NUMERICAL TIME-SCHEMES FOR AN OCEAN RELATED SYSTEM OF PDEs.

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ABSTRACT

In this article we consider a system of equations related to the δ–Primitive Equations of the ocean and the atmosphere, linearized around a stratified flow, and we supplement the equations with transparent boundary conditions. We study the stability of different numerical schemes and we show that for each case, letting the vertical viscosity δ go to 0, the stability conditions are the same as the classical CFL conditions for the transport equation.

1. INTRODUCTION

The issue of open boundary conditions for the Primitive Equations (PEs) of the ocean and the atmosphere is fundamental in the field of computational fluid dynamics (see e.g. [Ped87, TZ04, TT03]). The PEs, supplemented with any set of local boundary conditions, were shown to be ill-posed (see [OS78], [TT03]). To overcome this difficulty, the so-called δ-PEs were introduced with different motivations in [TT03] and [Sal98]. This new model consists in the addition of a friction term of the form δw in the hydrostatic equation, which is sufficient to ensure well-posedness (see [PR05]).

In a recent article [RTT04], the authors make a modal analysis of the δ-PEs linearized around a stratified flow, and perform numerical simulations of the so-called subcritical modes, that are the most challenging ones (see [RTT04]). In the case of classical Dirichlet boundary conditions, some reflexions of waves and boundary layers occur, and thus the authors consider another set of boundary conditions, of transparent type, in order to avoid these boundary layers as δ goes to zero. For

these models, some energy estimates are given, and a full proof of well-posedness and convergence as δ goes to zero are given in [RTT05]. In the present article we intend to study the stability of the schemes considered in these articles. Hereafter we consider different discretizations of the equations and boundary conditions that have been proposed in [RTT04, RTT05], and present the stability results. In the case when a stability condition occurs (e.g. in Section 4), we notice that if δ is taken equal to zero, the condition matches with the classical CFL condition for the transport equation.

The article is organized as follows: in Section 2, we recall the equations and boundary conditions introduced in [RTT05], and set the functional framework of our study. We then start the stability studies in Section 3 with an implicit Euler scheme, which is proved to be unconditionally stable. For the explicit scheme, we derive in Section 4 a stability condition involving $\Delta t, \Delta x$, and δ. We then prove in Section 5 the stability of the Crank-Nicholson scheme, with no condition on the parameters, and end this article with a study in Section 6 of the so-called fractional scheme method, which is shown to be easier to implement in the numerical computations, while it remains consistent and stable without any additional stability condition on the parameters $\Delta t, h$, and ε. The approach for the study of stability is the classical one, based on energy estimates, which is more appropriate than the von-Neumann spectral method for nonperiodic boundary value problems.

2. Equations and functional framework

The δ-PEs of the ocean with no Coriolis force, in a 2D domain $\mathcal{M} = (-H, 0) \times (0, L)$, and linearized around a constant stratified flow

$\overline{U}_0 \, \mathbf{e_x} = (\overline{U}_0, 0)$ with $\overline{U}_0 > 0$, read:

$$(2.1) \qquad \frac{\partial u}{\partial t} + \overline{U}_0 \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = F_u,$$

$$(2.2) \qquad \frac{\partial v}{\partial t} + \overline{U}_0 \frac{\partial v}{\partial x} = F_v,$$

$$(2.3) \qquad \frac{\partial \psi}{\partial t} + \overline{U}_0 \frac{\partial \psi}{\partial x} + N^2 w = F_\psi,$$

$$(2.4) \qquad \delta w + \frac{\partial \phi}{\partial z} = \psi,$$

$$(2.5) \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Here u, v, w, ϕ and ρ are all perturbed quantities; (u, v) is the horizontal velocity, w the vertical velocity, ϕ the pressure, ρ the density, and ψ the temperature. The constant g is the gravitational acceleration.

We perform the so-called normal mode decomposition, that is we look for some solutions of the form[1]:

$$(2.6) \qquad (u, v, \phi) = \sum_{n \geq 0} \cos(N\lambda_n z) \, (u_n, v_n, \phi_n) \, (x, t),$$

$$(2.7) \qquad (w, \psi) = \sum_{n \geq 1} \sin(N\lambda_n z) \, (w_n, \psi_n) \, (x, t).$$

Here $N\lambda_n = n\pi/H$, where N is the constant Brunt-Väisälä (or buoyancy) frequency, and n is the number of the considered mode.

We obtain for each mode $n \geq 1$ the following system of equations:

$$(2.8) \qquad \begin{cases} \dfrac{\partial u_n}{\partial t} + \overline{U}_0 \dfrac{\partial u_n}{\partial x} + \dfrac{\partial \phi_n}{\partial x} & = F_{u,n}, \\[2mm] \dfrac{\partial v_n}{\partial t} + \overline{U}_0 \dfrac{\partial v_n}{\partial x} & = F_{v,n}, \\[2mm] \dfrac{\partial \psi_n}{\partial t} + \overline{U}_0 \dfrac{\partial \psi_n}{\partial x} + N^2 w_n & = F_{\psi,n}, \\[2mm] \phi_n = -\dfrac{1}{N\lambda_n}(\psi_n - \delta w_n), \\[2mm] w_n = -\dfrac{1}{N\lambda_n} \dfrac{\partial u_n}{\partial x}. \end{cases}$$

Dropping the equation on v_n that can be solved independently (in the absence of Coriolis force), and replacing ϕ_n and w_n by their expression

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[1]See [RTT04, RTT05] for more details.

in the equations for u_n and ψ_n, we obtain:
(2.9)
$$\begin{cases} \dfrac{\partial u_n}{\partial t} + \overline{U}_0 \dfrac{\partial u_n}{\partial x} - \dfrac{1}{N\lambda_n}\dfrac{\partial \psi_n}{\partial x} - \dfrac{\delta}{N^2\lambda_n^2}\dfrac{\partial^2 u_n}{\partial x^2} = F_{u,n}, \\[2mm] \dfrac{\partial \psi_n}{\partial t} + \overline{U}_0 \dfrac{\partial \psi_n}{\partial x} - \dfrac{N}{\lambda_n}\dfrac{\partial u_n}{\partial x} = F_{\psi,n}. \end{cases}$$

Finally, we set $\xi = u_n - \psi_n/N$, $\eta = u_n + \psi_n/N$, and we find for every $(x,t) \in (0,L) \times (0,T)$:

(2.10)
$$\begin{cases} \dfrac{\partial \xi}{\partial t}(x,t) + \alpha\dfrac{\partial \xi}{\partial x}(x,t) - \varepsilon(\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial^2 \eta}{\partial x^2})(x,t) = f(x,t), \\[2mm] \dfrac{\partial \eta}{\partial t}(x,t) - \beta\dfrac{\partial \eta}{\partial x}(x,t) - \varepsilon(\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial^2 \eta}{\partial x^2})(x,t) = g(x,t), \end{cases}$$

where $\alpha = \overline{U}_0 + \lambda_n^{-1}$, $\beta = -\overline{U}_0 + \lambda_n^{-1}$ are some constants depending on the mode that we consider. We restrict ourselves to the subcritical modes that are the most important and the most challenging and, in that case n is such that $\beta > 0$. The reader is referred to the articles quoted before for more discussions about these modes. The parameter $\varepsilon = \delta/2N^2\lambda_n^2$ is proportional to δ, hence is devoted to tend to zero. We supplement these equations with the following (nonreflecting) boundary conditions:

(2.11)
$$\begin{cases} \xi(0,t) = 0, \\ \eta(L,t) = 0, \\ \dfrac{\partial \xi}{\partial t}(L,t) + \alpha(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial x})(L,t) = f(L,t) - \dfrac{\alpha}{\beta}g(L,t), \\[2mm] \dfrac{\partial \eta}{\partial t}(0,t) - \beta(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial x})(0,t) = g(0,t) - \dfrac{\beta}{\alpha}f(0,t), \end{cases}$$

for every $t > 0$.

Working with finite differences in space, for $0 \le j \le N$ we set:

(2.12)
$$\begin{cases} \xi_j(t) = \xi(jh,t), \\ \eta_j(t) = \eta(jh,t), \end{cases}$$

where $h = \Delta x = L/N$ is the mesh size. The discretization in space given in [RTT05] reads, for every $1 \le j \le N-1$, $t > 0$:
(2.13)
$$\begin{cases} \dfrac{d\xi_j}{dt}(t) + \alpha\,(\overline{\nabla}_h\xi)_j(t) - \varepsilon\,[\nabla_h\overline{\nabla}_h(\xi+\eta)]_j(t) = f_j(t), \\[2mm] \dfrac{d\eta_j}{dt}(t) - \beta\,(\nabla_h\eta)_j(t) - \varepsilon\,[\nabla_h\overline{\nabla}_h(\xi+\eta)]_j(t) = g_j(t), \end{cases}$$

where ∇_h and $\overline{\nabla}_h$ are the following discrete operators:

$$(\nabla_h \varphi)_j = \frac{\varphi_{j+1} - \varphi_j}{h}, \quad \forall j = 0..N - 1,$$

$$(\overline{\nabla}_h \varphi)_j = \frac{\varphi_j - \varphi_{j-1}}{h}, \quad \forall j = 1..N.$$

Finally, we have the following discrete boundary conditions:

$$(2.14) \qquad \begin{cases} \xi_0(t) = 0, \\ \eta_N(t) = 0, \end{cases} \quad \forall t > 0,$$

and for every $t > 0$,
(2.15)
$$\begin{cases} \dfrac{d\xi_N}{dt}(t) + \dfrac{\alpha}{h}(\xi_N - \xi_{N-1} - \eta_{N-1})(t) = f_N(t) - \dfrac{\alpha}{\beta} g_N(t), \\ \dfrac{d\eta_0}{dt}(t) - \dfrac{\beta}{h}(\xi_1 + \eta_1 - \eta_0)(t) = g_0(t) - \dfrac{\beta}{\alpha} f_0(t). \end{cases}$$

Let us now set the functional framework of the problem.

For $U = (\xi_1, ..., \xi_{N-1}, \eta_1, ..., \eta_{N-1}, \xi_N, \eta_0) \in H = \mathbb{R}^{2N}$, we define the following scalar product :

$$(2.16) \qquad (U, \tilde{U})_H = \sum_{j=1}^{N-1} h\, \xi_j \tilde{\xi}_j + \sum_{j=1}^{N-1} h\, \eta_j \tilde{\eta}_j + \frac{\varepsilon}{\alpha} \xi_N \tilde{\xi}_N + \frac{\varepsilon}{\beta} \eta_0 \tilde{\eta}_0.$$

Given some continuous functions (f, g), we set, using the same notation as in (2.12):

$$(2.17) \qquad F = (f_1, ..., f_{N-1}, g_1, ..., g_{N-1}, f_N - \frac{\alpha}{\beta} g_N, g_0 - \frac{\beta}{\alpha} f_0).$$

In the sequel, we will prove the stability of classical time discretisation schemes applied to equations (2.13)-(2.15), the consistency with the continuous equations (2.10)-(2.11) being easy. However, we have not been able to show that, in the limit $\Delta x \longrightarrow 0$, the boundary value problem (2.10)-(2.11) is well-posed and, in fact, a naive count of the number of equations indicates that the continuous system may be overdetermined if no precautions are taken. However, as we said in [RTT04, RTT05], this perturbed system has some computational advantages, and for $\Delta x > 0$ fixed it is well-posed, the limit system is also well-posed (but this is standard), and the perturbed system does converge to the expected limit when $\varepsilon \longrightarrow 0$ (Δx fixed).

3. The Implicit Euler time scheme

3.1. Discretization of the equations and boundary conditions.

We now give the time discretization for (2.13) and (2.15) based on the implicit Euler scheme. For each $m \leq M$, we denote by u^m the quantity $u(m\Delta t)$ where $\Delta t = T/M$ is the time meshsize. Inside the domain, we have, $\forall 1 \leq j \leq N-1$, $\forall 1 \leq m \leq M$:

$$(3.1) \quad \begin{cases} \dfrac{\xi_j^m - \xi_j^{m-1}}{\Delta t} + \alpha \dfrac{\xi_j^m - \xi_{j-1}^m}{h} - \varepsilon \dfrac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{h^2} &= f_j^m, \\[2ex] \dfrac{\eta_j^m - \eta_j^{m-1}}{\Delta t} - \beta \dfrac{\eta_{j+1}^m - \eta_j^m}{h} - \varepsilon \dfrac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{h^2} &= g_j^m, \end{cases}$$

For the sake of simplicity, in the relation (3.3) above and in the sequel, we denote by v_j^m the quantity $(\xi_j^m + \eta_j^m)$. On the boundary, equation (2.15) gives, $\forall 1 \leq m \leq M$:

$$(3.2)$$
$$\begin{cases} \dfrac{\xi_N^m - \xi_N^{m-1}}{\Delta t} &+& \alpha \dfrac{\xi_N^m - \xi_{N-1}^m}{h} &-& \dfrac{\alpha}{h} \eta_{N-1}^m &=& f_N^m - \dfrac{\alpha}{\beta} g_N^m, \\[2ex] \dfrac{\eta_0^m - \eta_0^{m-1}}{\Delta t} &-& \beta \dfrac{\eta_1^m - \eta_0^m}{h} &-& \dfrac{\beta}{h} \xi_1^m &=& g_0^m - \dfrac{\beta}{\alpha} f_0^m. \end{cases}$$

3.2. Proof of stability.

Let us prove that the implicit scheme (3.1)-(3.2) is stable, with no condition on Δt, $h = \Delta x$ or ε. In order to recover the scalar product in H, we multiply $(3.1)_1$ by $h\,\xi_j^m$ for $1 \leq j \leq N-1$, $(3.1)_2$ by $h\,\eta_j^m$ for $1 \leq j \leq N-1$, $(3.2)_1$ by $\varepsilon\,\xi_N^m/\alpha$, and $(3.2)_2$ by $\varepsilon\,\eta_0^m/\beta$; we sum all

these equations, and obtain:
(3.3)

$$
\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^m|^2 + \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^m|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m|^2
$$

$$
-\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^{m-1}|^2 - \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^{m-1}|^2
$$

$$
-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m-1}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m-1}|^2
$$

$$
+\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_j^{m-1}|^2 + \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^m - \eta_j^{m-1}|^2
$$

$$
+\frac{\alpha}{2} \sum_{j=1}^{N-1} |\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2} \sum_{j=1}^{N-1} |\eta_{j+1}^m - \eta_j^m|^2 + \frac{\alpha}{2}|\xi_{N-1}^m|^2 + \frac{\beta}{2}|\eta_1^m|^2
$$

$$
+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m - \xi_N^{m-1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m - \eta_0^{m-1}|^2
$$

$$
-\frac{\varepsilon}{h} \sum_{j=1}^{N-1} (v_{j+1}^m - v_j^m)\,v_j^m - \frac{\varepsilon}{h} \sum_{j=1}^{N-1} (v_{j-1}^m - v_j^m)\,v_j^m
$$

$$
+\frac{\varepsilon}{h}\,(\xi_N^m - \xi_{N-1}^m - \eta_{N-1}^m)\,\xi_N^m - \frac{\varepsilon}{h}\,(\eta_1^m - \eta_0^m + \xi_1^m)\,\eta_0^m
$$

$$
= \sum_{j=1}^{N-1} h\,f_j^m\,\xi_j^m + \sum_{j=1}^{N-1} h\,g_j^m\,\eta_j^m
$$

$$
+\frac{\varepsilon}{\alpha}\,(f_N^m - \frac{\alpha}{\beta}\,g_N^m)\,\xi_N^m + \frac{\varepsilon}{\beta}\,(g_0^m - \frac{\beta}{\alpha}\,f_0^m)\,\eta_0^m.
$$

Thanks to some easy computations, we find:

(3.4)
$$
-\frac{\varepsilon}{h} \sum_{j=1}^{N-1} (v_{j+1}^m - v_j^m)\,v_j^m - \frac{\varepsilon}{h} \sum_{j=1}^{N-1} (v_{j-1}^m - v_j^m)\,v_j^m
$$

$$
= \frac{\varepsilon}{h} \sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2
$$

$$
-\frac{\varepsilon}{h}\,(\eta_0^m - \xi_1^m - \eta_1^m)\,\eta_0^m + \frac{\varepsilon}{h}\,(\xi_{N-1}^m + \eta_{N-1}^m - \xi_N^m)\,\xi_N^m
$$

Using (3.4), equation (3.3) becomes:

(3.5)

$$\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^m|^2 + \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^m|^2 + \frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m|^2$$

$$-\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^{m-1}|^2 - \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^{m-1}|^2$$

$$-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m-1}|^2 - \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m-1}|^2$$

$$+\frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_j^{m-1}|^2 + \frac{1}{2\,\Delta t} \sum_{j=1}^{N-1} h\,|\eta_j^m - \eta_j^{m-1}|^2$$

$$+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m - \xi_N^{m-1}|^2 + \frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m - \eta_0^{m-1}|^2$$

$$+\frac{\alpha}{2} \sum_{j=1}^{N-1} |\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2} \sum_{j=1}^{N-1} |\eta_{j+1}^m - \eta_j^m|^2 + \frac{\alpha}{2}|\xi_{N-1}^m|^2 + \frac{\beta}{2}|\eta_1^m|^2$$

$$+\frac{\varepsilon}{h} \sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2$$

$$= \sum_{j=1}^{N-1} h\,f_j^m\,\xi_j^m + \sum_{j=1}^{N-1} h\,g_j^m\,\eta_j^m$$

$$+\frac{\varepsilon}{\alpha}\,(f_N^m - \frac{\alpha}{\beta}\,g_N^m)\,\xi_N^m + \frac{\varepsilon}{\beta}\,(g_0^m - \frac{\beta}{\alpha}\,f_0^m)\,\eta_0^m.$$

Using the functional framework defined in Section 2, equation (3.5) can be rewritten as:

(3.6)
$$\frac{1}{2\,\Delta t}\,|U^m|_H^2 - \frac{1}{2\,\Delta t}\,|U^{m-1}|_H^2 + \frac{1}{2\,\Delta t}\,|U^m - U^{m-1}|_H^2$$

$$+\frac{\alpha}{2\,h} \sum_{j=1}^{N-1} h\,|\xi_j^m - \xi_{j-1}^m|^2 + \frac{\beta}{2\,h} \sum_{j=1}^{N-1} h\,|\eta_{j+1}^m - \eta_j^m|^2$$

$$+\frac{\alpha}{2}\,|\xi_{N-1}^m|^2 + \frac{\beta}{2}\,|\eta_1^m|^2 + \frac{\varepsilon}{h} \sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2$$

$$= (F, U^m)_H.$$

After dropping some positive terms[2], and using the Cauchy-Schwarz inequality we find, with $F \in L^{\infty}(0, T; H)$:

$$(3.7) \qquad \frac{1}{2 \Delta t} |U^m|_H^2 \leq \frac{1}{2 \Delta t} |U^{m-1}|_H^2 + |F|_{\infty} |U^m|_H.$$

Thanks to some easy computations, (3.7) implies:

$$(3.8) \qquad |U^m|_H^2 \leq \frac{1}{1 - \Delta t} |U^{m-1}|_H^2 + \frac{\Delta t}{1 - \Delta t} |F|_{\infty}^2.$$

Writing (3.8) for $m = 1, ..., m_0$, $m_0 \leq M$, we find recursively:

$$
\begin{aligned}
|U^{m_0}|_H^2 \quad \leq \quad & \frac{1}{(1 - \Delta t)^{m_0}} |U^0|_H^2 \\
(3.9) \qquad & + \frac{\Delta t}{1 - \Delta t} |F|_{\infty}^2 \left(1 + \frac{1}{1 - \Delta t} + ... + \frac{1}{(1 - \Delta t)^{m_0 - 1}}\right).
\end{aligned}
$$

Assuming that $\Delta t < 1$, we obtain:

$$(3.10) \qquad |U^{m_0}|_H^2 \leq \frac{1}{(1 - \Delta t)^{m_0}} |U^0|_H^2 + \frac{1}{(1 - \Delta t)^{m_0}} |F|_{\infty}^2.$$

Finally, we use the classical inequality $e^{-2x} \leq 1 - x$ valid for every $x \in [0, x^*]$ where x^* is the positive root of $f(x) = e^{-2x} + x - 1$, $(0.79 < x^* < 0.80)$. Assuming then that $0 < \Delta t < min(1, x^*)$, we find for every $m \leq M$:

$$(3.11) \qquad |U^m|_H^2 \leq e^{2m\Delta t} (|U^0|_H^2 + |F|_{\infty}^2) \leq e^{2T} (|U^0|_H^2 + |F|_{\infty}^2),$$

which guarantees the stability of our scheme.

4. The explicit Euler time scheme

4.1. Discretization of the equations and boundary conditions.

We now give the time discretization of (2.13) and (2.15) using the explicit Euler scheme. Inside the domain, we have, $\forall 1 \leq j \leq N-1, 1 \leq$

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[2]We recall that the modes we consider are such that $\alpha, \beta > 0$.

$m \leq M$:
(4.1)
$$
\begin{cases}
\dfrac{\xi_j^{m+1} - \xi_j^m}{\Delta t} + \alpha \dfrac{\xi_j^m - \xi_{j-1}^m}{h} - \varepsilon \dfrac{v_{j+1}^m - 2\,v_j^m + v_{j-1}^m}{h^2} = f_j^m, \\[2ex]
\dfrac{\eta_j^{m+1} - \eta_j^m}{\Delta t} - \beta \dfrac{\eta_{j+1}^m - \eta_j^m}{h} - \varepsilon \dfrac{v_{j+1}^m - 2\,v_j^m + v_{j-1}^m}{h^2} = g_j^m.
\end{cases}
$$

On the boundary, equation (2.15) gives, $\forall 1 \leq m \leq M$:
(4.2)
$$
\begin{cases}
\dfrac{\xi_N^{m+1} - \xi_N^m}{\Delta t} + \alpha \dfrac{\xi_N^m - \xi_{N-1}^m}{h} - \dfrac{\alpha}{h} \eta_{N-1}^m = f_N^m - \dfrac{\alpha}{\beta} g_N^m, \\[2ex]
\dfrac{\eta_0^{m+1} - \eta_0^m}{\Delta t} - \beta \dfrac{\eta_1^m - \eta_0^m}{h} - \dfrac{\beta}{h} \xi_1^m = g_0^m - \dfrac{\beta}{\alpha} f_0^m.
\end{cases}
$$

4.2. **Proof of stability.**

Here we expect a condition on $(\Delta t, h, \varepsilon)$ for the scheme to be stable. Proceeding like in Section 3.2, we multiply $(4.1)_1$ by $h\,\xi_j^m$ for $1 \leq j \leq N-1$, $(4.1)_2$ by $h\,\eta_j^m$ for $1 \leq j \leq N-1$, $(4.2)_1$ by $\varepsilon\,\xi_N^m/\alpha$, and $(4.2)_2$

by $\varepsilon\,\eta_0^m/\beta$. We sum all these equations, and obtain:

(4.3)

$$\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}|^2+\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}|^2$$

$$+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}|^2+\frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}|^2$$

$$-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\xi_j^{m}|^2-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\eta_j^{m}|^2-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m}|^2-\frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m}|^2$$

$$-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}-\xi_j^{m}|^2-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}-\eta_j^{m}|^2$$

$$-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}-\xi_N^{m}|^2-\frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}-\eta_0^{m}|^2$$

$$+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m}-\xi_{j-1}^{m})^2+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m}-\eta_j^{m})^2+\frac{\alpha}{2}\,|\xi_{N-1}^{m}|^2+\frac{\beta}{2}\,|\eta_1^{m}|^2$$

$$-\frac{\varepsilon}{h}\sum_{j=1}^{N-1}(v_{j+1}^{m}-v_j^{m})\,v_j^{m}-\frac{\varepsilon}{h}\sum_{j=1}^{N-1}(v_{j-1}^{m}-v_j^{m})\,v_j^{m}$$

$$+\frac{\varepsilon}{h}\,(\xi_N^{m}-\xi_{N-1}^{m}-\eta_{N-1}^{m})\,\xi_N^{m}-\frac{\varepsilon}{h}\,(\eta_1^{m}-\eta_0^{m}+\xi_1^{m})\,\eta_0^{m}$$

$$=\sum_{j=1}^{N-1}h\,f_j^{m}\,\xi_j^{m}+\sum_{j=1}^{N-1}h\,g_j^{m}\,\eta_j^{m}$$

$$+\frac{\varepsilon}{\alpha}\,(f_N^{m}-\frac{\alpha}{\beta}\,g_N^{m})\,\xi_N^{m}+\frac{\varepsilon}{\beta}\,(g_0^{m}-\frac{\beta}{\alpha}\,f_0^{m})\,\eta_0^{m}.$$

where v_j^m has been defined above.

Using again (3.4) and the notations defined above, we obtain:

$$\frac{1}{2\,\Delta t}\,|U^{m+1}|_H^2-\frac{1}{2\,\Delta t}\,|U^{m}|_H^2-\frac{1}{2\,\Delta t}\,|U^{m+1}-U^{m}|_H^2$$

$$+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m}-\xi_{j-1}^{m})^2+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m}-\eta_j^{m})^2$$

(4.4)

$$+\frac{\alpha}{2}\,|\xi_{N-1}^{m}|^2+\frac{\beta}{2}\,|\eta_1^{m}|^2+\frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^{m}-v_j^{m})^2$$

$$=(F,U^{m})_H.$$

We now need to estimate the quantity $|U^{m+1}-U^{m}|_H^2$. To this aim, we multiply $(4.1)_1$ by $h\,(\xi_j^{m+1}-\xi_j^{m})/2$ for $1\le j\le N-1$, $(4.1)_2$ by

$h\left(\eta_j^{m+1} - \eta_j^m\right)/2$ for $1 \leq j \leq N-1$, $(4.2)_1$ by $\varepsilon\left(\xi_N^{m+1} - \xi_N^m\right)/2\,\alpha$, and $(4.2)_2$ by $\varepsilon\left(\eta_0^{m+1} - \eta_0^m\right)/2\,\beta$. We sum all these equations, and obtain:

(4.5)
$$\frac{1}{2\,\Delta t}\left|U^{m+1} - U^m\right|^2 =$$
$$-\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m - \xi_{j-1}^m)\left(\xi_j^{m+1} - \xi_j^m\right) + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m - \eta_j^m)\left(\eta_j^{m+1} - \eta_j^m\right)$$
$$+\frac{\varepsilon}{2\,h}\sum_{j=1}^{N-1}(v_{j+1}^m - 2\,v_j^m + v_{j-1}^m)\left(v_j^{m+1} - v_j^m\right)$$
$$-\frac{\varepsilon}{2\,h}\left(\xi_N^m - \xi_{N-1}^m - \eta_{N-1}^m\right)\left(\xi_N^{m+1} - \xi_N^m\right)$$
$$+\frac{\varepsilon}{2\,h}(\eta_1^m - \eta_0^m + \xi_1^m)\left(\eta_0^{m+1} - \eta_0^m\right) + \frac{1}{2}\left(F, U^{m+1} - U^m\right)_H.$$

Let us now bound terms that appear in the right hand side of equation (4.5). Firstly, we have:
(4.6)
$$-\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m - \xi_{j-1}^m)\left(\xi_j^{m+1} - \xi_j^m\right) + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m - \eta_j^m)\left(\eta_j^{m+1} - \eta_j^m\right)$$
$$\leq \frac{\alpha}{4}\sum_{j=1}^{N-1}(\xi_j^m - \xi_{j-1}^m)^2 + \frac{\alpha}{4\,h}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1} - \xi_j^m|^2$$
$$+\frac{\beta}{4}\sum_{j=1}^{N-1}(\eta_{j+1}^m j - \xi_j^m)^2 + \frac{\beta}{4\,h}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1} - \eta_j^m|^2$$

For the nonpositive terms involving ε, we have:

(4.7)
$$\frac{\varepsilon}{2\,h}\sum_{j=1}^{N-1}(v_{j+1}^m - v_j^m)\left(v_j^{m+1} - v_j^m\right)$$
$$\leq \frac{\varepsilon}{4\,h}\sum_{j=1}^{N-1}[(\xi_{j+1}^m - \xi_j^m) + (\eta_{j+1}^m - \eta_j^m)]^2$$
$$+\frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1} - \xi_j^m|^2 + \frac{\varepsilon}{2\,h^2}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1} - \eta_j^m|^2$$

and

$$\frac{\varepsilon}{2\,h} \sum_{j=1}^{N-1} (v_{j-1}^m - v_j^m)\,(v_j^{m+1} - v_j^m)$$

(4.8)
$$\leq \frac{\varepsilon}{4\,h} \sum_{j=1}^{N-1} [(\xi_j^m - \xi_{j-1}^m) + (\eta_j^m - \eta_{j-1}^m)]^2$$

$$+ \frac{\varepsilon}{2\,h^2} \sum_{j=1}^{N-1} h\,|\xi_j^{m+1} - \xi_j^m|^2 + \frac{\varepsilon}{2\,h^2} \sum_{j=1}^{N-1} h\,|\eta_j^{m+1} - \eta_j^m|^2$$

Similarly, for the boundary terms:

(4.9)
$$\begin{cases} -\dfrac{\varepsilon}{2\,h} (\xi_N^m - \xi_{N-1}^m - \xi_{N-1}^m)(\xi_N^{m+1} - \xi_N^m) & \leq & \dfrac{\varepsilon}{4\,h} (\xi_N^m - \xi_{N-1}^m - \xi_{N-1}^m)^2 \\ & & + \dfrac{\varepsilon}{4\,h} (\xi_N^{m+1} - \xi_N^m)^2, \\ \dfrac{\varepsilon}{2\,h} (\eta_1^m - \eta_0^m + \xi_1^m)(\eta_0^{m+1} - \eta_0^m) & \leq & \dfrac{\varepsilon}{4\,h} (\eta_1^m - \eta_0^m + \xi_1^m)^2 \\ & & + \dfrac{\varepsilon}{4\,h} (\eta_0^{m+1} - \eta_0^m)^2. \end{cases}$$

Finally, using inequalities (4.6)-(4.9) and the former equation (3.4), equation (4.5) becomes:

(4.10)
$$\frac{1}{2\,\Delta t} |U^{m+1} - U^m|^2 \leq \frac{1}{2} |F|_H |U^{m+1} - U^m|_H$$

$$+ \frac{\alpha}{4} \sum_{j=1}^{N-1} (\xi_j^m - \xi_{j-1}^m)^2 + (\frac{\alpha}{4\,h} + \frac{\varepsilon}{2\,h^2}) \sum_{j=1}^{N-1} h\,|\xi_j^{m+1} - \xi_j^m|^2$$

$$+ \frac{\beta}{4} \sum_{j=1}^{N-1} (\eta_{j+1}^m - \eta_j^m)^2 + (\frac{\beta}{4\,h} + \frac{\varepsilon}{2\,h^2}) \sum_{j=1}^{N-1} h\,|\eta_j^{m+1} - \eta_j^m|^2$$

$$+ \frac{\varepsilon}{2\,h} \sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2 + \frac{\varepsilon}{4\,h} (\xi_N^{m+1} - \xi_N^m)^2 + \frac{\varepsilon}{4\,h} (\eta_0^{m+1} - \eta_0^m)^2$$

Now, since $a\,b \leq a^2/2\,\mu + \mu\,b^2/2$ for every $(a, b) \in \mathbb{R}^2$ and $\mu > 0$, we find, thanks to the fact that $0 < \beta < \alpha$:

(4.11)
$$(\frac{1}{2\,\Delta t} - \frac{1}{4\,\mu\,\Delta t} - \frac{\alpha}{4\,h} - \frac{\varepsilon}{h^2}) |U^{m+1} - U^m|^2$$

$$\leq \frac{\alpha}{4} \sum_{j=1}^{N-1} (\xi_j^m - \xi_{j-1}^m)^2 + \frac{\beta}{4} \sum_{j=1}^{N-1} (\eta_{j+1}^m - \eta_j^m)^2$$

$$+ \frac{\varepsilon}{2\,h} \sum_{j=0}^{N-1} (v_{j+1}^m - v_j^m)^2$$

$$+ \frac{\mu\,\Delta t}{4} |F|_\infty^2$$

Let us assume that $R = 1 - \dfrac{1}{2\,\mu} - \dfrac{\alpha\,\Delta t}{2\,h} - \dfrac{2\,\varepsilon\,\Delta t}{h^2}$ is positive. Returning to (4.4), we obtain:

(4.12)

$$
\frac{1}{2\,\Delta t}\,|U^{m+1}|_H^2 - \frac{1}{2\,\Delta t}\,|U^m|_H^2 + \frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^m - v_j^m)^2
$$

$$
+ \frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^m - \xi_{j-1}^m)^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^m - \eta_j^m)^2 + \frac{\alpha}{2}\,|\xi_{N-1}^m|^2 + \frac{\beta}{2}\,|\eta_1^m|^2
$$

$$
\leq |F|_\infty\,|U^m|_H + \frac{\varepsilon}{2\,h\,R}\sum_{j=0}^{N-1}(v_{j+1}^m - v_j^m)^2
$$

$$
+ \frac{\alpha}{4\,R}\sum_{j=1}^{N-1}(\xi_j^m - \xi_{j-1}^m)^2 + \frac{\beta}{4\,R}\sum_{j=1}^{N-1}(\eta_{j+1}^m - \eta_j^m)^2
$$

We obtain the stability result if $R > 1/2$, that is[3]:

(4.13)
$$
0 < \frac{\alpha\,\Delta t}{h} + \frac{4\,\Delta t\,\varepsilon}{h^2} < 1
$$

Remark 4.1. *:* We note that the condition (4.13) matches the classical CFL condition if ε equals to zero.

5. THE CRANK NICHOLSON SCHEME

5.1. **Discretization of the equations and boundary conditions.**

The discretization in time of equations (2.10)-(2.11) using the C-N scheme reads:

(5.1)
$$
\begin{cases}
\dfrac{\xi_j^{m+1} - \xi_j^m}{\Delta t} + \alpha\,\dfrac{\xi_j^{m+1/2} - \xi_{j-1}^{m+1/2}}{h} \\[2mm]
\qquad\qquad - \varepsilon\,\dfrac{v_{j+1}^{m+1/2} - 2\,v_j^{m+1/2} + v_{j-1}^{m+1/2}}{h^2} = f_j^{m+1/2}, \\[4mm]
\dfrac{\eta_j^{m+1} - \eta_j^m}{\Delta t} - \beta\,\dfrac{\eta_{j+1}^{m+1/2} - \eta_j^{m+1/2}}{h} \\[2mm]
\qquad\qquad - \varepsilon\,\dfrac{v_{j+1}^{m+1/2} - 2\,v_j^{m+1/2} + v_{j-1}^{m+1/2}}{h^2} = g_j^{m+1/2},
\end{cases}
$$

where $u^{m+1/2}$ naturally denotes the quantity $(u^{m+1} + u^m)/2$.

---

[3]If the condition (4.13) is satisfied, one can easily find $\mu > 0$ such that $R > 1/2$.

The boundary conditions read:

(5.2)
$$\begin{cases} \dfrac{\xi_N^{m+1} - \xi_N^m}{\Delta t} + \alpha \dfrac{\xi_N^{m+1/2} - \xi_{N-1}^{m+1/2}}{h} \\ \qquad\qquad - \dfrac{\alpha}{h}\eta_{N-1}^{m+1/2} = f_N^{m+1/2} - \dfrac{\alpha}{\beta} g_N^{m+1/2}, \\ \dfrac{\eta_0^{m+1} - \eta_0^m}{\Delta t} - \beta \dfrac{\eta_1^{m+1/2} - \xi_0^{m+1/2}}{h} \\ \qquad\qquad + \dfrac{\beta}{h}\xi_1^{m+1/2} = g_0^{m+1/2} - \dfrac{\beta}{\alpha} f_0^{m+1/2}. \end{cases}$$

5.2. Proof of stability.

We claim that the stability holds for every set of parameters $(\Delta t, h, \varepsilon)$ (unconditional stability). This time we multiply $(5.1)_1$ by $h\,\xi_j^{m+1/2}$ for $1 \leq j \leq N-1$, $(5.1)_2$ by $h\,\eta_j^{m+1/2}$ for $1 \leq j \leq N-1$, $(5.2)_1$ by $\varepsilon\,\xi_N^{m+1/2}/\alpha$, and $(5.2)_2$ by $\varepsilon\,\eta_0^{m+1/2}/\beta$. We sum all these equations, and obtain:

(5.3)
$$\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^{m+1}|^2 + \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^{m+1}|^2$$
$$+\frac{\varepsilon}{\alpha}\frac{1}{2\,\Delta t}|\xi_N^{m+1}|^2 + \frac{\varepsilon}{\beta}\frac{1}{2\,\Delta t}|\eta_0^{m+1}|^2$$
$$-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\xi_j^m|^2 - \frac{1}{2\,\Delta t}\sum_{j=1}^{N-1} h\,|\eta_j^m|^2 - \frac{\varepsilon}{\alpha}\frac{1}{2\,\Delta t}|\xi_N^m|^2 - \frac{\varepsilon}{\beta}\frac{1}{2\,\Delta t}|\eta_0^m|^2$$
$$+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m+1/2} - \xi_{j-1}^{m+1/2})^2 + \frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m+1/2} - \eta_j^{m+1/2})^2$$
$$+\frac{\alpha}{2}|\xi_{N-1}^{m+1/2}|^2 + \frac{\beta}{2}|\eta_1^{m+1/2}|^2$$
$$-\frac{\varepsilon}{h^2}\sum_{j=1}^{N-1}[v_{j+1}^{m+1/2} - 2v_j^{m+1/2} + v_{j-1}^{m+1/2}]v_j^{m+1/2}$$
$$+\frac{\varepsilon}{h}(\xi_N^{m+1/2} - \xi_{N-1}^{m+1/2} - \eta_{N-1}^{m+1/2})\,\xi_N^{m+1/2}$$
$$-\frac{\varepsilon}{h}(\eta_1^{m+1/2} - \eta_0^{m+1/2} + \xi_1^{m+1/2})\,\eta_0^{m+1/2}$$

$$= (F, U^{m+1/2})_H$$

Again, we use (3.4) and obtain, for the Crank Nicholson scheme:

(5.4)

$$\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\xi_j^{m+1}|^2+\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\eta_j^{m+1}|^2$$

$$+\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^{m+1}|^2+\frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^{m+1}|^2$$

$$-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\xi_j^m|^2-\frac{1}{2\,\Delta t}\sum_{j=1}^{N-1}h\,|\eta_j^m|^2-\frac{\varepsilon}{\alpha}\,\frac{1}{2\,\Delta t}\,|\xi_N^m|^2-\frac{\varepsilon}{\beta}\,\frac{1}{2\,\Delta t}\,|\eta_0^m|^2$$

$$+\frac{\alpha}{2}\sum_{j=1}^{N-1}(\xi_j^{m+1/2}-\xi_{j-1}^{m+1/2})^2+\frac{\beta}{2}\sum_{j=1}^{N-1}(\eta_{j+1}^{m+1/2}-\eta_j^{m+1/2})^2$$

$$+\frac{\alpha}{2}\,|\xi_{N-1}^{m+1/2}|^2+\frac{\beta}{2}\,|\eta_1^{m+1/2}|^2+\frac{\varepsilon}{h}\sum_{j=0}^{N-1}(v_{j+1}^m-v_j^m)^2$$

$$=(F,U^{m+1/2})_H$$

Hence, we find:

(5.5) $$\qquad\frac{1}{\Delta t}\,|U^{m+1}|_H^2\leq\frac{1}{\Delta t}\,|U^m|_H^2+|F|_H\,|U^{m+1}+U^m|_H.$$

After some computations, we obtain:

(5.6) $$\qquad|U^{m+1}|_H^2\leq\frac{2+\Delta t}{2-\Delta t}\,|U^m|_H^2+\frac{2\,\Delta t}{2-\Delta t}\,|F|_\infty^2$$

Similarly to Section 4, we write these inequalities for every m, and finally obtain:

(5.7)
$$\begin{aligned}|U^{m+1}|_H^2&\leq&\left(\frac{2+\Delta t}{2-\Delta t}\right)^{m+1}|U^0|_H^2+\left(\frac{2+\Delta t}{2-\Delta t}\right)^m|F|_\infty^2\\&\leq&e^{2T}\,(|U^0|_H^2+|F|_\infty^2),\end{aligned}$$

which guarantees the finite time stability of the scheme with no additional condition.

6. FRACTIONAL SCHEME

6.1. **Discretization of the equations and boundary conditions.**

In this section we use the fractional scheme method, (see [Mar71] and, for the Navier-Stokes equations, [Tem69]), which consists in splitting each time step into several (here two) intermediate steps. The advantage is that the numerical computations for each intermediate step are easier, while the stability result does not require any condition on the parameters $(\Delta t, h, \varepsilon)$.

Let us now describe the two intermediate steps, and give the semi-discretized (in time) schemes. We consider the previous system (2.9) with the subscripts n dropped, and reintroduce the parameter ε:

(6.1)
$$\begin{cases} \dfrac{\partial u}{\partial t} + \overline{U}_0 \dfrac{\partial u}{\partial x} - \dfrac{1}{N\lambda}\dfrac{\partial \psi}{\partial x} - 2\varepsilon \dfrac{\partial^2 u}{\partial x^2} = F_u, \\[2mm] \dfrac{\partial \psi}{\partial t} + \overline{U}_0 \dfrac{\partial \psi}{\partial x} - \dfrac{N}{\lambda}\dfrac{\partial u}{\partial x} = F_\psi. \end{cases}$$

Since the second space derivative does not occur in the second equation (6.1), we choose the two intermediate steps as follows. The first intermediate step $m + 1/2$ (between m and $m + 1$) reads:

(6.2)
$$\begin{cases} \dfrac{u^{m+1/2} - u^m}{\Delta t} + \overline{U}_0 \, u_x^{m+1/2} - \dfrac{1}{N\lambda}\psi_x^{m+1/2} = F_u^m, \\[2mm] \dfrac{\psi^{m+1/2} - \psi^m}{\Delta t} + \overline{U}_0 \, \psi_x^{m+1/2} - \dfrac{N}{\lambda}u_x^{m+1/2} = F_\psi^m. \end{cases}$$

For the second intermediate step, we set:

(6.3)
$$\begin{cases} \dfrac{u^{m+1} - u^{m+1/2}}{\Delta t} - 2\,\varepsilon\, u_{xx}^{m+1} = 0, \\[2mm] \dfrac{\psi^{m+1} - \psi^{m+1/2}}{\Delta t} = 0. \end{cases}$$

Let us now go back to the notation (ξ, η), and rewrite (6.2) and (6.3), discretized in space. For every $1 \le j \le N - 1$:

(6.4a)
$$\frac{\xi_j^{m+1/2} - \xi_j^m}{\Delta t} + \alpha \frac{\xi_j^{m+1/2} - \xi_{j-1}^{m+1/2}}{h} = f_j^m,$$

(6.4b)
$$\frac{\eta_j^{m+1/2} - \eta_j^m}{\Delta t} - \beta \frac{\eta_{j+1}^{m+1/2} - \eta_j^{m+1/2}}{h} = g_j^m,$$

We supplement equations (6.4) with the following natural boundary conditions:

(6.5a)
$$\xi_0^{m+1/2} = 0,$$

(6.5b)
$$\eta_N^{m+1/2} = 0,$$

(6.5c)
$$\frac{\xi_N^{m+1/2} - \xi_N^m}{\Delta t} = f_N^m - \frac{\alpha}{\beta}g_N^m,$$

(6.5d)
$$\frac{\eta_0^{m+1/2} - \eta_0^m}{\Delta t} = g_0^m - \frac{\beta}{\alpha}f_0^m.$$

Given (ξ_j^m, η_j^m) with $0 \le j \le N$, one can easily compute from (6.4) and (6.5) the intermediate solution $(\xi_j^{m+1/2}, \eta_j^{m+1/2})$ with $0 \le j \le N$. For the second intermediate step $m + 1$, we have for $1 \le j \le N - 1$:

(6.6a)
$$\frac{\xi_j^{m+1} - \xi_j^{m+1/2}}{\Delta t} - \varepsilon \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{h^2} = 0,$$

(6.6b)
$$\frac{\eta_j^{m+1} - \eta_j^{m+1/2}}{\Delta t} - \varepsilon \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{h^2} = 0,$$

We supplement equations (6.6) with the following boundary conditions:

(6.7a) $\xi_0^{m+1} = 0,$

(6.7b) $\eta_N^{m+1} = 0,$

(6.7c)
$$\frac{\xi_N^{m+1} - \xi_N^{m+1/2}}{\Delta t} + \alpha \frac{\xi_N^{m+1} - \xi_{N-1}^{m+1} - \eta_{N-1}^{m+1}}{h} = 0,$$

(6.7d)
$$\frac{\eta_0^{m+1} - \eta_0^{m+1/2}}{\Delta t} - \beta \frac{\eta_1^{m+1} - \eta_0^{m+1} + \xi_1^{m+1}}{h} = 0.$$

Knowing $(\xi_j^{m+1/2}, \eta_j^{m+1/2})$ with $0 \le j \le N$, we can finally compute $(\xi_j^{m+1}, \eta_j^{m+1})$ with $0 \le j \le N$, using the relations (6.6) and (6.7).

Before going into the proof of stability, let us first observe that this two-steps scheme is consistent. To this aim, we express $(\xi^{m+1/2}, \eta^{m+1/2})$ from (6.6) and substitute it in (6.4). Eventually we recover the usual consistency, with the help of Taylor expansions.

6.2. Proof of stability.

We start by multiplying (6.4a) by $h\,\xi_j^{m+1/2}$ and (6.4b) by $h\,\eta_j^{m+1/2}$ for $1 \le j \le N-1$. We sum and obtain, thanks to (6.5a) and (6.5b):

(6.8)
$$\begin{cases} \dfrac{1}{2\Delta t} \displaystyle\sum_{j=1}^{N-1} h|\xi_j^{m+1/2}|^2 \le \dfrac{1}{2\Delta t} \displaystyle\sum_{j=1}^{N-1} h|\xi_j^m|^2 + \displaystyle\sum_{j=1}^{N-1} h f_j^m \xi_j^{m+1/2}, \\[2ex] \dfrac{1}{2\Delta t} \displaystyle\sum_{j=1}^{N-1} h|\eta_j^{m+1/2}|^2 \le \dfrac{1}{2\Delta t} \displaystyle\sum_{j=1}^{N-1} h|\eta_j^m|^2 + \displaystyle\sum_{j=1}^{N-1} h g_j^m \eta_j^{m+1/2}. \end{cases}$$

In order to recover the scalar product of H, we also multiply (6.5c) by $\varepsilon\xi_N^{m+1/2}/\alpha$, (6.5d) by $\varepsilon\eta_0^{m+1/2}/\beta$ and obtain:

(6.9)
$$\begin{cases} \dfrac{1}{2\Delta t}\dfrac{\varepsilon}{\alpha}|\xi_N^{m+1/2}|^2 \le \dfrac{1}{2\Delta t}\dfrac{\varepsilon}{\alpha}|\xi_N^m|^2 + \dfrac{\varepsilon}{\alpha}\xi_N^{m+1/2}(f_N^m - \dfrac{\alpha}{\beta}g_N^m), \\[2ex] \dfrac{1}{2\Delta t}\dfrac{\varepsilon}{\beta}|\eta_0^{m+1/2}|^2 \le \dfrac{1}{2\Delta t}\dfrac{\varepsilon}{\beta}|\eta_0^m|^2 + \dfrac{\varepsilon}{\beta}\eta_0^{m+1/2}(g_0^m - \dfrac{\beta}{\alpha}f_0^m). \end{cases}$$

We now use (6.8) together with (6.9), and find:

$$(6.10) \qquad \frac{1}{2\Delta t}|U^{m+1/2}|_H^2 \leq \frac{1}{2\Delta t}|U^m|_H^2 + |F|_\infty \, |U^{m+1/2}|_H.$$

For the second intermediate step, we multiply (6.6a) by $h\,\xi_j^{m+1}$ and (6.4b) by $h\,\eta_j^{m+1/2}$, for $1 \leq j \leq N-1$. We also multiply (6.7c) by $\varepsilon\,\xi_N^{m+1}/\alpha$, (6.7d) by $\varepsilon\,\eta_0^{m+1}/\beta$ and we add the resulting equations. Using (3.4) again, we finally have:

$$(6.11) \qquad |U^{m+1}|_H^2 \leq |U^{m+1/2}|_H^2.$$

From inequalities (6.10) and (6.11), we easily obtain:

$$(6.12) \qquad |U^{m+1}|_H^2 \leq \frac{\Delta t}{1-\Delta t}|F|_\infty^2 + \frac{1}{1-\Delta t}\,|U^m|_H^2,$$

and we are thus back to (3.8) of Section 3, which guarantees the stability of the scheme, with no condition on $(\Delta t, h, \varepsilon)$.

To conclude this section, we emphasize the fact that from the numerical point of view, the above fractional scheme method is more convenient; the first part (6.4)-(6.5) is quite easy to implement while the second part (6.6)-(6.7) can be written as follows, with $v = \xi + \eta$ $(= 2u)$ and $w = \xi - \eta$ $(= -2\psi/N)$:

$$(6.13) \qquad \frac{v_j^{m+1} - v_j^{m+1/2}}{\Delta t} - 2\varepsilon \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2} = 0,$$

with the boundary conditions:

$$(6.14) \qquad \begin{cases} \dfrac{v_N^{m+1} - v_N^{m+1/2}}{\Delta t} + \alpha \dfrac{v_N^{m+1} - v_{N-1}^{m+1}}{h} = 0, \\[2ex] \dfrac{v_0^{m+1} - v_0^{m+1/2}}{\Delta t} + \beta \dfrac{v_1^{m+1} - v_0^{m+1}}{h} = 0. \end{cases}$$

These equations on v are decoupled from those on w which are:

$$(6.15) \qquad \frac{w_j^{m+1} - w_j^{m+1/2}}{\Delta t} = 0, \ \ 1 \leq j \leq N-1.$$

The advantage for these equations in w is that they do not depend on the space discretization, so that there is no linear system to solve. Also, although we did not perform error analyses in this article, we conjecture that, alternating the steps $(m + 1/2, \ m + 1)$, using the classical procedure of Strang [Str68], we would obtain here a scheme of second order in time; these questions will be addressed elewhere.

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