33}$$

where $C_1 = K\|\omega^0\|_\infty$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t}|z(t)| \le C_1\chi(|z(t)|), \tag{3.34}$$

since by Stampacchia's Theorem (see e.g. [6]) we know that:

$$\left|\frac{\mathrm{d}|z(t)|}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right| \text{ a.e..} \tag{3.35}$$

In view of Lemma 3.8 below, we need the solution of the following ordinary differential equation:

$$\begin{aligned} &\frac{\mathrm{d}m}{\mathrm{d}t} = C_1 m(1 - \ln m), \\ &m(s) = |y_1 - y_2|, \end{aligned} \tag{3.36}$$

which is found to be $m(t) = e^{1-e^{C_1(s-t)}}|y_1 - y_2|^{e^{C_1(s-t)}}$.

We can apply Lemma 3.8 below as long as $m(t) < 1$ and we obtain:

$$|z(t)| \le e^{1-e^{C_1(s-t)}}|y_1 - y_2|^{e^{C_1(s-t)}} \le e|y_1 - y_2|^{e^{C_1(s-t)}} < 1, \text{ for } 0 < s, t < T, \tag{3.37}$$

if $|y_1 - y_2| < e^{-e^{C_1 T}} < 1$.

The interval I coincides with $[0, T]$. Indeed, if $I = (t_0, t_1)$ with $0 \le t_0 < t_1 < T$ is the maximal interval containing s on which $|z(t)| < 1$, then by the continuity of the solution, we have:

$$|z(t_1)| \le e^{1-e^{-C_1 T}}|y_1 - y_2|^{e^{-|s-t_1|C_1}} < 1. \tag{3.38}$$

This contradicts the maximality of the interval I.

Finally the restriction $|y_1 - y_2| < e^{-e^{C_1 T}} < 1$ can be removed by increasing the constant C_5 and thus (3.31) is valid for $t_1 = t_2$, $s_1 = s_2$, $|y_1 - y_2| \le 1$, and $\delta = e^{-KT\|\omega^0\|_\infty}$.

(ii) We need to estimate the trajectories in t. Let us consider two arbitrary points t_1 and t_2 and the corresponding trajectories $x(t_1; y, s)$ and $x(t_2; y, s)$. Taking the difference between these functions, we find:

$$\begin{aligned} |x(t_1; y, s) - x(t_2; y, s)| &= |\int_s^{t_1} \mathbf{u}(x(r; y, s), r)\, \mathrm{d}r - \int_s^{t_2} \mathbf{u}(x(r; y, s), r)\, \mathrm{d}r| \\ &= |\int_{t_1}^{t_2} \mathbf{u}(x(r; y, s), r)\, \mathrm{d}r| \\ &\le \|\mathbf{u}\|_\infty |t_1 - t_2| \le K\|\omega^0\|_\infty |t_1 - t_2|, \end{aligned} \tag{3.39}$$

where we used (3.8a) for the last inequality.

(iii) It now remains to estimate the trajectories in s. As before, let us consider two arbitrary instants of time s_1 and s_2 and the corresponding trajectories $x(t; y, s_1)$ and $x(t; y, s_2)$. We set $a = x(t; y, s_1)$, $b = x(t; y, s_2)$ and $c = x(s_1; y, s_2)$. Then, by Lemma 3.5, we also know that $b = x(t; c, s_1)$, and we find:

$$|a - b| = |x(t; y, s_1) - x(t; y, s_2)| = |x(t; y, s_1) - x(t; c, s_1)| \le C_3(\omega^0)|y - c|^\delta, \tag{3.40}$$

where for the last inequality we used (3.37) obtained at point (i); $\delta = e^{-C_1 T}$.

Using now (3.39) from point (ii), we compute:

$$|y - c| = |x(s_2; y, s_2) - x(s_1; y, s_2)| \le C_2(\omega^0)|s_2 - s_1|. \tag{3.41}$$

Equations (3.40) and (3.41) lead to:

$$|x(t; y, s_1) - x(t; y, s_2)| \le C_4(\omega^0)|s_1 - s_2|^\delta, \tag{3.42}$$

where $\delta = e^{-C_1 T}$.

Taking into account (i), (ii) and (iii), we obtain:

$$|x(t_1; y_1, s_1) - x(t_2; y_2, s_2)| \le C_5(\omega^0)(|t_1 - t_2|^\delta + |y_1 - y_2|^\delta + |s_1 - s_2|^\delta), \tag{3.43}$$

which proves the Lemma. □

In the proof above, we used the following result borrowed from [9] (see Appendix 2.1):

Lemma 3.8. *Let $u \in \mathcal{C}([0,T];\mathbf{R}_+)$ and let $\varphi \in \mathcal{C}(\mathbf{R}_+,\mathbf{R}_+)$ be a nondecreasing function, such that:*

$$u(t) \leq u(0) + \int_0^t \varphi(u(s))\,\mathrm{d}s, \quad t \leq T, \tag{3.44}$$

and let $v = v(t)$ be the solution of the initial value problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} v = \varphi(v), \\ &v(0) = u(0). \end{aligned} \tag{3.45}$$

Then:

$$u(t) \leq v(t) \text{ for any } t \in [0,T]. \tag{3.46}$$

It remains to estimate the vorticity. For the vorticity, the following result holds:

Lemma 3.9. *If $\omega^1 = \Lambda(\omega^0)$, $\omega_0 \in \mathcal{C}^\alpha(\bar{\Omega})$, $\mathbf{f} \in \mathcal{C}^{1+\alpha,0}(Q_T)$, then the following inequalities hold:*

$$|\omega^1(x,t)| \leq \|\omega_0\|_\infty + T\|\mathrm{curl}\,\mathbf{f}\|_\infty, \tag{3.47}$$

$$|\omega^1(x_1,t_1) - \omega^1(x_2,t_2)| \leq C_6(\|\omega^0\|_\infty)(|x_1-x_2|^{\delta'} + |t_1-t_2|^{\delta'}), \tag{3.48}$$

for $|x_1 - x_2| \leq 1$, $|t_1 - t_2| \leq 1$, where $\delta' = \alpha\delta$, and δ is the same as in Lemma 3.7.

Proof. Let us consider two points x_1 and x_2 and the corresponding trajectories $x(t;x_1,s)$ and $x(t;x_2,s)$. We find:

$$\begin{aligned} |\omega^1(x_1,t) - \omega^1(x_2,t)| \leq &|\omega_0(x(0;x_1,t)) - \omega_0(x(0;x_2,t))| \\ &+ \int_0^t |\mathrm{curl}\,\mathbf{f}(x(t';x_1,t),t') - \mathrm{curl}\,\mathbf{f}(x(t';x_2,t),t')|\,\mathrm{d}t'. \end{aligned} \tag{3.49}$$

Since $\omega_0 \in \mathcal{C}^\alpha(\bar{\Omega})$, we can estimate the first term of (3.49) as:

$$\begin{aligned} |\omega_0(x(0;x_1,t)) - \omega_0(x(0;x_2,t))| &\leq H_x^\alpha(\omega_0)|x(0;x_1,t) - x(0;x_2,t)|^\alpha \\ &\leq H_x^\alpha(\omega_0) C_5(\omega^0)|x_1 - x_2|^{\delta\alpha}, \end{aligned} \tag{3.50}$$

where in the last inequality we made use of Lemma 3.7.

For the second term we use the fact that $\mathrm{curl}\,\mathbf{f} \in \mathcal{C}^{\alpha,0}(Q_T)$ and find:

$$\begin{aligned} &\int_0^t |\mathrm{curl}\,\mathbf{f}(x(t';x_1,t),t') - \mathrm{curl}\,\mathbf{f}(x(t';x_2,t),t')|\,\mathrm{d}t' \\ &\quad \leq \int_0^t H_x^\alpha(\mathrm{curl}\,\mathbf{f})|x(t';x_1,t) - x(t';x_2,t)|^\alpha\,\mathrm{d}t' \leq H_x^\alpha(\mathrm{curl}\,\mathbf{f})|x_1-x_2|^{\alpha\delta} T. \end{aligned} \tag{3.51}$$

We then have, with the same δ as in Lemma 3.7:

$$|\omega^1(x_1,t) - \omega^1(x_2,t)| \leq C(|\omega_0|_\alpha, \|\omega^0\|_\infty, |\mathrm{curl}\,\mathbf{f}|_\alpha, T)|x_1 - x_2|^{\alpha\delta}. \tag{3.52}$$

We also need to derive Hölder estimates in time. Let us consider two arbitrary instants of time t_1 and t_2. We write:

$$|\omega^1(y,t_1) - \omega^1(y,t_2)| \leq I_1 + I_2, \tag{3.53}$$

where

$$I_1 = |\omega_0(x(0;y,t_1)) - \omega_0(x(0;y,t_2))|, \tag{3.54}$$

and

$$I_2 = |\int_0^{t_1} \text{curl } \mathbf{f}(x(t';y,t_1),t')\,\mathrm{d}t' - \int_0^{t_2} \text{curl } \mathbf{f}(x(t';y,t_2),t')\,\mathrm{d}t'|. \tag{3.55}$$

Using Lemma 3.7 and $\omega_0 \in \mathcal{C}^\alpha(\bar{\Omega})$, the term I_1 can be bounded as follows:

$$\begin{aligned} I_1 &\leq H_x^\alpha(\omega_0)|x(0;y,t_1) - x(0;y,t_2)|^\alpha \\ &\leq H_x^\alpha(\omega_0) C_5(\omega^0)|t_1 - t_2|^{\alpha\delta}. \end{aligned} \tag{3.56}$$

Assuming $t_1 < t_2$, the term I_2 is also estimated as:

$$\begin{aligned} I_2 \leq &\int_0^{t_1} |\text{curl } \mathbf{f}(x(t';y,t_1),t') - \text{curl } \mathbf{f}(x(t';y,t_2),t')|\,\mathrm{d}t' \\ &+ \int_{t_1}^{t_2} |\text{curl } \mathbf{f}(x(t';y,t_2),t')|\,\mathrm{d}t' \\ \leq &H_x^\alpha(\text{curl } \mathbf{f}) \int_0^{t_1} |x(t';y,t_1) - x(t';y,t_2)|^\alpha\,\mathrm{d}t' + \|\text{curl } \mathbf{f}\|_\infty (t_2 - t_1) \\ \leq &\,\text{with(3.31)} \leq C(\|\omega^0\|_\infty, |\text{curl } \mathbf{f}|_{\alpha,0})\,|t_1 - t_2|^{\alpha\delta}. \end{aligned} \tag{3.57}$$

Gathering the estimates for I_1 and I_2, we find:

$$|\omega^1(y,t_1) - \omega^1(y,t_2)| \leq C(\|\omega^0\|_\infty, |\omega_0|_\alpha, |\text{curl } \mathbf{f}|_{\alpha,0})|t_1 - t_2|^{\alpha\delta}, \tag{3.58}$$

and we set $\delta' = \alpha\delta$.

The lemma follows from (3.52) and (3.58). □

3.3. **Application of the fixed point theorem.** It follows from (3.47) that:

$$\|\omega^1\|_\infty \leq \|\omega_0\|_\infty + T\|\text{curl } \mathbf{f}\|_\infty = M. \tag{3.59}$$

We define S as the subset of $\mathcal{C}(Q_T)$ consisting of the functions ω satisfying $\|\omega\|_\infty \leq M$ and

$$|\omega(x_1,t_1) - \omega(x_2,t_2)| \leq C(M)(|t_1 - t_2|^{\delta''} + |x_1 - x_2|^{\delta''}), \tag{3.60}$$

when $|x_1 - x_2| \leq 1$, $|t_1 - t_2| \leq 1$; δ'' was obtained by replacing $\|\omega^0\|_\infty$ by M in the definition of δ' and $C(M)$ was obtained by the same manner from (3.48).

The set S is a convex compact subset of $\mathcal{C}(Q_T)$ and the function Λ defined by (3.3) maps the set S into itself.

In order to apply the the Schauder fixed point theorem, we observe the following:

Lemma 3.10. *The function Λ is continuous on S for the topology of $\mathcal{C}(Q_T)$.*

Proof. Consider a sequence ω_n^0, $n \in \mathbf{N}$ and ω^0, all belonging to S and assume that ω_n^0 converges to ω^0 in the topology of $\mathcal{C}(Q_T)$. Lemma 3.1 implies that the corresponding $\mathbf{u}_n$ converge to $\mathbf{u}$ in the topology of $\mathcal{C}(Q_T)$.

We need to study the behavior of the trajectories corresponding respectively to $\mathbf{u}_n$ and $\mathbf{u}$. We write, assuming $s < t$:

$$\begin{aligned} |x_n(t;y,s) - x(t;y,s)| &\leq \int_s^t |\mathbf{u}_n(x_n(t';y,s),t') - \mathbf{u}(x(t';y,s),t')| \, \mathrm{d}t' \\ &\leq \int_s^t |\mathbf{u}_n(x_n(t';y,s),t') - \mathbf{u}_n(x(t';y,s),t')| \, \mathrm{d}t' \\ &+ \int_s^t |\mathbf{u}_n(x(t';y,s),t') - \mathbf{u}(x(t';y,s),t')| \, \mathrm{d}t'. \end{aligned} \tag{3.61}$$

For the last term in the right hand side of (3.61), we write:

$$\int_s^t |\mathbf{u}_n(x(t';y,s),t') - \mathbf{u}(x(t';y,s),t')| \, \mathrm{d}t' \leq T \|\mathbf{u}_n - \mathbf{u}\|_\infty. \tag{3.62}$$

Using (3.8) in the term $J = \int_s^t |\mathbf{u}_n(x_n(t';y,s),t') - \mathbf{u}_n(x(t';y,s),t')| \, \mathrm{d}t'$, we obtain:

$$J \leq K \|\omega_n^0\|_\infty \int_s^t \chi(|x_n(t';y,s) - x(t';y,s)|) \, \mathrm{d}t'. \tag{3.63}$$

Using (3.14), we obtain an estimate similar to (3.15):

$$|x_n(t;y,s) - x(t;y,s)| \leq T(\|\mathbf{u}_n - \mathbf{u}\|_\infty + C'\epsilon) \frac{e^{C' L_\epsilon T}}{C' L_\epsilon}, \tag{3.64}$$

where C' is a constant depending only on $\|\omega^0\|_\infty$.

Taking the limit when $n \to 0$, we find:

$$\lim_{n\to\infty} \sup_{t,y,s} |x_n(t;y,s) - x(t;y,s)| \leq C \frac{\epsilon^{1-C'T}}{L_\epsilon}, \ \forall \epsilon > 0.$$

For T small enough ($T < C'^{-1}$), this quantity is arbitrary small, so that, for $n \to \infty$

$$|x_n(t;y,s) - x(t;y,s)| \to 0, \tag{3.65}$$

uniformly in t, y and s. The argument can be extended for arbitrary large times because C' depends only on the L^∞ norm of ω_0.

Since $\omega_n^1 = \Lambda(\omega_n^0)$ and $\omega^1 = \Lambda(\omega^0)$, we find by (3.5) that ω_n^1 converges to ω^1 in the topology of $\mathcal{C}(Q_T)$. □

We can now apply the Schauder fixed point theorem, and conclude that there exists a unique fixed point $\omega \in S$ of Λ, $\Lambda(\omega) = \omega$.

Since $\omega \in \mathcal{C}^{\delta''}(Q_T)$, we apply Lemma 3.2 and obtain $\mathbf{u} \in \mathcal{C}^{1+\epsilon,\epsilon}(\Omega \times [0,T])$ for $0 < \epsilon < \delta''$.

Moreover, the following result holds true:

Lemma 3.11. *The derivative in time of the velocity* $\mathbf{u}$ *exists and belongs to* $\mathcal{C}(Q_T)$.

Proof. By the definition of $\mathbf{u}$ we have:

$$\mathbf{u}(t,x) = \int_\Omega \operatorname{curl} G(x,y)\omega(y)\,\mathrm{d}y. \tag{3.66}$$

For the convenience of notation we set $\mathcal{G}(\omega) = \int_\Omega G(x,y)\omega(y)\,\mathrm{d}y$; $\mathbf{u} = \operatorname{curl} \mathcal{G}(\omega)$.

For a function $\psi \in \mathcal{C}^2(\Omega)$, ψ periodic in x_1, we have:

$$(\mathbf{u},\psi) = (\operatorname{curl} \mathcal{G}(\omega),\psi) = (\mathcal{G}(\omega), \operatorname{curl} \psi) = (\omega, \mathcal{G}(\operatorname{curl} \psi)). \tag{3.67}$$

Taking the time derivative of (3.67) and using Lemma 3.6, we find:

$$\begin{aligned} D_t(\mathbf{u},\psi) = D_t(\omega,\mathcal{G}(\operatorname{curl} \psi)) &= (\operatorname{curl} \mathbf{f}, \mathcal{G}(\operatorname{curl} \psi)) + (\omega, \mathbf{u}\cdot \operatorname{grad} \mathcal{G}(\operatorname{curl} \psi)) \\ &= (\operatorname{curl} \mathcal{G}(\operatorname{curl} \mathbf{f}),\psi) + (\omega, \mathbf{u}\cdot \operatorname{grad} \mathcal{G}(\operatorname{curl} \psi)). \end{aligned} \tag{3.68}$$

It remains to estimate the second term from the right hand side of (3.68):

$$(\omega, \mathbf{u}\cdot \operatorname{grad} \mathcal{G}(\operatorname{curl} \psi)) = (\mathbf{u}\omega, \operatorname{grad} \mathcal{G}(\operatorname{curl} \psi)) = -(\operatorname{curl} \mathcal{G}(\operatorname{div}\,(\mathbf{u}\omega)),\psi), \tag{3.69}$$

where by $\mathcal{G}(\operatorname{div}\, f)$ we understand the function which, for $f \in \mathcal{C}^1(\Omega)$, gives:

$$\mathcal{G}(\operatorname{div}\, f)(x) = -\int_\Omega \operatorname{grad}_y G(x,y)\cdot f(y)\,\mathrm{d}y. \tag{3.70}$$

For the details regarding the validity of the last relation in (3.69), we refer the interested reader to [7] and the references herein.

We thus find:

$$D_t\mathbf{u} = \operatorname{curl} \mathcal{G}(\operatorname{div}\,(\mathbf{u}\omega)) + \operatorname{curl} \mathcal{G}(\operatorname{curl} \mathbf{f}), \tag{3.71}$$

and the right-hand-side of (3.71) belongs to $\mathcal{C}(Q_T)$. □

We also need to prove the existence of the pressure. This result is given by the following Lemma:

Lemma 3.12. *There exists a scalar function* $p \in \mathcal{C}^{1,0}(Q_T)$ *such that* $(\mathbf{u}, p)$ *is solution of equation (2.1).*

Proof. For a function $\theta \in \mathcal{C}^1(\Omega)$, θ periodic in x_1 with $\theta = 0$ at $x_2 = 0$ and $x_2 = L_2$, we write, using (3.25):

$$\begin{aligned} D_t(\mathbf{u}, \operatorname{curl} \theta) = D_t(\operatorname{curl} \mathbf{u}, \theta) &= (\operatorname{curl} \mathbf{f}, \theta) + (\omega, \mathbf{u}\cdot \operatorname{grad} \theta) \\ &= (\mathbf{f}, \operatorname{curl} \theta) - ((\mathbf{u}\cdot \operatorname{grad})\,\mathbf{u}, \operatorname{curl} \theta), \end{aligned} \tag{3.72}$$

which can be also written as:

$$(D_t\mathbf{u} + (\mathbf{u}\cdot \operatorname{grad})\,\mathbf{u} - \mathbf{f}, \operatorname{curl} \theta) = 0. \tag{3.73}$$

We obtain, by the Hopf theorem, that there exists a function $p \in \mathcal{C}^{1,0}(\Omega)$ such that:

$$D_t\mathbf{u} + (\mathbf{u}\cdot\operatorname{grad})\,\mathbf{u} + \operatorname{grad} p = \mathbf{f}. \tag{3.74}$$

□

As we announced in Theorem 2.1, the solution $(\mathbf{u}, p)$ is unique. The proof of the uniqueness is classical, one can chose two different solutions and taking the difference between them, prove that the difference vanishes.

4. Euler equation in a $3D$ channel

As we already mentioned at the beginning of this article, in this section we prove the existence and uniqueness of solution for the Euler equations in a three dimensional channel. The proof extends the approach of Temam [12].

The Euler equation in a $3D$ channel $\Omega_\infty = \mathbb{R}^2 \times (0, L_3)$ reads:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \operatorname{grad} p = \mathbf{f}, \tag{4.1a}$$

$$\operatorname{div}\ \mathbf{u} = 0, \tag{4.1b}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity of the fluid and p is the pressure. As for the $2D$ case, we also consider $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$, where L_1, L_2 are respectively the periods in the $0x_1$ and $0x_2$ directions.

We supplement the system (4.1) with the following boundary conditions:

$$\begin{aligned}&\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_3 = \{x_3 = 0,\ x_3 = L_3\},\\ &\text{and the functions are periodic in } x_1 \text{ and } x_2,\end{aligned} \tag{4.2}$$

where $\mathbf{n}$ is the outward normal on Γ_3.

Remark 4.1. As for the $2D$ case, we can suppose that the horizontal velocity is of average zero. However, taking into account the method adopted here, this does not represent a simplification, and this hypothesis is not made here. See a different situation in [10].

We introduce here the following function spaces:

$$\begin{aligned}X_m =&\{\mathbf{v} \in (H^m(\Omega))^3;\ \operatorname{div}\ \mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_3,\\ &\frac{\partial^j \mathbf{v}}{\partial x_i^j}\Big|_{x_i=0} = \frac{\partial^j \mathbf{v}}{\partial x_i^j}\Big|_{x_i=L_i} \text{ for } i = 1,2,\ j = 1,..,m\},\end{aligned} \tag{4.3}$$

$$H_m = \{\mathbf{v} \in (H^m(\Omega))^3;\ \frac{\partial^j \mathbf{v}}{\partial x_i^j}\Big|_{x_i=0} = \frac{\partial^j \mathbf{v}}{\partial x_i^j}\Big|_{x_i=L_i} \text{ for } i = 1,2,\ j = 1,..,m\}, \tag{4.4}$$

where m is a positive number.

The spaces X_m and H_m are endowed with the usual norm on H^m, denoted here by $\|\cdot\|_m$. We recall that if $m > 3/2$ then H^m is an algebra for the pointwise multiplication of functions.

The main result of this section is:

Theorem 4.1. *Let L_1, L_2, L_3, T be given positive and arbitrary, and let $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$. For $m > 5/2$, let $\mathbf{u_0}$ be in X_m and $\mathbf{f}$ in $L^2(0, t; H_m)$. Then there exists $T_\star$ depending on the initial data, $T_\star \leq T$, and there exists a unique solution $\mathbf{u}$ and p of problem (4.1) such that:*

$$\mathbf{u} \in L^\infty(0, T_\star; X_m), \quad p \in L^\infty(0, T_\star; H_{m+1}).$$

Remark 4.2. Theorem 4.1 is in fact valid in any space dimension, taking $m > 1 + N/2$, where N is the dimension of the space.

The proof of existence of solutions stated in Theorem 4.1 is based on a priori estimates. It is useful to express the pressure p in terms of $\mathbf{u}$, the result needed is given by the following Lemma:

Lemma 4.1. *If $\mathbf{u}$ and p satisfy (4.1), then:*

$$\Delta p = \text{div } \mathbf{f} - \sum_{i,j} D_i u_j D_j u_i, \tag{4.5a}$$

$$\frac{\partial p}{\partial n} = \mathbf{f} \cdot \mathbf{n} \text{ on } \Gamma_3, \tag{4.5b}$$

$$p \text{ periodic in } x_1 \text{ and } x_2. \tag{4.5c}$$

Proof. Equation (4.5) is immediately obtained by applying the divergence operator to both sides of (4.5a) and taking the scalar product of (4.5a) with $\mathbf{n}$ on Γ_3. □

We can then estimate the pressure p in terms of $\mathbf{u}$:

Lemma 4.2. *If $\mathbf{u}$ and p satisfy (4.1), then for $m > 5/2$ the following estimate holds:*

$$\|\text{grad} p(t)\|_m \leq c_1\{\|\mathbf{f}(t)\|_m + \|\mathbf{u}(t)\|_m^2\}, \tag{4.6}$$

where c_1 is a constant depending only on m and Ω.

Proof. By classical regularity results for elliptic problems, we have:

$$\|\text{grad} p(t)\|_m \leq c_0\{\|\text{div } \mathbf{f} - \sum_{i,j} D_i u_j D_j u_i\|_{m-1} + \|\mathbf{f} \cdot \mathbf{n}\|_{m-1/2,\Gamma_3}\}. \tag{4.7}$$

We need to estimate $\| \sum_{i,j} D_i u_j D_j u_i\|_{m-1}$. Since $m > 5/2$, $H^{m-1}(\Omega)$ is an algebra, and:

$$\|D_i u_j D_j u_i\|_{m-1} \leq c_2 \|D_i u_j\|_{m-1} \|D_j u_i\|_{m-1}, \tag{4.8}$$

and this leads to:

$$\| \sum_{i,j} D_i u_j D_j u_i\|_{m-1} \leq c_2' \|\mathbf{u}\|_m^2. \tag{4.9}$$

Using the trace theorem, we also have:

$$\|\mathbf{f} \cdot \mathbf{n}\|_{m-1/2,\Gamma_3} \leq \|\mathbf{f}\|_m. \tag{4.10}$$

Estimate (4.6) then follows. □

We now derive the necessary estimates for the velocity.

A priori estimates for the velocity:

Let α be a multi-index, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{R}$, with $|\alpha| \le m$, where $|\alpha| = \alpha_1+\alpha_2+\alpha_3$. We apply the operator D^α on each side of (4.1a). We then multiply the resulting equation by $D^\alpha \mathbf{u}$, integrate over Ω and add these equations for $|\alpha| \le m$. We find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_m^2 = -\sum_{j=1}^{3}((u_j\frac{\mathbf{u}}{x_j}, \mathbf{u}))_m - ((\mathrm{grad}p, \mathbf{u}))_m + ((\mathbf{f}, \mathbf{u}))_m. \tag{4.11}$$

For the last term of (4.11), we have:

$$((\mathbf{f}, \mathbf{u}))_m \le \|\mathbf{f}\|_m \|\mathbf{u}\|_m.$$

For the second term from the right hand side of (4.11), we use (4.6) and we find:

$$((\mathrm{grad}p, \mathbf{u}))_m \le \|\mathrm{grad}p\|_m \|\mathbf{u}\|_m \le c_1\{\|\mathbf{f}\|_m + \|\mathbf{u}\|_m^2\}\|\mathbf{u}\|_m. \tag{4.12}$$

It now remains to estimate the first term from the right hand side of (4.11). We set $\psi = \sum_j u_j \partial\mathbf{u}/\partial x_j$ and we write:

$$(((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u}))_m = \sum_{|\alpha|\le m}(D^\alpha\phi, D^\alpha\mathbf{u}). \tag{4.13}$$

Applying the Leibnitz rule for ϕ, we find:

$$D^\alpha\psi = (\mathbf{u}\cdot\mathrm{grad})D^\alpha\mathbf{u} + \sum_{0<\beta\le\alpha} c_{\alpha,\beta}(D^\beta\mathbf{u}\cdot\mathrm{grad})D^{\alpha-\beta}\mathbf{u}. \tag{4.14}$$

The contribution of the first term from (4.14) is zero, since $\mathbf{u} \in X_m$.
It remains to estimate the second term from (4.14):

$$\begin{aligned} &|\sum_{0<\beta\le\alpha} c_{\alpha,\beta}((D^\beta\mathbf{u}\cdot\mathrm{grad})D^{\alpha-\beta}\mathbf{u}, D^\alpha\mathbf{u})| \\ &\quad\le \sum_{0<\alpha\le\beta}|c_{\alpha,\beta}||(D^\beta\mathbf{u}\cdot\mathrm{grad})D^{\alpha-\beta}\mathbf{u}|_{L^2}|D^\alpha\mathbf{u}|_{L^2}. \end{aligned} \tag{4.15}$$

We will show that:

$$|(D^\beta\mathbf{u}\cdot\mathrm{grad})D^{\alpha-\beta}\mathbf{u}|_{L^2} \le c\|\mathbf{u}\|_m^2. \tag{4.16}$$

We set $g = D^\beta u_i$ and $h = D_i D^{\alpha-\beta} u_j$ and we need to bound $|gh|_{L^2}$. We will use the Sobolev embeddings $H^m(\Omega) \subset L^r(\Omega)$, where $1/r = 1/2 - m/3$ if $m < 3/2$, $1 \le r < \infty$ arbitrary if $m = 3/2$ and $r = \infty$ if $m > 3/2$.
We have:

$$g \in H^{m-|\beta|}(\Omega) \subset L^p(\Omega), \quad |g|_{L^p} \le c\|g\|_{m-|\beta|} \le c\|\mathbf{u}\|_m, \tag{4.17}$$

and

$$h \in H^{m-|\alpha|+|\beta|-1}(\Omega) \subset L^q(\Omega), \quad |h|_{L^q} \leq c\|h\|_{m-|\alpha|+|\beta|-1} \leq c\|\mathbf{u}\|_m, \tag{4.18}$$

where p and q are given by the Sobolev inclusion theorems.

If p or q is infinite (we take for example $p = \infty$, since the case $q = \infty$ is similar), then:

$$|gh|_{L^2} \leq |g|_{L^\infty}|h|_{L^2} \leq c\|\mathbf{u}\|_m|h|_{L^2} \leq c\|\mathbf{u}\|_m^2.$$

If $|\beta| = m - 3/2$, then $p \geq 1$ is arbitrary. On the other hand,

$$m - |\alpha| + |\beta| - 1 = 2m - |\alpha| - 5/2 \geq m - 5/2 > 0,$$

so $q > 2$ from the Sobolev embedding. Choosing p such that $1/2 = 1/p + 1/q$, we write:

$$|gh|_{L^2} \leq |g|_{L^p}|h|_{L^q} \leq c\|\mathbf{u}\|_m^2.$$

We apply the same reasoning for the case $m - |\alpha| + |\beta| - 1 = 3/2$, and we obtain $|gh|_{L^2} \leq c\|\mathbf{u}\|_m^2$.

The last case we need to consider is when both p and q are finite and given by:

$$\frac{1}{p} = \frac{1}{2} - \frac{m - |\beta|}{3}, \quad \frac{1}{q} = \frac{1}{2} - \frac{m - |\alpha| + |\beta| - 1}{3}.$$

We then have $1/p + 1/q \leq 1/2$, since $m \geq 5/2$ and $|\alpha| \leq m$. We can apply the Hölder inequality and find again $|gh|_{L^2} \leq c\|\mathbf{u}\|_m^2$.

At this point we can conclude that:

$$(((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}))_m \leq c\|\mathbf{u}\|_m^3. \tag{4.19}$$

Returning now to (4.11), we obtain the following a priori estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_m^2 \leq c_1'\|\mathbf{u}\|_m^3 + c_2'\|\mathbf{f}\|_m\|\mathbf{u}\|_m, \tag{4.20}$$

which implies that there exists a time $T_\star$ with $T_\star \leq T$, depending on $\|\mathbf{u_0}\|_m$ and $\|\mathbf{f}\|$ such that:

$$\|\mathbf{u}\|_m^2 \leq 2\|\mathbf{u_0}\|_m^2, \quad \forall\, t \leq T_\star. \tag{4.21}$$

The proof of the existence of solution is obtained by classical methods (Galerkin method), following the same arguments as in [12]. As observed in [13], we need a regularity result for the spectral Galerkin functions and this regularity result can be proved by the methods used by Ghidaglia, see [5], as for the bounded case. The uniqueness of the solution is standard.

Remark 4.3. The a priori estimate proved above can be improved by noticing that the time $T_\star$ depends only on the norm H^3 of $\mathbf{u_0}$ but not on the higher norm H^m of the initial data $\mathbf{u_0}$. The new estimate leads us to the existence of $\mathcal{C}^\infty$ solution for the Euler equations. The main point in the proof is to use the following inequalities,

$$\begin{aligned} &|fg|_{L^2(\Omega)} \leq c_1|f|_{H^2(\Omega)}|g|_{L^2(\Omega)}, \\ &|fg|_{L^2(\Omega)} \leq c_1|f|_{H^1(\Omega)}|g|_{H^1(\Omega)}, \end{aligned} \tag{4.22}$$

so that we obtain, instead of (4.19), the following estimate:

$$(((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u}))_m \leq c\{\|\mathbf{u}\|_m^2\|\mathbf{u}\|_3 + \|\mathbf{u}\|_m\|\mathbf{u}\|_{m-1}^2\}. \tag{4.23}$$

For more details we refer the interested reader to [11].

5. Appendix: Properties of the Green function for a 2D channel

In order to prove the existence of solutions for the Euler equation, we must prove the properties of the Green function, as stated in Theorem 5.1 below. We know that the Green function depends on the form of the domain. In our case, as we will see, the Green function can be written in terms of a series of trigonometric functions in x_1 and x_2.

The Green function for the channel is the solution $G = G(x, y)$ of the following system:

$$\begin{aligned} &\Delta G = \delta(x-y) \text{ in } \Omega, \\ &G \text{ is periodic in the } 0x_1 \text{ direction}, \\ &G = 0 \text{ at } x_2 = 0 \text{ and } x_2 = L_2, \end{aligned} \tag{5.1}$$

where the Laplacian Δ operates in the x variable and $y \in \Omega$ is fixed, δ is the Dirac function on Ω.

We will express the solution in terms of a series of spectral functions corresponding to the eigenvalue problem:

$$\begin{aligned} &\Delta G + \lambda G = 0 \text{ in } \Omega, \\ &G \text{ is periodic in the } 0x_1 \text{ direction}, \\ &G = 0 \text{ at } x_2 = 0 \text{ and } x_2 = L_2. \end{aligned} \tag{5.2}$$

We classically solve (5.2) by separation of variables meaning that, for y fixed, we look for $G = G(x_1, x_2) = X(x_1)Y(x_2)$; we find that there exist a constant κ such that:

$$\begin{aligned} &X'' + (\lambda - \kappa^2)X = 0, \\ &X(0) = X(L_1),\ X'(0) = X'(L_1), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &Y'' + \kappa^2 Y = 0, \\ &Y(0) = Y(L_2) = 0. \end{aligned} \tag{5.4}$$

The well–known solutions of (5.3) and (5.4) read:

$$X(x_1) = M\sin\big(\frac{2m\pi}{L_1}x_1\big) + N\cos\big(\frac{2m\pi}{L_1}x_1\big), \text{ with } m \geq 0, \tag{5.5}$$

and

$$Y(x_2) = P\sin(\frac{n\pi x_2}{L_2}) \text{ with } n \geq 1; \tag{5.6}$$

as usual $\kappa^2 = (n\pi/L_2)^2$ and λ from relation (5.3) is:

$$\lambda = \lambda_{m,n} = \big(\frac{2m\pi}{L_1}\big)^2 + \big(\frac{n\pi}{L_2}\big)^2, \text{ where } m \geq 0,\ n \geq 1. \tag{5.7}$$

From the computations above, we deduce that the eigenfunctions corresponding to the eigenvalue problem (5.2) are:

$$\phi_{m,n}(x) = \sin\big(\frac{2m\pi x_1}{L_1}\big)\sin\big(\frac{n\pi x_2}{L_2}\big),\ m \geq 1,\ n \geq 1, \tag{5.8}$$

and

$$\tilde{\phi}_{m,n}(x) = \cos\big(\frac{2m\pi x_1}{L_1}\big)\sin\big(\frac{n\pi x_2}{L_2}\big),\ m \geq 0,\ n \geq 1. \tag{5.9}$$

We then expand G in terms of these eigenfunctions:

$$G(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}(y_1,y_2)\phi_{m,n}(x_1,x_2) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} b_{m,n}(y_1,y_2)\tilde{\phi}_{m,n}(x_1,x_2). \tag{5.10}$$

The Dirac function $\delta(x-y) = \delta(x_1-y_1)\otimes\delta(x_2-y_2)$ is also expanded as:

$$\delta(x-y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}(y_1,y_2)\phi_{m,n}(x_1,x_2) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} d_{m,n}(y_1,y_2)\tilde{\phi}_{m,n}(x_1,x_2), \tag{5.11}$$

these series converging in the space of distributions (or measures). The $c_{m,n}$ and $d_{m,n}$ are computed as follows:

$$c_{m,n} = \frac{(\delta,\phi_{m,n})}{(\phi_{m,n},\phi_{m,n})} = \frac{\phi_{m,n}}{(\phi_{m,n},\phi_{m,n})},\ d_{m,n} = \frac{\tilde{\phi}_{m,n}}{(\tilde{\phi}_{m,n},\tilde{\phi}_{m,n})}. \tag{5.12}$$

By elementary computations we find:

$$\begin{aligned} &(\phi_{m,n},\phi_{m,n}) = \frac{L_1L_2}{4}, \\ &(\tilde{\phi}_{m,n},\tilde{\phi}_{m,n}) = \frac{L_1L_2}{4} \text{ when } m>0 \text{ and } (\tilde{\phi}_{m,n},\tilde{\phi}_{m,n}) = \frac{L_1L_2}{2} \text{ when } m=0. \end{aligned} \tag{5.13}$$

We now substitute these expressions of G and δ in the first equation (5.1) and identify the coefficients (using the fact that $\Delta\phi_{m,n} = \lambda_{m,n}\phi_{m,n}$ and $\Delta\tilde{\phi}_{m,n} = \lambda_{m,n}\tilde{\phi}_{m,n}$). We find:

$$a_{m,n}(y_1,y_2) = \frac{1}{\lambda_{m,n}}c_{m,n}(y_1,y_2) = \frac{1}{\lambda_{m,n}}\phi_{m,n}(y_1,y_2), \tag{5.14a}$$

$$b_{m,n}(y_1,y_2) = \frac{1}{\lambda_{m,n}}d_{m,n}(y_1,y_2) = \frac{1}{\lambda_{m,n}}\tilde{\phi}_{m,n}(y_1,y_2). \tag{5.14b}$$

Introducing these coefficients in (5.10), we obtain the following expansion of the Green function G:

$$G(x,y)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\epsilon_m\frac{4}{L_1L_2}\frac{1}{\lambda_{m,n}}\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)\sin\big(\frac{n\pi x_2}{L_2}\big)\sin\big(\frac{n\pi y_2}{L_2}\big), \tag{5.15}$$

where we define ϵ_m as being equal to 1 when $m\geq 1$ and to 1/2 when $m=0$. From (5.15) we notice that the function G is symmetrical in x and y. The double series in(5.15), is convergent in the classical sense for $x_1\neq y_1$ and $x_2\neq y_2$. To prove the convergence of the double series, we use a lemma of Hardy (see for example [2]), which we now recall:

Lemma 5.1. *If in a double series $\sum\alpha_{m,n}$ the partial sums $s_{m,n}$ is bounded in absolute value by a constant C for all values of m and n, the double series $\sum\alpha_{m,n}u_{m,n}$ converges, provided that the expressions*

$$u_{m,n}-u_{m+1,n},\quad u_{m,n}-u_{m,n+1},\quad u_{m,n}-u_{m+1,n}-u_{m,n+1}+u_{m+1,n+1},$$

are all positive and that $u_{m,n}$ tends to zero as either m or n tends to ∞.

We apply Lemma 5.1 with:

$$\alpha_{m,n}=\epsilon_m\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)\sin\big(\frac{n\pi x_2}{L_2}\big)\sin\big(\frac{n\pi y_2}{L_2}\big),$$
$$u_{m,n}=\frac{1}{\lambda_{m,n}}.$$

One can easily check that the assumptions on $u_{m,n}$ are satisfied. For $\alpha_{m,n}$ we can also see that:

$$\begin{aligned}&\Big|\sum_{m=m_1,\,n=n_1}^{m_2,\,n_2}\epsilon_m\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)\sin\big(\frac{n\pi x_2}{L_2}\big)\sin\big(\frac{n\pi y_2}{L_2}\big)\Big|\\ &\quad=\Big|\sum_{m=m_1}^{m_2}\epsilon_m\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)\Big|\Big|\sum_{n=n_1}^{n_2}\sin\big(\frac{n\pi x_2}{L_2}\big)\sin\big(\frac{n\pi y_2}{L_2}\big)\Big|\leq C,\end{aligned}\tag{5.16}$$

since $\sum_{m=m_1}^{m_2}\epsilon_m\cos\big(2m\pi(x_1-y_1)/L_1\big)$ and $\sum_{n=n_1}^{n_2}\sin\big(n\pi x_2/L_2\big)\sin\big(n\pi y_2/L_2\big)$ are bounded for all m_1, m_2, n_1, n_2, if $x\neq y$. In fact, by classical trigonometrical computations, we recall that, for example, for $m_1\geq 1$ we have:

$$\sum_{m=m_1}^{m_2}\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)=\frac{\sin\big(\frac{\pi}{L_1}(m_2-m_1+1)(x_1-y_1)\big)}{\sin\big(\frac{\pi}{L_1}(x_1-y_1)\big)}\cos\big(\frac{\pi}{L_1}(m_1+m_2)(x_1-y_1)\big),$$

so, for $x_1\neq y_1$ we have:

$$\Big|\sum_{m=m_1}^{m_2}\cos\big(\frac{2m\pi}{L_1}(x_1-y_1)\big)\Big|\leq\frac{1}{\sin\big(\frac{\pi}{L_1}(x_1-y_1)\big)}.\tag{5.17}$$

By the same kind of reasoning, we find a bound for $\sum_{n=n_1}^{n_2} \sin\big(n\pi x_2/L_2\big)\sin\big(n\pi y_2/L_2\big)$.

We now have this expression of G and we need to determine the properties of the Green function in a channel, as stated in Theorem 5.1.

We write the Green function (5.15) as:

$$\begin{aligned} G(x,y) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \epsilon_m \frac{4}{L_1 L_2}\Big(\int_0^\infty e^{-\lambda_{m,n}s}\,\mathrm{d}s\Big) \\ &\qquad \cos\Big(\frac{2m\pi}{L_1}(x_1-y_1)\Big)\sin\Big(\frac{n\pi x_2}{L_2}\Big)\sin\Big(\frac{n\pi y_2}{L_2}\Big) \\ &= \int_0^\infty \theta_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s, \end{aligned} \tag{5.18}$$

where we used the expression (5.7) of $\lambda_{m,n}$ and where we have set:

$$\theta_1(x_1,y_1,s) = \frac{2}{L_1}\sum_{m=0}^{\infty} \epsilon_m \cos\Big(\frac{2m\pi}{L_1}(x_1-y_1)\Big)e^{-(2m\pi/L_1)^2 s}, \tag{5.19}$$

$$\theta_2(x_2,y_2,s) = \frac{2}{L_2}\sum_{n=1}^{\infty} \sin\Big(\frac{n\pi x_2}{L_2}\Big)\sin\Big(\frac{n\pi y_2}{L_2}\Big)e^{-(n\pi/L_2)^2 s}. \tag{5.20}$$

We will need below alternative expressions of θ_1 and θ_2. For θ_1 we write:

$$\begin{aligned} \theta_1(x_1,y_1,s) &= \frac{2}{L_1}\sum_{m=0}^{\infty} \epsilon_m \cos\frac{2m\pi(x_1-y_1)}{L_1}\exp\Big\{-\Big(\frac{2m\pi}{L_1}\Big)^2 s\Big\} \\ &= \frac{1}{L_1}\sum_{m=-\infty}^{\infty} \cos\frac{2m\pi(x_1-y_1)}{L_1}\exp\Big\{-\Big(\frac{2m\pi}{L_1}\Big)^2 s\Big\} \\ &= \frac{1}{2L_1}\sum_{m=-\infty}^{\infty} \exp\Big\{\frac{2m\pi i}{L_1}(x_1-y_1)-\Big(\frac{2m\pi}{L_1}\Big)^2 s\Big\} \\ &= \frac{1}{4(\pi s)^{1/2}}\sum_{p=-\infty}^{\infty} \exp\Big\{-\frac{(x_1-y_1-pL_1)^2}{4s}\Big\}. \end{aligned} \tag{5.21}$$

On the last line we used as argument the Jacobi imaginary transformation, since the function $\theta_1(x_1,y_1,s)$ is an elliptic theta-function. We recall that the Jacobi imaginary transformation relates elliptic functions to other elliptic functions of the same type but having different arguments. In our case we used the fact that, by the Jacobi imaginary transform, the Jacobi theta function

$$\nu(z,r) = \sum_{n=-\infty}^{\infty} e^{i\pi r n^2} e^{2n\pi i z}, \text{ with } z\in\mathbb{C},$$

becomes:

$$\nu(z,r) = (-ir)^{-1/2} e^{ir'z^2/\pi} \nu(zr', r'), \text{ where } r' = 1/r'.$$

For more details regarding the elliptic theta-functions and the Jacobi imaginary transformation, we refer the interested reader to Whittaker and Watson [15] (pp. 474-476 and 505).

For θ_2 we similarly obtain that:

(5.22)

$$\begin{aligned}\theta_2(x_2,y_2,s) &= \frac{2}{L_2}\sum_{n=1}^{\infty}\sin\frac{n\pi x_2}{L_2}\sin\frac{n\pi y_2}{L_2}\exp\Big\{-\big(\frac{n\pi}{L_2}\big)^2 s\Big\}\\ &= \frac{1}{2(\pi s)^{1/2}}\sum_{p=-\infty}^{\infty}\Big[\exp\Big\{-\frac{(x_2-y_2-2pL_2)^2}{4s}\Big\} - \exp\Big\{-\frac{(x_2+y_2-2pL_2)^2}{4s}\Big\}\Big].\end{aligned}$$

We need to study the behavior of the Green function G, when $\rho = |x-y| \to 0$. We split the integral in (5.18) into two integrals:

$$G(x,y) = G^{(1)}(x,y) + G^{(2)}(x,y), \tag{5.23}$$

where

$$G^{(1)}(x,y) = \int_0^1 \theta_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s, \tag{5.24}$$

and

$$G^{(2)}(x,y) = \int_1^{\infty} \theta_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s. \tag{5.25}$$

Using the expressions (5.19) and (5.20) of θ_1, θ_2, the term $G^{(2)}$ is bounded by:

$$\begin{aligned}|G^{(2)}| &\le \Big|\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\epsilon_m\frac{4}{L_1L_2}\int_1^{\infty}e^{\lambda_{m,n}s}\,\mathrm{d}s\cos\frac{2m\pi}{L_1}(x_1-y_1)\sin\frac{n\pi x_2}{L_2}\sin\frac{n\pi y_2}{L_2}\Big|\\ &\le \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{4}{L_1L_2}\frac{\exp\{-\big(\frac{2m\pi}{L_1}\big)^2-\big(\frac{n\pi}{L_1}\big)^2\}}{\big(\frac{2m\pi}{L_1}\big)^2+\big(\frac{n\pi}{L_1}\big)^2},\end{aligned} \tag{5.26}$$

and the series from the right hand side of (5.26) is absolutely convergent.

It remains to study the behavior of $G^{(1)}$. It appears from (5.21) that θ_1 is a periodic function of $x_1 - y_1$ (and thus of x_1 and y_1) of period L_1. We want to study the behavior on a period which we choose for convenience to be $(-L_1/2, L_1/2)$, and using (5.21) we write:

$$\theta_1(x_1,y_1,s) = \frac{1}{2(\pi s)^{1/2}}\exp\{-\frac{(x_1-y_1)^2}{4s}\} + \tilde{\theta}_1(x_1,y_1,s), \tag{5.27}$$

with

$$\tilde{\theta}_1(x_1,y_1,s)=\frac{1}{2(\pi s)^{1/2}}\sum_{p=-\infty,\,p\neq 0}^{p=\infty}\exp\{-\frac{(x_1-y_1-pL_1)}{4s}\}.$$

Since $e^{-x}\leq 1/x$ for all $x>0$, we see that

$$\tilde{\theta}_1(x_1,y_1,s)\leq\frac{1}{2(\pi s)^{1/2}}\sum_{p=-\infty,\,p\neq 0}^{p=\infty}\frac{4s}{(x_1-y_1-pL_1)^2}\leq s^{1/2}\bar{\theta}_1(x_1,y_1), \tag{5.28}$$

where

$$\bar{\theta}_1(x_1,y_1)=\frac{2}{\pi^{1/2}}\sum_{p=-\infty,\,p\neq 0}^{p=\infty}\frac{1}{(x_1-y_1-pL_1)^2}\leq\frac{4}{\pi^{1/2}L_1^2}\sum_{p=1}^{\infty}\frac{1}{(p-\frac{1}{2})^2},$$

is a bounded $\mathcal{C}^\infty$ function of x_1-y_1, for $|x_1-y_1|\leq L_1/2$.

Similarly for θ_2, we write:

$$\theta_2(x_2,y_2,s)=\frac{1}{2(\pi s)^{1/2}}\exp\{-\frac{(x_2-y_2)^2}{4s}\}+\tilde{\theta}_2(x_2,y_2,s), \tag{5.29}$$

where $\tilde{\theta}_2$ is a function which can be bounded like $\tilde{\theta}_1$ by

$$\tilde{\theta}_2(x_2,y_2,s)\leq s^{1/2}\bar{\theta}_2(x_2,y_2).$$

We can now return to $G^{(1)}$ and write:

$$\begin{aligned}G^{(1)}(x,y)=&\int_0^1\theta_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s+\int_0^1\theta_1(x_1,y_1,s)\tilde{\theta}_2(x_2,y_2,s)\,\mathrm{d}s\\&+\int_0^1\tilde{\theta}_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s+\int_0^1\tilde{\theta}_1(x_1,y_1,s)\tilde{\theta}_2(x_2,y_2,s)\,\mathrm{d}s.\end{aligned} \tag{5.30}$$

The last term from (5.30) is bounded, since:

$$|\int_0^1\tilde{\theta}(x_1,y_1,s)\tilde{\theta}_2(x_2,y_2,s)\,\mathrm{d}s|\leq\bar{\theta}(x_1,y_1)\bar{\theta}(x_2,y_2)\int_0^1 s\,\mathrm{d}s. \tag{5.31}$$

The second and the third terms from (5.30) are of the same type, and we estimate only one of them:

$$|\int_0^1\theta_1(x_1,y_1,s)\tilde{\theta}_2(x_2,y_2,s)\,\mathrm{d}s|\leq\bar{\theta}_2(x_2,y_2)\frac{1}{2\pi^{1/2}}\int_0^1\exp\{-\frac{(x_2-y_2)^2}{4s}\}\,\mathrm{d}s,$$

which is also a bounded term, for all x, y in Ω with $|x_2-y_2|\leq L_2/2$.

For the first term from (5.30), we have:

(5.32)

$$\begin{aligned}\int_0^1 \theta_1(x_1,y_1,s)\theta_2(x_2,y_2,s)\,\mathrm{d}s &= \int_0^1 \frac{1}{4\pi s}\exp\{-\frac{|x-y|^2}{4s}\}\,\mathrm{d}s\\ &= \frac{1}{4\pi}\int_0^1 \frac{1}{s}\exp\{-\frac{\rho^2}{4s}\}\,\mathrm{d}s\\ &= \frac{1}{4\pi}\int_{\rho^2/4}^1 \frac{1}{t}\,\mathrm{d}t + \frac{1}{4\pi}\int_{\rho^2/4}^1 \frac{e^{-t}-1}{t}\,\mathrm{d}t + \frac{1}{4\pi}\int_1^\infty e^{-t}t^{-1}\,\mathrm{d}t\\ &= \frac{1}{2\pi}\log\frac{1}{\rho} + A,\end{aligned}$$

where A is the sum of the two last integrals from (5.32) and it is bounded as $\rho \to 0$.

Thus, as $\rho \to 0$, the behavior of the Green function G is the same as the behavior of $\log 1/\rho$, that is:

$$G(x,y) \sim \frac{1}{2\pi}\log\frac{1}{|x-y|}. \tag{5.33}$$

We also need to study the behavior of $\partial G/\partial\rho$, in absolute value, as $\rho \to 0$. Proceeding as before we find that, as $\rho \to 0$, we have:

$$\frac{\partial G}{\partial \rho}(x,y) \sim -\frac{1}{2\pi\rho}.$$

We conclude by stating the main result of the section:

Theorem 5.1. *(a) The Green function G for the 2D channel with periodic and homogeneous Dirichlet boundary conditions is continuous on $\bar\Omega \times \bar\Omega$, except at $x = y$.*

(b) G has continuous first-order partial derivatives with respect to x, except at $x = y$, in the neighborhood of which the following estimate holds:

$$|D_x G(x,y)| \le K|x-y|^{-1}. \tag{5.34}$$

(c) G is symmetrical in x and y.

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References

[1] J. T. Beale, T. Kato, and A. Majda, (1984): Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations. *Commun. Math. Phys.* **9**4; 61-66

[2] T. J. I. Bromwich, (1992): *Introduction in the theory of infinite series.* American Mathematical Society

[3] D. G. Ebin and J. Marsden, (1970): Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. of Math. (2)* **9**2; 102-163

[4] D. G. Ebin and J. Marsden, (1971): On the motion of incompressible fluids. *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2* 211-214

[5] J.-M. Ghidaglia. Régularité des solutions de certains problèmes aux limites linéaires liés aux équations d'Euler. *Comm. Partial Differential Equations*, 9(13):1265–1298, 1984.

[6] P. Hartman and G. Stampacchia. On some non-linear elliptic differential-functional equations. *Acta Math.*, 115:271–310, 1966.

[7] T. Kato. On classical solutions of two dimensional nonstationary euler equations. *Arch. Rational Mech. Anal.*, 25:188–200, 1967.

[8] L. Lichtenstein. *Grundlagen der Hydromechanik*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 30. Springer-Verlag, Berlin, 1968.

[9] C. Marchioro and M. Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[10] M. Petcu. Euler equation in a 3d channel. 2004. Article in preparation.

[11] R. Temam. *Local existence of $\mathcal{C}^\infty$ solutions of the Euler equations of incompressible perfect fluids*, volume 565 of *Lecture Notes in Math.* Springer-Verlag, 1975.

[12] R. Temam. On the Euler equations of incompressible perfect fluids. *J. Functional Analysis*, 20(1):32–43, 1975.

[13] R. Temam. Remarks on the Euler equations. In *Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983)*, volume 45 of *Proc. Sympos. Pure Math.*, pages 429–430. Amer. Math. Soc., Providence, RI, 1986.

[14] M. Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)*, 32(6):769–812, 1999.

[15] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[16] W. Wolibner. Un théorème sur l'existence du movement plan d'un fluide parfait, homogène, incompressible, pendant un temps infinitement longue. *Mathematische Zeitschrift*, pages 698–726, 1933.