AN INTERFACE PROBLEM: THE TWO-LAYER SHALLOW WATER EQUATIONS

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Abstract. The aim of this article is to study a model of two superposed layers of fluid governed by the shallow water equations in space dimension one. Under some suitable hypotheses the governing equations are hyperbolic. We introduce suitable boundary conditions and establish a result of existence and uniqueness of smooth solutions for a limited time for this model.

1. Introduction

The determination of suitable boundary conditions has been recognized as an important problem in geophysical fluid dynamics, in particular for the problem of Limited Areas Models (LAMs); see e.g. [41]. The difficulty is that the domains considered in limited areas models do not have in general a physical significance, and therefore there are no physical laws which can provide the conditions at the boundary. Hence the choice of the boundary conditions for such models must be based on other considerations, physical intuition and computational relevance. For hyperbolic equations the direction of the characteristics at the boundary usually provides a valuable information, the idea being that we want the waves to enter and leave freely the computational domain without generating undesirable reflections at the boundary. See in [29] and the references therein a discussion about the boundary conditions for the equations of the geophysical fluid mechanics, Euler equations, primitive equations and shallow water (SW) equations.

In earlier work we have addressed the question of the boundary conditions for the primitive equations, rigorously for the linearized equations in [35, 36], and computationally for the full nonlinear equations in [6], [7]. We have also addressed the question of the boundary conditions for the one-dimensional shallow water equation for one layer, rigorously in [32] and computationally in [38]. In this article we want to address the question of the boundary conditions which are suitable for two superposed layers of fluid governed by the shallow water equations. The case of two layers of shallow water equations is substantially different from the case of one layer, because the corresponding system may not be hyperbolic; see [1], [27], [4] and below. See additional comments on the two layer shallow water equations in e.g. [8], [37], [40]. Also, beside the intrinsic interest of studying certain forms of the two-layer SW equations, another motivation of this work is that they can serve as a guide for the study of two coupled modes of the primitive equations [35, 36], a question that we intend to address in the future.

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Let us expand on the last point in order to better position this work. The two-layer system that we consider essentially reduces to a one-layer shallow water equation for the so-called barotropic mode, and another similar system for the so-called baroclinic mode. Both systems relate to isentropic gas dynamics which has been extensively studied following the program of Di Perna [9, 10]; see e.g. [21], [19], [20], [5] and the references therein. To the best of our knowledge most of these works do not address the issue of boundary conditions in a limited domain and their preoccupation do not relate to those of geophysical fluid dynamics as evoked above. The well-posedness of initial value problems for one dimensional hyperbolic systems is extensively addressed in [22]. However this work is based on a transformation of the initial problem which is remarkable on the theoretical side, but which makes it difficult to apply to concrete problems in the context of geophysical fluid dynamics, in particular in view of discretisation which raises many additional difficulties. We recall also that, in the long range, we aim to study initial boundary value problems for nonlinear primitive equations, following the linear case studied in [35, 36], and we believe that the present work is a useful step in that direction since two-layer shallow water equations essentially correspond to two modes of the vertical expansion of the primitive equations ([29], [39], [35, 36]).

Returning to the need, mentioned at the beginning of this Introduction of boundary conditions that let the waves move freely out of the domain, the boundary conditions that we study here have proven to be fully satisfactory with that respect; they appeared as fully transparent in the numerical simulations that we have performed in [38] and [2], although our primary preoccupation has been well-posedness of the initial and boundary problems rather than transparency. We might attribute this transparency property to the fact that the initial and boundary value problems are mildly dissipative which is needed in establishing a priori estimates and well-posedness. For a systematic (and totally different) study of transparent boundary conditions, we refer the reader to the well-known article [11]; see also [12] and further developements in [18], [15, 16], [13], [25], [28], [26]. For questions concerning initial and boundary value problems for hyperbolic equations, see in particular [24], [34], [33], [3], [14], [42] and the references therein.

This article is organized as follows. In Section 2 we introduce the two-layer model. In Section 3 we recall the main results applying to the barotropic mode. The main results of this article concern the baroclinic mode and appear in Section 4. Finally in Section 5 we briefly present the results of numerical simulations using the boundary condition that we described.

2. The two-layer model

In the present work we are interested in studying the well-posedness of the two-layers Shallow Water equations under appropriate open boundary conditions in space dimension one. We start by introducing the Shallow Water equations for a flow along a simple channel, see [1], [27], [4]. The channel is of length L, having a rectangular cross-section with variable width $b(x)$ and the elevation of the bottom is $h_0(x)$. We assume we have two distinct immiscible fluid layers, each layer being described by the velocity u_i that depends upon horizontal but not vertical position, the constant density ρ_i and the height h_i, with $i = 1, 2$.

The continuity equation for each layer is:

$$\frac{\partial h_i}{\partial t} = -\frac{1}{b}\frac{\partial}{\partial x}(bu_ih_i),\ i = 1, 2. \tag{2.1}$$

The momentum equations are:

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = -g\frac{\partial}{\partial x}(h_2 + h_1 + h_0) + g'\frac{\partial h_2}{\partial x}, \tag{2.2}$$

and

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} = -g\frac{\partial}{\partial x}(h_2 + h_1 + h_0), \tag{2.3}$$

where g' is the reduced gravity, defined as $g' = g(\rho_1 - \rho_2)/\rho_1$. We can write this system under the compact form:

$$\frac{\partial U}{\partial t} + \tilde{A}(U)\frac{\partial U}{\partial x} = F(U), \tag{2.4}$$

where $U = (h_1, h_2, u_1, u_2)$ and

$$\tilde{A}(U) = \begin{pmatrix} u_1 & 0 & h_1 & 0 \\ 0 & u_2 & 0 & h_2 \\ g & g - g' & u_1 & 0 \\ g & g & 0 & u_2 \end{pmatrix}$$

and

$$F(U) = (-u_1h_1/b\partial b/\partial x, -u_2h_2/b\partial b/\partial x, -g\partial h_0/\partial x, -g\partial h_0/\partial x).$$

The difficulty of the two-layer Shallow Water model written under this form is that the system is not hyperbolic; see [1]. In order to overcome this difficulty we consider as in [27] a suitable approximation of this model; the way the approximate problem is obtained is explained below. Assuming for simplicity that the channel has straight walls and a flat bottom, we place ourselves in the case where the width b is constant and the bottom elevation h_0 equals zero. Assuming that $g' \ll g$ (that is $\rho_1 \approx \rho_2$), the barotropic and baroclinic modes couple very weakly. Thus, by writing $h = h_1 + h_2$ and $u = \dfrac{h_1u_1 + h_2u_2}{h}$, we obtain a system in (h, u) describing the barotropic mode by neglecting the term in which g' appears and assuming $u_1 \approx u_2$. After some simple computations, we obtain:

$$\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0. \end{aligned} \tag{2.5}$$

We notice that the equations for the barotropic mode are exactly the equations for the single-layer Shallow Water model and the well-posedness of this problem under open boundary conditions has been considered in [32].

The baroclinic mode is then described by $v = u_1 - u_2$ and h_1 and the equations read:

$$\begin{aligned} &\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(uh_1 + v\frac{h_1h_2}{h}) = 0, \\ &\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv + \frac{v^2(h_2 - h_1)}{2h} + g'h_1) = 0, \end{aligned} \tag{2.6}$$

where $h_2 = h - h_1$ but we use h_2 in order to simplify the notations.

We can thus consider these models separately; we first solve the barotropic mode and once (h, u) are known, we can solve the baroclinic mode. Since (h, u) are independently determined, once inserted into the baroclinic mode we consider them as given functions. In fact as explained below, for the sake of simplicity we will assume that h and u are constants. System (2.6) thus becomes equivalent to:

$$\begin{aligned}&\frac{\partial h_1}{\partial t} + (u + v - 2v\frac{h_1}{h})\frac{\partial h_1}{\partial x} + (h_1 - \frac{h_1^2}{h})\frac{\partial v}{\partial x} = 0,\\ &\frac{\partial v}{\partial t} + (u + v - 2\frac{h_1 v}{h})\frac{\partial v}{\partial x} + (g' - \frac{v^2}{h})\frac{\partial h_1}{\partial x} = 0,\end{aligned} \tag{2.7}$$

which can be also written in the compact form:

$$\frac{\partial V}{\partial t} + A(V)\frac{\partial V}{\partial x} = 0, \tag{2.8}$$

where here $V = (h_1, v)$ and

$$A(V) = \begin{pmatrix} u + v - 2v\dfrac{h_1}{h} & h_1 - \dfrac{h_1^2}{h} \\ g' - \dfrac{v^2}{h} & u + v - 2v\dfrac{h_1}{h_2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix}. \tag{2.9}$$

The eigenvalues associated with $A(V)$ are solutions of the equation:

$$(a - \lambda)^2 - bc = 0; \tag{2.10}$$

so the eigenvalues are:

$$\lambda_{1,2} = a \pm \sqrt{bc} = u + v - 2v\frac{h_1}{h} \pm \frac{1}{h}\sqrt{h_1 h_2 (g'h - v^2)}, \tag{2.11}$$

and the condition we will impose in what follows in order to ensure the hyperbolicity of the baroclinic mode is:

$$g'h - v^2 > 0. \tag{2.12}$$

Performing the combinations $(1/\sqrt{b}, \pm 1/\sqrt{c})$ of the equations (2.7), we arrive at the system

$$\left(\frac{h_{1,t}}{\sqrt{h_1 h - h_1^2}} \mp \frac{v_{1,t}}{\sqrt{g'h - v^2}}\right) + \lambda_{1,2}\left(\frac{h_{1,x}}{\sqrt{h_1 h - h_1^2}} \mp \frac{v_{1,x}}{\sqrt{g'h - v^2}}\right) = 0, \tag{2.13}$$

which we can rewrite as

$$\begin{aligned}&\frac{\partial \xi_1}{\partial t} + \lambda_1 \frac{\partial \xi_1}{\partial x} = 0,\\ &\frac{\partial \xi_2}{\partial t} + \lambda_2 \frac{\partial \xi_2}{\partial x} = 0,\end{aligned} \tag{2.14}$$

with

$$\xi_1 = \arcsin\frac{h-2h_1}{h} - \arcsin\frac{v}{\sqrt{g'h}},$$
$$\xi_2 = \arcsin\frac{h-2h_1}{h} + \arcsin\frac{v}{\sqrt{g'h}}. \tag{2.15}$$

In relation with the criterion (4.1), (4.2) below which determines the sign of λ_1 and λ_2, the system (3.3) leads to the boundary conditions (4.7) below.

3. The barotropic mode

As we mentioned before, we first start by studying the barotropic mode that corresponds to the single layer Shallow Water model. We give here a summary of the results available on the well-posedness of the model; a detailed proof of these results can be found in [32]. System (2.5) can be written as:

$$\frac{\partial U}{\partial t} + B(U)\frac{\partial U}{\partial x} = 0, \tag{3.1}$$

with $U = (h, u)$ and $B(U) = \begin{pmatrix} u & h \\ g & u \end{pmatrix}$.

The eigenvalues for the matrix $B(U)$ are $\mu_{1,2} = u \pm \sqrt{gh}$; the eigenvalues are also called the characteristic velocities. If $u^2 - gh < 0$ then we are in the barotropically subcritical case, while the case $u^2 - gh > 0$ is called barotropically supersonic. Let us consider a constant stationary solution for (3.1), $U_s = (u_s, h_s)$ such that $u_s^2 - gh_s < 0$, that is this stationary solution is subcritical (subsonic). More exactly we suppose that u_s and h_s satisfy $u_s^2 - gh_s \leq -c_0^2$ and $2h_0 \leq h_s \leq 2\bar{h}_0$, with $c_0, \underline{h}_0$ and $\bar{h}_0$ given, positive constants.

We endow equation (3.1) with the initial conditions:

$$U(0, x) = (u_1(x), h_0(x)), \tag{3.2}$$

where u_0 and h_0 satisfy the following conditions:

$$\begin{cases} u_0^2(x) - gh_0(x) \leq -c_0^2, & \forall\, x \in (0, L), \\ 2\underline{h}_0 \leq h_0(x) \leq 2\bar{h}_0, & \forall\, x \in (0, L). \end{cases} \tag{3.3}$$

The natural open boundary conditions for this problem are:

$$\begin{cases} u + 2\sqrt{gh} = u_s + 2\sqrt{gh_s} + g_0(t) \text{ at } x = 0, \\ u - 2\sqrt{gh} = u_s - 2\sqrt{gh_s} + g_L(t) \text{ at } x = L. \end{cases} \tag{3.4}$$

The boundary and the initial data satisfy certain compatibility conditions; for more details on the natural compatibility conditions for this problem see, e.g. [32].

We can thus prove the following result (see [32] for a proof):

Theorem 3.1. *Let there be given u_0 and h_0 in $H^{7/2}(0, L)$ satisfying* (3.3) *and g_0, g_L in $H^{7/2}(0, L)$ satisfying certain compatibility conditions. Then there exists $T_* > 0$ depending on the initial and boundary data and a unique solution U of problem* (3.1), (3.2), (3.4) *on $(0, T_*)$ such that:*

$$U \in (H^3((0, L) \times (0, T_*)))^2,$$

and

$$U|_{x=0} \in (H^3(0,T_*))^2, U|_{x=L} \in (H^3(0,T_*))^2.$$

For the barotropically supercritical (supersonic) case the analysis is similar, (see [17]). Let us consider a constant stationary solution $U_s = (u_s, h_s)$ such that $u_s^2 - 2gh_s > 0$. As before, we suppose in fact that

$$u_s^2 - gh_s \geq c_0^2 \text{ and } 2\underline{h}_0 \leq h_s \leq 2\bar{h}_0,$$

with $c_0, \underline{h}_0$ and $\bar{h}_0$ given, positive constants.

Equation (3.1) is endowed with the initial condition:

$$U(0,x) = (u_0(x), h_0(x)), \tag{3.5}$$

where u_0 and h_0 satisfy the following conditions:

$$\begin{cases} u_0^2(x) - gh_0(x) \geq c_0^2, \ \ \forall x \in (0,L), \\ 2\underline{h}_0 \leq h_0(x) \leq 2\bar{h}_0 \ \ \forall x \in (0,L). \end{cases} \tag{3.6}$$

Since in the supercritical case both eigenvalues for the matrix $B(U)$ are positive, we need to prescribe two boundary conditions at $x = 0$. Thus, the natural open boundary conditions for this problem are:

$$u = u_s + g_u(t), h = h_s + g_h(t), \text{ at } x = 0. \tag{3.7}$$

As before, the boundary data and the initial data have to satisfy some natural compatibility conditions. We can prove that problem (3.1), (3.5), (3.7) is locally well-posed; more exactly an analogous of Theorem 3.1 holds for this problem, [17].

4. The baroclinic mode

Once the barotropic mode is solved, we can start studying the baroclinic mode. The barotropic variables (u, h) are known at this stage and they are considered from now on as constants for the sake of simplicity; the non constant case would only induce additional technical difficulties.

Now whether the (constant) barotropic flow is sub or supercritical, the baroclinic flow can be itself a sub- or supercritical flow, and the four combinations are possible. The criterion for baroclinically subcritical flow is:

$$(u + v - 2v\frac{h_1}{h})^2 < \frac{h_1 h_2}{h^2}(g'h - v^2), \tag{4.1}$$

while a flow is called baroclinically supercritical when:

$$(u + v - 2v\frac{h_1}{h})^2 > \frac{h_1 h_2}{h^2}(g'h - v^2). \tag{4.2}$$

Note that these criterions relate to the sign of λ_1, λ_2 in (2.11); see also [27].

We start by considering a constant stationary solution of the baroclinic mode (2.7) that is $V_s = (h_{1,s}, v_s)$ such that $g'h_{1,s} - v_s^2 > 0$. Since the baroclinic velocity v is in fact the difference between u_1 and u_2 and we supposed that this difference is small, it is natural to actually consider $v_s = 0$. So, the stationary constant solution is $V_s = (h_{1,s}, 0)$.

We first consider that the stationary solution is baroclinically subsonic, meaning with (4.1) and $v_s = 0$:

$$u^2 < g'\frac{h_{1,s}(h - h_{1,s})}{h}; \tag{4.3}$$

the baroclinically supersonic flow is considered at the end of this section.

For the baroclinically subsonic flow we enforce (4.2) by imposing:

$$u^2 - g'\frac{h_{1,s}(h - h_{1,s})}{h} \leq -c_0^2 \text{ and } 2\underline{h}_0 \leq h_{1,s} \leq 2\bar{h}_0, \tag{4.4}$$

with $c_0, \underline{h}_0, \bar{h}_0$ given positive constants.

We consider the initial conditions:

$$V_0 = (h_{1,0}, v_0), \tag{4.5}$$

with $h_{1,0}$ and v_0 satisfying the following conditions (compare to (4.1) and (4.3)):

$$\begin{cases} \left(u + v_0(x) - 2v_0(x)\dfrac{h_{1,0}(x)}{h}\right)^2 - \dfrac{h_{1,0}(x)(h - h_{1,0}(x))}{h^2}\big(g'h - v_0^2(x)\big) \leq -c_0^2, \\ g'h - v_0^2(x) \geq \dfrac{3}{4}g'h, \forall x \in (0, L), \\ 2\underline{h}_0 \leq h_{1,0}(x) \leq 2\bar{h}_0, \forall x \in (0, L), \end{cases} \tag{4.6}$$

with $\bar{h}_0$ chosen such that $h - 3\bar{h}_0 > 0$.

Since the characteristic velocity λ_1 is positive, the characteristic variable ξ_1 is entering the domain and we only prescribe a boundary condition for ξ_1 at $x = 0$. The second characteristic velocity λ_2 is negative so we prescribe a boundary condition for ξ_2 at $x = L$. Thus equations (2.7) are supplemented with the following boundary conditions (see also [27]):

(4.7)
$$\begin{cases} \arcsin\left(\dfrac{v}{\sqrt{g'h}}\right) + \arcsin\left(\dfrac{h - 2h_1}{h}\right) = \arcsin\left(\dfrac{h - 2h_{1,s}}{h}\right) + g_L(t), \text{ at } x = L \\ \arcsin\left(\dfrac{v}{\sqrt{g'h}}\right) - \arcsin\left(\dfrac{h - 2h_1}{h}\right) = -\arcsin\left(\dfrac{h - 2h_{1,s}}{h}\right) + g_0(t), \text{ at } x = 0. \end{cases}$$

Intuitively, the meaning of these boundary conditions is that at the boundary the flow is a perturbation by (g_0, g_L) from the stationary state.

In order to be able to solve the initial boundary value problem given by (4.7), we need to impose some technical conditions. The first one is that $V_s = (h_{1,s}, 0)$ is a stationary solution of equations (2.7) supplemented with the boundary conditions (4.7) written at $t = 0$. This implies with $h_1 = h_{1,s}$ and $v = 0$:

$$g_0(0) = 0, g_L(0) = 0. \tag{4.8}$$

The second condition requires that the boundary data and the initial data satisfy some suitable compatibility conditions, consistent with the fact that the solution we construct is $\mathcal{C}^p$ up to the boundary, with $p = 2$. Thus, we write:

$$b_0(V) = \arcsin(\frac{v}{\sqrt{g'h}}) - \arcsin(\frac{h-2h_1}{h}),$$
$$b_L(V) = \arcsin(\frac{v}{\sqrt{g'h}}) + \arcsin(\frac{h-2h_1}{h}). \tag{4.9}$$

Then, for $p = 0$ the compatibility conditions are:

$$b_0(V_0) = b_0(V_s) \text{ at } x = 0,\ b_L(V_0) = b_L(V_s) \text{ at } x = L; \tag{4.10}$$

for $p = 1$ the compatibility conditions read:

$$\begin{aligned} g_0'(0) &= db_0(V_0) \cdot \partial_t V(x,0) \\ &= db_0(V_0) \cdot (-A(V_0)V_{0,x}) \text{ at } x = 0, \\ g_L'(0) &= db_L(V_0) \cdot \partial_t V(x,0) = db_L(V_0) \cdot (-A(V_0)V_{0,x}) \text{ at } x = L, \end{aligned} \tag{4.11}$$

and similarly for $p = 2$. Here db_0, db_L denote the Frechet differentials of b_0, b_L.

The main result of this work is the following:

Theorem 4.1. *Given $V_0 = (h_{1,0}, v_0)$ belonging to $\left(H^{7/2}(0,L)\right)^2$ and satisfying (4.6) and g_0, g_L in $H^{7/2}(0,T)$ satisfying the compatibility conditions (4.10), (4.11) and the analog for $p = 2$. Then there exists $T_* > 0$ depending on the initial and boundary data and a unique solution $V = (h_1, v)$ of problem (2.7), (4.5), (4.7) on $(0, T_*)$ such that:*

$$V \in \left(H^3((0,L) \times (0,T_*))\right)^2,$$

and

$$V\,|_{x=0}, V\,|_{x=L} \in (H^3(0,T_*))^2.$$

4.1. **Equivalent problems.** Our problem (2.7), (4.3), (4.7) is equivalent to solving the following system:

$$\begin{cases} \dfrac{\partial \tilde{h}_1}{\partial t} + (u + \tilde{v} - \dfrac{2\tilde{v}}{h}(\tilde{h}_1 + h_{1,s}))\dfrac{\partial \tilde{h}_1}{\partial x} + (h_{1,s} + \tilde{h}_1 - (\dfrac{h_{1,s} + \tilde{h}_1}{h})^2)\dfrac{\partial \tilde{v}}{\partial x} = 0, \\ \dfrac{\partial \tilde{v}}{\partial t} + (g' - \dfrac{\tilde{v}^2}{h})\dfrac{\partial \tilde{h}_1}{\partial x} + (u + \tilde{v} - \dfrac{2\tilde{v}}{h}(h_{1,s} + \tilde{h}_1))\dfrac{\partial \tilde{v}}{\partial x} = 0, \end{cases} \tag{4.12}$$

where $\tilde{V} = (\tilde{h}_1, \tilde{v})$ is a perturbation from the constant profile V_s, meaning that $v = \tilde{v}, h_1 = \tilde{h}_1 + h_{1,s}$.

System (4.12) can be written in a compact form as:

$$\frac{\partial \tilde{V}}{\partial t} + A(\tilde{V} + V_s)\frac{\partial \tilde{V}}{\partial x} = 0. \tag{4.13}$$

The initial conditions for (4.13) are:

$$\tilde{V}(0,x) = \tilde{V}_0(x) = (\tilde{h}_{1,0}(x), \tilde{v}_0(x)), \tag{4.14}$$

and the boundary condition become:

$$\begin{cases} b_0(\tilde{V} + V_s) = b_0(V_s) + g_0(t) \text{ at } x = 0 \\ b_L(\tilde{V} + V_s) = b_L(V_s) + g_L(t), \text{ at } x = L. \end{cases} \tag{4.15}$$

We thus want to solve the equivalent problem (4.13)-(4.15) but the inconvenience with this problem is that we do not have zero initial value data. This difficulty is overcome by considering an appropriate lifting function $V_a = (h_{1,a}, v_a)$ of the initial data $\tilde{V}_0 = (\tilde{h}_{1,0}, \tilde{v}_0)$. The solution of problem (4.13) will thus be constructed as $\tilde{V} = V_a + \bar{V}$, where $\bar{V}$ is the new unknown.

We construct V_a such that $V_a(x,0) = \tilde{V}_0$ and such that:

$$\begin{aligned} &| h_{1,a}(x,t) - h_{1,0}(x) | \leq \delta \quad \forall (x,t) \in (0,L) \times (0,T), \\ &| v_a(x,t) - v_0(x) | \leq \delta, \quad \forall (x,t) \in (0,L) \times (0,T), \end{aligned}$$

with $\delta > 0$ chosen small enough. More exactly, δ is chosen such that if

$$| \bar{h}_1(x,t) | \leq \delta, \quad | \bar{v}(x,t) | \leq \delta \quad \forall (x,t) \in (0,L) \times (0,T), \tag{4.16a}$$

then $V_s + V_a + \bar{V}$ satisfies a suitable form of (4.6), that is:

$$\begin{aligned} &\left\{ u + v_a + \bar{v} - 2(v_a + \bar{v}) \frac{(h_{1,a} + \bar{h}_a + h_{1,s})}{h} \right\}^2 - \\ &- \frac{(h_{1,a} + \bar{h}_1 + h_{1,s})(h - h_{1,s} - h_{1,a} - \bar{h}_1)}{h^2} (g'h - (v_a + \bar{v})^2) \leq -\frac{c_0{}^2}{2}, \\ &\forall (x,t) \in (0,L) \times (0,T), \end{aligned} \tag{4.16b}$$

$$g'h - (v_a(x,t) + \bar{v}(x,t)) \geq \frac{g'h}{2}, \forall (x,t) \in (0,L) \times (0,T). \tag{4.17}$$

$$\underline{h}_0 \leq h_{1,a}(x,t) + \bar{h}_1(x,t) + h_{1,s} \leq 3\bar{h}_0, \forall (x,t) \in (0,L) \times (0,T), \tag{4.18}$$

Using this lifting function V_a and the stationary solution V_s, the Shallow Water equations are now equivalent to the problem:

$$\frac{\partial(\bar{V} + V_a)}{\partial t} + A(\bar{V} + V_a + V_s) \frac{\partial(\bar{V} + V_a)}{\partial x} = 0, \tag{4.19}$$

with initial conditions:

$$\bar{V}(x,0) = 0, \tag{4.20}$$

and the boundary conditions:

$$\begin{cases} b_0(\bar{V} + V_a + V_s) = b_0(V_s) + g_0(t) \text{ at } x = 0, \\ b_L(\bar{V} + V_a + V_s) = b_L(V_s) + g_L(t) \text{ at } x = L. \end{cases} \tag{4.21}$$

In order to simplify the notations, in all that follows we write $V = (h_1, v)$ instead of $\bar{V} = (\bar{h}_1, \bar{v})$. Thus (4.19) becomes:

$$\frac{\partial V}{\partial t} + A(V + V_a + V_s) \frac{\partial V}{\partial x} = -\frac{\partial V_a}{\partial t} - A(V + V_a + V_s) \frac{\partial V_a}{\partial x}. \tag{4.22}$$

The existence of the appropriate lifting V_a is given by the following lemma (for details, see [3] Lemma 11.2).

Lemma 4.1. *Given $\tilde{V}_0$ in $(H^{7/2}(0,L))^2$ there exist $T_0 > 0$ and $V_a \in (H^4((0,L) \times \mathbb{R}))^2$ vanishing for $| t | \geq 2T_0$, such that:*

$$\begin{aligned} &V_a(x,0) = \tilde{V}_0(x) \quad \forall x \in (0,L), \\ &| V_a(x,t) - \tilde{V}_0(x) | \leq \delta, \quad \forall (x,t) \in [0,L] \times [-T_0, T_0], \end{aligned}$$

and such that if

$$\mathcal{F} = -\frac{\partial V_a}{\partial t} - A(V_a + V_s)\frac{\partial V_a}{\partial x}, \tag{4.23}$$

and

$$\begin{aligned} \mathcal{G}_1 &= g_0(t) - b_0(V_a + V_s) + b_0(V_s), \text{ at } x = 0, \\ \mathcal{G}_2 &= g_L(t) - b_L(V_a + V_s) + b_L(V_s) \text{ at } x = L, \end{aligned} \tag{4.24}$$

then $\partial_t^p \mathcal{F} = 0$ *and* $\partial_t^p \mathcal{G}_i = 0$ *at* $t = 0$*, for all* $p = 0, 1, 2$ *and* $i = 1, 2$*.*

Moreover, $\mathcal{F}$ *belongs to* $(H^3((0, L) \times \mathbb{R}))^2, \mathcal{G}_1, \mathcal{G}_2$ *belong to* $H^3(\mathbb{R})$ *and all the functions vanish for* $|\, t \,| \geq 2T_0$*.*

4.2. **Iterative Scheme.** The solution of problem (4.19)-(4.21) is constructed as the limit of a sequence of solutions $V^0 = 0, \ldots, V^{k+1}$ recursively defined by solving the linear problems:

$$\frac{\partial V^{k+1}}{\partial t} + A(V^k + V_a + V_s)\frac{\partial V^{k+1}}{\partial x} = \mathcal{F}(V), \tag{4.25}$$

with $\mathcal{F}(V) = -\frac{\partial V_a}{\partial t} - A(V + V_a + V_s)\frac{\partial V_a}{\partial x}$. The initial condition for (4.25) is:

$$V^{k+1}|_{t=0} = 0, \tag{4.26}$$

and the boundary conditions are:

$$\begin{cases} db_L(V^{k+1} + V_a + V_s) \cdot V^{k+1} & = db_L(V^k + V_a + V_s) \cdot V^k - b_L(V^k + V_a + V_s) \\ & \quad + b_L(V_s) + g_L(t), \text{ at } x = L, \\ db_0(V^k + V_a + V_s) \cdot V^{k+1} & = db_0(V^k + V_a + V_s) \cdot V^k - b_0(V^k + V_a + V_s) \\ & \quad + b_0(V_s) + g_0(t), \text{ at } x = 0. \end{cases} \tag{4.27}$$

Thus, the boundary conditions for this linear problem are:

$$\begin{aligned} &\frac{1}{\sqrt{g'h - (v_a + v^k)^2}} v^{k+1} - \frac{1}{\sqrt{(h_1^k + h_{1,a} + h_{1,s})(h - h_1^k - h_{1,a} - h_{1,s})}} h^{k+1} = \mathcal{G}_L(V^k), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{at } x = L, \\ &\frac{1}{\sqrt{g'h - (v_a + v^k)^2}} v^{k+1} - \frac{1}{\sqrt{(h_1^k + h_{1,a} + h_{1,s})(h - h_1^k - h_{1,a} - h_{1,s})}} h^{k+1} = \mathcal{G}_0(V^k), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{at } x = 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{G}_L(V) &= db_L(V + V_a + V_s) \cdot V - b_L(V + V_a + V_s) + b_L(V_s) + g_L, \\ \mathcal{G}_0(V) &= db_0(V + V_a + V_s) \cdot V - b_0(V + V_a + V_s) + b_0(V_s) + g_0. \end{aligned}$$

The properties of the functions $\mathcal{F}, \mathcal{G}_L, \mathcal{G}_0$ are given by the following technical result (see [3] for more details):

Lemma 4.2. *For all* $T \in (0, T_0]$ *and all* $V = (h_1, v) \in H^3(\Omega_T)$ *of norm less than* δ/ν_3 *such that* $V_{|x=0}$ *and* $V_{|x=L}$ *belong to* $H^3(-\infty, T)$ *and* $V\,|_{t\leq 0}= 0$*, we have:*

$$\partial_t^p \mathcal{F}(V)_{|t=0} = 0, \partial_t^p \mathcal{G}_0(V)_{|t=0} = 0, \partial_t^p \mathcal{G}_L(V)_{|t=0} = 0, \tag{4.28}$$

for $p = 0, 1, 2$. Here δ is as in(4.16)-(4.18) and ν_3 in the norm of the Sobolev imbedding $H^3((0,L)\times\mathbb{R}) \subset L^\infty((0,L)\times\mathbb{R})$ and $\Omega_T = (0,L)\times(-\infty,T)$.

Furthermore, for all $M \in (0, \delta/\nu_3)$ there exist two constants $C_1 = C_1(M)$ and $C_2 = C_2(M)$ such that for all $T \in (0, T_0]$, the relation $\|V\|_{H^3(\Omega_T)} \le M$ implies:

$$\|\mathcal{F}(V)\|_{H^3(\Omega_T)} \le C_1(M), \tag{4.29}$$

$$\|\mathcal{G}(V)\|_{H^3(-\infty,T)} \le TC_2(M) + \epsilon(M), \|\mathcal{G}_L(V)\|_{H^3(-\infty,T)} \le TC_2(M) + \epsilon(T), \tag{4.30}$$

where $\epsilon(T)$ is independent of M and goes to zero as T goes to zero.

4.3. **Study of the linear problem.** In this subsection we fix $M \in (0, \delta/\nu_3)$ and considering that we know $V^0, \dots, V^k$, we want to prove the existence and uniqueness of V^{k+1} in $H^3(\Omega_T)$. We also deduce a priori estimates on the $H^3(\Omega_T)-$ norm of V^{k+1} as well as on the $H^3(-\infty, T)-$ norm of $V^{k+1}|_{x=0}, V^{k+1}|_{x=L}$, assuming that V^k is such that for a certain $T \in (0, T_0]$ we have:

Hypothesis 1.

$$V^k \in H^3(\Omega_T), V^k|_{x=0} \in H^3(-\infty,T), V^k|_{x=L} \in H^3(-\infty,T),$$
$$\|V^k\|_{H^3(\Omega_T)} \le M, \|V^k|_{x=0}\|_{H^3(-\infty,T)} \le M, \|V^k|_{x=L}\|_{H^3(-\infty,T)} \le M \text{ and } V^k|_{t<0} \equiv 0.$$

Before obtaining the a priori estimates for V^{k+1}, we start with the following remark:

Remark 4.1. The operator $A(V^k + V_a + V_s)$ defined on Ω_T:

$$A(V^k+V_a+V_s) = \begin{pmatrix} u + v_a + v^k - \frac{2(v_a+v_k)(h_{1,a}+h_1^k+h_{1,s})}{h} & h_{1,s} + h_{1,a} + h_1^k - \frac{(h_{1,s}+h_{1,a}+h_1^k)^2}{h} \\ g' - \frac{(v_a+v^k)^2}{h} & u + v_a + v^k - \frac{2(v_a+v^k)(h_{1,a}+h_1^k+h_{1,s})}{h} \end{pmatrix} \tag{4.31}$$

is Friedrichs symmetrizable with:

$$S_0^k = \begin{pmatrix} g' - \dfrac{(v_a+v^k)^2}{h} & 0 \\ 0 & h_{1,s} + h_{1,a} + h_1^k - \dfrac{(h_{1,s}+h_{1,a}+h_1^k)}{h} \end{pmatrix} \tag{4.32}$$

as the Friedrichs symmetrizer.

We also need to study the properties of the boundary conditions:

Remark 4.2. The boundary conditions (4.27) associated with the linear problem (4.25) are strictly dissipative, meaning that there exists $\alpha_0 > 0$ and $\beta_0 > 0$ such that:

$$\begin{aligned}(S_0^k A(V^k+V_a+V_s)W, W) \ge &\alpha_0||W||^2 - \beta_0||db_L(V^k+V_a+V_s)\cdot W||^2, \\ &\forall W \in \mathbb{R}^2 \text{ at } x = L,\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}(S_0^k A(V^k+V_a+V_s)W, W) \le &-\alpha_0||W||^2 + \beta_0||db_0(V^k+V_a+V_s)\cdot W||^2, \\ &\forall W \in \mathbb{R}^2 \text{ at } x = 0;\end{aligned} \tag{4.34}$$

here $||\cdot||$ is the Euclidian norm in $\mathbb{R}^2$.

Proof. The strict dissipativity of the boundary conditions is proved by direct computation. In order to simplify the notations, we write:

$$\begin{aligned} m_1 &= \frac{g'h-(v_a+v^k)^2}{h}, \\ m_2 &= u+v_a+v^k-2\frac{(v_a+v^k)(h_{1,a}+h_1^k+h_{1,s})}{h}, \\ m_3 &= \frac{(h_1^k+h_{1,a}+h_{1,s})(h-h_1^k-h_{1,a}-h_{1,s})}{h}, \end{aligned}$$

and thus:

$$S_0^k A(V^k+V_a+V_s) = \begin{pmatrix} m_1m_2 & m_1m_3 \\ m_1m_3 & m_2m_3 \end{pmatrix}. \tag{4.35}$$

The boundary condition at $x=L$ becomes:

$$db_L(V^k+V_a+V_s)\cdot W = -\frac{1}{\sqrt{m_3h}}w_1+\frac{1}{\sqrt{m_1h}}w_2, \text{ with } W=(w_1,w_2). \tag{4.36}$$

Thus, we obtain:

$$\begin{aligned} (SAW,W) =&\alpha||W||^2+\beta||db_L(V^k+V_a+V_s)\cdot W||^2 = \\ =&(m_1m_2-\alpha+\frac{\beta}{m_3h})w_1^2+2w_1w_2(m_1m_3-\sqrt{\frac{m_3}{m_1}}\frac{\beta}{m_3h}) \\ &-w_2^2(m_2m_3-\alpha+\frac{\beta}{m_1h}). \end{aligned} \tag{4.37}$$

Taking $\beta=m_1m_3h\sqrt{m_1m_3}$, relation (4.37) becomes:

$$(SAW,W) = (m_1m_2+m_1\sqrt{m_1m_3}-\alpha)w_1^2+(m_2m_3+m_3\sqrt{m_1m_3}-\alpha)w_2^2, \tag{4.38}$$

and this is positive for all $W\in\mathbb{R}^2$ if:

$$m_1m_2+m_1\sqrt{m_1m_3}-\alpha>0 \text{ and } m_2m_3+m_3\sqrt{m_1m_3}-\alpha>0,$$

which implies that:

$$\alpha<m_1(m_2+\sqrt{m_1m_3}) \text{ and } \alpha<m_3(m_2+\sqrt{m_1m_3}).$$

The fact that $m_2+\sqrt{m_1m_3}$ is positive is insured by the baroclinic subsonic mode condition and by the fact that we assumed that for all (x,t) in Ω_T (4.16) holds (thanks to hypothesis 1 on V^k), so we find:

$$m_2^2-m_1m_3\le-\frac{c_0^2}{2},$$

which also implies:

$$\min_{x,t}(m_2+\sqrt{m_1m_3})>0.$$

Thanks to hypothesis 1 on V^k, (4.17) and (4.18) also hold and this implies that $\min_{x,t}m_1>0$ and $\min_{x,t}m_2>0$. Thus, there exists a constant $\alpha_0>0$ depending only on g,g',$\underline{h}_0$,$\overline{h}_0$,h,c_0 but not on $(x,t)\in\Omega_T$ for which relation (4.38) is true, for all $(x,t)\in\Omega_T$.

Instead of $\beta=m_1m_3h\sqrt{m_1m_3}$ which has the disadvantage of not being an absolute constant, we take $\beta_0=\sup_{x,t}m_1m_3h\sqrt{m_1m_3}$.

The same reasoning works for the boundary condition at $x=0$.

Adapting Theorem 9.21 from [3] to our case (Ω a bounded domain), we find that the linear problem (4.25), (4.26), (4.27) has a unique solution V^{k+1} in $H^3(\Omega_T)$, having a trace at $x = 0$ and at $x = L$ which belong to $H^3(-\infty, T)$. The solution satisfies the following energy estimate:

$$
\begin{aligned}
&\frac{1}{T}||V^{k+1}||^2_{H^3(\Omega_T)} + ||V^{k+1}|_{x=0}||^2_{H^3(-\infty,T)} + ||V^{k+1}|_{x=L}||^2_{H^3(-\infty,T)} \leq \\
&c(T||\mathcal{F}(V^k)||^2_{H^3(\Omega_T)} + ||\mathcal{G}_0(V^k)||^2_{H^3(-\infty,T)} + ||\mathcal{G}_L(V^k)||^2_{H^3(-\infty,T)}),
\end{aligned} \tag{4.39}
$$

where C is a constant depending on $V_a, V_s, c_0, \underline{h}_0, \overline{h}_0$ and M but independent of k. Thus, using Lemma 4.2, we obtain:

$$
\begin{aligned}
\frac{1}{T}||V^{k+1}||^2_{H^3(\Omega_T)} + &||V^{k+1}|_{x=0}||^2_{H^3(-\infty,T)} + ||V^{k+1}|_{x=L}||^2_{H^3(-\infty,T)} \leq \\
&\leq C(TC_1(M) + T^2C_2(M) + \varepsilon(T)),
\end{aligned} \tag{4.40}
$$

with $\varepsilon(T)$ independent of M and going to zero as T goes to zero. Choosing T small enough (T depending on M but not on k), we construct the sequence of approximate solutions satisfying hypothesis 1.

4.4. **Convergence of the sequence of approximate solutions.** The proof for the existence of a solution for the nonlinear baroclinic mode of the two-layers Shallow Water model consists in proving the strong convergence of the sequence V^k of approximate solutions. In order to do so, we estimate the difference between the solution of problem (4.25)-(4.27) at step $k+1$ and the solution at step k and we prove that the sequence $(V^k)_k$ is Cauchy and therefore strongly convergent. Writing $W^{k+1} = V^{k+1} - V^k$, we find that W^{k+1} satisfies the following equation:

$$
\frac{\partial W^{k+1}}{\partial t} + A(V^k + V_a + V_s)\frac{\partial W^{k+1}}{\partial x} = \mathcal{F}^W, \tag{4.41}
$$

where the forcing term $\mathcal{F}^W$ is given by:

$$
\mathcal{F}^W = \mathcal{F}(V^k) - \mathcal{F}(V^{k-1}) - \{A(V^k + V_a + V_s) - A(V^{k-1} + V_a + V_s)\}\frac{\partial V^k}{\partial x}. \tag{4.42}
$$

The initial condition associated with equation (4.41) is:

$$
W^{k+1}(x, 0) = 0, \tag{4.43}
$$

and the boundary conditions are:

$$
\begin{cases}
db_L(V^k + V_a + V_s) \cdot W^{k+1} = \mathcal{G}^W_L, \text{ at } x = L, \\
db_0(V^k + V_a + V_s) \cdot W^{k+1} = \mathcal{G}^W_0, \text{ at } x = 0,
\end{cases} \tag{4.44}
$$

with

$$
\begin{aligned}
\mathcal{G}^W_L &= db_L(V^{k-1} + V_a + V_s) \cdot W^k - b_L(V^k + V_a + V_s) + b_L(V^{k-1} + V_a + V_s), \\
\mathcal{G}^W_0 &= db_L(V^{k-1} + V_a + V_s) \cdot W^k - b_0(V^k + V_a + V_s) + b_0(V^{k-1} + V_a + V_s).
\end{aligned}
$$

Problem (4.41)-(4.44) is similar to (4.25)-(4.27) and thus L^2-estimate similar to (4.39) is available:

$$\begin{aligned}\frac{1}{T}\|W^{k+1}\|^2_{L^2(\Omega_T)}+\|W^{k+1}|_{x=0}\|^2_{L^2(-\infty,T)}+\|W^{k+1}|_{x=L}\|^2_{L^2(-\infty,T)}\\ \leq c(T\|\mathcal{F}^W\|^2_{L^2(\Omega_T)}+\|\mathcal{G}^W_0\|^2_{L^2(-\infty,T)}+\|\mathcal{G}^W_L\|^2_{L^2(-\infty,T)}).\end{aligned}\tag{4.45}$$

We thus only need to estimate the L^2-norms for $\mathcal{F}^W,\mathcal{G}^W_0$ and $\mathcal{G}^W_L$. We have:

$$\mathcal{F}^W=-\{A(V^k+V_a+V_s)-A(V^{k-1}+V_a+V_s)\}\frac{\partial(V_a+V^k)}{\partial x},\tag{4.46}$$

and so we can estimate $\mathcal{F}^W$ as follows:

$$\|\mathcal{F}^W\|_{L^2(\Omega_T)}\leq\|\frac{\partial(V_a+V_k)}{\partial x}\|_{L^\infty(\Omega_T)}\|A(V^k+V_a+V_s)-A(V^{k-1}+V_a+V_s)\|_{L^2(\Omega_T)}$$

$$\leq c\|V_a+V^k\|_{H^3(\Omega_T)}\|W^k\|_{L^2(\Omega_T)},\tag{4.47}$$

since $H^2(\Omega_T)$ is continuously embedded into $L^\infty(\Omega_T)$. The relation:

$$\|A(V^k+V_a+V_s)-A(V^{k-1}+V_a+V_s)\|_{L^2(\Omega_T)}\leq c(V^k,V_a,V_s)\|W^k\|_{L^2(\Omega_T)},$$

follows from the fact that the matrix $A(W)$ is $\mathcal{C}^\infty$ with respect to W in $\mathbb{R}^2$; $c(V^k,V_a,V_s)$ is a constant depending on V_s and increasingly on the $H^3(\Omega_T)$- norms of V^k and V_a.

From (4.47) we thus conclude that:

$$\|\mathcal{F}^W\|_{L^2(\Omega_T)}\leq c_1(M)\|W^k\|_{L^2(\Omega_T)}.\tag{4.48}$$

In order to estimate the boundary terms, we write a second order Taylor expansion for b_L and b_0 and we find:

$$|\ \mathcal{G}^W_0\ |_{L^2(-\infty,T)}\leq c\ |\ W^k|_{x=0}|_{L^2(-\infty,T)}\ |\ W^k|_{x=0}|_{L^\infty(-\infty,T)}.$$

Since $H^3(-\infty,T)\subset L^\infty(-\infty,T)$, we obtain:

$$|\ \mathcal{G}^W_0\ |_{L^2(-\infty,T)}\leq c\ |\ W^k|_{x=0}\ |_{H^3(-\infty,T)}\ |W^k|_{x=0}\ |_{L^2(-\infty,T)},\tag{4.49}$$

Similarly, we obtain:

$$|\ \mathcal{G}^W_L\ |_{L^2(-\infty,T)}\leq c\ |\ W^k|_{x=L}\ |_{H^3(-\infty,T)}|\ W^k|_{x=L}\ |_{L^2(-\infty,T)}\tag{4.50}$$

and $|\ W^k|_{x=0}\ |_{H^3(-\infty,T)}, |\ W^k|_{x=L}\ |_{H^3(-\infty,T)}$ are bounded by $c(\sqrt{T}c_1(M)+Tc_2(M)+\epsilon(T))$, thanks to (4.40).

Using (4.48)-(4.50) into (4.45), we obtain:

$$\begin{aligned}&\frac{1}{T}\|W^{k+1}\|^2_{L^2(\Omega_T)}+\|W^{k+1}|_{x=0}\|^2_{L^2(-\infty,T)}+\|W^{k+1}|_{x=L}\|^2_{L^2(-\infty,T)}\\ &\quad\leq c(Tc_1(M)\|W^k\|^2_{L^2(\Omega T)}+\epsilon(T,M)\|W^k|_{x=0}\|^2_{L^2(-\infty,T)}+\epsilon(T,M)\|W^k|_{x=L}\|^2_{L^2(-\infty,T)},\end{aligned}\tag{4.51}$$

with $c_1(M),c_2(M)$ are constants depending on T and M, going to zero when T goes to zero.

We take T sufficiently small so that:

$$\begin{aligned}\|W^{k+1}\|^2_{L^2(\Omega_T)}+\|W^{k+1}\|^2_{x=0}\|_{L^2(-\infty,T)}+\|W^{k+1}|_{x=L}\|^2_{L^2(-\infty,T)}\\ \leq\frac{1}{2}(\|W^k\ |^2_{L^2(\Omega_T)}+\|W^k|_{x=0}\|^2_{L^2(-\infty,T)}+\|W^k|_{x=L}\|^2_{L^2(-\infty,T)}),\end{aligned}\tag{4.52}$$

which implies:

$$\begin{aligned}\|W^{k+1}\|^2_{L^2(\Omega_T)} &+ \|W^{k+1}|_{x=0}\|^2_{L^2(-\infty,T)} + \|W^{k+1}|_{x=L}\|^2_{L^2(-\infty,T)} \\ &\leq 2^{-(k+1)}(\|W^0\|^2_{L^2(\Omega_T)} + \|W^0|_{x=0}\|^2_{L^2(-\infty,T)} + \|W^0|_{x=L}\|^2_{L^2(-\infty,T)}).\end{aligned} \tag{4.53}$$

We prove in this way that $(V^k)_k, (V^k(0,\cdot))_k, (V^k(L,\cdot))_k$ are Cauchy sequences in L^2. Let V be the $L^2(\Omega_T)$- limit of $(V^k)_k$ and $V_{|0}, V_{|L}$ respectively the L^2- limits of $(V^k(0,\cdot))_k$ and $(V^k(L,\cdot))_k$. Since V^k is bounded in $H^3(\Omega_T)$ and $V^k(0,\cdot)$, $V^k(L,\cdot)$ are bounded in $H^3(-\infty,T)$, we deduce that V belongs to $H^3(\Omega_T)$ and $V|_0, V|_L$ belong to $H^3(-\infty,T)$. By an L^2-H^3 interpolation argument, we infer that $(V^k)_k$ converges strongly in $H^s(\Omega_T)$, for any $s<3$ which implies that

$$V(0,\cdot) = V|_0, V(L,\cdot) = V|_L.$$

With this we conclude that V is the solution of the initial boundary value problem (4.19)-(4.21).

The uniqueness of this solution is immediate by taking W to be the difference of two solutions satisfying the same initial and boundary conditions and deducing similar L^2- estimates for W.

With this Theorem 3.1 is proved. □

Let us briefly mention the baroclinically supersonic case, meaning that we consider a constant stationary solution $V_s = (h_{1,s}, 0)$ for which we have:

$$u^2 > \frac{g' h_{1,s}(h - h_{1,s})}{h}. \tag{4.54}$$

More exactly, we assume that:

$$u^2 - g'\frac{h_{1,s}(h-h_{1,s})}{h} \geq c_0^2 \text{ and } 2\underline{h}_0 \leq h_{1,s} \leq 2\overline{h}_0, \tag{4.55}$$

with c_0,$\underline{h}_0$,$\overline{h}_0$ given, positive constants. We consider the initial conditions:

$$V_0 = (h_{1,0}, v_0), \tag{4.56}$$

with $h_{1,0}, v_0$ satisfying the following conditions:

$$\begin{aligned}&\Big(u + v_1(x) - 2v_0(x)\frac{h_{1,0}(x)}{h}\Big)^2 - \frac{h_{1,0}(x)(h-h_{1,0}(x))}{h^2} \geq c_0^2, \ \ \forall\, x \in (0,L), \\ &g'h - v_0^2(x) \geq \frac{3}{4}g'h, \ \ \forall\, x \in (0,L), \\ &2\underline{h}_0 \leq h_{1,0}(x) \leq 2\overline{h}_0, \ \ \forall\, x \in (0,L),\end{aligned} \tag{4.57}$$

with $\overline{h}_0$ chosen such that $h - 3\overline{h}_0 > 0$.

Since in the supersonic case both eigenvalues of the operator $A(V)$ are positive, it means that we actually need two boundary conditions at $x=0$, the natural boundary condition being to prescribe V at $x=0$:

$$V = V_s + g(t), \text{ at } x = 0, \tag{4.58}$$

with $g(t) = (g_1(t), g_2(t))$. As for the subsonic case, we require that V_s is a stationary solution of the initial boundary value problem with V_s as initial data and satisfying the boundary condition at $t = 0$, which implies:

$$g(t = 0) = 0. \tag{4.59}$$

We also need to impose the following compatibility conditions for the boundary data and the initial data. The compatibility conditions are consistent with the fact that the solution is $\mathcal{C}^p$ up to the boundary, with $p = 2$. Thus, we have for $p = 0$:

$$V(x, t) = V_s + g(t) \text{ at } x = 0, \tag{4.60}$$

which implies that $V_0(x) = V_s + g(0)$ at $x = 0$ and we find:

$$V_0(x = 0) = V_s. \tag{4.61}$$

For $p = 1$, we differentiate relation (4.60) in time and we obtain:

$$\partial_t V(x, t) = g'(t) \text{ at } x = 0,$$

which implies

$$-A(V)\partial_x V(x, t) = g'(t) \text{ at } x = 0. \tag{4.62}$$

Taking $t = 0$ we obtain the compatibility condition:

$$-A(V_0)V_{0,x} = g'(0) \text{ at } x = 0. \tag{4.63}$$

The compatibility condition for $p = 2$ is obtained by differentiating twice in time relation (4.62). We thus obtain:

$$-A(-A(V)\partial_x V)\partial_x V - A(V)(-A(\partial_x V)\partial_x V - A(V)\partial_{xx} V) = G''(t) \text{ at } x = 0. \tag{4.64}$$

Setting $t = 0$ in (4.64) we find the following compatibility condition:

$$-A(-A(V_0)V_{0,x})V_{0,x} - A(V_0)(-A(V_{0,x})V_{0,x} - A(V_0)V_{o,xx}) = g''(0). \tag{4.65}$$

Thus, an analog of (4.1) for the baroclinically supersonic case can be proved for the initial data and boundary data satisfying (4.59), (4.61), (4.63), (4.65).

5. Numerical experiments

We finally display the results of some numerical simulations corresponding to our two-layers model. Note that the bottom is not flat, which renders the flow more interesting. In this article we have taken $h_0 = 0$ for the sake of simplicity. It is interesting to see that the boundary conditions are actually transparent and that all waves generated by the initial conditions (and the topography) move freely out of the domain until we reach a stationary state; see Figure 1.

These calculations were performed by Ming-Cheng Shiue and they involve quite delicate numerical issues for the implementation of the boundary conditions, beside those related to the equations themselves. Related situations will be addressed in [2].

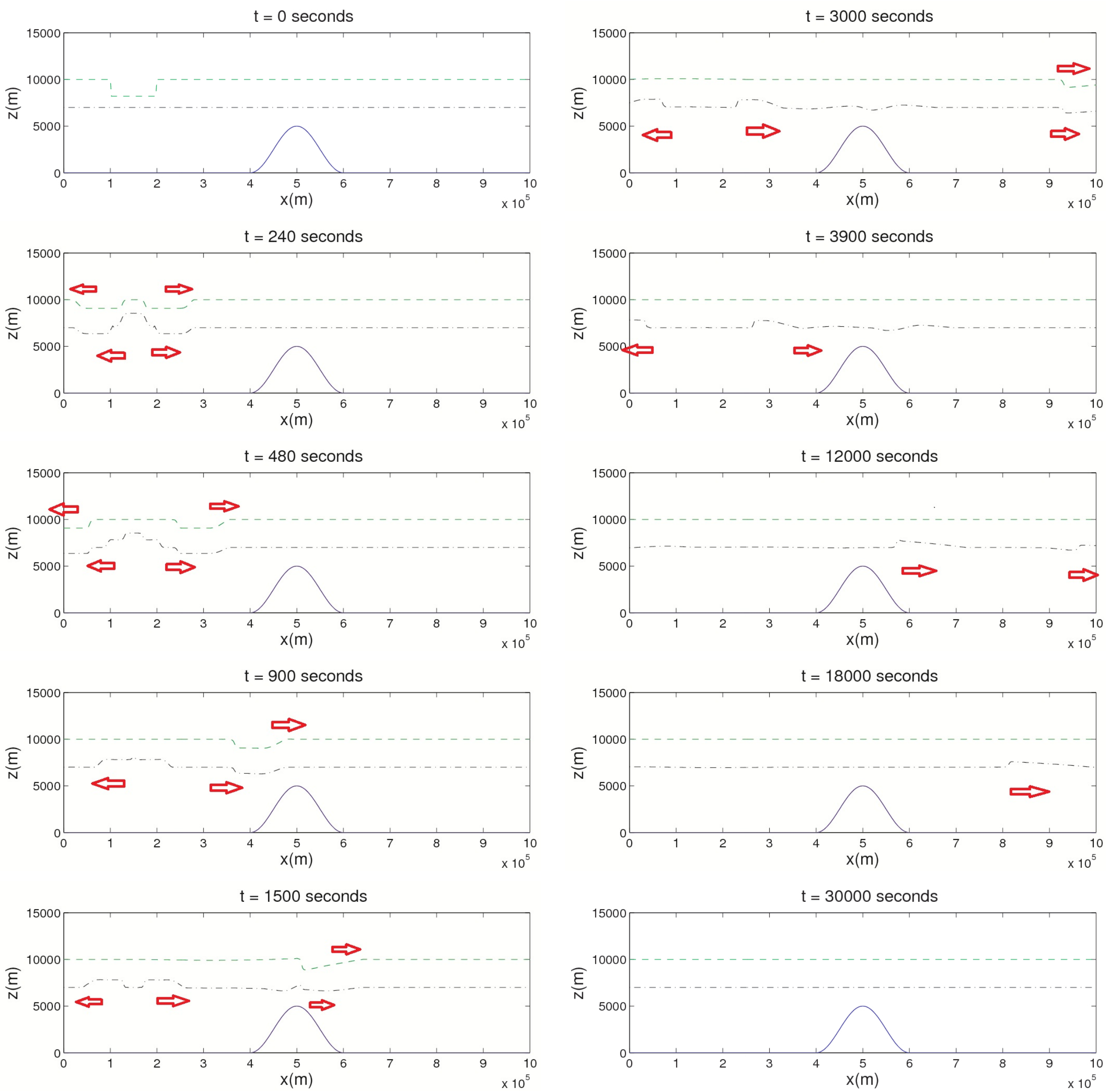

FIGURE 1. The height of the free surface $h + h_0$ and $h_1 + h_0$ at different times. The solid line represents the height of the bottom topography, the dashed line represents the height of the free surface $h+h_0$ and the dashdot line represents the height of the free surface $h_1 + h_0$.

6. Conclusion

In this article we have studied the local in time well-posedness for a two-layer Shallow Water model with open boundary conditions. The classical multilayer Shallow Water model is not hyperbolic but under the assumption that $\rho_1 \approx \rho_2$ and $u_1 \approx u_2$, we can approximate the classical model by two hyperbolic models: one for the barotropic mode (and this mode coincides with the single layer Shallow Water problem) and one for the baroclinic mode. The single layer Shallow Water problem with open boundary conditions has been already studied (see [32]). So we recalled the available results for this case and we completely studied the baroclinic mode. Suitable open boundary conditions were implemented for the baroclinically subcritical and baroclinically supercritical cases and the local in time well-posedness was proved. In total there are four possible types of flow, depending on whether the barotropic flow and baroclinic flow are each (separately), subcritical or supercritical.

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