Exponential decay of the power spectrum and finite dimensionality for solutions of the three dimensional primitive equations

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Abstract In this article we estimate the number of modes, volumes and nodes, sufficient to describe well the solution of the three dimensional primitive equations; the physical meaning of these estimates is also discussed. We also study the exponential decay of the spatial power spectrum for the three dimensional primitive equations.

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1 Introduction

The primitive equations in their dimensional form read:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial x_2} + w \frac{\partial u}{\partial x_3} - f v + \frac{1}{\rho_0} \frac{\partial p}{\partial x_1} = \nu \Delta u + F_u, \tag{1.1a}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial x_2} + w \frac{\partial v}{\partial x_3} + f u + \frac{1}{\rho_0} \frac{\partial p}{\partial x_2} = \nu \Delta v + F_v, \tag{1.1b}
\]

\[
\frac{\partial p}{\partial x_3} = -\rho g, \tag{1.1c}
\]

\[
\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0, \tag{1.1d}
\]
\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x_1} + v \frac{\partial T}{\partial x_2} + w \frac{\partial T}{\partial x_3} = \mu \Delta T + F_T. \tag{1.1e}
\]

In our model, \((u, v, w)\) are the three components of the velocity vector and \(p, \rho \) and \(T\) are respectively the perturbations of the pressure, of the density and of the temperature from the reference (average) constant states \(p_0, \rho_0\), and \(T_0\). The relation between the full temperature and the full density is given by the equation of state, and we consider here a version of this equation, linearized around the reference state \(\rho_0\) and \(T_0\),

\[
\rho_{\text{full}} = \rho_0 (1 - \beta_T (T_{\text{full}} - T_0)). \tag{1.2}
\]

For the perturbations \(\rho\) and \(T\), the equation of state reads:

\[
\rho = -\beta_T \rho_0 T. \tag{1.3}
\]

The constant \(g\) is the gravitational acceleration and \(f\) the Coriolis parameter, \(v\) and \(\mu\) are the eddy diffusivity coefficients, \((F_u, F_v)\) represent body forces per unit of mass and \(F_T\) represents a heating source. In applications \(F_u, F_v\) and \(F_T\) vanish for the ocean, but we consider here nonzero forces for mathematical generality. We denote by \(F\) the forcing vector \((F_u, F_v, F_T)\). For more details regarding the derivation of these equations as well as on the available results on this model, we refer the interested reader to [10].

We work in a bounded domain:

\[
\mathcal{M} = (0, L) \times (0, L) \times (-L/2, -L/2), \tag{1.4}
\]

and we assume space periodicity with period \(\mathcal{M}\), meaning that all functions are taken to satisfy:

\[
f (x_1, x_2, x_3, t) = f (x_1 + L, x_2, x_3, t) = f (x_1, x_2 + L, x_3, t) = f (x_1, x_2, x_3 + L, t), \tag{1.5}
\]

when extended to \(\mathbb{R}^3\).

We work with periodic functions, so all the functions admit a Fourier series expansion:

\[
f (x_1, x_2, x_3, t) = \sum_{k \in \mathbb{R}^3} f_k (t) e^{i (k'_1 x_1 + k'_2 x_2 + k'_3 x_3)}, \tag{1.6}
\]

where, for notational conciseness, we set \(k'_j = 2\pi k_j / L\) for \(j = 1, 2, 3\).
We assume that the functions have the following symmetries:

\[ u(x_1, x_2, x_3, t) = u(x_1, x_2, -x_3, t), \quad F_u(x_1, x_2, x_3, t) = F_u(x_1, x_2, x_3, t), \]
\[ v(x_1, x_2, x_3, t) = v(x_1, x_2, -x_3, t), \quad F_v(x_1, x_2, x_3, t) = F_v(x_1, x_2, -x_3, t), \]
\[ T(x_1, x_2, x_3, t) = -T(x_1, x_2, -x_3, t), \quad F_T(x_1, x_2, x_3, t) = -F_T(x_1, x_2, -x_3, t), \]
\[ w(x_1, x_2, x_3, t) = -w(x_1, x_2, -x_3, t), \quad p(x_1, x_2, x_3, t) = p(x_1, x_2, -x_3, t); \]

and we say that \( u, v, p \) are even and \( w, T \) odd in \( x_3 \). As explained in [9] for example, these symmetry properties are necessary for the space periodicity to be consistent with (1.1).

The variational formulation of the problem

We start by introducing the natural function spaces for this problem:

\[ V = \{ U = (u, v, T) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3 \mid u, v \text{ even in } x_3, \ T \text{ odd in } x_3, \] (1.7)
\[
\int_{-L/2}^{L/2} (u_{x_1}(x_1, x_2, x_3') + v_{x_2}(x_1, x_2, x_3')) \, dx_3' = 0, \]
\[ H = \text{closure of } V \text{ in } (L^2(\mathcal{M}))^3, \]
\[ V_2 = \text{the closure of } V \cap (\dot{H}^2_{\text{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\text{per}}(\mathcal{M}))^3. \] (1.8)

As in [11], we endow these spaces with the following scalar products: on \( H \) we consider:

\[ (U, \tilde{U})_H = (u, \tilde{u})_{L^2} + (v, \tilde{v})_{L^2} + \kappa(T, \tilde{T})_{L^2}, \] (1.9)

and on \( V \):

\[ ((U, \tilde{U}))_V = ((u, \tilde{u})) + ((v, \tilde{v})) + \kappa((T, \tilde{T})). \] (1.10)

Here the dots above \( \dot{H}^1_{\text{per}} \) and \( L^2 \) denote the functions with zero average over \( \mathcal{M} \) and \( \kappa \) is a positive constant that will be chosen later. Since we work with functions with zero average over \( \mathcal{M} \), we can use the generalized Poincaré inequality:

\[ c' |U|_H \leq ||U||_V, \forall U \in V, \] (1.11)

so we know that the norm \( || \cdot ||_V \) defined above is equivalent to the usual \( H^1 \)-norm, where \( c' \) is a constant related to the Poincaré constant (more exactly \( c' = 2\pi/L \)).

We briefly recall that the unknown functions are of two types: the prognostic variables \( u, v \) and \( T \), for which the initial values are prescribed, and the diagnostic variables \( \rho, w, p \) which can be defined, at each instant of time, as functions of the prognostic variables. More details regarding the way the diagnostic variables can be determined, are available in [11]. In order to simplify the notations, we also write \( \mathbf{u} = (u, v) \) for the horizontal velocity.
The variational formulation of this problem is obtained classically by considering a test function $U^b = (u^b, v^b, T^b)$ in $V$, multiplying (1.1a) by $u^b$, (1.1b) by $v^b$, (1.1e) by $\kappa T^b$, adding and integrating. We find the following problem:

**Find** $U : [0, t_0] \to V$, such that,

$$
\frac{d}{dt}(U, U^b)_H + a(U, U^b) + b(U, U, U^b) + e(U, U^b) = (F, U^b)_H, \quad \forall U^b \in V, \quad (1.12)
$$

$U(0) = U_0$.

In (1.12) we introduced the following forms:

$a : V \times V \to \mathbb{R}$ bilinear, continuous:

$$
a(U, U^b) = v((u, u^b)) + v((v, v^b)) + \kappa \mu((T, T^b)), \quad (1.13)
$$

$b : V \times V \times V^2 \to \mathbb{R}$ trilinear:

$$
b(U, U^z, U^b) = \int_M \left( u \frac{\partial u^z}{\partial x_1} u^b + v \frac{\partial u^z}{\partial x_2} u^b + w(U) \frac{\partial u^z}{\partial x_3} u^b \right) \, dM
+ \int_M \left( u \frac{\partial v^z}{\partial x_1} v^b + v \frac{\partial v^z}{\partial x_2} v^b + w(U) \frac{\partial v^z}{\partial x_3} v^b \right) \, dM
+ \kappa \int_M \left( u \frac{\partial T^z}{\partial x_1} T^b + v \frac{\partial T^z}{\partial x_2} T^b + w(U) \frac{\partial T^z}{\partial x_3} T^b \right) \, dM, \quad (1.14)
$$

$e : V \times V \to \mathbb{R}$ bilinear, continuous:

$$
e(U, U^b) = f \int_M (uv^b - vu^b) \, dM - g \beta_T \int_M Tw(U^b) \, dM. \quad (1.15)
$$

As shown in [11], we choose a $\kappa > 0$ sufficiently large such that $a + e$ is coercive on $V$, more exactly we choose $\kappa$ such that:

$$
a(U, U) + e(U, U) \geq \frac{1}{2} v|u|^2 + \frac{1}{2} v|v|^2 + \frac{1}{2} \mu \kappa |T|^2. \quad (1.16)
$$

Regarding the properties of $b$, in [11] we proved the following property:

**Lemma 1** The form $b$ is trilinear continuous from $V \times V \times V$ into $\mathbb{R}$ and from $V \times V \times V^2$ into $\mathbb{R}$, and

$$
|b(U, U^z, U^b)| \leq c_2 \|U\|_H^{1/2} \|U^z\|^{1/2} \|U^b\|_2, \quad \forall U, U^z \in V, \ U^b \in V_2, \quad (1.17)
$$

$$
|b(U, U^z, U^b)| \leq c_3 \|U\|^{1/2} \|U^z\|^{1/2} \|U^b\|_H^{1/2} \|U^b\|^{1/2}, \quad |b(U, U^z, U^b)| \leq c_4 \|U\|_H^{1/2} \|U^z\|^{1/2} \|U^b\|_2^{1/2} \|U^b\|^{1/2}. \quad (1.18)
$$
We also have:

\[ |b(U, U^z, U^b)| \leq c\|U\||\nabla U^z|\infty|U^b|_H. \]  

(1.19)

Furthermore,

\[
\begin{align*}
 b(U, U^b, U^b) &= 0 \quad \forall U \in V, U^b \in V_2, \\
 b(U, U^b, U^z) &= -b(U, U^z, U^b), \quad \forall U, U^b, U^z \in V \text{ with } U^b \text{ or } U^z \in V_2.
\end{align*}
\]  

(1.20)

Problem (1.12) can also be written as an operator evolution equation in \( V_2' \):

\[
\frac{dU}{dt} + AU + B(U, U) + EU = F, \\
U(0) = U_0,
\]  

(1.21)

where we introduced the following operators:

- A linear continuous from \( V \) into \( V' \), defined by
  \[ \langle AU, U^b \rangle = a(U, U^b), \quad \forall U, U^b \in V. \]

- B bilinear, continuous from \( V \times V \) into \( V_2' \), defined by
  \[ \langle B(U, U^b), U^z \rangle = b(U, U^b, U^z), \quad \forall U, U^b, U^z \in V, \forall U^z \in V_2. \]

- E linear continuous from \( V \) into \( V' \), defined by
  \[ \langle EU, U^b \rangle = e(U, U^b), \quad \forall U, U^b \in V. \]

2 Finite dimensionality of the turbulent flows described by the primitive equations

In this section we are interested in studying the local and global finite dimensionality of the flows described by the primitive equations. We are interested in determining how many points (called determining nodes) or sub-cubes (called determining volumes) in the configuration space are necessary in order to follow the evolution of the flow and to obtain a good representation. Another interesting question is to determine how many low modes (called determining modes) are necessary to determine the behavior of the remaining modes.

We start by studying the existence of the determining nodes. A set of points in the configuration space is called a set of determining nodes if whenever the difference between the measurements at those points of the prognostic variables of any two flows goes to zero as time goes to infinity, then the difference between the prognostic variables goes to zero uniformly on the whole configuration space.

The question of the existence of a finite number of determining nodes for the case of flows described by the Navier–Stokes equations was first treated by C. Foias and R. Temam in [5]. More recent refinements of these results can be also found in [2,7] and [4].
We are interested in studying the existence of the determining nodes for the flows described by the primitive equations and in checking if the result coincides with the case of Navier–Stokes equations. For that we start by considering the forcing terms $F$ and $G$ and we suppose that in a certain norm $F$ and $G$ have the same asymptotic behavior, more exactly that either

$$\lim_{t \to \infty} \int_M |F(x, t) - G(x, t)| \, dM \to 0, \quad (2.1)$$

or

$$\lim_{t \to \infty} \int_M \|F(x, t) - G(x, t)\| \, dM \to 0. \quad (2.2)$$

We consider a set of $N$ points, uniformly distributed into the domain, denoted by

$$\mathcal{E} = \{x^1, x^2, \ldots, x^N\}. \quad (2.3)$$

We say that $\mathcal{E}$ is a set of determining nodes if for $U_1$ and $U_2$ that are respectively solutions for the equation (1.12) with the forcing $F$ and $G$ having the same asymptotic behavior (expressed by (2.1) or (2.2)), such that the solutions $U_1$ and $U_2$ have the same time-asymptotic behavior at the points of $\mathcal{E}$, then we have

$$\lim_{t \to \infty} \int_M |U_1(x, t) - U_2(x, t)| \, dM \to 0. \quad (2.4)$$

The assumption that the solutions $U_1$ and $U_2$ have the same time-asymptotic behavior at the points of $\mathcal{E}$ can be expressed by the condition:

$$\max_{j=1, 2, \ldots, N} |U_1(x^j, t) - U_2(x^j, t)| \to 0 \quad \text{as } t \to \infty. \quad (2.5)$$

We will actually prove a stronger result than (2.4), that is:

$$\lim_{t \to \infty} \int_M |(-\Delta)U_1(x, t) - (-\Delta)U_2(x, t)| \, dM \to 0. \quad (2.6)$$

The proof of this result necessitates a technical result that we state and prove here:

**Lemma 2** Let the domain $\mathcal{M}$ be covered by $N$ identical cubes of the type $Q = (0, l) \times (0, l) \times (0, l)$, with $l > 0$ and consider the set $\mathcal{E} = \{x^1, \ldots, x^N\}$ of points in $\mathcal{M}$, distributed one in each cube. Then, for each vector field $f \in D((-\Delta)^{3/2})$ the following inequality holds:

$$|\Delta f|^2_{L^2(\mathcal{M})} \leq c_1 l^{-1} \eta(f)^2 + c_2 l^2 |(-\Delta)^{3/2} f|^2_{L^2(\mathcal{M})},$$
where we used the following notation: $\eta(f) = \max_{1,\ldots,N} |f(x^j)|$.

**Proof** Let us fix a point $x^0$ in $Q$ and let $f$ be a smooth function defined on $Q$. For any $x$ in $Q$ we have:

$$f(x) - f(x^0) = \int_{x_1^0}^{x_1} \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \, d\theta + \int_{x_2^0}^{x_2} \frac{\partial f}{\partial x_2} (x_1, \theta, x_3^0) \, d\theta + \int_{x_3^0}^{x_3} \frac{\partial f}{\partial x_3} (x_1, x_2, \theta) \, d\theta,$$

which implies:

$$|f(x) - f(x^0)|^2 \leq 2l^4 \int_0^l \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \right|^2 \, d\theta + 2l^4 \int_0^l \left| \frac{\partial f}{\partial x_2} (x_1, \theta, x_3^0) \right|^2 \, d\theta + 2l^2 \int_{\mathcal{M}} \left| \frac{\partial f}{\partial x_3} (x_1, x_2, \theta) \right|^2 \, d\mathcal{M}.$$  

(2.7)

Integrating over $Q$ in $x_1, x_2, x_3$, we find:

$$\int_{\mathcal{M}} \left| f(x) - f(x^0) \right|^2 \, d\mathcal{M} \leq 2l^4 \int_0^l \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \right|^2 \, d\theta + 2l^4 \int_0^l \left| \frac{\partial f}{\partial x_2} (x_1, \theta, x_3^0) \right|^2 \, d\theta + 2l^2 \int_{\mathcal{M}} \left| \frac{\partial f}{\partial x_3} (x_1, x_2, \theta) \right|^2 \, d\mathcal{M}.$$  

(2.8)

We need to estimate the first two terms from the right hand side of (2.8). For the second term we write:

$$\left| \frac{\partial f}{\partial x_2} (x_1, \theta, x_3^0) \right|^2 = \left| \frac{\partial f}{\partial x_2} (x_1, \theta, \psi) \right|^2 + 2 \int_\psi^{x_3^0} \frac{\partial f}{\partial x_2} (x_1, \theta, \eta) \frac{\partial^2 f}{\partial x_2 \partial x_3} (x_1, \theta, \eta) \, d\eta \leq \left| \frac{\partial f}{\partial x_2} (x_1, \theta, \psi) \right|^2 + 2 \int_0^l \left| \frac{\partial f}{\partial x_2} (x_1, \theta, \eta) \right| \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} (x_1, \theta, \eta) \right| \, d\eta.$$  

(2.9)
Integrating (2.10) in \( x_1, \theta, \psi \) over \( Q \) and using the Cauchy Schwarz inequality, we find:

\[
l \int_0^l \int_0^l \left| \frac{\partial f}{\partial x_2} (x_1, \theta, x_3^0) \right|^2 \, d\theta \, dx_1 \leq |f_{x_2}|^2_{L^2(Q)} + 2l \int_Q \left| \frac{\partial f}{\partial x_2} \right| \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} \right| \, dQ \leq 2|f_{x_2}|^2_{L^2(Q)} + l^2 \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} \right|^2_{L^2(Q)}. \quad (2.10)
\]

For the first term of the right hand side of (2.8), we write:

\[
\left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \right|^2 = \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, \psi) \right|^2 + 2 \int_0^l \frac{\partial f}{\partial x_1} (\theta, \eta, x_3^0) \frac{\partial^2 f}{\partial x_2 \partial x_1} (\theta, \eta, x_3^0) \, d\eta \leq \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, \psi) \right|^2 + 2 \int_0^l \left| \frac{\partial f}{\partial x_1} (\theta, \eta, x_3^0) \right| \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} (\theta, \eta, x_3^0) \right| \, d\eta. \quad (2.11)
\]

Integrating (2.11) in \( \theta \) and \( \psi \) over \((0, l) \times (0, l)\) and using the Cauchy Schwarz inequality as before, we find:

\[
l \int_0^l \int_0^l \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \right|^2 \, d\theta \leq 2l \int_0^l \int_0^l \left| \frac{\partial f}{\partial x_1} (\theta, x_2^0, x_3^0) \right|^2 \, d\theta \, d\psi + l^2 \int_0^l \int_0^l \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} (\theta, \eta, x_3^0) \right|^2 \, d\theta \, d\eta. \quad (2.12)
\]

Gathering all these estimates in (2.8), we find:

\[
\int_Q |f|^2 \, dQ \leq l^3 \left| f (x_1^0) \right|^2 + 2l^2 |f_{x_3}|^2_{L^2(Q)} + 4l^2 |f_{x_2}|^2_{L^2(Q)} + 8l^2 |f_{x_1}|^2_{L^2(Q)} + 2l^4 |f_{x_2x_3}|^2_{L^2(Q)} + 4l^4 |f_{x_1x_3}|^2_{L^2(Q)} + 2l^5 \int_0^l \int_0^l \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} (\theta, \eta, x_3^0) \right|^2 \, d\theta \, d\eta. \quad (2.13)
\]
It only remains to estimate the last term in (2.13). Using the same reasoning as before, we find:

\[
\int_0^l \int_0^l \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} \left( \theta, \eta, x^0 \right) \right| \, d\theta d\eta \leq 2 \left| \frac{\partial^2 f}{\partial x_2 \partial x_3} \right|^2_{L^2(Q)} + l^2 \left| \frac{\partial^3 f}{\partial x_2 \partial^2 x_3} \right|^2_{L^2(Q)}.
\]

(2.14)

We can thus conclude that:

\[
|f|^2_{L^2(Q)} \leq l^3 |f(x^0)|^2 + 8l^2 \|f\|^2 + 4c_1 l^4 |\Delta f|^2_{L^2(Q)} + 2c_2 l^6 (-\Delta)^{3/2} f|^2_{L^2(Q)},
\]

(2.15)

where \(c_1, c_2\) are respectively constants resulting from the norm equivalences \(|\cdot|_{H^2} \sim |\Delta \cdot|_{L^2}\) and \(|\cdot|_{H^3} \sim |(-\Delta)^{3/2} \cdot|_{L^2}\).

Taking \(x^0 = x^i, x^i \in E\) in (2.15), and adding over all the cubes \(Q_i\), we find:

\[
|f|^2_{L^2(M_0)} \leq l^3 \eta(f)^2 + 8l^2 \|f\|^2 + 4c_1 l^4 |\Delta f|^2_{L^2(M_0)} + 2c_2 l^6 (-\Delta)^{3/2} f|^2_{L^2(M_0)},
\]

(2.16)

Using the interpolation relations:

\[
|(-\Delta)^{1/2} f|^2 \leq c |f||\Delta f|, \quad |\Delta f|^2 \leq c |(-\Delta)^{1/2} f| |(-\Delta)^{3/2} f|,
\]

and by the Young inequality, we find:

\[
|\Delta f|^2_{L^2(M_0)} \leq c_1 l^{-1} \eta(f)^2 + c_2 l^2 |(-\Delta)^{3/2} f|^2_{L^2(M_0)}.
\]

(2.17)

The proofs of the required results are also based on a generalization of the classical Gronwall lemma. We state here the result and we refer the interested reader to [4] for a complete proof.

**Lemma 3** Let \(\alpha = \alpha(t)\) and \(\beta = \beta(t)\) be locally integrable real-valued functions on \([0, \infty)\) satisfying, for some \(T > 0\) the following conditions:

\[
\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) \, d\tau > 0,
\]

\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha^{-}(\tau) \, d\tau < \infty,
\]

\[
\lim_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^{+}(\tau) \, d\tau = 0,
\]

where \(\alpha^{-}(t) = \max(-\alpha(t), 0)\) and \(\beta^{+}(t) = \max(\beta(t), 0)\). Suppose that \(\xi = \xi(t)\) is an absolutely continuous non-negative function on \([0, \infty)\) which satisfies the following
inequality almost everywhere on \([0, \infty)\):

\[
\frac{d\xi}{dt} + \alpha \xi \leq \beta. \tag{2.18}
\]

Then, \(\xi(t) \to 0\) as \(t \to \infty\).

In order to obtain the required result, we write \(U = U_1 - U_2\) and we take the difference between equation (1.12) for \(U_1\) and the corresponding one for \(U_2\). Taking the test function \(U^\flat = U_1 - U_2\) we obtain:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} |\Delta U|^2_H &+ \frac{\nu}{2} |(-\Delta)^{3/2} u|^2 + \frac{\kappa \mu}{2} |(-\Delta)^{3/2} T|^2 \\
&\leq |(F - G, (-\Delta)^2 U)|_H + |b(U_1, U, (-\Delta)^2 U)|_H \\
&\quad + |b(U, U_1, (-\Delta)^2 U)|_H + |b(U, U, (-\Delta)^2 U)|. \tag{2.19}
\end{align*}
\]

In order to be able to estimate (2.19), we need to know how to bound the terms of the form \(b(U_1, U_2, (-\Delta)^2 U_3)\), meaning that we need to consider integrals of the type:

\[
\begin{align*}
\int_M u_1 \frac{\partial u_2}{\partial x_1} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u_3 \, dM, \\
\int_M v_1 \frac{\partial u_2}{\partial x_2} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u_3 \, dM, \\
\int_M w(U_1) \frac{\partial u_2}{\partial x_3} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u_3 \, dM. \tag{2.20}
\end{align*}
\]

where \(\alpha_i \in \mathbb{N}\) with \([\alpha] = \alpha_1 + \alpha_2 + \alpha_3 = 2\). Integrating by parts and using periodicity, the integrals become:

\[
\begin{align*}
\int_M D^\alpha \left( u_1 \frac{\partial u_2}{\partial x_1} \right) D^\alpha u_3 \, dM, \\
\int_M D^\alpha \left( v_1 \frac{\partial u_2}{\partial x_2} \right) D^\alpha u_3 \, dM, \\
\int_M D^\alpha \left( w(U_1) \frac{\partial u_2}{\partial x_3} \right) D^\alpha u_3 \, dM. \tag{2.21}
\end{align*}
\]

Integrating by parts and using periodicity, the integrals become:

\[
\begin{align*}
\int_M D^\alpha \left( u_1 \frac{\partial u_2}{\partial x_1} \right) D^\alpha u_3 \, dM, \\
\int_M D^\alpha \left( v_1 \frac{\partial u_2}{\partial x_2} \right) D^\alpha u_3 \, dM, \\
\int_M D^\alpha \left( w(U_1) \frac{\partial u_2}{\partial x_3} \right) D^\alpha u_3 \, dM. \tag{2.22}
\end{align*}
\]
Using Leibniz’ formula, we see that the integrals can be written as sums of integrals of the form

\[
\int_M u_1 D^\alpha \frac{\partial u_2}{\partial x_1} D^\alpha u_3 \, dM, \quad \int_M v_1 D^\alpha \frac{\partial u_2}{\partial x_2} D^\alpha u_3 \, dM, \quad \int_M w(U_1) D^\alpha \frac{\partial u_2}{\partial x_3} D^\alpha u_3 \, dM,
\]

(2.23)

and of integrals of the form

\[
\int_M \delta^k u_1 \delta^{2-k} \frac{\partial u_2}{\partial x_1} D^\alpha u_3 \, dM, \quad \int_M \delta^k v_1 \delta^{2-k} \frac{\partial u_2}{\partial x_2} D^\alpha u_3 \, dM,
\]

\[
\int_M \delta^k w(U_1) \delta^{2-k} \frac{\partial u_2}{\partial x_3} D^\alpha u_3 \, dM,
\]

(2.24)

where \( k = 1, 2 \) and \( \delta^k \) is some differential operator \( D^\alpha \) with \([\alpha] = k\).

The terms from (2.23) are estimated as follows, using the Sobolev embeddings:

\[
\left| \int_M u_1 D^\alpha \frac{\partial u_2}{\partial x_1} D^\alpha u_3 \, dM \right| \leq |u_1|_{L^6} \left| D^\alpha \frac{\partial u_2}{\partial x_1} \right|_{L^2} |D^\alpha u_3|_{L^3} \\
\leq c |U_1|_1 |U_2|_3 |U_3|_2^{1/2} |U_3|_3^{1/2},
\]

(2.25)

where by \(| \cdot |_m\) we understand the classical norm on \( H^m(M) \) and by \( c \) we denote an independent constant. We also have:

\[
\left| \int_M w(U_1) D^\alpha \frac{\partial u_2}{\partial x_3} D^\alpha u_3 \, dM \right| \leq |w(U_1)|_{L^6} \left| D^\alpha \frac{\partial u_2}{\partial x_3} \right|_{L^2} |D^\alpha u_3|_{L^3} \\
\leq c |U_1|_2 |U_2|_3 |U_3|_2^{1/2} |U_3|_3^{1/2}.
\]

(2.26)

The first terms from (2.24) are estimated as follows (we give here just the first estimate, the second one is identical):

\[
\left| \int_M \delta^k u_1 \delta^{2-k} \frac{\partial u_2}{\partial x_1} D^\alpha u_3 \, dM \right| \leq |\delta^k u_1|_{L^3} \left| \delta^{2-k} \frac{\partial u_2}{\partial x_1} \right|_{L^2} |D^\alpha u_3|_{L^6} \\
\leq c |U_1|_{k}^{1/2} |U_1|_{k+1}^{1/2} |U_2|_3^{1-k} |U_3|_3.
\]

(2.27)
For the last term we have, for $k = 1$:

$$
\left| \int_M \delta^k w(U_1) \delta^{2-k} \frac{\partial u_2}{\partial x_3} D^\alpha u_3 \, dM \right| \leq |\delta^k w(U_1)|_{L^2} \left| \delta^{2-k} \frac{\partial u_2}{\partial x_1} \right|_{L^3} |D^\alpha u_3|_{L^6}
$$

$$
\leq c |U_1|_2 |U_2|_2^{1/2} |U_3|_3^{1/2} |U_3|_3, \quad (2.28)
$$

and for $k = 2$:

$$
\left| \int_M \delta^k w(U_1) \delta^{2-k} \frac{\partial u_2}{\partial x_3} D^\alpha u_3 \, dM \right| \leq |\delta^k w(U_1)|_{L^2} \left| \delta^{2-k} \frac{\partial u_2}{\partial x_1} \right|_{L^6} |D^\alpha u_3|_{L^3}
$$

$$
\leq c |U_1|_3 |U_2|_2^{1/2} |U_3|_3^{1/2}. \quad (2.29)
$$

Gathering all these inequalities, we can estimate the trilinear terms as:

$$
|b(U, U, (-\Delta)^2 U)| \leq c |U_2|^{3/2} |U_3|^{3/2} \leq \frac{c_0}{8} \left| (-\Delta)^{3/2} U \right|_H + \frac{c}{c_0} |U|_2^6, \quad (2.30)
$$

where we wrote $c_0 = \min(\nu, \mu)$. We also have:

$$
|b(U_1, U, (-\Delta)^2 U)|
$$

$$
\leq c \left( |U_1|_2 |U_2|_2^{1/2} |U_3|_3^{3/2} + |U_1|_2^{1/2} |U_1|_3^{1/2} |U_1|_3 |U_3|_3 + |U_1|_3 |U_2|_2^{3/2} |U_3|_3^{1/2} \right)
$$

$$
\leq \frac{c_0}{8} \left| (-\Delta)^{3/2} U \right|_H + \frac{c}{c_0} |U_1|_2^{1/2} |U_2|_2^{1/2} + \frac{c}{c_0} |U_1|_3^{4/3} |U_2|_2^{1/2} + \frac{c}{c_0} |U_1|_2 |U_1|_3 |U_1|_2^{1/2}, \quad (2.31)
$$

and

$$
|b(U, U_1, (-\Delta)^2 U)|
$$

$$
\leq c \left( |U_1|_3 |U_2|_2^{1/2} |U_3|_3^{3/2} + |U_1|_2 |U_2|_2^{1/2} |U_3|_3^{3/2} + |U_1|_2^{1/2} |U_1|_3^{1/2} |U_2|_2 |U_3|_3 \right)
$$

$$
\leq \frac{c_0}{8} \left| (-\Delta)^{3/2} U \right|_H + \frac{c}{c_0} |U_1|_3^{4/3} |U_2|_2^{1/2} + \frac{c}{c_0} |U_1|_2^{1/2} |U_2|_2^{1/2} + \frac{c}{c_0} |U_1|_2 |U_1|_3 |U_2|_2. \quad (2.32)
$$

For the term containing the forcing terms we easily find:

$$
|F - G, (-\Delta)^2 U|_H \leq c \| F - G \| \left| (-\Delta)^{3/2} U \right|_H
$$

$$
\leq \frac{c_0}{8} \left| (-\Delta)^{3/2} U \right|_H + \frac{c}{c_0} \| F - G \|^2. \quad (2.33)
$$
We thus find:

\[
\frac{d}{dt} |\Delta U|^2_H + \frac{c_0}{2} |(-\Delta)^{3/2} U|^2_H \leq \frac{c}{c_0^3} |U|^6_2 + \frac{c}{c_0^3} |U_1|^4_2 |U|^2_2 \\
+ \frac{c}{c_0^3} |U_1|^4 / |U|^2_2 + \frac{c}{c_0} |U_1|^2 |U|^2_2 + \frac{c}{c_0} \|F - G\|^2.
\] (2.34)

From (2.17) we can deduce that:

\[
\left| (-\Delta)^{3/2} f \right|^2 \geq \frac{1}{c_2 l^2} \left( |\Delta f|^2 - c_1 l^{-1} \eta(f)^2 \right), \text{ with } c_1, c_2 \text{ absolute constants},
\]

and inserting this property into (2.34), we find:

\[
\frac{d}{dt} |\Delta U|^2_H + |\Delta U|^2_H \left( c_0 c_3 l^{-2} - c_4 c_0^{-3} |U|^4_2 - c_5 c_0^{-3} |U_1|^4 - c_6 c_0^{-3} |U_1|^4 / \right) \\
- c_7 c_0^{-1} |U_1|^2 / \leq c_8 c_0^{-1} \|F - G\|^2 + c_9 c_0 l^{-3} \eta(U)^2.
\] (2.35)

We want to apply (2.18) to (2.35), with \( \xi(t) = |\Delta U(t)|^2_H \). From (2.2) and (2.5), we infer that in our case \( \beta \) does satisfy the condition that \( \beta(t) \) goes to zero as \( t \) goes to infinity. It remains to verify the conditions on \( \alpha \). They can be deduced from the fact that for \( m \) a positive integer, the solution is bounded uniformly in every Sobolev space \( H^m \) and the time average of the square of their norm in \( H^m \) is also uniformly bounded, the bounds being independent on the initial condition but depending on the viscosity, the domain, the forcing terms and the value of \( m \) (for more details see [8] or [10]).

Since

\[
\lim \inf_{t \to \infty} \frac{1}{s} \int_t^{t+s} \alpha(\tau) \, d\tau \geq c_0 c_3 l^{-2} - C(F, G, c_0, M),
\]

taking \( l \) small enough, the condition on \( \alpha \) is assured. We thus proved the desired result:

**Theorem 1** Let the domain \( M \) be covered by \( N \) identical cubes of volume \( l^3 \) and consider a set \( E = \{x^1, x^2, \ldots, x^N\} \) of points in \( M \), distributed one in each cube. Let \( F \) and \( G \) be two forcing terms in \( L^\infty(0, \infty; V) \) satisfying (2.2). Then, there exists a constant \( C = C(F, G, \nu, \mu, M) \) such that if

\[
l^{-2} \geq C(F, G, \nu, \mu, M),
\]

the set \( E \) is a set of determining nodes for the three dimensional primitive equations with periodic boundary conditions.

Following the work of Constantin, Doering and Titi [2] for the 3D Navier Stokes equations, we derive similar bounds on the length scale of determining local volume elements. A study on the global volume elements, meaning derivation of estimates
independent on the individual solution, can be obtained identically as before. The disadvantage of these estimates is that they do not give us constants of physical relevance, while here we will obtain estimates related to a modified mean rate of dissipation.

In what follows we will need a version of the Poincaré inequality that we enunciate here (a proof of this result can be found in [2]):

**Lemma 4** For every \( u \in H^1(M) \) we have:

\[
|u|^2_{L^2(Q_j)} \leq 2|\tilde{u}_j|^2 l^3 + c_1 l^2 |\nabla u|^2_{L^2(Q_j)},
\]

\[
|u|^2_{L^2(M)} \leq 2(\tilde{\eta}(u))^2 l^3 + c_1 l^2 |\nabla u|^2_{L^2(M)},
\]

where we denoted by

\[
\tilde{u}_j = \frac{1}{l^3} \int_{Q_j} u(x) \, dx,
\]

the average of the function \( u \) on each cube \( Q_j \) and we set \( \tilde{\eta}(u) = \left( \sum_{j=1}^N |\tilde{u}_j|^2 \right)^{1/2} \).

Let \( U_1, U_2 \) respectively be two solutions of (1.12) corresponding to the forcing terms \( F \) and \( G \) and we write \( U = U_1 - U_2 \). The equation satisfied by \( U \) reads as:

\[
\frac{d}{dt} (U, U^b)_H + a(U, U^b) + b(U_1, U, U^b) + b(U, U_1, U^b) - b(U, U, U^b) + e(U, U^b) = (F - G, U^b)_H, \quad \forall U^b \in V,
\]

\( U(0) = U_0 \). \hspace{1cm} (2.36)

Taking \( U^b = U \) and using (1), we find:

\[
\frac{1}{2} \frac{d}{dt} \left[ |u|^2 + |T|^2 \right] + \nu |u|^2 + \mu \kappa |T|^2 \leq |(F - G, U)_H| + |b(U, U_1, U)|. \hspace{1cm} (2.37)
\]

The terms from \( b(U, U_1, U) \) are estimated as follows:

\[
\left| \int_{M} \left[ (u \cdot \nabla_2)u_1 \right] \cdot u \, dM \right| \leq |u|^2 |\nabla_2 u_1|_{L^\infty} \leq |u|^2 |\nabla u_1|_{L^\infty},
\]

\[
\left| \int_{M} w \frac{\partial u_1}{\partial x_3} \cdot u \, dM \right| \leq |u||w||\nabla u_1|_{L^\infty}, \hspace{1cm} (2.38)
\]
and similarly for the terms in $T$. The $L^2$-norm of the vertical velocity is easily estimated as:

$$|w|^2 = \int_{\mathcal{M}} \left( \int_0^z \text{div}^2 u \, dz' \right)^2 \, d\mathcal{M} \leq L \int_{\mathcal{M}} \left( \int_{-L/2}^{L/2} |\text{div}^2 u|^2 \, dz' \right)^2 \, d\mathcal{M} = L^2 |\text{div}^2 u|^2.$$  

Gathering all these, we find:

$$\frac{1}{2} \frac{d}{dt} \left( |u|^2 + \kappa |T|^2 \right) + v |u|^2 + \kappa \mu |T|^2 \leq |u|^2 |\nabla u_1|_{L^\infty} + \sqrt{2} L |u| |\nabla u|_{L^\infty}$$

$$+ \kappa |u||T||\nabla T_1|_{L^\infty} + \sqrt{2} \kappa L |u||\nabla T_1|_{L^\infty}$$

$$\leq \frac{v}{2} |u|^2 + \frac{L^2}{v} \left( 4 + \frac{1}{2\pi^2} \right) |u|^2 |\nabla u_1|^2_{L^\infty} + \frac{L^2}{v} \kappa^2 \left( 4 + \frac{1}{2\pi^2} \right) |T|^2 |\nabla T_1|^2_{L^\infty}.$$  

(2.39)

where we used the fact that $|\text{div}^2 u| \leq \sqrt{2} |u|$. From Lemma 4 we know that:

$$|\nabla w|^2 \geq \frac{1}{c_1 l^2} |w|^2 - \frac{2l}{c_1} (\bar{\eta}(w))^2,$$

so we obtain:

$$\frac{d}{dt} \left( |u|^2 + \kappa |T|^2 \right) + \frac{v}{2} |u|^2 \left( \frac{1}{c_1 l^2} - \frac{c_1 L^2}{v^2} |\nabla u_1|^2_{L^\infty} \right)$$

$$+ \frac{\kappa \mu}{2} |T|^2 \left( \frac{1}{c_1 l^2} - \frac{c_1 L^2}{v \mu} |\nabla T_1|^2_{L^\infty} \right) \leq \frac{lv}{c_1} (\bar{\eta}(u))^2 + \frac{l \mu \kappa}{c_1} (\bar{\eta}(T))^2,$$  

(2.40)

where $c_1' = 4 + 1/2\pi^2$.

If we have $l$ such that:

$$l^{-2} \geq \inf_{s > 0} \left( \limsup_{t \to \infty} \frac{c_1 c_1' L^2}{v^2 s} \int_t^{t+s} |\nabla u_1|^2_{L^\infty} \, ds' \right),$$

$$l^{-2} \geq \inf_{s > 0} \left( \limsup_{t \to \infty} \frac{c_1 c_1' L^2 \kappa}{v \mu s} \int_t^{t+s} |\nabla T_1|^2_{L^\infty} \, ds' \right),$$  

(2.41)

then we can apply (2) and we obtain that for every solution $U$ for which $\bar{\eta}(U(t) - U_1(t)) \to 0$ as $t \to \infty$, we have $|U - U_1|_{H} \to 0$, as $t \to \infty$.

We thus proved the following result:
**Theorem 2** Let the domain $\mathcal{M}$ be covered by $N$ identical cubes of volume $l^3$ and suppose that $l$ is small enough so that

\[
l^{-2} \geq \inf_{s > 0} \left( \lim_{t \to \infty} \sup \frac{c_1 c'_1 L^2}{v^2 s} \int_t^{t+s} |\nabla u|_{L^\infty}^2 \, ds' \right),
\]

\[
l^{-2} \geq \inf_{s > 0} \left( \lim_{t \to \infty} \sup \frac{c_1 c'_1 L^2 \kappa}{v \mu s} \int_t^{t+s} |\nabla T|_{L^\infty}^2 \, ds' \right),
\]

(2.42)

with $U$ a solution of the primitive equations (1.12). Then for any solution $V$ of the primitive equations for which $\tilde{\eta}(U(t) - V(t)) \to 0$ as $t \to \infty$, we have $|U - V|_H \to 0$, as $t \to \infty$.

This result also reads as follows: we define a modified mean rate of energy dissipation as $\epsilon_\infty = \max(\epsilon_1, \epsilon_2)$, where

\[
\epsilon_1 = v \inf_{s > 0} \left( \lim_{t \to \infty} \sup \frac{1}{s} \int_t^{t+s} |\nabla u|_{L^\infty}^2 \, ds' \right),
\]

and

\[
\epsilon_2 = v \inf_{s > 0} \left( \lim_{t \to \infty} \sup \frac{v^2 \kappa}{\mu s} \int_t^{t+s} |\nabla T|_{L^\infty}^2 \, ds' \right),
\]

and a modified small length scale:

\[
l_\infty = \frac{1}{\sqrt{c_1 c'_1 L}} \left( \frac{v^3}{\epsilon_\infty} \right)^{1/2}.
\]

(2.43)

If the side of the cubes $Q_j$ is small enough, more exactly if $l \leq l_\infty$, then the cubes are determining volume elements.

**Remark 1** We note here that the modified length scale (2.43) is in fact similar to the classical Kolmogorov dissipation length for the two dimensional Navier Stokes equations. Even if we work with a three dimensional model, the result is not so surprising since the global in time well posedness of the three dimensional primitive equations was proved using a hidden “2D structure” of the model (see [1] where the well-posedness of the 3D primitive equations was proved).

**Remark 2** The estimate that we obtained in (2.43) is a local one since it depends on an individual solution. Global estimates can be easily obtained by simply noticing that in a three dimensional space we have the Sobolev embedding $|f|_{L^\infty(\mathcal{M})} \leq c |f|_{H^2(\mathcal{M})}$ and by the fact that we have uniform bounds (independent of the initial condition) in any Sobolev space for a solution $U$ of (1.12) (see [8]).
As announced before, we also study the number of low modes that can be considered in order to have an accurate description of the whole flow. To be more precise, let us consider the complete orthonormal set of three-dimensional vector-valued eigenfunctions \((W_n)_{n=1}^\infty\) satisfying the GFD-Stokes operator. Let also \(P_m\) be the orthogonal projection associated with the first \(m\) modes of the GFD-Stokes operator and we write \(Q_m = I - P_m\).

Let \(U\) and \(V\) be two solutions of the three-dimensional primitive equations corresponding to two possibly different forcing terms \(F\) and \(G\), where \(F\) and \(G\) are given forces in \(L^\infty(0, \infty, H)\). We can expand each solution as:

\[
U(t, x) = \sum_{k=1}^\infty U_k(t)W_k(x), \quad V(t, x) = \sum_{k=1}^\infty V_k(t)W_k(x).
\]

We assume that the forcing terms have the same asymptotic behavior at large time, meaning:

\[
\int_M |F(t, x) - G(t, x)|^2 \, dM \to 0, \quad \text{as } t \to \infty. \tag{2.44}
\]

Then, a set of modes \(\{W_j\}_{j=1}^m\) is called a set of determining modes if the condition

\[
\int_M |P_mU(t, x) - P_mV(t, x)|^2 \, dM \to 0 \quad \text{as } t \to \infty, \tag{2.45}
\]

implies

\[
\int_M |U(t, x) - V(t, x)|^2 \, dM \to 0 \quad \text{as } t \to \infty. \tag{2.46}
\]

Let \(W = U - V\). We start by assuming that \(|P_m W| \to 0\) as \(t \to \infty\). The equation for \(W\) is:

\[
\frac{dW}{dt} + AW + B(U, W) + B(W, U) - B(W, W) + EW = F(t) - G(t), \quad W(0) = 0. \tag{2.47}
\]

Taking the scalar product in \(H\) of (2.47) with \(Q_m W\), we find:

\[
\frac{1}{2} \frac{d}{dt} |Q_m W|^2_H + c_0 \|Q_m W\|^2 \leq |(F - G, Q_m W)_H| + |b(U, W, Q_m W)| + |b(W, U, Q_m W)| + |b(W, W, Q_m W)|. \tag{2.48}
\]
We need to estimate the RHS of (2.48). Using (1.18), we get:

\[
b(U, W, Q_m W) = b(U, P_m W, Q_m W) + b(U, Q_m W, Q_m W)
\]
\[
= b(U, P_m W, Q_m W)
\]
\[
\leq c \|U\| \|P_m W\|^{1/2} \|P_m W\|_H^{1/2} \|Q_m W\|_H^{1/2} \|Q_m W\|^{1/2}
\]
\[
\leq c \lambda_m^{3/4} \|U\| \|P_m W\|_H \|Q_m W\|_H^{1/2} \|Q_m W\|^{1/2}
\]
\[
\leq \frac{c_0}{12} \|Q_m W\|^2 + \frac{c}{c_0^3} \lambda_m \|U\|^{4/3} P_m W_H^{4/3} \|Q_m W\|^{2/3},
\]

where \(\{\lambda_m\}_m\) are the eigenvalues of the GFD-Stokes operator and we used the fact that \(\|P_m W\| \leq \lambda_m^{1/2} \|P_m W\|_H\) and \(\|P_m W\|_2 \leq \lambda_m \|P_m W\|_H\).

We also write:

\[
b(W, W, Q_m W) = b(P_m W, U, Q_m W) + b(Q_m W, U, Q_m W),
\]

and we estimate the first term using (1.18):

\[
|b(P_m W, U, Q_m W)| \leq \frac{c_0}{12} \|Q_m W\|^2 + \frac{c}{c_0^3} \lambda_m \|U\|^{4/3} P_m W_H^{4/3} \|Q_m W\|^{2/3},
\]

and the second term using (1.18):

\[
|b(Q_m W, U, Q_m W)| \leq c \|Q_m W\| \|U\|^{1/2} \|U\|_2^{1/2} \|Q_m W\|_H^{1/2} \|Q_m W\|^{1/2} \geq \frac{c_0}{12} \|Q_m W\|^2
\]
\[
+ \frac{c}{c_0^3} \|U\|^2 \|U\|_2^2 \|Q_m W\|_H^2.
\]

We also need to estimate:

\[
b(W, W, Q_m W) = b(P_m W, P_m W, Q_m W) + b(Q_m W, P_m W, Q_m W).
\]

For the first term we have:

\[
|b(P_m W, P_m W, Q_m W)| \leq c \|P_m W\|^{3/2} \|P_m W\|_H^{1/2} \|P_m W\|_H^{1/2} \|Q_m W\|^{1/2}
\]
\[
\leq c \lambda_m^{7/4} \|P_m W\|_H^2 \|Q_m W\|_H^{1/2} \|Q_m W\|^{1/2}
\]
\[
\leq \frac{c_0}{12} \|Q_m W\|^2 + \frac{c}{c_0^3} \lambda_m^{7/3} \|P_m W\|_H^{8/3} \|Q_m W\|^{2/3},
\]

and the second term gives (using (1.18)):

\[
|b(Q_m W, P_m W, Q_m W)| \leq c \|Q_m W\| \|P_m W\|^{1/2} \|P_m W\|_2^{1/2} \|Q_m W\|_H^{1/2} \|Q_m W\|^{1/2}
\]
\[
\leq \frac{c_0}{12} \|Q_m W\|^2 + \frac{c}{c_0^3} \lambda_m^{3} \|P_m W\|_H^{4} \|Q_m W\|_H^2.
\]
For the forcing term we have:

\[
| (F - G, Q_m W) |_H \leq |F - G|_H |Q_m W|_H \leq \frac{c_0}{12} \| Q_m W \|^2 + \frac{c}{c_0} |F - G|^2_H, \tag{2.56}
\]

Gathering all the estimates, we obtain:

\[
\frac{d}{dt} |Q_m W|_H^2 + c_0 \| Q_m W \|^2 \leq \frac{c}{c_0} \lambda_m |U|^{4/3} |P_m W|^{4/3}_H |Q_m W|^{2/3}_H \\
+ \frac{c}{c_0} \| U \|^2 \| U \|^2_2 |Q_m W|^{2/3}_H + \frac{c}{c_0} \lambda_m^{7/3} |P_m W|^{8/3}_H |Q_m W|^{2/3}_H \\
+ \frac{c}{c_0} \lambda_m |P_m W|^{4}_H |Q_m W|^{2}_H + \frac{c}{c_0} |F - G|^2_H, \tag{2.57}
\]

which can be also written as:

\[
\frac{d}{dt} |Q_m W|_H^2 + \alpha(t) |Q_m W|_H^2 \leq \beta(t), \tag{2.58}
\]

with

\[
\alpha(t) = c_0 \lambda_{m+1} - \frac{c}{c_0} \| U \|^2 \| U \|^2_2,
\]

and

\[
\beta(t) = \frac{c}{c_0} \lambda_m |U|^{4/3} |P_m W|^{4/3}_H |Q_m W|^{2/3}_H + \frac{c}{c_0} \lambda_m^{7/3} |P_m W|^{8/3}_H |Q_m W|^{2/3}_H \\
+ \frac{c}{c_0} \lambda_m |P_m W|^{4}_H |Q_m W|^{2}_H + \frac{c}{c_0} |F - G|^2.
\tag{2.59}
\]

In order to define the function \( \alpha \), we used the relation \( \lambda_{m+1}^{1/2} |Q_m W| \leq \| Q_m W \| \).

We intend to apply the generalized Gronwall Lemma 3 to (2.58), but for that we need to check the conditions on \( \alpha \) and \( \beta \). From the initial assumptions, \( |P_m W| \) and \( |F - G|^2 \) tend to zero as \( t \) goes to infinity, and \( |Q_m W| \) is bounded since \( |Q_m W| \leq |U| + |V| \) and \( U, V \) are bounded for all time in \( H \).

The condition that:

\[
\liminf_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \alpha^{-} (\tau) \, d\tau \leq \infty,
\]

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is assured by the fact that there exists a time $t_1(U_0, F, \mathcal{M}, \nu, \mu)$ and the constants $K_0, K_1$ independent of the initial condition, such that:

$$|U(t)|_1 \leq K_0(F), \quad \forall t \geq t_1(U_0, F, \mathcal{M}, \nu, \mu)$$

$$\limsup_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} |U(s)|^2 \, ds \leq K_1(F, T), \quad \forall t \geq t_1(U_0, F, \mathcal{M}, \nu, \mu).$$

(2.60)

The proof of (2.60) can be found in [8] where it was proved the existence of absorbing sets in all Sobolev spaces, for the case of periodic boundary conditions.

In order to have that

$$\liminf_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \alpha(\tau) \, d\tau > 0,$$

we ask that:

$$\lambda_{m+1} \geq \frac{c}{c_0} K_0(F) K_1(F, T).$$

(2.61)

Thus, we proved the following result:

**Theorem 3** Suppose that $m \in \mathbb{N}$ is such that:

$$\lambda_{m+1} \geq \frac{c}{c_0} K_0(F) K_1(F, T),$$

where $c$ is a constant depending only on the shape of the domain $\mathcal{M}$ and $c_0$, $K_0(F)$, $K_1(F, T)$ are constants defined before. Then, the first $m$ modes are determining in the sense of (2.44)–(2.46).

**Remark 3** Estimating $b(Q_m W, U, Q_m W)$ by using (1.19) instead of (1.18), we have:

$$|b(Q_m W, U, Q_m W)| \leq c|\nabla U|_\infty |Q_m W||Q_m W| \leq \frac{c_0}{12} \|Q_m W\|^2 + \frac{c}{c_0} |\nabla U|_\infty^2 |Q_m W|^2.$$

We want to apply the generalized Gronwall lemma (3) to a different differential inequality, in which $\alpha$ is the function:

$$\alpha(t) = c_0 \lambda_{m+1} - \frac{c}{c_0} |\nabla U|_\infty^2.$$

Thus, we define a modified mean rate of energy dissipation as:

$$\epsilon_\infty = c_0 \inf_{T > 0} \left( \limsup_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} |\nabla U(s)|^2 \, ds \right),$$
and we find that if \( m \in \mathbb{N} \) is such that:

\[
\lambda_{m+1} \geq \frac{c}{c_0^3} \epsilon_\infty, 
\]

then the first \( m \) modes are determining in the sense of (2.44)–(2.46).

### 3 Exponential decay of the power spectrum for the 3D primitive equations

The exponential decay of the solutions of the 3D primitive equations has been shown in [8]. In what follows we are interesting in deducing an explicit lower bound on the decay rate. For the case of Navier–Stokes equations, this question was addressed by Foias and Temam (see [6]) where the authors proved the exponential decay of the spatial power spectrum for the solutions and by Doering and Titi (see [3]) where an explicit lower bound on a small length scale defined by the exponential decay rate is found.

We first start by introducing the main notations:

\[
[U_k]_k^2 = |u_k|^2 + |v_k|^2 + \kappa |T_k|^2.
\]

We also write \( k = (\tilde{k}, k_3) \), with \( \tilde{k} \in \mathbb{Z}^2 \).

We define the Gevrey space \( D(e^{\tau(-\Delta)^{1/2}}), \alpha > 0 \), as the set of functions \( U \) in \( H \) satisfying

\[
|M| \sum_{k \in \mathbb{Z}^3} e^{2\alpha |k'|} [U_k]_k^2 = \left| e^{\alpha(-\Delta)^{1/2}} U \right|_H^2 < \infty.
\] (3.1)

The Hilbert norm of \( D(e^{\alpha(-\Delta)^{1/2}}) \) is given by

\[
|U|_\alpha := |U|_{D(e^{\alpha(-\Delta)^{1/2}})} = \left| e^{\alpha(-\Delta)^{1/2}} U \right|_H, \text{ for } U \in D(e^{\alpha(-\Delta)^{1/2}}), \] (3.2)

and the associated scalar product is

\[
(U, V)_\alpha := (U, V)_{D(e^{\alpha(-\Delta)^{1/2}})} = (e^{\alpha(-\Delta)^{1/2}} U, e^{\alpha(-\Delta)^{1/2}} V)_H, \text{ for } U, V \in D(e^{\alpha(-\Delta)^{1/2}}). \] (3.3)

Another Gevrey space that we will use is \( D((-\Delta)^{m/2} e^{\alpha(-\Delta)^{1/2}}) \) with \( m \geq 1 \) integer, which is a Hilbert space when endowed with the inner product:

\[
(U, V)_{D((-\Delta)^{m/2} e^{\alpha(-\Delta)^{1/2}})} = \left( (-\Delta)^{m/2} e^{\alpha(-\Delta)^{1/2}} U, (-\Delta)^{m/2} e^{\alpha(-\Delta)^{1/2}} V \right)_H. \] (3.4)

In order to be able to estimate the norm of the solution \( U \) into a Gevrey space, we first need a technical result that we prove here:
Lemma 5 Let $U, U^u$ and $U^b$ be given in $D \left( -\Delta e^{\tau(-\Delta)^{1/2}} \right)$, for $\alpha \geq 0$. Then the following inequality holds:

$$|(B(U, U^u), \Delta U^b)_{\alpha}| \leq c |(-\Delta)^{1/2}U|_{\alpha}^{1/2} |\Delta U^u|_{\alpha}^{1/2} |(-\Delta)^{1/2}U^b|_{\alpha}^{1/2} |\Delta U^b|_{\alpha}^{1/2}. \quad (3.5)$$

Proof In Fourier modes the first term of (3.5) writes as:

$$|(B(U, U^u), \Delta U^b)_{\alpha}| = \left| \sum_{j+l+k=0} i (l'_1 - \delta_{j,1} l'_3) u_j + (l'_2 - \delta_{j,2} l'_3) v_j \right| e^{2\alpha |k'|} |k'|^2 \left( u_k^u + v_k^v + \kappa T_k^v T_k^b \right), \quad (3.6)$$

where we defined, for each $j \in \mathbb{Z}^3$, $\delta_{j,n}$ as $j'_n/j'_3$ when $j'_3 \neq 0$ and as 0 when $j'_3 = 0$, for $n = 1, 2$.

The most difficult term to estimate is the one containing $\delta_{j,n}$, so we start estimating this term:

$$\sum_{j+l+k=0} \frac{|j'_1|}{|j'_3|} |l'_3||k'|^2 |u_j||u_k^u||u_k^v| e^{2\alpha |k'|} \leq \sum_{j+l+k=0} \frac{|j'_1|}{|j'_3|} |l'_3||k'|^2 |\tilde{u}_j||u_k^u||u_k^v|$$

$$\leq \sum_{j+l+k=0} \frac{|j'_1|}{|j'_3|^2} \left( \sum_{k_3} |k'|^4 |u_k^v|^2 \right)^{1/2} \left( \sum_{l_3} |l'|^2 |u_l^v|^2 \right)^{1/2}$$

$$\leq \sum_{j+l+k=0} \frac{|j'_1|}{|j'_3|^2} \left( \sum_{j_3} \frac{1}{|j'_3|^2} \right)^{1/2} \left( \sum_{j_3} |\tilde{u}_j|^2 \right)^{1/2} \times \left( \sum_{k_3} |k'|^4 |u_k^v|^2 \right)^{1/2} \left( \sum_{l_3} |l'|^2 |u_l^v|^2 \right)^{1/2}$$

$$\leq c \sum_{j+l+k=0} \frac{|j'_1|}{|j'_3|^2} \left( \sum_{j_3} |\tilde{u}_j|^2 \right)^{1/2} \left( \sum_{k_3} |k'|^4 |u_k^v|^2 \right)^{1/2} \left( \sum_{l_3} |l'|^2 |u_l^v|^2 \right)^{1/2} = I, \quad (3.7)$$

where, in order to simplify the computations, we used the notation:

$$\tilde{u} = \sum_{j \in \mathbb{Z}^3} \tilde{u}_j e^{i(j'_1 x_1 + j'_2 x_2 + j'_3 x_3)} = \sum_{j \in \mathbb{Z}^3} \tilde{u}_j e^{i j' \cdot x}, \quad \text{where} \quad \tilde{u}_j = e^{\tau |j'|} |u_j|. \quad (3.8)$$
We continue to estimate $I$ as:

$$\begin{align*}
I \leq c \int (0,L) \times (0,L) f g h \, d\mathcal{M}' & \leq c |f|_{L^4(M')} |g|_{L^2(M')} |h|_{L^4(M')} \\
& \leq c |f|^{1/2}_{L^2(M')} |f|^{1/2}_{H^1(M')} |g|_{L^2(M')} |h|^{1/2}_{L^2(M')} |h|^{1/2}_{H^1(M')} \\
& \leq c |(-\Delta)^{1/2} U|^2_{\alpha} |\Delta U|^2_{\alpha} |\Delta U|^2_{\alpha} |(-\Delta)^{1/2} U|^2_{\alpha} |\Delta U|^2_{\alpha},
\end{align*}$$

(3.9)

where we wrote:

$$\begin{align*}
f(x_1, x_2) &= \sum_{j \in \mathbb{Z}^2} |j|^2 \left( \sum_{j} \hat{u}_{j}^2 \right)^{1/2} e^{i j \cdot x}, \\
& \quad g(x_1, x_2) = \sum_{k \in \mathbb{Z}^2} \left( \sum_{k} |k|^4 \hat{u}_{k}^2 \right)^{1/2} e^{i k \cdot x}, \\
h(x_1, x_2) &= \sum_{l \in \mathbb{Z}^2} \left( \sum_{l} |l|^2 \hat{u}_{l}^2 \right)^{1/2} e^{i l \cdot x}.
\end{align*}$$

(3.10)

With this, we proved (3.5).

In what follows, we want to derive a differential equation for $|e^{tt(-\Delta)^{1/2}} (-\Delta)^{1/2} U|_H$. In order to obtain that we apply the operator $e^{tt(-\Delta)^{1/2}}$ to (1.1a) and respectively (1.1b), we take the scalar product by $e^{tt(-\Delta)^{1/2}} U$ and by $e^{tt(-\Delta)^{1/2}} \Delta U$; we also apply the operator $e^{tt(-\Delta)^{1/2}}$ to (1.1e) and take the scalar product by $\kappa e^{tt(-\Delta)^{1/2}} \Delta T$. We add the resulting equations and be notice that the nonlinear term will be bound, according to (3.5), by $|e^{tt(-\Delta)^{1/2}} (-\Delta)^{1/2} U|_H |e^{tt(-\Delta)^{1/2}} \Delta U|_H^2$ which is a weak estimate since the term $|e^{tt(-\Delta)^{1/2}} U|_H^2$ is at power 2 and we will be obliged to work with small initial data. In order to avoid this we split the solution $U$ of (1.12) into $U = U^* + \tilde{U}$, where $U^*$ is the solution of the linear problem:

$$\begin{align*}
\frac{dU^*}{dt} + AU^* + EU^* &= F, \\
U^*(0) &= U_0,
\end{align*}$$

(3.11)

and $\tilde{U}$ is the solution of the following nonlinear problem, in which $U^*$ is now known:

$$\begin{align*}
\frac{d\tilde{U}}{dt} + A\tilde{U} + B(\tilde{U}, U^*) + B(\tilde{U}, U^*) + B(U^*, \tilde{U}) + E\tilde{U} &= -B(U^*, U^*), \\
\tilde{U}(0) &= 0.
\end{align*}$$

(3.12)

*Estimates for the linear part:* In order to obtain the a priori estimates for $U^*$ we apply the operator $e^{tt(-\Delta)^{1/2}}$ to (3.11) and we take the scalar product by $e^{tt(-\Delta)^{1/2}} \Delta U^*$.
in \( H \). We thus obtain:

\[
\left( e^{\tau t (-\Delta)^{1/2}} \frac{d}{dt} \mathbf{U}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} + \left( e^{\tau t (-\Delta)^{1/2}} A \mathbf{U}^{*}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} + \left( e^{\tau t (-\Delta)^{1/2}} E \mathbf{U}^{*}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} = 0. \tag{3.13}
\]

The first term from the right hand side of (3.13) is estimated as follows:

\[
\left( \frac{d}{dt} \left( e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right) - \tau (-\Delta)^{1/2} e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} \geq \frac{1}{2} \frac{d}{dt} \left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2} - \frac{1}{2} \left| (-\Delta)^{1/2} e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{H} \left| e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right|_{H}. \tag{3.14}
\]

For the other two terms from the right hand side of (3.13) we use the coercivity and we find:

\[
\left( e^{\tau t (-\Delta)^{1/2}} A \mathbf{U}^{*}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} + \left( e^{\tau t (-\Delta)^{1/2}} E \mathbf{U}^{*}, e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right)_{H} \geq \frac{c_{0}}{2} \left| e^{\tau t (-\Delta)^{1/2}} (-\Delta) \mathbf{U}^{*} \right|_{H}^{2}. \tag{3.15}
\]

Gathering all these estimates and using the Young inequality, we have:

\[
\frac{d}{dt} \left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2} + \frac{c_{0}}{2} \left| e^{\tau t (-\Delta)^{1/2}} \Delta \mathbf{U}^{*} \right|_{H}^{2} \leq \frac{2 \tau^{2}}{c_{0}} \left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2}. \tag{3.16}
\]

We choose \( \tau \) such that \( 2 \tau^{2}/c_{0} = c_{0}/4c' \), with \( c' \) the Poincaré constant. Thus, we obtain:

\[
\frac{d}{dt} \left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2} + \frac{c_{0}}{4c'} \left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2} \leq 0,
\]

which implies, using the Gronwall lemma, that:

\[
\left| e^{\tau t (-\Delta)^{1/2}} \mathbf{U}^{*} \right|_{V}^{2} \leq e^{-\frac{c_{0}}{4c'}t} \left| \mathbf{U}_{0} \right|_{V}^{2}. \tag{3.17}
\]

By the same arguments, we can also obtain:

\[
\left| e^{\tau t (-\Delta)^{1/2}} \Delta \mathbf{U}^{*} \right|_{H}^{2} \leq e^{-\frac{c_{0}}{4c'}t} \left| \Delta \mathbf{U}_{0} \right|_{H}^{2}. \tag{3.18}
\]

Estimates for the non-linear part: In order to estimate the non-linear part \( \tilde{U} \), we do the same kind of estimates: we apply the operator \( e^{\tau t (-\Delta)^{1/2}} \) to (3.12) and we take
the scalar product by $e^{\tau t(-\Delta)^{1/2}} \Delta \tilde{U}$ in $H$. The nonlinear terms are estimated using Lemma 5. We have:

\[
\left| (e^{\tau t(-\Delta)^{1/2}} B(\tilde{U}, \tilde{U}), e^{\tau t(-\Delta)^{1/2}} \Delta \tilde{U})_H \right| \leq c \left| (-\Delta)^{1/2} e^{\tau t(-\Delta)^{1/2}} \tilde{U} \right|_H \\
\times |\Delta e^{\tau t(-\Delta)^{1/2}} \tilde{U}|_H^2, \tag{3.19}
\]

and

\[
|(B(\tilde{U}, U^*), \Delta \tilde{U})_{\tau t}| \leq c \left| (-\Delta)^{1/2} \tilde{U} \right|_{\tau t}^{1/2} |\Delta \tilde{U}|_{\tau t}^{3/2} \left| (-\Delta)^{1/2} U^* \right|_{\tau t}^{1/2} |\Delta U^*|_{\tau t}^{1/2} \\
\leq \frac{c_0}{12} |\Delta e^{\tau t(-\Delta)^{1/2}} \tilde{U}|_{\tau t}^{2} + \frac{c}{c_0} \left| (-\Delta)^{1/2} U^* \right|_{\tau t}^{2} |\Delta U^*|_{\tau t}^{2} \left| (-\Delta)^{1/2} \tilde{U} \right|_{\tau t}^{2}. \tag{3.20}
\]

We also have:

\[
|(B(U^*, \tilde{U}), \Delta \tilde{U})_{\tau t}| \leq c \left| (-\Delta)^{1/2} \tilde{U} \right|_{\tau t}^{1/2} |\Delta \tilde{U}|_{\tau t}^{3/2} \left| (-\Delta)^{1/2} U^* \right|_{\tau t}^{1/2} |\Delta U^*|_{\tau t}^{1/2} \\
\leq \frac{c_0}{12} |\Delta e^{\tau t(-\Delta)^{1/2}} \tilde{U}|_{\tau t}^{2} + \frac{c}{c_0} \left| (-\Delta)^{1/2} U^* \right|_{\tau t}^{2} |\Delta U^*|_{\tau t}^{2} \left| (-\Delta)^{1/2} \tilde{U} \right|_{\tau t}^{2}. \tag{3.21}
\]

and

\[
|(B(U^*, U^*), \Delta \tilde{U})_{\tau t}| \leq c \left| (-\Delta)^{1/2} U^* \right|_{\tau t} \left| \Delta U^* \right|_{\tau t} |\Delta \tilde{U}|_{\tau t} \\
\leq \frac{c_0}{12} |\Delta e^{\tau t(-\Delta)^{1/2}} \tilde{U}|_{\tau t}^{2} + \frac{c}{c_0} \left| (-\Delta)^{1/2} U^* \right|_{\tau t}^{2} |\Delta U^*|_{\tau t}^{2}. \tag{3.22}
\]

The linear terms are estimated exactly as before. Gathering all these estimates, we find:

\[
\frac{d}{dt} \left| e^{\tau t(-\Delta)^{1/2}} \tilde{U} \right|_V^2 + \frac{c_0}{2} \left| e^{\tau t(-\Delta)^{1/2}} \Delta \tilde{U} \right|_H^2 \leq c \left| e^{\tau t(-\Delta)^{1/2}} \tilde{U} \right|_V \left| e^{\tau t(-\Delta)^{1/2}} \Delta \tilde{U} \right|_H^2 \\
+ f(t) \left| e^{\tau t(-\Delta)^{1/2}} \tilde{U} \right|_V^2 + g(t), \tag{3.23}
\]

where we wrote:

\[
f(t) = \frac{c \tau^2}{c_0} + \frac{1}{c_0^2} g(t), \tag{3.24}
\]

and

\[
g(t) = \frac{c}{c_0} \left| e^{\tau t(-\Delta)^{1/2}} U^* \right|_V \left| e^{\tau t(-\Delta)^{1/2}} \Delta U^* \right|_H^2. \tag{3.25}
\]
Since \( \hat{U}(0) = 0 \), we assume that \( |e^{rt(-\Delta)^{1/2}} \hat{U}|_V \leq \frac{c_0}{4c} \) for some finite interval \((0, t'_*).\) On this interval we have the following inequality:

\[
\frac{d}{dt} \left| e^{rt(-\Delta)^{1/2}} \hat{U}\right|_V^2 + \frac{c_0}{4} \left| e^{rt(-\Delta)^{1/2}} \Delta \hat{U}\right|_H^2 \leq f(t) \left| e^{rt(-\Delta)^{1/2}} \hat{U}\right|_V^2 + g(t). \tag{3.26}
\]

Taking into account the a priori estimates obtained for \( U^* \), we find that \( f \) and \( g \) are functions locally integrable. So, we can apply the Gronwall lemma and deduce the following estimate on \((0, t'_*):\)

\[
\left| e^{rt(-\Delta)^{1/2}} \hat{U}\right|_V^2 \leq \int_0^t g(s) \exp \left( \int_s^t f(\xi) \, d\xi \right) \, ds. \tag{3.27}
\]

Since \( f \) and \( g \) are locally integrable, we can define \( t'_* \) as the first time for which we have:

\[
\int_0^{t'_*} g(s) \exp \left( \int_s^{t'_*} f(\xi) \, d\xi \right) \, ds = \frac{c_0^2}{16c^2}. \tag{3.28}
\]

Then, on the interval \((0, t'_*),\) we find:

\[
\left| e^{rt(-\Delta)^{1/2}} \hat{U}\right|_V \leq \frac{c_0}{4c}.
\]

Thus, using the integration by parts, we find:

\[
\int_0^{t'_*} g(s) \exp \left( \int_s^{t'_*} f(\xi) \, d\xi \right) \, ds = \int_0^{t'_*} g(s) \exp \left( \frac{c\tau^2}{c_0}(t'_* - s) \right) \exp \left( \frac{1}{c_0^2} \int_s^{t'_*} g(\xi) \, d\xi \right) \, ds \\
= c_0^2 \exp \left( \frac{c\tau^2}{c_0} t'_* \right) \exp \left( \frac{1}{c_0^2} \int_0^{t'_*} g(\xi) \, d\xi \right) - c_0^2 \\
- \frac{c_0^3}{c\tau^2} \int_0^{t'_*} \exp \left( \frac{c\tau^2}{c_0}(t'_* - s) \right) \exp \left( \frac{1}{c_0^2} \int_s^{t'_*} g(\xi) \, d\xi \right) \, ds. \tag{3.29}
\]

Since the last term is positive, we will search for a \( t'_* \) such that:

\[
c_0^2 \exp \left( \frac{c\tau^2}{c_0} t'_* \right) \exp \left( \frac{1}{c_0^2} \int_0^{t'_*} g(\xi) \, d\xi \right) - c_0^2 \leq \frac{c_0^2}{16c^2}.
\]
Exponential decay for the 3D primitive equations

This translates in solving the inequation in \( t' \):

\[
\exp \left( \frac{c\tau^2}{c_0 t'} \right) \exp \left( \frac{1}{c_0^2} \int_{0}^{t'} g(\xi) \, d\xi \right) \leq k_1,
\]

with \( k_1 > 1 \) an absolute constant. Using (3.17) and (3.18), we find:

\[
\int_{0}^{t'} g(\xi) \, d\xi \leq |U_0|^2_{V} \Delta U_0^2_{H} \frac{2c'}{c_0} \left( 1 - e^{-\frac{c_0}{2c} t'} \right) \leq |U_0|^2_{V} \Delta U_0^2_{H} t'.
\]

We thus have a rough estimate for \( t'_* \) as:

\[
t'_* = \frac{k_2}{c_0/8c' + |U_0|^2_{V} \Delta U_0^2_{H}/c_0^2},
\]

with \( k_2 \) an absolute constant.

So, over the interval \((0, t'_*)\), we have the following estimate:

\[
|e^{\tau t(-\Delta)^{1/2}} U|_{V}^2 \leq 2 \left\{ \left| e^{\tau t(-\Delta)^{1/2}} U \right|_{V}^2 + \left| e^{\tau t(-\Delta)^{1/2}} U_* \right|_{V}^2 \right\} \leq 2 \left( \frac{2c'}{c_0} |U_0|^2_{V} \Delta U_0^2_{H} \left( 1 - e^{-\frac{c_0}{2c} t'} \right) + e^{-\frac{c_0}{4c} t} |U_0|^2_{V} \right). \tag{3.30}
\]

This implies the following decay on the Fourier coefficients of the solution:

\[
|U_k(t)|^2 \leq 2 \frac{L^3}{|k'|^2} e^{-2\tau t |k'|} |U_0|^2_{V} \left\{ \frac{2c'}{c_0} |\Delta U_0|^2_{H} \left( 1 - e^{-\frac{c_0}{2c} t} \right) + e^{-\frac{c_0}{4c} t} \right\}, \quad \forall t \leq t_* \tag{3.31}
\]

On \( t_* \) we thus have:

\[
|U_k(t'_*)|^2 \leq 2 \frac{L^3}{|k'|^2} e^{-\frac{c_0}{\sqrt{2}c'} t_* |k'|} |U_0|^2_{V} \left\{ \frac{2c'}{c_0} |\Delta U_0|^2_{H} \left( 1 - e^{-\frac{c_0}{2c} t_*} \right) + e^{-\frac{c_0}{4c} t_*} \right\}, \tag{3.32}
\]

and the exponential decay is:

\[
\lambda_* = \frac{c_0}{\sqrt{2}c'} t_* = \frac{k_2}{\sqrt{2}c'} \frac{c_0}{c_0/8c' + |U_0|^2_{V} \Delta U_0^2_{H}/c_0^2}. \tag{3.33}
\]

Since we know that \( |U|^2_{V} \) and \( |\Delta U|^2_{H} \) are bounded uniformly in time, we can reiterate the argument in order to obtain uniform bound on the decay of the Fourier coefficients.
spectrum for \( t \geq t_\ast \). The decay after a transient time \( t_\ast \) is thus estimated as:

\[
\lambda = \frac{c_0}{\sqrt{2}c'} \frac{k_2}{c_0/8c' + \sup_t |U|^2 V \sup_t |\Delta U|^2_H/c_0^2}.
\]

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