ON THE BACKWARD UNIQUENESS OF THE PRIMITIVE EQUATIONS

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Abstract. In this article we prove the backward uniqueness (as well as the uniqueness) for a class, defined in the article, of solutions of the two dimensional Primitive Equations that we call *$z-$weak solutions*. We also prove the backward uniqueness for the strong solutions in the two and three dimensional cases. By backward uniqueness we understand that once we know that two solutions are equal at a time $t > 0$, then we can conclude that they are equal everywhere on the interval $(0, t)$.

Résumé. On considère les équations primitives en dimension deux et trois d'espace et on étudie l'unicité rétrograde des solutions. Pour l'unicité rétrograde on prouve que si deux solutions coincident à un instant $t > 0$, alors elles sont égales sur tout l'intervalle $(0, t)$. Pour le système 2D, on montre l'unicité rétrograde des solutions *z-faibles* . On montre aussi l'unicité rétrograde des solutions fortes pour le cas 2D et 3D.

1. Introduction

In this article we consider the primitive equations of the ocean, in a two dimensional and then a three dimensional domain, with periodic boundary conditions. The question to which we want to respond is: for which kind of solutions can we prove the backward uniqueness. Lions and Malgrange treated the problem of the backward uniqueness in [6] for certain parabolic problems and later Bardos and Tartar in [1] proved in particular that the weak solutions for the $2D$ Navier-Stokes equations have this property. In this article we will prove that the $2D$ primitive equations possess the backward uniqueness property for a special class of weak solutions, that we call the z-weak solutions. The terminology here is the standard one for fluid mechanics: the weak solutions are those bounded in the L^2-norm, and the strong solutions are those bounded in the H^1-norm. Below we call $z-$weak solutions the weak solutions for which the z derivative is also bounded in L^2 for all finite time; we also call $z-$strong solutions the strong solutions for which the z derivative is bounded in H^1 for all finite time.

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The primitive equations are the equations modelling the motion of the ocean and of the atmosphere, and they are deduced from the fundamental law of physics with simple hypotheses.

On the subject of the well-posedness of the primitive equations much work has been done: we cite here the pioneering work of Lions, Temam and Wang, where they started to study in a mathematical framework the behavior of the solutions for the primitive equations (see e.g. [7], [8]). In this work the authors considered the primitive equations in a three-dimensional domain and they proved the existence, globally in time, of a weak solution. The existence and uniqueness, locally in time of a strong solution was proved by Guillén-González, Masmoudi and Rodríguez-Bellido [3] (see also [17]). On a thin domain, Hu, Temam and Ziane [4], proved the global existence of strong solutions for the primitive equations. The same result but working in a cylindrical domain of arbitrary depth, has recently been proved by Cao and Titi [2] and independently, by Kobelkov [5].

For a $2D$ domain, Petcu, Temam and Wirosoetisno [12] proved the existence, globally in time, of very regular solutions, in fact they proved the existence of absorbing sets in each Sobolev space H^m, and in [11] the Gevrey regularity of such solutions was proved in the space periodic case.

The model we are working with reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + v\frac{\partial u}{\partial x_2} + w\frac{\partial u}{\partial x_3} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x_1} = \nu\Delta u + F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} + v\frac{\partial v}{\partial x_2} + w\frac{\partial v}{\partial x_3} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial x_2} = \nu\Delta v + F_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial x_3} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0, \tag{1.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x_1} + v\frac{\partial \rho}{\partial x_2} + w\frac{\partial \rho}{\partial x_3} - \frac{\rho_0 N^2}{g}w = \mu\Delta\rho + F_\rho. \tag{1.1e}$$

Here, $(u,\, v,\, w)$ are the three components of the velocity vector. In order to obtain this model we wrote the full density ρ_{full} as

$$\rho_{\text{full}}(x_1,\, x_2,\, x_3,\, t) = \rho_0 + \bar\rho(x_3) + \rho(x_1,\, x_2,\, x_3,\, t), \tag{1.2}$$

where ρ_0 is the reference (average) value of the density and $\bar\rho = \bar\rho(x_3)$ is a stratification profile of the density. Similarly, we wrote the pressure as,

$$p_{\text{full}}(x_1,\, x_2,\, x_3,\, t) = p_0 + \bar p(x_3) + p(x_1,\, x_2,\, x_3,\, t), \tag{1.3}$$

where p_0 is a $\partial p_0/\partial x_3 = -g\rho_0$ and $\partial\bar p/\partial x_3 = -g\bar\rho$. From here we obtained (1.1c).

In equation (1.1e) we introduced the (constant) Brunt–Väisälä frequency N, defined by

$$N^2 = -\frac{g}{\rho_{\mathrm{ref}}} \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x_3}\,. \tag{1.4}$$

The constant g is the gravitational acceleration and f the Coriolis parameter, ν and μ are the eddy diffusivity coefficients, (F_u, F_v) represent body forces per unit of mass and F_ρ represents a heating source. In the applications F_u and F_v vanish for the ocean, but we consider here nonzero forces for mathematical generality. We denote by F the vector (F_u, F_v, F_ρ). For more details regarding the derivation of these equations, we refer the interested reader to [9] or the physical appendix of [13].

In what follows we work in a bounded domain:

$$\mathcal{M} = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2), \tag{1.5}$$

and we assume space periodicity with period $\mathcal{M}$, meaning that all functions are taken to satisfy:

$$f(x_1, x_2, x_3, t) = f(x_1 + L_1, x_2, x_3, t) = f(x_1, x_2 + L_2, x_3, t) = f(x_1, x_2, x_3 + L_3, t), \tag{1.6}$$

when extended to $\mathbb{R}^3$.

Working with periodic functions, all functions admit a Fourier series expansion:

$$f(x_1, x_2, x_3, t) = \sum_{k \in \mathbb{R}^3} f_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)}, \tag{1.7}$$

where, for notational conciseness, we set $k_j' = 2\pi k_j / L_j$ for $j = 1, 2, 3$.

We also assume as in [15], [12], that the functions have the following symmetries:

$$\begin{aligned}
u(x_1, x_2, x_3, t) &= u(x_1, x_2, -x_3, t), & F_u(x_1, x_2, x_3, t) &= F_u(x_1, x_2, x_3, t),\\
v(x_1, x_2, x_3, t) &= v(x_1, x_2, -x_3, t), & F_v(x_1, x_2, x_3, t) &= F_v(x_1, x_2, -x_3, t),\\
\rho(x_1, x_2, x_3, t) &= -\rho(x_1, x_2, -x_3, t), & F_\rho(x_1, x_2, x_3, t) &= -F_\rho(x_1, x_2, -x_3, t),\\
w(x_1, x_2, x_3, t) &= -w(x_1, x_2, -x_3, t), & p(x_1, x_2, x_3, t) &= p(x_1, x_2, -x_3, t);
\end{aligned} \tag{1.8}$$

and we say that u, v, p are even and w, ρ odd in x_3. As explained in [12], these symmetry properties are necessary for the space periodicity to be consistent with (1.1). Space periodicity in x_1 and x_2 only (and without the symmetry properties (1.8)) will be considered elsewhere.

The variational formulation of the problem

We start by introducing the natural function spaces for this problem:

$$
\begin{aligned}
(1.9)\qquad & V=\{U=(u,\,v,\,\rho)\in(\dot{H}^1_{\mathrm{per}}(\mathcal{M}))^3,\ u,\,v\text{ even in }x_3,\ \rho\text{ odd in }x_3,\\
& \qquad \int_{-L_3/2}^{L_3/2}(u_{x_1}(x_1,x_2,x_3')+v_{x_2}(x_1,x_2,x_3'))\,\mathrm{d}x_3'=0\},\\
& H=\text{closure of } V \text{ in } (\dot{L}^2(\mathcal{M}))^3,\\
(1.10)\qquad & V_2=\text{ the closure of } V\cap(\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3.
\end{aligned}
$$

As in [15], we endow these spaces with the following scalar products:
on H we consider:

$$
(U,\,\tilde{U})_H=(u,\,\tilde{u})_{L^2}+(v,\,\tilde{v})_{L^2}+\kappa(\rho,\,\tilde{\rho})_{L^2}, \tag{1.11}
$$

and on V:

$$
((U,\,\tilde{U}))_V=((u,\,\tilde{u}))+((v,\,\tilde{v}))+\kappa((T,\,\tilde{T})). \tag{1.12}
$$

Here the dots above $\dot{H}^1_{\mathrm{per}}$ and $\dot{L}^2$ denote the functions with zero average over $\mathcal{M}$. Since we work with functions with zero average over $\mathcal{M}$, we can use the generalized Poincaré inequality:

$$
c_0|U|_H\le\|U\|_V,\ \forall\,U\in V, \tag{1.13}
$$

where c_0 is a constant related to the Poincaré constant.

The variational formulation of this problem is obtained classically by considering a test function $U^\flat=(u^\flat,v^\flat,\rho^\flat)$ in V, multiplying (1.1a) by $u^\flat$, (1.1b) by $v^\flat$, (1.1e) by $\kappa\rho^\flat$, adding and integrating over $\mathcal{M}$. We find the following problem:

Find $U:[0,t_0]\to V$, such that,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(U,U^\flat)_H+a(U,U^\flat)+b(U,U,U^\flat)+e(U,U^\flat)=(F,U^\flat)_H,\quad\forall\,U^\flat\in V,\\
&U(0)=U_0.
\end{aligned} \tag{1.14}
$$

In (1.14) we introduced the following forms:
$a:V\times V\to\mathbb{R}$ *bilinear, continuous, coercive:*

$$
a(U,U^\flat)=\nu((u,u^\flat))+\nu((v,v^\flat))+\kappa\mu((\rho,\rho^\flat)), \tag{1.15}
$$

with $\kappa = g^2/N^2\rho_0^2$,
$b : V \times V \times V_2 \to \mathbb{R}$ *trilinear:*

$$\begin{aligned} b(U, U^\sharp, U^\flat) = &\int_{\mathcal{M}} (u\frac{\partial u^\sharp}{\partial x}u^\flat + v\frac{\partial u^\sharp}{\partial y}u^\flat + w(U)\frac{\partial u^\sharp}{\partial z}u^\flat)\,\mathrm{d}\mathcal{M} \\ &+ \int_{\mathcal{M}} (u\frac{\partial v^\sharp}{\partial x}v^\flat + v\frac{\partial v^\sharp}{\partial y}v^\flat + w(U)\frac{\partial v^\sharp}{\partial z}v^\flat)\,\mathrm{d}\mathcal{M} \\ &+ \kappa\int_{\mathcal{M}} (u\frac{\partial \rho^\sharp}{\partial x}\rho^\flat + v\frac{\partial \rho^\sharp}{\partial y}\rho^\flat + w(U)\frac{\partial \rho^\sharp}{\partial z}\tilde{\rho})\,\mathrm{d}\mathcal{M}, \end{aligned} \tag{1.16}$$

$e : V \times V \to \mathbb{R}$ *bilinear, continuous:*

$$e(U, U^\flat) = f\int_{\mathcal{M}} (uv^\flat - vu^\flat)\,\mathrm{d}\mathcal{M} + \frac{g}{\rho_0}\int_{\mathcal{M}} \rho w(U^\flat)\,\mathrm{d}\mathcal{M} - \frac{g}{\rho_0}\int_{\mathcal{M}} \rho^\flat w(U)\,\mathrm{d}\mathcal{M}, \tag{1.17}$$

with $e(U, U) = 0$ for all $U \in V$.

We also have the following properties for b:

Lemma 1.1. *The form b is trilinear continuous from $V \times V_2 \times V$ into $\mathbb{R}$ and from $V \times V \times V_2$ into $\mathbb{R}$, and*

$$|b(U, U^\sharp, U^\flat)| \leq c_2\|U\||U^\sharp|_H^{1/2}\|U^\sharp\|^{1/2}\|U^\flat\|_{V_2}, \quad \forall\, U, U^\sharp \in V,\ U^\flat \in V_2. \tag{1.18}$$

Furthermore,

$$b(U, U^\flat, U^\flat) = 0 \quad \forall\, U \in V,\ U^\flat \in V_2,$$

and

$$b(U, U^\flat, U^\sharp) = -b(U, U^\sharp, U^\flat), \quad \forall\, U, U^\flat, U^\sharp \in V \text{ with } U^\flat \text{ or } U^\sharp \in V_2.$$

These properties and other properties of these forms are proved in detail in [12] and [13].

Problem (1.14) can also be written as an operator evolution equation in V_2':

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U, U) + EU = F, \\ &U(0) = U_0, \end{aligned} \tag{1.19}$$

where we introduced the following operators:

A linear continuous from V into V', defined by

$\langle AU, U^\flat\rangle = a(U, U^\flat), \quad \forall U, U^\flat \in V,$

B bilinear, continuous from $V \times V$ into V_2', defined by

$\langle B(U, U^\flat),\, U^\sharp\rangle = b(U, U^\flat, U^\sharp) \quad \forall\, U, U^\flat \in V,\ \forall\, U^\sharp \in V_2,$

E linear continuous from V into V', defined by

$\langle EU, U^\flat\rangle = e(U, U^\flat), \quad \forall\, U, U^\flat \in V,$ *with* $\langle EU, U\rangle = 0.$

In this article we would like to know in what class of solutions we have the backward uniqueness. That is, when can we conclude that two solutions that are equal at a time $t > 0$, are equal everywhere on the interval $(0, t)$. In what follows we will prove that for the two-dimensional model of primitive equations we have the backward uniqueness for a class of solutions which are weak in the horizontal direction and strong in the vertical direction, solutions that we call $z-$weak. For the three dimensional model, the backward uniqueness of (usual) strong solutions is proved.

We start, in Section 2 and 3, by proving the necessary results of existence and (forward) uniqueness in space dimensions 2 and 3. Then, in Section 4 and 5, we address the question of backward uniqueness in dimension 2 and then 3.

2. Existence and uniqueness of z-weak solutions in dimension 2

In this section, we consider the $2D$ version of (1.1): all the functions are independent of the x_2-variable but the velocity v is not zero, so we still model a three dimensional motion. The equations read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + w\frac{\partial u}{\partial x_3} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x_1} = \nu_{\boldsymbol{v}}\,\Delta u + F_u, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} + w\frac{\partial v}{\partial x_3} + fu = \nu_{\boldsymbol{v}}\,\Delta v + F_v, \tag{2.1b}$$

$$\frac{\partial p}{\partial x_3} = -g\rho, \tag{2.1c}$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial w}{\partial x_3} = 0, \tag{2.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x_1} + w\frac{\partial \rho}{\partial x_3} - \frac{\rho_0 N^2}{g}w = \nu_\rho\,\Delta\rho + F_\rho. \tag{2.1e}$$

In [12] we proved the existence, globally in time, of a weak solution for this model, as well as the existence and uniqueness of a strong solution. In this section we prove an intermediate result, that is the existence and uniqueness, globally in time, of solutions which are weak in the horizontal direction and strong in the vertical direction (the so-called $z-$weak solutions). We start by introducing the function spaces necessary for this problem:

$$\mathcal{V} = \{U = (u,\, v,\, \rho) \in V,\ \frac{\partial U}{\partial x_3} \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3\},$$

which is a Hilbert space when endowed with the following norm:

$$|U|^2_{\mathcal{V}} = \|U\|^2 + \Big\|\frac{\partial U}{\partial x_3}\Big\|^2.$$

Another useful function space is:

$$\mathcal{H} = \{U = (u,\, v,\, \rho) \in H,\, \frac{\partial U}{\partial x_3} \in (\dot{L}^2_{\text{per}}(\mathcal{M}))^3\},$$

which is a Hilbert space when endowed with the norm:

$$|U|^2_{\mathcal{H}} = |U|^2_{L^2} + \left|\frac{\partial U}{\partial x_3}\right|^2_{L^2}.$$

We now prove the existence and uniqueness, globally in time, of a z-weak solution for (2.1) (see [14] and [18]).

Theorem 2.1. (z-weak solutions in dimension two) *Given $U_0 \in \mathcal{H}$ and $F \in L^\infty(0,T;\mathcal{H})$, there exists a unique solution U of problem* (2.1) *satisfying the initial condition $U(0) = U_0$ and:*

$$U \in \mathcal{C}([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}). \tag{2.2}$$

Proof. The existence of a weak solution for problem (2.1) was proved in [12] and [18]. It remains to prove that starting with an initial data and a forcing more regular (satisfying the hypotheses of Theorem 2.1), the solution is strong in the vertical direction. In order to prove that, we need to obtain a priori estimates for $U_{x_3} = \partial U/\partial x_3$. We formally differentiate (2.1a), (2.1b) and (2.1e) in x_3 and then multiply respectively by u_{x_3}, v_{x_3} and ρ_{x_3}, and integrate over $\mathcal{M}$. We find:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U_{x_3}|^2_{L^2} &+ \int_{\mathcal{M}} (u_{x_1} + w_{x_3}) u^2_{x_3}\,\mathrm{d}\mathcal{M} + \frac{1}{\rho_0}\int_{\mathcal{M}} p_{x_1 x_3} u_{x_3}\,\mathrm{d}\mathcal{M}\\
&+ \int_{\mathcal{M}} (u_{x_3} v_{x_1} + v_{x_3} w_{x_3}) v_{x_3}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (u_{x_3}\rho_{x_1} + w_{x_3}\rho_{x_3})\rho_{x_3}\,\mathrm{d}\mathcal{M} + \nu\|U_{x_3}\|^2\\
&= (F_{x_3}, U_{x_3})_{L^2}.
\end{aligned} \tag{2.3}$$

The second term of (2.3) is zero because of the mass conservation equation. The pressure term can be estimated, using the hydrostatic equation (2.1c) and integrating by parts:

$$\int_{\mathcal{M}} p_{x_1 x_3} u_{x_3}\,\mathrm{d}\mathcal{M} = -g\int_{\mathcal{M}} \rho_{x_1} u_{x_3}\,\mathrm{d}\mathcal{M} \leq g|\rho|\|U_{x_3}\|. \tag{2.4}$$

We also estimate:

$$\int_{\mathcal{M}} (u_{x_3} v_{x_1} + v_{x_3} w_{x_3}) v_{x_3}\,\mathrm{d}\mathcal{M} \leq |u_{x_3}|_{L^4}|v_{x_1}|_{L^2}|v_{x_3}|_{L^4} + |u_{x_1}|_{L^2}|v_{x_3}|^2_{L^4} \leq c|U_{x_3}|_{L^2}\|U_{x_3}\|\|U\|,$$

and

$$\int_{\mathcal{M}} (u_{x_3}\rho_{x_1} + w_{x_3}\rho_{x_3})\rho_{x_3}\,\mathrm{d}\mathcal{M} \leq |u_{x_3}|_{L^4}|\rho_{x_1}|_{L^2}|\rho_{x_3}|_{L^4} + |u_{x_1}|_{L^2}|\rho_{x_3}|^2_{L^4} \leq c|U_{x_3}|\|U_{x_3}\|\|U\|.$$

Using the above estimates into (2.3), we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U_{x_3}|_{L^2}^2 + \nu\|U_{x_3}\|^2 \le |F|_{L^2}\|U_{x_3}\| + c|\rho|_{L^2}\|U_{x_3}\| + c|U_{x_3}|_{L^2}\|U_{x_3}\|\|U\|,$$

which, by the Young inequality, implies:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U_{x_3}|_{L^2}^2 + \frac{\nu}{2}\|U_{x_3}\|^2 \le c|S|_{L^2}^2 + c|U_{x_3}|_{L^2}^2\|U\|^2 + c|U|_{L^2}^2. \tag{2.5}$$

Applying the Gronwall lemma to (2.5) and using the estimates valid for weak solutions (U in $L^2(0,T,V)$ $\forall T$), we find a bound for U_{x_3} in $L^\infty(0,T;L^2(\mathcal{M}))$ and $L^2(0,T;\dot{H}^1_{\mathrm{per}}(\mathcal{M}))$.

Using all these estimates and the Galerkin method, we can prove the existence of a $z-$weak solution that is with U and U_{x_3} belonging to $L^\infty(0,T;L^2(\mathcal{M}))\cap L^2(0,T;\dot{H}^1_{\mathrm{per}}(\mathcal{M}))$.

The forward uniqueness of a $z-$weak solution is then proved classically: we suppose that U_1 and U_2 are two $z-$weak solutions for (2.1), satisfying the same initial condition. Then, $\tilde{U} = U_1 - U_2$ satisfies the following equation:

$$\tilde{U}' + A\tilde{U} + E\tilde{U} + B(U_1,\tilde{U}) + B(\tilde{U},U_2) = 0, \tag{2.6}$$

with $\tilde{U}(0) = 0$.

We take the $(V',V)-$duality product of (2.6) with $\tilde{U}$. We find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{U}|_H^2 + c_0\|\tilde{U}\|_V^2 + b(U_1,\tilde{U},\tilde{U}) + b(\tilde{U},U_2,\tilde{U}) \le 0. \tag{2.7}$$

From the orthogonality property we know that $b(U_1,\tilde{U},\tilde{U}) = 0$, under the hypotheses of Lemma 1.1. But we note here that in our case U_1 and $\tilde{U}$ do not satisfy the conditions in Lemma 1.1; however, the same result can be easily obtained for the case $U_1 \in V$ and $\tilde{U} \in \mathcal{V}$, using the same kind of reasoning as before. It remains to estimate $b(\tilde{U},U_2,\tilde{U})$:

$$b(\tilde{U},U_2,\tilde{U}) = \int_{\mathcal{M}} \tilde{u}\frac{\partial U_2}{\partial x_1}\cdot\tilde{U}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w(\tilde{U})\frac{\partial U_2}{\partial x_3}\cdot\tilde{U}\,\mathrm{d}\mathcal{M}. \tag{2.8}$$

The first term of (2.8) is estimated using the Holder inequality and the Sobolev embeddings:

$$\Big|\int_{\mathcal{M}} \tilde{u}\frac{\partial U_2}{\partial x_1}\cdot\tilde{U}\,\mathrm{d}\mathcal{M}\Big| \le |\tilde{u}|_{L^4}\Big|\frac{\partial U_2}{\partial x_1}\Big|_{L^2}|\tilde{U}|_{L^4} \le c|\tilde{U}|\|\tilde{U}\|\|U_2\|. \tag{2.9}$$

For the second term, we find:

$$\Big|\int_{\mathcal{M}} w(\tilde{U})\frac{\partial U_2}{\partial x_3}\cdot\tilde{U}\,\mathrm{d}\mathcal{M}\Big| \le |w(\tilde{U})|_{L^2}\Big|\frac{\partial U_2}{\partial x_3}\Big|_{L^4}|\tilde{U}|_{L^4} \le c|\tilde{U}|^{1/2}\|\tilde{U}\|^{3/2}\Big|\frac{\partial U_2}{\partial x_3}\Big|^{1/2}\Big\|\frac{\partial U_2}{\partial x_3}\Big\|^{1/2}. \tag{2.10}$$

Using the above estimates into (2.8), we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{U}|_H^2 + c_0\|\tilde{U}\|_V^2 \le g(t)|\tilde{U}|_H^2, \tag{2.11}$$

where

$$g(t) = c\|U_2\|^2 + c\Big|\frac{\partial U_2}{\partial x_3}\Big|^2\Big\|\frac{\partial U_2}{\partial x_3}\Big\|^2.$$

Since U_2 is a $z-$weak solution, the function g belongs to $L^1(0,T)$ for any $T > 0$. So applying the Gronwall lemma to (2.11), we find that $\tilde{U}(t) = 0$ for all $t > 0$.

It remains to prove that the $z-$weak solution U belongs to $\mathcal{C}([0,T],\mathcal{H})$. We start by proving that $B(U,U)$ belongs to $L^2(0,T,\mathcal{V}')$. Let $\tilde{U}$ be in $\mathcal{V}$. Then:

$$< B(U,U),\tilde{U} >_{\mathcal{V}',\mathcal{V}} = b(U,U,\tilde{U}) = \int_{\mathcal{M}} u\frac{\partial U}{\partial x_1}\cdot\tilde{U}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w(U)\frac{\partial U}{\partial x_3}\cdot\tilde{U}\,\mathrm{d}\mathcal{M}. \tag{2.12}$$

The first term is estimated as:

$$\begin{aligned}\int_{\mathcal{M}} u\frac{\partial U}{\partial x_1}\cdot\tilde{U}\,\mathrm{d}\mathcal{M} &= \int_0^{L_1}\int_{-L_3/2}^{L_3/2} u\frac{\partial U}{\partial x_1}\cdot\tilde{U}\,\mathrm{d}x_3\,\mathrm{d}x_1 \le \int_0^{L_1}|u|_{L^2_{x_3}}\Big|\frac{\partial U}{\partial x_1}\Big|_{L^4_{x_3}}|\tilde{U}|_{L^4_{x_3}}\,\mathrm{d}x_1\\ &\le \int_0^{L_1}|U|_{L^2_{x_3}}\Big(\Big|\frac{\partial U}{\partial x_1}\Big|^{1/2}_{L^2_{x_3}} + \Big|\frac{\partial^2 U}{\partial x_1\partial x_3}\Big|^{1/2}_{L^2_{x_3}}\Big)|\tilde{U}|^{1/2}_{L^2_{x_3}}\Big(|\tilde{U}|^{1/2}_{L^2_{x_3}} + \Big|\frac{\partial\tilde{U}}{\partial x_3}\Big|^{1/2}_{L^2_{x_3}}\Big).\end{aligned} \tag{2.13}$$

Here and below $L^q_{x_1}$ is $L^q(0,L_1)$ and $L^q_{x_3}$ is $L^q(-L_3/2,L_3/2)$.

The most difficult term of (2.13) is:

$$\begin{aligned}&\int_0^{L_1}|U|_{L^2_{x_3}}\Big|\frac{\partial U}{\partial x_1}\Big|^{1/2}_{L^2_{x_3}}\Big|\frac{\partial^2 U}{\partial x_1\partial x_3}\Big|^{1/2}_{L^2_{x_3}}|\tilde{U}|^{1/2}_{L^2_{x_3}}\Big|\frac{\partial\tilde{U}}{\partial x_3}\Big|^{1/2}_{L^2_{x_3}}\,\mathrm{d}x_1\\ &\le c|U|_{L^2(\mathcal{M})}\Big|\frac{\partial U}{\partial x_1}\Big|^{1/2}_{L^2(\mathcal{M})}\Big|\frac{\partial^2 U}{\partial x_1\partial x_3}\Big|^{1/2}_{L^2(\mathcal{M})}\big||\tilde{U}|_{L^2_{x_3}}|^{1/2}_{L^\infty_{x_1}}\Big|\Big|\frac{\partial\tilde{U}}{\partial x_3}\Big|_{L^2_{x_3}}\Big|^{1/2}_{L^\infty_{x_1}}\\ &\le c|U|_{L^2(\mathcal{M})}\Big|\frac{\partial U}{\partial x_1}\Big|^{1/2}_{L^2(\mathcal{M})}\Big|\frac{\partial^2 U}{\partial x_1\partial x_3}\Big|^{1/2}_{L^2(\mathcal{M})}\|\tilde{U}\|^{1/2}\Big\|\frac{\partial\tilde{U}}{\partial x_3}\Big\|^{1/2}\\ &\le c|U|_{L^2(\mathcal{M})}\|U\|_{\mathcal{V}}\|\tilde{U}\|_{\mathcal{V}};\end{aligned} \tag{2.14}$$

we used the fact that, in dimension one, we have the Sobolev embedding $H^1_{x_1}\subset L^\infty_{x_1}$, which implies that:

$$|\tilde{U}|_{L^\infty_{x_1}(L^2_{x_3})} \le c|\tilde{U}|_{H^1_{x_1}(L^2_{x_3})} \le c\|\tilde{U}\|. \tag{2.15}$$

We also need to estimate the second term from (2.12):

$$
\begin{aligned}
\Big| \int_{\mathcal{M}} w(U) \frac{\partial U}{\partial x_3} \cdot \tilde{U} \, \mathrm{d}\mathcal{M} \Big| &\le \int_0^{L_1} |w(U)|_{L^\infty_{x_3}} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2_{x_3}} |\tilde{U}|_{L^2_{x_3}} \, \mathrm{d}x_1 \\
&\le c \int_0^{L_1} |U_{x_1}|_{L^2_{x_3}} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2_{x_3}} |\tilde{U}|_{L^2_{x_3}} \, \mathrm{d}x_1 \\
&\le c |U_{x_1}|_{L^2_{x_1}(L^2_{x_3})} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2_{x_1}(L^2_{x_3})} |\tilde{U}|_{L^\infty_{x_1}(L^2_{x_3})} \\
&\le c |U_{x_1}|_{L^2(\mathcal{M})} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2(\mathcal{M})} |\tilde{U}|_{H^1_{x_1}(L^2_{x_3})} \\
&\le c |U_{x_1}|_{L^2(\mathcal{M})} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2(\mathcal{M})} \|\tilde{U}\|.
\end{aligned} \tag{2.16}
$$

Combining (2.14) and (2.16), we find that:

$$
\|B(U,U)\|_{\mathcal{V}'} \le c|U|_{L^2(\mathcal{M})} \|U\|_{\mathcal{V}} + c|U_{x_1}|_{L^2(\mathcal{M})} \Big| \frac{\partial U}{\partial x_3} \Big|_{L^2(\mathcal{M})}, \tag{2.17}
$$

which, taking into account that $U \in L^2(0,T,\mathcal{V})$, implies that $B(U,U) \in L^2(0,T,\mathcal{V}')$.

Then one can easily conclude from (1.19) that $U' \in L^2(0,T,\mathcal{V}')$. We know that $U \in L^2(0,T,\mathcal{V})$ and $\mathcal{V} \subset V \subset H \subset V' \subset \mathcal{V}'$ where each space is dense into the other. We can then conclude, using a technical result (see [16] for more details), that U belongs to $\mathcal{C}([0,T],\mathcal{H})$, observing that $\mathcal{H} = [\mathcal{V},\mathcal{V}']_{1/2}$ is the $1/2-$interpolate between $\mathcal{V}$ and $\mathcal{V}'$. □

3. Existence and uniqueness of $z-$strong solutions in dimension 3

In what follows, we also need the existence globally in time as well as the uniqueness of $z-$strong solutions. We can prove the following result:

Theorem 3.1. (z-strong solution in dimension two and three) *Given $U_0 \in \mathcal{V}$ and $F \in L^\infty(0,T;\mathcal{V})$, there exists a unique solution U of problem* (2.1)*, satisfying the initial condition $U(0) = U_0$ and:*

$$
U \in L^\infty([0,T];\mathcal{V}) \cap L^2(0,T;\dot{H}^2_{\mathrm{per}}(\mathcal{M})), \quad \frac{\partial U}{\partial x_3} \in L^2(0,T;\dot{H}^2_{\mathrm{per}}(\mathcal{M})). \tag{3.1}
$$

Proof. We start by mentioning that the following reasoning is related to dimension 3; the dimension 2 is similar and much easier.

In [17] the authors proved, using the Galerkin approximation, the existence and uniqueness of a strong solution, locally in time. We are now interested in obtaining a priori estimates for the z-strong solution, so that, using the Galerkin method, to prove the existence locally in time and the uniqueness of a z-strong solution.

We assume U is a smooth solution for the primitive equations and we first derive here some a priori estimates on U_{x_3}. At the end of the proof we explain how these estimates provide the existence of the z-strong solution, globally in time.

We start by differentiating the evolution equation (1.19) in x_3; we find:

$$U'_{x_3} + AU_{x_3} + EU_{x_3} + (B(U,U))_{x_3} = F_{x_3}. \tag{3.2}$$

Multiplying (3.2) by $-\Delta U_{x_3}$ and integrating over $\mathcal{M}$, we find:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_{x_3}\|^2 + c_0|\Delta U_{x_3}|^2 \le \Big|\int_{\mathcal{M}} u_{x_3}U_{x_1}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big| + \Big|\int_{\mathcal{M}} v_{x_3}U_{x_2}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big| \\
&\quad + \Big|\int_{\mathcal{M}} w(U)_{x_3}U_{x_3}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big| + \Big|\int_{\mathcal{M}} uU_{x_1x_3}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M} \\
&\quad + \int_{\mathcal{M}} vU_{x_2x_3}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w(U)U_{x_3x_3}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big| + \Big|\int_{\mathcal{M}} F_{x_3}\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big|.
\end{aligned} \tag{3.3}$$

We need to estimate the terms from the right-hand-side of (3.3). The first three terms are similar so we will estimate just one of them:

$$\begin{aligned}
\Big|\int_{\mathcal{M}} u_{x_3}U_{x_1}\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M}\Big| &\le |U_{x_3}|_{L^4}|U_{x_1}|_{L^4}|\Delta U_{x_3}|_{L^2} \\
&\le c|\Delta U_{x_3}||U_{x_1}|^{1/4}\|U_{x_1}\|^{3/4}|U_{x_3}|^{1/4}\|U_{x_3}\|^{3/4} \\
&\le c|\Delta U_{x_3}|\|U\|^{1/4}|U|_{H^2}^{3/4}|U_{x_3}|^{1/4}\|U_{x_3}\|^{3/4} \\
&\le \frac{c_0}{8}|\Delta U_{x_3}|^2 + c|U|_{H^1}^{1/2}|U|_{H^2}^{3/2}|U_{x_3}|^{1/2}\|U_{x_3}\|^{3/2}.
\end{aligned} \tag{3.4}$$

By integration by parts we also find for the other terms:

$$\begin{aligned}
\int_{\mathcal{M}}(uU_{x_1x_3} + vU_{x_2x_3} + w(U)U_{x_3x_3})\cdot\Delta U_{x_3}\,\mathrm{d}\mathcal{M} &= -\int_{\mathcal{M}} u\nabla U_{x_1x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} \\
&\quad - \int_{\mathcal{M}} v\nabla U_{x_2x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} - \int_{\mathcal{M}} w(U)\nabla U_{x_3x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} \\
&\quad - \int_{\mathcal{M}}[(\nabla u\cdot\nabla)U_{x_3}]\cdot U_{x_1x_3}\,\mathrm{d}\mathcal{M} - \int_{\mathcal{M}}[(\nabla v\cdot\nabla)U_{x_3}]\cdot U_{x_2x_3}\,\mathrm{d}\mathcal{M} \\
&\quad - \int_{\mathcal{M}}[(\nabla w(U)\cdot\nabla)U_{x_3}]\cdot U_{x_3x_3}\,\mathrm{d}\mathcal{M}.
\end{aligned} \tag{3.5}$$

We first notice that by integration by parts and using the mass conservation, we find:

$$\int_{\mathcal{M}} u\nabla U_{x_1x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} v\nabla U_{x_2x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w(U)\nabla U_{x_3x_3}\cdot\nabla U_{x_3}\,\mathrm{d}\mathcal{M} = 0. \tag{3.6}$$

We need to estimate the remaining terms, which are of two types: containing or not $w(U)$. We find:

$$\begin{aligned}\Big|\int_{\mathcal{M}}[(\nabla u\cdot\nabla)U_{x_3}]\cdot U_{x_1x_3}\,\mathrm{d}\mathcal{M}\Big| &\le |\nabla U|_{L^2}|\nabla U_{x_3}|^2_{L^4}\le c\|U\|\|U_{x_3}\|^{1/2}|U_{x_3}|^{3/2}_{H^2}\\ &\le \frac{c_0}{8}|\Delta U_{x_3}|^2_{L^2}+c\|U\|^4\|U_{x_3}\|^2,\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\int_{\mathcal{M}}&[(\nabla w(U)\cdot\nabla)U_{x_3}]\cdot U_{x_3x_3}\,\mathrm{d}\mathcal{M}=\int_{\mathcal{M}'}\int_{-L_3/2}^{L_3/2}[(\nabla w(U)\cdot\nabla)U_{x_3}]\cdot U_{x_3x_3}\,\mathrm{d}x_3\,\mathrm{d}\mathcal{M}'\\ &\le\int_{\mathcal{M}'}|\nabla w(U)|_{L^\infty_{x_3}}|\nabla U_{x_3}|^2_{L^2_{x_3}}\,\mathrm{d}\mathcal{M}'\le c\int_{\mathcal{M}'}|\Delta U|_{L^2_{x_3}}|\nabla U_{x_3}|^2_{L^2_{x_3}}\,\mathrm{d}\mathcal{M}'\\ &\le c||\Delta U|_{L^2_{x_3}}|_{L^2(\mathcal{M}')}||\nabla U_{x_3}|_{L^2_{x_3}}|^2_{L^4(\mathcal{M}')}\\ &\le c|\Delta U|_{L^2(\mathcal{M})}||\nabla U_{x_3}|_{L^2_{x_3}}|_{L^2(\mathcal{M}')}||\nabla U_{x_3}|_{L^2_{x_3}}|_{H^1(\mathcal{M}')},\end{aligned}\tag{3.8}$$

where $\mathcal{M}'=(0,L_1)\times(0,L_2)$.

One can easily show, by direct differentiation and classical estimates, that:

$$||\nabla U_{x_3}|_{L^2_{x_3}}|^2_{H^1(\mathcal{M}')}\le c(|\nabla U_{x_3}|^2_{L^2(\mathcal{M})}+|U_{x_3}|^2_{H^2(\mathcal{M})})\le c|U_{x_3}|^2_{H^2(\mathcal{M})}.\tag{3.9}$$

Using (3.9) into (3.8), we find:

$$\begin{aligned}\int_{\mathcal{M}}[(\nabla w(U)\cdot\nabla)U_{x_3}]\cdot U_{x_3x_3}\,\mathrm{d}\mathcal{M}&\le c|\Delta U||\nabla U_{x_3}||\Delta U_{x_3}|\\ &\le\frac{c_0}{8}|\Delta U_{x_3}|^2+c|\Delta U|^2|\nabla U_{x_3}|^2.\end{aligned}\tag{3.10}$$

The forcing term is easy to estimate, and gathering all the above estimates we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_{x_3}\|^2+c_0|\Delta U_{x_3}|^2\le f(t)\|U_{x_3}\|^2+g(t),\tag{3.11}$$

with

$$f(t)=c(\|U\|^2+|\Delta U|^2_{L^2}),\quad g(t)=|F_{x_3}|^2_{L^2}.$$

Using these a priori estimates and the Galerkin method, we prove that the z-weak solution exists on an interval $(0,t_\star)$, with $t_\star\le T$. But the recent improvements due to C. Cao and E. Titi [2] and to G. Kobelkov [5] showed the existence of a global strong solution (meaning $t_\star=T$) and since the estimates in (3.11) depend only on ΔU, we conclude that the z-strong solution exists globally in time.

□

4. Backward uniqueness for the $z-$weak solutions in dimension two

In what follows we prove that the $z-$weak solutions for the $2D$ primitive equations have the backward uniqueness property. This means that if two $z-$weak solutions U_1 and U_2 defined on the interval $[0,T]$ coincide at a point $t_\star \in (0,T)$, then we can conclude that the solutions coincide on the whole interval $[0,t_\star]$. The arguments we use are similar to the case of Navier-Stokes equations considered in [1], [6].

In fact we can prove that:

Theorem 4.1. (z-weak solutions in dimension two) *Let F be in $L^2(0,T,\mathcal{V})$ and let U_1, U_2 be two $z-$weak solutions for the primitive equations* (2.1)*, U_1, U_2 belonging to $\mathcal{C}([0,T];\mathcal{H}) \cap L^2(0,T,\mathcal{V})$, such that $U_1(t_\star) = U_2(t_\star)$. Then $U_1 = U_2$ on the interval $[0,t_\star]$.*

Before starting to prove the result announced, we give the following useful result:

Proposition 4.1. *Let F be in $L^2(0,T;\mathcal{V})$ and U_0 in $\mathcal{V}$. Let us also consider U solution of the linear primitive equations:*

$$\begin{aligned} &U'(t) + AU(t) + EU(t) = F, \\ &U(0) = U_0. \end{aligned} \tag{4.1}$$

For all time t such that $U(t) \neq 0$, we define the following function:

$$\phi(t) = \frac{((A+E)U(t),U(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}}. \tag{4.2}$$

Then, ϕ is differentiable for almost every t where it is defined (meaning where $U(t) \neq 0$) and

$$\phi'(t) \le \frac{|F(t)|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}}. \tag{4.3}$$

Proof. By classical methods, one can immediately show (compare to Theorem 3.1) that the solutions U of the linear primitive equations satisfy:

$$U \in L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;\dot{H}^2_{\text{per}}(\mathcal{M})), \quad \frac{\partial U}{\partial x_3} \in L^2(0,T;\dot{H}^2_{\text{per}}(\mathcal{M})), \quad U \in \mathcal{C}([0,T],\mathcal{H}).$$

We first note that the function ϕ is defined on the open subset of $(0,T)$ where $|U(t)|_{\mathcal{H}} > 0$; the set where $|U(t)|_{\mathcal{H}} > 0$ is open because $U \in \mathcal{C}([0,T],\mathcal{H})$.

Then, all the computations below, performed formally, can be fully justified by using a Galerkin approximation. We first note that, since E is an skewsymmetric operator, we have:

$$\phi(t) = \frac{((A+E)U(t),U(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} = \frac{(AU(t),U(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}}.$$

We find:

(4.4)
$$
\begin{aligned}
\phi'(t) =& 2\frac{< AU'(t), U(t) >_{V',V} + < AU'_{x_3}(t), U_{x_3}(t) >_{V',V}}{|U(t)|^2_{\mathcal{H}}} \\
& - 2\frac{(AU(t), U(t))_{\mathcal{H}}}{|U(t)|^4_{\mathcal{H}}}\{< U'(t), U(t) >_{V',V} + < U'_{x_3}(t), U_{x_3}(t) >_{V',V}\} \\
=& 2\frac{(F - AU(t) - EU(t), AU(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - 2\frac{(AU(t), U(t))_{\mathcal{H}}}{|U(t)|^4_{\mathcal{H}}}(F - AU(t) - EU(t), U(t))_{\mathcal{H}} \\
=& 2\frac{(F, AU(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - 2\frac{|AU(t)|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - 2\frac{(AU(t), U(t))_{\mathcal{H}}}{|U(t)|^4_{\mathcal{H}}}(F, U(t))_{\mathcal{H}} \\
& + 2\frac{|(AU(t), U(t))_{\mathcal{H}}|^2}{|U(t)|^4_{\mathcal{H}}},
\end{aligned}
$$

where, in the computations above, we used the fact that:

$$< AU(t), EU(t) >_{V',V} = 0.$$

The relation above can be formally checked as follows (rigorous justifications can be derived):

$$
\begin{aligned}
< AU(t), EU(t) >_{V',V} =& - f\int_{\mathcal{M}}(u\Delta v - v\Delta u)\mathrm{d}\mathcal{M} \\
& - \frac{g}{\rho_0}\int_{\mathcal{M}}\rho w(\Delta U)\mathrm{d}\mathcal{M} + \frac{g}{\rho_0}\int_{\mathcal{M}}\Delta\rho w(U)\mathrm{d}\mathcal{M} \\
=& - \frac{g}{\rho_0}\sum_{l+m=0, l_3\neq 0}|l|^2\rho_l\frac{m_1}{m_3}u_m + \frac{g}{\rho_0}\sum_{l+m=0, m_3\neq 0}\rho_l\frac{m_1}{m_3}|m|^2u_m \\
=& 0,
\end{aligned} \tag{4.5}
$$

where we used the definition of $w(U)$ as $-k_1/k_3u_k$ for $k_3 \neq 0$, and 0 when $k_3 = 0$.

We have the following relation:

$$
\begin{aligned}
|(AU(t), U(t))_{\mathcal{H}}|^2 - (AU(t), U(t))_{\mathcal{H}}(F, U(t))_{\mathcal{H}} + \frac{1}{4}|(F, U(t))_{\mathcal{H}}|^2 \\
= |(AU(t) - F/2, U(t))_{\mathcal{H}}|^2 \leq |AU(t) - F/2|^2_{\mathcal{H}}|U|^2_{\mathcal{H}}.
\end{aligned} \tag{4.6}
$$

Continuing to estimate ϕ' in (4.4), we can conclude:

$$
\begin{aligned}
\phi'(t) \leq & 2\frac{(F, AU(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - 2\frac{|AU(t)|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} + 2\frac{|AU(t) - F/2|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - \frac{1}{2}\frac{|(F, U(t))_{\mathcal{H}}|^2}{|U(t)|^4_{\mathcal{H}}} \\
\leq & 2\frac{(F, AU(t))_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - 2\frac{|AU(t)|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}} - \frac{1}{2}\frac{|(F, U(t))_{\mathcal{H}}|^2}{|U(t)|^4_{\mathcal{H}}} \\
& + \frac{2}{|U|^2_{\mathcal{H}}}\{|AU|^2_{\mathcal{H}} - (F, AU(t))_{\mathcal{H}} + \frac{1}{4}|F|^2_{\mathcal{H}}\} \\
\leq & \frac{|F(t)|^2_{\mathcal{H}}}{|U(t)|^2_{\mathcal{H}}}.
\end{aligned} \tag{4.7}
$$

□

We can now start to prove the main result of this section.

Remark 4.1. A similar result is also true in dimension three but in other spaces. More exactly, let F be in $L^2(0, T; V)$ and U_0 in V. Let us also consider U as the solution of the linear primitive equations:

$$
\begin{aligned}
& U'(t) + AU(t) + EU(t) = F, \\
& U(0) = U_0.
\end{aligned} \tag{4.8}
$$

For all time t such that $U(t) \neq 0$, we define the following function:

$$
\phi(t) = \frac{((A + E)U(t), U(t))_H}{|U(t)|^2_H}.
$$

Then, ϕ is differentiable for almost all t where it is defined and

$$
\phi'(t) \leq \frac{|F(t)|^2_H}{|U(t)|^2_H}. \tag{4.9}
$$

Proof of Theorem 4.1. We notice that since U_1 and U_2 are $z-$weak solutions, U_1 and U_2 belong to $L^2(0, T, \mathcal{V})$ and we can thus find a δ arbitrarily small such that $U_1(\delta)$ and $U_2(\delta)$ belong to $\mathcal{V}$. Considering the primitive equations having $U_1(\delta)$ and $U_2(\delta)$ as initial condition at $t = \delta$, one obtains, using Theorem 3.1 (for the dimension 2), the existence of $z-$strong solutions $\tilde{U}_1$ and $\tilde{U}_2$. We note here that Theorem 3.1 was stated in a more general case, for the three dimensional primitive equations, but in this article we need just the two dimensional case. By the uniqueness of the solution we conclude that $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$ on the interval $[\delta, T]$, so U_1, U_2 belong to $L^\infty(\delta, T, \mathcal{V}) \cap L^2(\delta, T, \dot{H}^2_{\text{per}}(\mathcal{M}))$, and $\partial U_1/\partial x_3$, $\partial U_2/\partial x_3$ belong to $L^2(\delta, T, \dot{H}^2_{\text{per}}(\mathcal{M}))$ for $\delta > 0$ arbitrarily small.

We write $U^\sharp = U_1 - U_2$ and $\tilde{U} = U_1 + U_2$. Combining the equations for U_1 and U_2, we find that $U^\sharp$ satisfies the following equation:

$$U^{\sharp\prime} + AU^\sharp + EU^\sharp + \frac{1}{2}B(\tilde{U}, U^\sharp) + \frac{1}{2}B(U^\sharp, \tilde{U}) = 0, \tag{4.10}$$

with $U^\sharp(t_\star) = 0$.

We define the following operator:

$$\begin{aligned} M(t)U^\sharp &= \frac{1}{2}B(\tilde{U}, U^\sharp) + \frac{1}{2}B(U^\sharp, \tilde{U}) \\ &= \frac{1}{2}\big(\tilde{u}\frac{\partial U^\sharp}{\partial x_1} + w(\tilde{U})\frac{\partial U^\sharp}{\partial x_3}\big) + \frac{1}{2}\big(u^\sharp\frac{\partial \tilde{U}}{\partial x_1} + w(U^\sharp)\frac{\partial \tilde{U}}{\partial x_3}\big). \end{aligned} \tag{4.11}$$

In what follows, the task is to prove that $\|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$ belongs to $L^2(\delta, T)$. We thus compute:

$$\begin{aligned} \Big|w(\tilde{U})\frac{\partial U^\sharp}{\partial x_3}\Big|_{L^2} &\leq |w(\tilde{U})|_{L^4}\Big|\frac{\partial U^\sharp}{\partial x_3}\Big|_{L^4} \leq c|w(\tilde{U})|_{L^2}^{1/2}\|w(\tilde{U})\|^{1/2}\Big|\frac{\partial U^\sharp}{\partial x_3}\Big|_{L^2}^{1/2}\Big\|\frac{\partial U^\sharp}{\partial x_3}\Big\|^{1/2} \\ &\leq c\|\tilde{U}\|^{1/2}|\tilde{U}|_{H^2}^{1/2}\|U^\sharp\|_{\mathcal{V}}, \\ |\tilde{u}\frac{\partial U^\sharp}{\partial x_1}|_{L^2} &\leq |\tilde{U}|_{L^\infty}\|U^\sharp\| \leq c|\tilde{U}|_{H^2}^{1/2}\|\tilde{U}\|^{1/2}\|U^\sharp\|_{\mathcal{V}}. \end{aligned} \tag{4.12}$$

We also find:

$$\begin{aligned} \Big|w(U^\sharp)\frac{\partial \tilde{U}}{\partial x_3}\Big|_{L^2} &\leq |w(U^\sharp)|_{L^2}\Big|\frac{\partial \tilde{U}}{\partial x_3}\Big|_{H^2} \leq c\|U^\sharp\|_{\mathcal{V}}\Big|\frac{\partial \tilde{U}}{\partial x_3}\Big|_{H^2}, \\ |u^\sharp\frac{\partial \tilde{U}}{\partial x_1}|_{L^2} &\leq \Big|U^\sharp\Big|_{L^4}\Big|\frac{\partial \tilde{U}}{\partial x_1}\Big|_{L^4} \leq c\|U^\sharp\|_{\mathcal{V}}\|\tilde{U}\|^{1/2}|\tilde{U}|_{H^2}^{1/2}. \end{aligned} \tag{4.13}$$

Gathering the above estimates, we find:

$$|M(t)U^\sharp|_{L^2} \leq c\|\tilde{U}\|^{1/2}|\tilde{U}|_{H^2}^{1/2}\|U^\sharp\|_{\mathcal{V}} + c\Big|\frac{\partial \tilde{U}}{\partial x_3}\Big|_{H^2}\|U^\sharp\|_{\mathcal{V}}. \tag{4.14}$$

We now need to estimate the L^2-norm of the x_3-derivative of $M(t)U^\sharp$, in fact we need to estimate the following expression:

$$\begin{aligned} 2(M(t)U^\sharp)_{x_3} =& \tilde{u}_{x_3}\frac{\partial U^\sharp}{\partial x_1} + w(\tilde{U})_{x_3}\frac{\partial U^\sharp}{\partial x_3} + \tilde{u}\frac{\partial^2 U^\sharp}{\partial x_1 \partial x_3} + w(\tilde{U})\frac{\partial^2 U^\sharp}{\partial x_3^2} \\ &+ u^\sharp_{x_3}\frac{\partial \tilde{U}}{\partial x_1} + w(U^\sharp)_{x_3}\frac{\partial \tilde{U}}{\partial x_3} + u^\sharp\frac{\partial^2 \tilde{U}}{\partial x_1 \partial x_3} + w(U^\sharp)\frac{\partial^2 \tilde{U}}{\partial x_3^2}, \end{aligned} \tag{4.15}$$

and we separately bound each of the terms.

We easily find:

$$
\begin{aligned}
&\big|\tilde{u}_{x_3}\frac{\partial U^\sharp}{\partial x_1}\big|_{L^2} \le |\tilde{u}_{x_3}|_{L^\infty}\big|\frac{\partial U^\sharp}{\partial x_1}\big|_{L^2} \le c|\tilde{u}_{x_3}|_{H^2}\|U^\sharp\|_{\mathcal{V}},\\
&\big|w(\tilde{U})_{x_3}\frac{\partial U^\sharp}{\partial x_3}\big|_{L^2} \le |\tilde{u}_{x_1}|_{L^4}\big|\frac{\partial U^\sharp}{\partial x_3}\big|_{L^4} \le c|\tilde{U}_{x_1}|_{L^2}^{1/2}\|\tilde{U}_{x_1}\|^{1/2}\|U^\sharp\|_{\mathcal{V}},\\
&\big|\tilde{u}\frac{\partial^2 U^\sharp}{\partial x_1\partial x_3}\big|_{L^2} \le |\tilde{u}|_{L^\infty}\big|\frac{\partial^2 U^\sharp}{\partial x_1\partial x_3}\big|_{L^2} \le c|\tilde{u}|_{H^2}\|U^\sharp\|_{\mathcal{V}},
\end{aligned}
\tag{4.16}
$$

as well as:

$$
\begin{aligned}
&\big|u^\sharp\frac{\partial^2 \tilde{U}}{\partial x_1\partial x_3}\big|_{L^2} \le c|U^\sharp|_{H^1}\big|\frac{\partial^2 \tilde{U}}{\partial x_1\partial x_3}\big|_{H^1} \le c\|U^\sharp\|_{\mathcal{V}}\big|\frac{\partial \tilde{U}}{\partial x_3}\big|_{H^2},\\
&\big|u^\sharp_{x_3}\frac{\partial \tilde{U}}{\partial x_1}\big|_{L^2} \le c|U^\sharp_{x_3}|_{H^1}|\tilde{U}|_{H^2} \le c\|U^\sharp\|_{\mathcal{V}}|\tilde{U}|_{H^2},\\
&\big|w(U^\sharp)_{x_3}\frac{\partial \tilde{U}}{\partial x_3}\big|_{L^2} \le |U^\sharp_{x_1}|_{L^2}\big|\frac{\partial \tilde{U}}{\partial x_3}\big|_{L^\infty} \le c\|U^\sharp\|_{\mathcal{V}}\big|\frac{\partial \tilde{U}}{\partial x_3}\big|_{H^2}.
\end{aligned}
\tag{4.17}
$$

We remain with some more delicate terms to estimate, which need anisotropic estimates:

$$
\begin{aligned}
\big|w(\tilde{U})\frac{\partial^2 U^\sharp}{\partial x_3^2}\big|_{L^2} &\le \big||w(\tilde{U})|_{L^\infty_{x_3}}\big|\frac{\partial^2 U^\sharp}{\partial x_3^2}\big|_{L^2_{x_3}}\big|_{L^2_{x_1}} \le c\big||\tilde{U}_{x_1}|_{L^2_{x_3}}\big|_{L^\infty_{x_1}}\big|\frac{\partial^2 U^\sharp}{\partial x_3^2}\big|_{L^2}\\
&\le c|\tilde{U}|_{H^2}\big|\frac{\partial^2 U^\sharp}{\partial x_3^2}\big|_{L^2} \le c|\tilde{U}|_{H^2}\|U^\sharp\|_{\mathcal{V}},\\
\big|w(U^\sharp)\frac{\partial^2 \tilde{U}}{\partial x_3^2}\big|_{L^2} &\le \big||w(U^\sharp)|_{L^\infty_{x_3}}\big|\frac{\partial^2 \tilde{U}}{\partial x_3^2}\big|_{L^2_{x_3}}\big|_{L^2_{x_1}} \le c\big||U^\sharp_{x_1}|_{L^2_{x_1}(L^2_{x_3})}\big|\big|\frac{\partial^2 \tilde{U}}{\partial x_3^2}\big|_{L^\infty_{x_1}(L^2_{x_3})}\big|_{L^2_{x_1}}\\
&\le c\|U^\sharp\|_{\mathcal{V}}\big\|\frac{\partial^2 \tilde{U}}{\partial x_3^2}\big\|_{H^2}.
\end{aligned}
\tag{4.18}
$$

From the computations above we can now conclude that:

$$
|M(t)U^\sharp|_{\mathcal{H}} \le c\{\|\tilde{U}\|^{1/2}|\tilde{U}|_{H^2}^{1/2} + \big|\frac{\partial \tilde{U}}{\partial x_3}\big|_{H^2} + |\tilde{U}|_{H^2}\}\|U^\sharp\|_{\mathcal{V}}. \tag{4.19}
$$

Thus $\|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$ is bounded by the expression between brackets in (4.19) and, we conclude, taking into account the properties of $\tilde{U}$, that $\|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$ belongs to $L^2(\delta, T)$ for $\delta > 0$ arbitrarily small.

We now need to prove that if $|U(t_\star)|_{\mathcal{H}} = 0$, then $|U(t)|_{\mathcal{H}} = 0$ for all $t \in [\delta, t_\star]$, $0 < \delta < t_\star$. The equivalent relation that we prove is that if there exists a time $t \in (\delta, t_\star)$ such that $|U^\sharp(t)|_{\mathcal{H}} > 0$, then $|U^\sharp(t_\star)|_{\mathcal{H}} > 0$. Since we proved that $U^\sharp \in \mathcal{C}([0, T], \mathcal{H})$, it is enough to show that $\log|U^\sharp(t)|_{\mathcal{H}}$ is bounded from below on $[\delta, t_\star]$.

Writing (4.10) as:

$$
U^{\sharp\prime} + AU^\sharp + EU^\sharp + M(t)U^\sharp = 0,
$$

we can use Proposition 4.1 where ϕ is defined as in (4.2) for $U^\sharp$. We find:

$$\phi'(t) \leq \frac{|M(t)U^\sharp(t)|^2_{\mathcal{H}}}{|U^\sharp(t)|^2_{\mathcal{H}}} \leq \|M(t)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{H})} \frac{|U^\sharp(t)|^2_{\mathcal{V}}}{|U^\sharp(t)|^2_{\mathcal{H}}} \leq \frac{1}{c_0} \|M(t)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{H})} \phi(t); \tag{4.20}$$

in (4.20) we used the fact that

$$((A+E)U^\sharp(t), U^\sharp(t))_{\mathcal{H}} = (AU^\sharp(t), U^\sharp(t))_{\mathcal{H}} \geq c_0 \|U^\sharp\|^2_{\mathcal{V}}.$$

Since $\|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$ belongs to $L^2(\delta, T)$, we can apply the Gronwall lemma to (4.20) and find:

$$\phi(t) \leq \phi(\delta) \exp(\int_\delta^t c_0^{-1} \|M(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{H})} \, \mathrm{d}s) \leq K, \tag{4.21}$$

with K a constant independent of t.

Considering the function $\log |U^\sharp(t)|^2_{\mathcal{H}}$, we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\log |U^\sharp(t)|^2_{\mathcal{H}}) &= 2\frac{(U^\sharp, U^{\sharp\prime})_{\mathcal{H}}}{|U^\sharp(t)|^2_{\mathcal{H}}} = -2\frac{(U^\sharp, (A+E)U^\sharp)_{\mathcal{H}}}{|U^\sharp(t)|^2_{\mathcal{H}}} - 2\frac{(U^\sharp, M(t)U^\sharp)_{\mathcal{H}}}{|U^\sharp(t)|^2_{\mathcal{H}}} \\ &\geq -2\phi(t) - 2c' \|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \phi(t), \end{aligned} \tag{4.22}$$

since we can estimate:

$$\begin{aligned} (U^\sharp, M(t)U^\sharp)_{\mathcal{H}} &\leq |U^\sharp|_{\mathcal{H}} |M(t)U^\sharp|_{\mathcal{H}} \\ &\leq |U^\sharp|_{\mathcal{H}} \|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \|U^\sharp\|_{\mathcal{V}} \leq c' |U^\sharp(t)|^2_{\mathcal{H}} \|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \phi(t). \end{aligned} \tag{4.23}$$

Using (4.21) into (4.22), we find that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log |U^\sharp(t)|^2_{\mathcal{H}}) \geq -2K(1 + c' \|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}), \tag{4.24}$$

and since $\|M(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$ is in $L^1(\delta, T)$, we find that

$$\log |U^\sharp(t)|^2_{\mathcal{H}} \geq -2K(t_\star - t) + \log |U^\sharp(\delta)|^2_{\mathcal{H}} \geq K_1, \quad \forall t \in [\delta, t_\star],$$

with K_1 a constant independent of t. This gives that $|U^\sharp(t_\star)|^2_{\mathcal{H}} \neq 0$, which implies that if $U^\sharp(t_\star)|^2_{\mathcal{H}} = 0$, then $|U^\sharp(t)|^2_{\mathcal{H}} = 0$ on the interval $[\delta, t_\star]$. But we know that this relation can be proved for almost all δ in $[0, t_\star]$ and from the fact that $U^\sharp \in \mathcal{C}([0,T], \mathcal{H})$, the desired result follows. □

5. Backward uniqueness for the strong solutions of the three dimensional primitive equations

The purpose of this section is to prove the backward uniqueness for the strong solutions of the three dimensional primitive equations (1.1)-(1.14). This model was considered in [10], where we have shown the existence and uniqueness of the strong solutions, as well as the existence and (forward) uniqueness of very strong solutions (solutions with values in H^m, $m \geq 2$). These results will be used in what follows and we refer the interested reader to [10] for more details.

The result we will prove here is the following one:

Theorem 5.1. *Let F be in $L^2(0,T,V)$ and let U_1, U_2 be two strong solutions for the primitive equations* (1.1)*, U_1, U_2 belonging to $\mathcal{C}([0,T];V)\cap L^2(0,T,\dot{H}^2_{\rm per}(\mathcal{M}))$, such that $U_1(t_\star)=U_2(t_\star)$. Then $U_1=U_2$ on the interval $[0,t_\star]$.*

The proof of the theorem follows the main steps as in Theorem 4.1 so we only emphasize the points which are different.

Proof of Theorem 5.1: Let U_1 and U_2 be two strong solutions. We can then find a δ arbitrarily small such that $U_1(\delta)$ and $U_2(\delta)$ belong to $\dot{H}^2_{\rm per}(\mathcal{M})$. This implies, with the results of [10], that:

$$U_1,U_2\in\mathcal{C}(\delta,T,\dot{H}^2_{\rm per}(\mathcal{M}))\cap L^2(\delta,T,\dot{H}^3_{\rm per}(\mathcal{M})).$$

As in the previous section, we write $U^\sharp=U_1-U_2$ and $\tilde{U}=U_1+U_2$. Combining the equations for U_1 and U_2, we find that $U^\sharp$ satisfies the same equation as (4.10) with $U^\sharp(t_\star)=0$.

We need again to prove that the operator $M(t)$ defined by:

$$\begin{aligned}M(t)U^\sharp&=\frac{1}{2}B(\tilde{U},U^\sharp)+\frac{1}{2}B(U^\sharp,\tilde{U})\\&=\frac{1}{2}\big(\tilde{u}\frac{\partial U^\sharp}{\partial x_1}+w(\tilde{U})\frac{\partial U^\sharp}{\partial x_3}\big)+\frac{1}{2}\big(u^\sharp\frac{\partial\tilde{U}}{\partial x_1}+w(U^\sharp)\frac{\partial\tilde{U}}{\partial x_3}\big),\end{aligned}\tag{5.1}$$

has the property that $|M(t)|_{\mathcal{L}(V,H)}$ belongs to $L^2(\delta,T)$.

Here we estimate each term of (5.1) as follows:

$$\begin{aligned}&\big|\tilde{u}\frac{\partial U^\sharp}{\partial x_1}\big|_H\le|\tilde{u}|_{L^\infty}\big|\frac{\partial U^\sharp}{\partial x_1}\big|_H\le c|\tilde{U}|_{H^2}|U^\sharp|_V,\\&\big|\tilde{v}\frac{\partial U^\sharp}{\partial x_2}\big|_H\le c|\tilde{U}|_{H^2}|U^\sharp|_V,\\&\big|w(\tilde{U})\frac{\partial U^\sharp}{\partial x_3}\big|_H\le|w(\tilde{U})|_{L^\infty}\big|\frac{\partial U^\sharp}{\partial x_3}\big|_H\le c|\tilde{U}|_{H^3}|U^\sharp|_V,\end{aligned}\tag{5.2}$$

and also:

$$\begin{aligned}&\big|u^\sharp\frac{\partial\tilde{U}}{\partial x_1}\big|_H\le|u^\sharp|_{L^4}\big|\frac{\partial\tilde{U}}{\partial x_1}\big|_{L^4}\le c|\tilde{U}|_{H^2}|U^\sharp|_V,\\&\big|v^\sharp\frac{\partial\tilde{U}}{\partial x_2}\big|_H\le c|\tilde{U}|_{H^2}|U^\sharp|_V,\\&\big|w(U^\sharp)\frac{\partial\tilde{U}}{\partial x_3}\big|_H\le|w(U^\sharp)|_{L^2}\big|\frac{\partial\tilde{U}}{\partial x_3}\big|_{L^\infty}\le c|\tilde{U}|_{H^3}|U^\sharp|_V.\end{aligned}\tag{5.3}$$

Gathering the estimates above, we find:

$$|M(t)U^\sharp|_H\le c|\tilde{U}|_{H^3}|U^\sharp|_V,\tag{5.4}$$

which implies that $|M(t)|_{\mathcal{L}(V,H)}$ belongs to $L^2(\delta, T)$. We can now perform the same kind of reasoning as in Theorem 4.1 in order to prove the desired result. $\square$

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