Numerical analysis of a Cahn-Hilliard type equation with dynamic boundary conditions

Haydi Israel[1], Alain Miranville[1] and Madalina Petcu[1,2]

[1]UMR 7348 CNRS, Laboratoire de Mathématiques et Applications - Université de Poitiers - SP2MI
Boulevard Marie et Pierre Curie, Téléport 2
BP30179 - 86962 Futuroscope Chasseneuil Cedex - FRANCE.

[2]Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Abstract

We study space and time discretizations of a Cahn-Hilliard type equation with dynamic boundary conditions. We first study a semi-discrete version of the equation and we prove optimal error estimates in energy norms and weaker norms. Then, we study the stability of the fully discrete scheme obtained by applying the Euler backward scheme to the space semi-discrete problem. In particular, we show that this fully discrete problem is unconditionally stable. Some numerical results in two space dimensions conclude the paper.

1 Introduction

We consider the following problem in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$:

$$\begin{aligned} u_t &= \Delta w - w, \qquad t > 0,\ x \in \Omega, && (1.1)\\ w &= f(u) - \Delta u, \qquad t > 0,\ x \in \Omega, && (1.2)\\ u_t &= \Delta_\Gamma u - \lambda u - g(u) - \partial_n u, \qquad t > 0,\ x \in \Gamma, && (1.3)\\ \partial_n w &= 0, \qquad t > 0,\ x \in \Gamma, && (1.4) \end{aligned}$$

where Δ_Γ is the Laplace-Beltrami operator on the boundary Γ, f and g are given nonlinear interaction functions and λ is some given positive constant. The boundary condition (1.3) will be interpreted as an additional second-order parabolic equation on the boundary Γ.
Problem (1.1)-(1.2) was introduced by Karali and Katsoulakis in [10] as a simplification of a mesoscopic model for multiple microscopic mechanisms in surface processes such that surface diffusion and adsorption-desorption and studied in [9], [11] and [12]; questions related to the well-posedness and to the asymptotic behavior, such as the existence of the global attractor and an exponential attractor have been answered under various assumptions on the nonlinearities and when the system is endowed with Dirichlet and Neumann boundary conditions;

Here u is the order parameter and corresponds to a rescaled density of atoms and w is the chemical potential. Now, the question of how the phase separation process is influenced by the presence of walls has recently gained much attention and was mainly studied for polymer mixtures when the boundary conditions are given by (1.3)-(1.4). The well-posedness and the long time behavior of problem (1.1)-(1.4) were studied in [7] and [8], respectively with singular and regular potentials.
The chapter is organized as follows. We first introduce in Section 2 some notation and assumptions and, in Section 3, we study a space discretization of (1.1)-(1.4) by a Galerkin method. In Section 4, we prove optimal error estimates for the difference between the approximate and the exact solution $u^h - u$ in energy norms and weaker norms as the mesh step h tends to 0, where u^h is the solution of the space semi-discrete scheme and u is the solution of the continuous problem. In Section 5, we study the numerical stability of the fully discrete problem obtained by applying the Euler implicit method to the space semi-discrete problem. In particular, we show that this fully discrete problem is unconditionally stable and the solution converges to equilibrium as $h \to 0$. Finally, numerical simulations in two space dimensions are presented in Section 6.

2 Assumptions and notation

In what follows, we consider Ω to be a $2d$ or $3d$ slab, i.e.

$$\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times (0, L_d),\ L_i > 0, i = 1, ..., d,\ d = 2 \text{ or } 3,$$

with smooth boundary

$$\Gamma = \partial\Omega = \Pi_{i=1}^{d-1}(\mathbb{R}/(L_i\mathbb{Z})) \times \{0, L_d\}\,.$$

More precisely, when $d = 2$, Ω is the rectangle $(0, L_1) \times (0, L_2)$ and u, w are periodic in the x_1-direction while the boundary conditions in problem (1.1) are valid for $x_2 = 0$ and $x_2 = L_2$; when $d = 3$, Ω is a parallelepiped $(0, L_1) \times (0, L_2) \times (0, L_3)$, u and w are periodic in the x_1 and x_2-directions and the boundary conditions in problem (1.1) are valid for $x_3 = 0$ and $x_3 = L_3$.
We assume that the nonlinearities f and g belong to $C^2(\mathbb{R}, \mathbb{R})$ and satisfy the following standard dissipativity assumptions

$$\liminf_{|v|\to\infty} f'(v) > 0,\ \liminf_{|v|\to\infty} g'(v) > 0. \tag{2.1}$$

Typical choices are

$$f(v) = v^3 - v \text{ and } g(v) = kv - h \quad (v \in \mathbb{R}), \tag{2.2}$$

where $k > 0$ and $h \in \mathbb{R}$ are constants. The evolution boundary value problem (1.1)-(1.4) is completed by the initial condition $u(0) = u_0$.
We introduce the space

$$V = \left\{ v \in H^1_p(\Omega) \text{ and } v(\cdot, 0),\ v(\cdot, L_d) \in H^1_{per}\left(\Pi_{i=1}^{d-1}]0, L_i[\right) \right\},$$

where H^1_{per} is the classical space of periodic functions and

$$H^1_p(\Omega) = \left\{ v \in H^1(\Omega),\ v \text{ is periodic in the } x_1, \cdots, x_{d-1} - \text{directions} \right\}.$$

Then, V is a Hilbert space for the Hilbertian norm

$$\|v\|_V = \left(\|v\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)} \right)^{1/2}.$$

The space V can also be seen as the closure for the $\|\cdot\|_V-$norm of $C^1(\overline{\Omega})$ with the periodicity condition; V is continuously and densely embedded in $H^1(\Omega)$ and is isometric to the closed subspace $\tilde{V}$ of $H^1_p(\Omega) \times H^1_{per}(\Gamma)$ defined by:

$$\tilde{V} = \left\{ (u, \varphi) \in H^1_p(\Omega) \times H^1_{per}(\Gamma),\ u_{|\Gamma} = \varphi \text{ in the sense of traces} \right\}.$$

We denote by $(\cdot,\cdot)_\Omega$ the $L^2(\Omega)-$scalar product and by $|\cdot|_0$ the $L^2(\Omega)-$norm. Similarly, $(\cdot,\cdot)_\Gamma$ denotes the $L^2(\Gamma)-$scalar product and $|\cdot|_{0,\Gamma}$ the associated norm. As a shortcut, we will denote by $\|\cdot\|_k$ the Hilbertian norm in $H^k(\Omega)$ and $|\cdot|_k$ the associated seminorm, i.e.

$$\forall v \in H^k(\Omega), \|v\|^2_k = \sum_{|\alpha| \le k} |\partial^\alpha v|^2_0 \text{ and } |v|^2_k = \sum_{|\alpha| = k} |\partial^\alpha v|^2_0.$$

Similarly, $\|\cdot\|_{k,\Gamma}$ denotes the $H^k(\Gamma)-$norm and $|\cdot|_{k,\Gamma}$ the associated seminorm.

We note that from (2.1), we can deduce that (see [1])

$$F(v) \ge c_1 v^2 - c_2, \quad G(v) \ge c_1 v^2 - c_2, \quad \forall v \in \mathbb{R}, \tag{2.3}$$

for some constants $c_1 > 0$ and $c_2 \ge 0$ and where F is an antiderivative of f and G is an antiderivative of g.

3 The semi-discrete scheme

The variational formulation of (1.1)-(1.4) reads

$$\left\{ \begin{array}{ll} (u_t, \varphi)_\Omega & = -(\nabla w, \nabla \varphi)_\Omega - (w, \varphi)_\Omega, \\ (w, \chi)_\Omega & = (f(u), \chi)_\Omega + (\nabla u, \nabla \chi)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma \chi)_\Gamma + \lambda (u, \chi)_\Gamma + (g(u), \chi)_\Gamma \\ & \quad +(u_t, \chi)_\Gamma, \end{array} \right. \tag{3.1}$$

for all $\varphi \in H^1_p(\Omega)$ and for all $\chi \in V$.
We introduce the functional $\mathcal{E} : V \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x + \int_\Gamma \left(\frac{1}{2} |\nabla_\Gamma u|^2 + \frac{\lambda}{2} |u|^2 + G(u) \right) \mathrm{d}\sigma. \tag{3.2}$$

If u is a regular solution of (1.1)-(1.4), then u dissipates $\mathcal{E}$. Indeed, choosing $\varphi = w$ and $\chi = u_t$ in (3.1) and subtracting the two equations, we obtain

$$\frac{d}{dt} \mathcal{E}(u(t)) = -\int_\Omega |\nabla w|^2 \mathrm{d}x - \int_\Omega |w|^2 \mathrm{d}x - \int_\Gamma |u_t|^2 \mathrm{d}\sigma, \quad \forall t \ge 0. \tag{3.3}$$

For the space discretization, we consider a quasiuniform family of decompositions $\left\{ \Omega^h \right\}_h$ of $\Pi_{i=1}^d [0, L_i]$ into $d-$simplices which take into account the periodic boundary conditions on Ω, so that $\{\Omega^h\}_h$ is also a triangulation of $\overline{\Omega}$. The triangulation Ω^h of $\overline{\Omega}$ induces a triangulation Γ^h of Γ into $d-1$ simplices in a natural way. For a given triangulation $\Omega^h = \cup_{T\in\Omega^h} T$, we define V^h as the usual P^1 conforming finite element space

$$V^h = \left\{ v^h \in C^0(\overline{\Omega}), v^h|_T \text{ is affine } \forall T \in \Omega^h \right\}.$$

For $u \in C^0(\overline{\Omega})$, let $I^h u$ denote the P^1 interpolate of u on Ω^h, i.e. $I^h u$ is the unique function in V^h which takes the same values as u on the nodes of the triangulation. We have the following standard approximation results, where $C > 0$ denotes a constant which depends only on $\{\Omega^h\}_h$

$$\forall u \in H^2_p(\Omega),\ |u - I^h u|_0 + h|u - I^h u|_1 \le Ch^2 |u|_2 \tag{3.4}$$

and

$$\forall \varphi \in H^2_{per}(\Gamma),\ |\varphi - I^h\varphi|_{0,\Gamma} + h|\varphi - I^h\varphi|_{1,\Gamma} \le Ch^2|\varphi|_{2,\Gamma}. \tag{3.5}$$

Moreover, we have the following inverse estimate (see [5])

$$\forall v^h \in V^h,\ \|v^h\|_{C^0(\overline{\Omega})} \le Ch^{-d/2}|v^h|_0, \tag{3.6}$$

where d is the space dimension.

The space semi-discrete version of (3.1) reads:
Find $(u^h, w^h) : [0,T] \to V^h \times V^h$ such that

$$\left\{ \begin{array}{ll} (u^h_t, \varphi)_\Omega & = -(\nabla w^h, \nabla\varphi)_\Omega - (w^h, \varphi)_\Omega, \\ (w^h, \chi)_\Omega & = (f(u^h), \chi)_\Omega + (\nabla u^h, \nabla\chi)_\Omega + (\nabla_\Gamma u^h, \nabla_\Gamma \chi)_\Gamma + \lambda(u^h, \chi)_\Gamma + (g(u^h), \chi)_\Gamma \\ & \quad +(u^h_t, \chi)_\Gamma, \end{array} \right. \tag{3.7}$$

for all $\varphi,\ \chi \in V^h$.
We define the operator $G^h : L^2(\Omega) \to V^h$, $v \to G^h v$, where $G^h v$ is the unique solution of the problem

$$(\nabla G^h v, \nabla\chi)_\Omega + (G^h v, \chi)_\Omega = (v, \chi)_\Omega,\ \forall \chi \in V^h. \tag{3.8}$$

We also define the discrete norm

$$|v|_{-1,h} = (G^h v, v)^{1/2}_\Omega = \left(|\nabla G^h v|^2_0 + |G^h v|^2_0\right)^{\frac{1}{2}},\ \forall v \in L^2(\Omega).$$

The norm $|\cdot|_{-1,h}$ is a discrete version of the H^{-1}-norm. We note that G^h is selfadjoint and positive definite on $L^2(\Omega)$. Indeed, for $\chi = G^h v$ in (3.8), we have

$$(G^h v, v)_\Omega = |\nabla G^h v|^2_0 + |G^h v|^2_0 \ge 0$$

and

$$(v, G^h v')_\Omega = (\nabla G^h v, \nabla G^h v')_\Omega + (G^h v, G^h v')_\Omega = (v', G^h v)_\Omega,$$

for all $v, v' \in L^2(\Omega)$. Moreover, the following interpolation inequalities hold

$$|v^h|^2_0 \le |v^h|_{-1,h}\|v^h\|_1,\ \forall v^h \in V^h, \tag{3.9}$$

and

$$|v|_{-1,h} \le |v|_0,\ \forall v \in L^2(\Omega). \tag{3.10}$$

In order to prove (3.9), we write

$$\begin{aligned}|v^h|_0^2 &= (\nabla G^h v^h, \nabla v^h)_\Omega + (G^h v^h, v^h)_\Omega \\ &\leq |\nabla G^h v^h|_0 |\nabla v^h|_0 + |G^h v^h|_0 |v^h|_0 \\ &\leq (|\nabla G^h v^h|_0^2 + |G^h v^h|_0^2)^{\frac{1}{2}} (|\nabla v^h|_0^2 + |v^h|_0^2)^{\frac{1}{2}} \\ &= |v^h|_{-1,h} \|v^h\|_1.\end{aligned}$$

To prove (3.10), we write

$$|v|_{-1,h}^2 = (G^h v, v)_\Omega \leq |G^h v|_0 |v|_0$$

and

$$|G^h v|_0^2 \leq \|G^h v\|_1^2 = |\nabla G^h v|_0^2 + |G^h v|_0^2 = |v|_{-1,h}^2.$$

Proposition 1. *For every $u_0^h \in V^h$, problem (3.7) has a unique solution*

$$(u^h, w^h) \in C^1([0, +\infty); V^h \times V^h),$$

such that $u^h(0) = u_0^h$. Moreover,

$$\mathcal{E}(u^h(t)) + \int_0^t (|w^h|_1^2 + |w^h|_0^2 + |u_t^h|_{0,\Gamma}^2) \mathrm{d}s \leq \mathcal{E}(u^h(0)), \ \forall t \geq 0, \tag{3.11}$$

where $\mathcal{E}$ is defined by (3.2).

Proof. Let $(\varphi_1, \ldots, \varphi_m)$ be an orthonormal basis of V^h for the $L^2(\Omega)$-scalar product. We seek for $u^h(t) = \sum_{i=1}^m u_i(t)\varphi_i$ and $w^h(t) = \sum_{i=1}^m w_i(t)\varphi_i$. We define the matrices

$$A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)_\Omega, \ (M_\Gamma)_{ij} = (\varphi_i, \varphi_j)_\Gamma \text{ and } (A_\Gamma)_{ij} = (\nabla_\Gamma\varphi_i, \nabla_\Gamma\varphi_j)_\Gamma,$$

for $1 \leq i, j \leq m$, the vectors $U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$, and the functions

$$F^h(U) = \begin{pmatrix} (f(u^h), \varphi_1)_\Omega \\ \vdots \\ (f(u^h), \varphi_m)_\Omega \end{pmatrix}, \ G_\Gamma^h(U) = \begin{pmatrix} (\lambda u^h + g(u^h), \varphi_1)_\Gamma \\ \vdots \\ (\lambda u^h + g(u^h), \varphi_m)_\Gamma \end{pmatrix}.$$

Then (3.7) can be written as

$$\begin{pmatrix} (A+I) & I \\ -I & M_\Gamma \end{pmatrix} \begin{pmatrix} W \\ U' \end{pmatrix} = - \begin{pmatrix} 0 \\ AU + F^h(U) + A_\Gamma U + G_\Gamma^h(U) \end{pmatrix}. \tag{3.12}$$

Let B denote the square matrix of size $2M$ in the left-hand side of (3.12). We claim that B is invertible. Indeed, let $X, Y \in \mathbb{R}^M$. We have

$$\begin{aligned}(X^t, Y^t) B \begin{pmatrix} X \\ Y \end{pmatrix} &= X^t(A+I)X + X^tY - Y^tX + Y^t M_\Gamma Y \\ &= (\nabla x^h, \nabla x^h)_\Omega + (x^h, x^h)_\Omega + (y^h, y^h)_\Gamma \\ &\geq 0,\end{aligned} \tag{3.13}$$

where $x^h = \sum_{i=1}^m x_i\varphi_i$ and $y^h = \sum_{i=1}^m y_i\varphi_i$. This shows that B is positive semidefinite. Now, if $X, Y \in \mathbb{R}^M$ satisfy

$$B\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} (A+I)X + BY = 0 \\ -X + M_\Gamma Y = 0, \end{cases}$$

then by multiplying this equality on the left by (X^t, Y^t), we find that $(\nabla x^h, \nabla x^h)_\Omega + (x^h, x^h)_\Omega + (y^h, y^h)_\Gamma = 0$. This implies $X = Y = 0$ which yields that B is invertible, as claimed. Thus, problem (3.7) has a unique maximal solution $(u^h, w^h) \in C^1([0, T^+); V^h \times V^h)$ such that $u^h(0) = u_0^h$. Choosing $\varphi = w^h$ and $\chi = u_t^h$ in (3.7), we find

$$\frac{d}{dt}\mathcal{E}(u^h(t)) + |w^h|_1^2 + |w^h|_0^2 + |u_t^h|_{0,\Gamma}^2 = 0. \tag{3.14}$$

Integrating with respect to t, we deduce (3.11). Using (2.3), equation (3.11) leads to the a priori bound $\|u^h\|_1 \le C(R)$ provided that $u_0^h \in V^h$ with $\|u_0^h\|_1 \le R$.
Since $u^h \in L^\infty(0, T^+; H^1(\Omega))$, i.e. the bound is independent of time, we find that the solution is global, i.e. $T^+ = +\infty$, and the proof is complete. □

Exclusively for the next Theorem 3.1, we assume that f has a subcritical growth. In other words, we assume that there exists a positive constant c_3 such that

$$|f(s)| \le c_3(1 + |s|^{p-1}), \ \forall s \in \mathbb{R}, \tag{3.15}$$

with $p \in [2, 6]$ when $d = 3$ and $p \ge 2$ arbitrary when $d = 2$. When $d = 3$, we also assume that there exists a positive constant c_4 such that

$$|g(s)| \le c_4(1 + |s|^{q-1}), \ \forall s \in \mathbb{R}, \tag{3.16}$$

where $q \ge 2$ is arbitrary. The typical choices (2.2) satisfy these assumptions with $p = 4$ and $q = 2$. We have the following theorem:

Theorem 3.1. *Assume that $f, g \in C^1(\mathbb{R})$ satisfy (2.1), (3.15) and (3.16). Let $u_0 \in V$ and let $u_0^h \in V^h$ be such that $u_0^h \to u_0$ in V as $h \to 0$. Then, for all $T > 0$, we have*

$$u^h \to u \text{ weak * in } L^\infty(0, T; H_p^1(\Omega)) \text{ and strongly in } C^0([0, T]; L^2(\Omega)),$$

$$(u^h)_{|\Gamma} \to u_{|\Gamma} \text{ weak * in } L^\infty(0, T; H_{per}^1(\Gamma)) \text{ and strongly in } C^0([0, T]; L^2(\Gamma)),$$

$$w^h \to w \text{ weakly in } L^2(0, T; H_p^1(\Omega)),$$

where (u, w) is the unique solution of (3.1) such that $u(0) = u_0$ and

$$u \in L^\infty(0, T; V), \ u_{|\Gamma} \in W^{1,2}(0, T; L^2(\Gamma)) \text{ and } w \in L^2(0, T; H_p^1(\Omega)). \tag{3.17}$$

Proof. By (3.15) and (3.16), we have that

$$|F(\sigma)| \le c_5|\sigma|^p + c_6 \text{ and } |G(\sigma)| \le c_7|\sigma|^q + c_8, \forall \sigma \in \mathbb{R}, \tag{3.18}$$

where c_5, c_6, c_7 and c_8 are positive constants. Since $u_0^h \to u_0$ in V, using (3.18) and the Sobolev embeddings $H_p^1(\Omega) \subset L^p(\Omega)$ and $H_{per}^1(\Gamma) \subset L^q(\Gamma)$ (with $q = +\infty$ when $d = 2$), we know that

$\mathcal{E}(u_0^h)$ is bounded by a constant independent of h. The discrete energy estimate (3.11) implies that $(u^h)_h$ is bounded in $L^\infty(0,T;V)$, $(\nabla w^h)_h,\ (w^h)_h$ are bounded in $L^2(0,T;L^2(\Omega))$ and that $((u^h_{|\Gamma})_t)_h$ is bounded in $L^2(0,T;L^2(\Gamma))$. Thus, we obtain that, up to a subsequence, $u^h \to u$ weak * in $L^\infty(0,T;V)$ and $w^h \to w$ weakly in $L^2(0,T;H^1(\Omega))$.
If $\varphi \in H^1_p(\Omega)$ and $\chi \in V$, then choosing sequences $\varphi^h \in V^h$ and $\chi^h \in V^h$ such that $\varphi^h \to \varphi$ strongly in $H^1(\Omega)$ and $\chi^h \to \chi$ strongly in V. Using standard compactness results, we can pass to the limit in (3.7) and we obtain that (u,w) satisfies (1.1)-(1.4) and (3.17). By uniqueness, the whole sequence (u^h,w^h) converges to (u,w).
For the strong convergence of $(u^h_{|\Gamma})$ to $u_{|\Gamma}$, we use the fact that the space

$$\left\{v \in L^\infty(0,T;H^1_{per}(\Gamma)),\ v_t \in L^2(0,T;L^2(\Gamma))\right\}$$

is compactly embedded into $C^0([0,T];L^2(\Gamma))$. Finally, for the strong convergence of $(u^h)_h$, we use the fact that for all $0 \le s \le t \le T$, we have

$$\begin{aligned}
|u^h(t)-u^h(s)|_0^2 &= 2\int_s^t \left(u^h_t(\sigma), u^h(\sigma)-u^h(s)\right)_\Omega \mathrm{d}\sigma \\
&\le -2\int_s^t \left(\nabla w^h(\sigma), \nabla(u^h(\sigma)-u^h(s))\right)_\Omega \mathrm{d}\sigma - 2\int_s^t \left(w^h(\sigma), u^h(\sigma)-u^h(s)\right)_\Omega \mathrm{d}\sigma \\
&\le 4c\|u^h\|_{L^\infty(0,T;H^1_p(\Omega))}\left(\|\nabla w^h\|_{L^2(0,T;L^2(\Omega))} + \|w^h\|_{L^2(0,T;L^2(\Omega))}\right)|t-s|^{1/2}.
\end{aligned} \tag{3.19}$$

Thus, the sequence $(u^h)_h$ is uniformly equicontinuous in $C^0([0,T];L^2(\Omega))$. Since (u^h) is bounded in $C^0([0,T];H^1_p(\Omega))$ with $H^1_p(\Omega)$ compactly embedded into $L^2(\Omega)$, the Ascoli theorem implies that $u^h \to u$ strongly in $C^0([0,T];L^2(\Omega))$. □

4 Error estimates for the space semi-discrete scheme

In order to estimate the errors $u^h - u$ and $w^h - w$ in appropriate norms, we follow a standard approach (see [1], [2] and [18]) and we write

$$\begin{aligned}
&u^h(t) - u(t) = \theta^u(t) + \rho^u(t), \text{ with } \theta^u = u^h - \tilde{u}^h,\ \rho^u = \tilde{u}^h - u, \\
&w^h(t) - w(t) = \theta^w(t) + \rho^w(t), \text{ with } \theta^w = w^h - \tilde{w}^h,\ \rho^w = \tilde{w}^h - w,
\end{aligned}$$

for all $t \in [0,T]$, where $\tilde{u}^h = \tilde{u}^h(t)$ and $\tilde{w}^h = \tilde{w}^h(t)$ are the elliptic projections of $u = u(t)$ and $w = w(t)$, defined by

$$(\nabla \tilde{w}^h, \nabla\chi)_\Omega + (\tilde{w}^h, \chi)_\Omega = (\nabla w, \nabla \chi)_\Omega + (w,\chi)_\Omega,\ \forall \chi \in V^h, \tag{4.1}$$

$$\begin{aligned}
(\nabla \tilde{u}^h, \nabla\chi)_\Omega + (\nabla_\Gamma \tilde{u}^h, \nabla_\Gamma \chi)_\Gamma + \lambda(\tilde{u}^h,\chi)_\Gamma = (\nabla u, \nabla\chi)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma\chi)_\Gamma \\
+ \lambda(u,\chi)_\Gamma,\ \forall\chi \in V^h.
\end{aligned} \tag{4.2}$$

For a given $w \in H^1(\Omega)$, equation (4.1) defines a unique $\tilde{w}^h \in V^h$. Indeed, the bilinear form defined by

$$\tilde{a}(\varphi,\chi) = (\nabla\varphi, \nabla\chi)_\Omega + (\varphi,\chi)_\Omega \tag{4.3}$$

is the scalar product on $H^1(\Omega)$. Thus, applying the Lax-Milgram theorem, we obtain the result. Similarly, for a given $u \in V$, equation (4.2) defines a unique $\tilde{u}^h \in V^h$. Indeed, the norm $v \mapsto |\nabla v|_0^2 + |v|_{0,\Gamma}^2$ is equivalent to the H^1-norm, so that the bilinear form defined by

$$a(\varphi, \chi) = (\nabla\varphi, \nabla\chi)_\Omega + (\nabla_\Gamma\varphi, \nabla_\Gamma\chi)_\Gamma + \lambda(\varphi, \chi)_\Gamma \tag{4.4}$$

is coercive on V, i.e. there exists $c_0 > 0$ such that

$$a(\varphi, \varphi) = (\nabla\varphi, \nabla\varphi)_\Omega + (\nabla_\Gamma\varphi, \nabla_\Gamma\varphi)_\Gamma + \lambda(\varphi, \varphi)_\Gamma \geq c_0\|\varphi\|_V^2, \ \forall\varphi \in V.$$

The bilinear continuous form $a(\cdot, \cdot)$ is a fortiori coercive on $V^h \subset V$ and the Lax-Milgram theorem applies.

Lemma 4.1. *For all $w \in H^2(\Omega)$, the function $\tilde{w}^h \in V^h$ defined by*

$$(\nabla\tilde{w}^h, \nabla\chi)_\Omega + (\tilde{w}^h, \chi)_\Omega = (\nabla w, \nabla\chi)_\Omega + (w, \chi)_\Omega, \ \forall\chi \in V^h,$$

satisfies

$$|\tilde{w}^h - w|_0 + h|\tilde{w}^h - w|_1 \leq Ch^2|w|_2, \tag{4.5}$$

where C is a positive constant, independent of h.

Proof. By definition, we have

$$\tilde{a}(\tilde{w}^h, \chi) = \tilde{a}(w, \chi), \quad \forall\chi \in V^h, \tag{4.6}$$

where $\tilde{a}(\cdot, \cdot)$ is defined by (4.3). Since $\tilde{w}^h - I^h w \in V^h$, we have that $\tilde{a}(\tilde{w}^h - w, \tilde{w}^h - I^h w) = 0$, which yields

$$\begin{aligned} \tilde{a}(\tilde{w}^h - w, \tilde{w}^h - w) &= \tilde{a}(\tilde{w}^h - w, \tilde{w}^h - I^h w) + \tilde{a}(\tilde{w}^h - w, I^h w - w), \\ &= \tilde{a}(\tilde{w}^h - w, I^h w - w), \end{aligned}$$

implying

$$\|\tilde{w}^h - w\|_1^2 = \tilde{a}(\tilde{w}^h - w, \tilde{w}^h - w) \leq \|\tilde{w}^h - w\|_1\|I^h w - w\|_1.$$

By (3.4), we have

$$\|\tilde{w}^h - w\|_1 \leq \|I^h w - w\|_1 \leq Ch|w|_2, \tag{4.7}$$

which gives the H^1-estimate. In order to have the L^2-estimate, we set $\varphi \in H^1(\Omega)$ to be the unique solution of

$$\tilde{a}(\varphi, \chi) = (z, \chi)_\Omega, \quad \forall\chi \in V, \tag{4.8}$$

for a given function $z \in L^2(\Omega)$. Then, $\varphi \in H_p^2(\Omega)$ and thanks to the elliptic regularity, we have

$$|\varphi|_2 \leq C|z|_0, \tag{4.9}$$

for some constant $C > 0$ independent of z.
Choosing $\chi = \tilde{w}^h - w$ in (4.8) and using the fact that $\tilde{a}(\tilde{w}^h - w, I^h\varphi) = 0$, we find

$$\begin{aligned} (z, \tilde{w}^h - w)_\Omega =& \tilde{a}(\varphi, \tilde{w}^h - w) \\ =& \tilde{a}(\varphi - I^h\varphi, \tilde{w}^h - w) \\ \leq& \|\varphi - I^h\varphi\|_1\|\tilde{w}^h - w\|_1. \end{aligned} \tag{4.10}$$

Choosing $z = \tilde{w}^h - w$, using (3.4), (4.7) and (4.9), we obtain

$$\begin{aligned} |\tilde{w}^h - w|_0^2 \leq & \|\varphi - I^h\varphi\|_1 \|\tilde{w}^h - w\|_1 \\ \leq & Ch|\varphi|_2 Ch|w|_2 \\ \leq & Ch^2|\tilde{w}^h - w|_0 |w|_2, \end{aligned} \tag{4.11}$$

which gives

$$|\tilde{w}^h - w|_0 \leq Ch^2|w|_2. \tag{4.12}$$

By (4.7) and (4.12), we can conclude and the proof is complete. □

Lemma 4.2. *For all $u \in H_p^2(\Omega)$ with $u_{|\Gamma} \in H^2_{per}(\Gamma)$ the function $\tilde{u}^h \in V^h$ defined by (4.2) satisfies*

$$|\tilde{u}^h - u|_0 + |\tilde{u}^h - u|_{0,\Gamma} + h|\tilde{u}^h - u|_1 + h|\tilde{u}^h - u|_{1,\Gamma} \leq Ch^2(|u|_2 + |u|_{2,\Gamma}), \tag{4.13}$$

where the positive constant C is independent of h.

Proof. Arguing as above, we have

$$a(\tilde{u}^h, \chi) = a(u, \chi), \quad \forall \chi \in V^h, \tag{4.14}$$

where $a(\cdot,\cdot)$ is defined by (4.4). Since $\tilde{u}^h - I^h u \in V^h$, we have $a(\tilde{u}^h - u, \tilde{u}^h - I^h u) = 0$, which yields

$$\begin{aligned} a(\tilde{u}^h - u, \tilde{u}^h - u) &= a(\tilde{u}^h - u, \tilde{u}^h - I^h u) + a(\tilde{u}^h - u, I^h u - u), \\ &= a(\tilde{u}^h - u, I^h u - u). \end{aligned} \tag{4.15}$$

From (4.15) and using the coercivity of a, we obtain

$$c_0\|\tilde{u}^h - u\|_V^2 \leq a(\tilde{u}^h - u, \tilde{u}^h - u) \leq c\|\tilde{u}^h - u\|_V \|I^h u - u\|_V.$$

By (3.5), we have

$$\|\tilde{u}^h - u\|_V \leq c\|I^h u - u\|_V \leq ch(|u|_2 + |u|_{2,\Gamma}), \tag{4.16}$$

which gives the H^1-estimates. In order to have the L^2-estimates, we set $\varphi \in V$ to be the unique solution of

$$a(\varphi, \chi) = (z, \chi)_\Omega + (\psi, \chi)_\Gamma, \quad \forall \chi \in V, \tag{4.17}$$

for some given functions $(z, \psi) \in L^2(\Omega) \times L^2(\Gamma)$. Then, thanks to an elliptic regularity result (see [15]), we have $\varphi \in H_p^2(\Omega)$, $\varphi_{|\Gamma} \in H^2_{per}(\Gamma)$ and

$$|\varphi|_2 + |\varphi|_{2,\Gamma} \leq C(|z|_0 + |\psi|_{0,\Gamma}), \tag{4.18}$$

for some constant $C > 0$ independent of z and ψ.
Choosing $\chi = \tilde{u}^h - u$ in (4.17) and using the fact that $a(\tilde{u}^h - u, I^h\varphi) = 0$, we find

$$\begin{aligned} (z, \tilde{u}^h - u)_\Omega + (\psi, \tilde{u}^h - u)_\Gamma =& a(\varphi, \tilde{u}^h - u) \\ =& a(\varphi - I^h\varphi, \tilde{u}^h - u) \\ \leq& c\|\varphi - I^h\varphi\|_V \|\tilde{u}^h - u\|_V. \end{aligned} \tag{4.19}$$

Choosing $z = \tilde{u}^h - u$ and $\psi = (\tilde{u}^h - u)_\Gamma$, using (3.5), (4.16) and (4.18), we obtain

$$
\begin{aligned}
|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2 \leq & c\|\varphi - I^h\varphi\|_1 \|\tilde{u}^h - u\|_1 \\
\leq & Ch(|\varphi|_2 + |\varphi|_{2,\Gamma}) Ch(|u|_2 + |u|_{2,\Gamma}) \\
\leq & Ch^2(|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2)^{1/2}(|u|_2 + |u|_{2,\Gamma}).
\end{aligned} \tag{4.20}
$$

Estimate (4.20) leads to

$$
(|\tilde{u}^h - u|_0^2 + |\tilde{u}^h - u|_{0,\Gamma}^2)^{1/2} \leq Ch^2(|u|_2 + |u|_{2,\Gamma}). \tag{4.21}
$$

By (4.16) and (4.21), we can conclude and the proof is complete. $\square$

Lemma 4.3. *Let (u, w) the a solution of (3.1) with the initial condition $u(0) = u_0 \in V$ and (u^h, w^h) be the solution of (3.7) with the initial condition $u^h(0) = u_0^h \in V^h$. Assume that*

$$
\sup_{t\in[0,T]} \|u(t)\|_{C^0(\overline{\Omega})} < R, \quad \sup_{t\in[0,T]} \|u_t(t)\|_{C^0(\overline{\Omega})} < R, \quad \|u^h(0)\|_{C^0(\overline{\Omega})} < R,
$$

for some constant $R < +\infty$, and let $T^h \in (0, T]$ be the maximal time such that $\|u^h(t)\|_{L^\infty(\Omega)} \leq R$ for all $t \in [0, T^h]$. Then, the following estimate holds:

$$
\begin{aligned}
&\mathcal{N}(t) + \int_0^t (|\theta_t^u|_1^2 + \|\theta^w\|_1^2 + \|\theta_t^u\|_{1,\Gamma}^2) \mathrm{d}s \\
\leq & C\mathcal{N}(0) + C' \int_0^t (|\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho^u|_{0,\Gamma}^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^w|_0^2 + |\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2) \mathrm{d}s,
\end{aligned} \tag{4.22}
$$

for some positive constants C and C' which are independent of u, u^h and h, where

$$
\mathcal{N}(t) = |\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2.
$$

Proof. Subtracting the first equation of (3.1) from the first equation of (3.7), we obtain

$$
(u_t^h - u_t, \varphi)_\Omega + (\nabla(w^h - w), \nabla\varphi)_\Omega + (w^h - w, \varphi)_\Omega = 0, \ \forall \varphi \in V^h. \tag{4.23}
$$

Using the definitions of θ^u and θ^w as well as (4.1), we find

$$
(\theta_t^u, \varphi)_\Omega + (\nabla\theta^w, \nabla\varphi)_\Omega + (\theta^w, \varphi)_\Omega = -(\rho_t^u, \varphi)_\Omega, \ \forall \varphi \in V^h. \tag{4.24}
$$

Choosing $\varphi = \theta^w$ in (4.24), we obtain

$$
(\theta_t^u, \theta^w)_\Omega + |\theta^w|_1^2 + |\theta^w|_0^2 = -(\rho_t^u, \theta^w)_\Omega. \tag{4.25}
$$

Now, subtracting the second equation of (3.1) from the second equation of (3.7) and using (4.2), we find

$$
\begin{aligned}
&-(\theta^w, \chi)_\Omega + (\nabla\theta^u, \nabla\chi)_\Omega + (\nabla_\Gamma\theta^u, \nabla_\Gamma\chi)_\Gamma + \lambda(\theta^u, \chi)_\Gamma + (\theta_t^u, \chi)_\Gamma \\
&\quad = (\rho^w, \chi)_\Omega - (f(u^h) - f(u), \chi)_\Omega - (g(u^h) - g(u), \chi)_\Gamma - (\rho_t^u, \chi)_\Gamma,
\end{aligned} \tag{4.26}
$$

for all $\chi \in V^h$.
Choosing $\chi = \theta^u_t$, estimate (4.26) gives

$$\begin{aligned}
&-(\theta^w, \theta^u_t)_\Omega + |\theta^u_t|^2_{0,\Gamma} + \frac{1}{2}\frac{d}{dt}(|\theta^u|^2_1 + \lambda|\theta^u|^2_{0,\Gamma} + |\theta^u|^2_{1,\Gamma}) \\
&\quad = (\rho^w, \theta^u_t)_\Omega - (f(u^h) - f(u), \theta^u_t)_\Omega - (g(u^h) - g(u), \theta^u_t)_\Gamma - (\rho^u_t, \theta^u_t)_\Gamma.
\end{aligned} \tag{4.27}$$

Summing (4.25) and (4.27), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\theta^u|^2_1 + \lambda|\theta^u|^2_{0,\Gamma} + |\theta^u|^2_{1,\Gamma}) + |\theta^w|^2_1 + |\theta^w|^2_0 + |\theta^u_t|^2_{0,\Gamma} \\
&\quad = -(\rho^u_t, \theta^w)_\Omega + (\rho^w, \theta^u_t)_\Omega - (f(u^h) - f(u), \theta^u_t)_\Omega - (g(u^h) - g(u), \theta^u_t)_\Gamma - (\rho^u_t, \theta^u_t)_\Gamma.
\end{aligned} \tag{4.28}$$

We have:

$$\begin{aligned}
&|f(u^h) - f(u)|_0 \le L_f|u^h - u|_0, \\
&|g(u^h) - g(u)|_{0,\Gamma} \le L_g|u^h - u|_{0,\Gamma},
\end{aligned} \tag{4.29}$$

on $[0, T^h]$, where L_f and L_g are respectively the Lipschitz constants of f and g on $[-R, R]$. Thus, using (4.29) and the Hölder inequality, estimate (4.28) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\theta^u|^2_1 + \lambda|\theta^u|^2_{0,\Gamma} + |\theta^u|^2_{1,\Gamma}) + |\theta^w|^2_1 + |\theta^w|^2_0 + |\theta^u_t|^2_{0,\Gamma} \\
\le&|\rho^u_t|_0|\theta^w|_0 + |\rho^w|_0|\theta^u_t|_0 + L_f(|\theta^u|_0 + |\rho^u|_0)|\theta^u_t|_0 \\
&+ L_g(|\theta^u|_{0,\Gamma} + |\rho^u|_{0,\Gamma})|\theta^u_t|_{0,\Gamma} + |\rho^u_t|_{0,\Gamma}|\theta^u_t|_{0,\Gamma}.
\end{aligned} \tag{4.30}$$

Using the inequality

$$ab \le \epsilon a^2 + 1/(4\epsilon)b^2, \quad \forall a, b \ge 0, \quad \forall \epsilon > 0,$$

with ε conveniently chosen, estimate (4.30) gives

$$\begin{aligned}
&\frac{d}{dt}(|\theta^u|^2_1 + \lambda|\theta^u|^2_{0,\Gamma} + |\nabla_\Gamma\theta^u|^2_{0,\Gamma}) + |\theta^w|^2_1 + |\theta^w|^2_0 + |\theta^u_t|^2_{0,\Gamma} \\
&\le C_1(|\rho^u_t|^2_0 + |\rho^u|^2_0 + |\rho^u|^2_{0,\Gamma} + |\rho^u_t|^2_{0,\Gamma} + |\rho^w|^2_0) + C_2(|\theta^u|^2_{0,\Gamma} + |\theta^u|^2_0 + |\theta^u_t|^2_0),
\end{aligned} \tag{4.31}$$

for some positive constants C_1 and C_2 which depend on $|\Omega|, |\Gamma|, L_f$ and L_g.
To estimate θ^u_t, we differentiate (4.24) and (4.26) with respect to t. We obtain

$$(\theta^u_{tt}, \varphi)_\Omega + (\nabla\theta^w_t, \nabla\varphi)_\Omega + (\theta^w_t, \varphi)_\Omega = -(\rho^u_{tt}, \varphi)_\Omega, \forall\varphi \in V^h \tag{4.32}$$

and

$$\begin{aligned}
&-(\theta^w_t, \chi)_\Omega + (\nabla\theta^u_t, \nabla\chi)_\Omega + (\nabla_\Gamma\theta^u_t, \nabla_\Gamma\chi)_\Gamma + \lambda(\theta^u_t, \chi)_\Gamma + (\theta^u_{tt}, \chi)_\Gamma \\
&\quad = (\rho^w_t, \chi)_\Omega - ([f(u^h) - f(u)]_t, \chi)_\Omega - ([g(u^h) - g(u)]_t, \chi)_\Gamma - (\rho^u_{tt}, \chi)_\Gamma, \forall\chi \in V^h.
\end{aligned} \tag{4.33}$$

Choosing $\varphi = G^h\theta^u_t$ in (4.32) and $\chi = \theta^u_t$ in (4.33), adding the resulting equations and using the fact that

$$(\nabla\theta^w_t, \nabla G^h\theta^u_t)_\Omega + (\theta^w_t, G^h\theta^u_t)_\Omega = (\theta^w_t, \theta^u_t)_\Omega,$$

we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 + \lambda|\theta_t^u|_{0,\Gamma}^2 \\
=& -(\rho_{tt}^u, G^h\theta_t^u)_\Omega + (\rho_t^w, \theta_t^u)_\Omega - ([f(u^h) - f(u)]_t, \theta_t^u)_\Omega - ([g(u^h) - g(u)]_t, \theta_t^u)_\Gamma \\
&- (\rho_{tt}^u, \theta_t^u)_\Gamma.
\end{aligned} \tag{4.34}$$

Employing

$$[f(u^h) - f(u)]_t = f'(u^h)[u_t^h - u_t] + [f'(u^h) - f'(u)]u_t$$

and

$$[g(u^h) - g(u)]_t = g'(u^h)[u_t^h - u_t] + [g'(u^h) - g'(u)]u_t,$$

we find

$$\begin{aligned}
([f(u^h) - f(u)]_t, \theta_t^u)_\Omega =& (f'(u^h)[u_t^h - u_t], \theta_t^u)_\Omega + ([f'(u^h) - f'(u)]u_t, \theta_t^u)_\Omega \\
\leq& \sup_{[-R,R]} |f'|(|\theta_t^u|_0 + |\rho_t^u|_0)|\theta_t^u|_0 + RL_{f'}(|\theta^u|_0 + |\rho^u|_0)|\theta_t^u|_0
\end{aligned}$$

and

$$\begin{aligned}
([g(u^h) - g(u)]_t, \theta_t^u)_\Gamma =& (g'(u^h)[u_t^h - u_t], \theta_t^u)_\Gamma + ([g'(u^h) - g'(u)]u_t, \theta_t^u)_\Gamma \\
\leq& \sup_{[-R,R]} |g'|(|\theta_t^u|_{0,\Gamma} + |\rho_t^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma} + RL_{g'}(|\theta^u|_{0,\Gamma} + |\rho^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma},
\end{aligned}$$

where $L_{f'}$ and $L_{g'}$ are respectively the Lipschitz constants of f' and g' on $[-R, R]$. Thus, (4.34) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 + \lambda|\theta_t^u|_{0,\Gamma}^2 \\
\leq& |\rho_{tt}^u|_{-1,h}|\theta_t^u|_{-1,h} + |\rho_{tt}^u|_{0,\Gamma}|\theta_t^u|_{0,\Gamma} + |\rho_t^w|_0|\theta_t^u|_0 \\
&+ \sup_{[-R,R]} |f'|(|\theta_t^u|_0 + |\rho_t^u|_0)|\theta_t^u|_0 + RL_{f'}(|\theta^u|_0 + |\rho^u|_0)|\theta_t^u|_0 \\
&+ \sup_{[-R,R]} |g'|(|\theta_t^u|_{0,\Gamma} + |\rho_t^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma} + RL_{g'}(|\theta^u|_{0,\Gamma} + |\rho^u|_{0,\Gamma})|\theta_t^u|_{0,\Gamma}.
\end{aligned} \tag{4.35}$$

Using (3.10), the fact that the norm $v \mapsto |\nabla v|_0^2 + |v|_{0,\Gamma}^2$ is equivalent to the H^1-norm and the interpolation property (3.9) applied to θ_t^u, (4.35) gives

$$\begin{aligned}
&\frac{d}{dt}(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 + \lambda|\theta_t^u|_{0,\Gamma}^2 \\
&\leq C_3(|\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta^u|_0^2 + |\theta^u|_{0,\Gamma}^2) \\
&+ C_4(|\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2 + |\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^u|_{0,\Gamma}^2), \text{ on } (0, T^h),
\end{aligned} \tag{4.36}$$

for some positive constants C_3 and C_4 which depend on $\lambda, R, c_p, L_{f'}, L_{g'}, \sup_{[-R,R]} |f'|$ and $\sup_{[-R,R]} |f'|$.

Adding (4.31) and (4.36), we find

$$\begin{aligned}
&\frac{d}{dt}(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2) + |\theta^w|_1^2 + |\theta^w|_0^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta_t^u|_1^2 + |\theta_t^u|_{1,\Gamma}^2 \\
&\leq C_5(|\rho_t^u|_0^2 + |\rho^u|_0^2 + |\rho^u|_{0,\Gamma}^2 + |\rho_t^u|_{0,\Gamma}^2 + |\rho^w|_0^2 + |\rho_{tt}^u|_0^2 + |\rho_{tt}^u|_{0,\Gamma}^2 + |\rho_t^w|_0^2) \\
&+ C_6(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\theta^u|_{1,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2 + |\theta_t^u|_0^2),
\end{aligned} \tag{4.37}$$

where we use the fact that $|\theta^u|_0^2 \leq c(|\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2)$ and for θ_t^u we use again the interpolation property (3.9). Applying Gronwall's lemma, we find estimate (4.22) with $C = e^{C_6 T}$ and $C' = C_5 e^{C_6 T}$. □

Theorem 4.4. *Let* (u, w) *be the solution of problem* (1.1)-(1.4) *with the initial condition* $u(0) = u_0$ *such that*

$$u, u_t, u_{tt}, w, w_t \in L^2(0, T; H_p^2(\Omega)) \tag{4.38}$$

and

$$u_{|\Gamma}, (u_t)_{|\Gamma}, (u_{tt})_{|\Gamma}, w_{|\Gamma}, (w_t)_{|\Gamma} \in L^2(0, T; H_{per}^2(\Gamma)) \tag{4.39}$$

and let (u^h, w^h) *be the solution of problem* (3.7) *with initial condition* $u^h(0) = u_0^h$. *If*

$$\theta^u(0) = 0 \text{ and } \theta^w(0) = 0, \tag{4.40}$$

then the following estimates hold, for h *small enough:*

$$\begin{aligned} \sup_{[0,T]} \left(|u^h - u|_0 + |u^h - u|_{0,\Gamma} + |u_t^h - u_t|_{-1,h} + |u_t^h - u_t|_{0,\Gamma}\right) &\leq Ch^2, \\ \sup_{[0,T]} \left(|u^h - u|_1 + |u^h - u|_{1,\Gamma}\right) &\leq Ch, \\ \int_0^T |w^h - w|_0^2 \mathrm{d}s &\leq Ch^4, \\ \int_0^T \left(|w^h - w|_1^2 + |u_t^h - u_t|_1^2 + |u_t^h - u_t|_{1,\Gamma}^2\right) \mathrm{d}s &\leq Ch^2. \end{aligned}$$

Proof. If we differentiate equations (4.1) and (4.2) with respect to t, we obtain that the elliptic projections of u_t and w_t are respectively $(\tilde{u})_t$ and $(\tilde{w})_t$. A similar statement holds for u_{tt}. Therefore, Lemma 4.1 applies with w replaced by w_t and Lemma 4.2 applies with u replaced by u_t, u_{tt}, i.e.

$$\begin{gathered} |\rho_t^u|_0 + |\rho_t^u|_{0,\Gamma} + h|\rho_t^u|_1 + h|\rho_t^u|_{1,\Gamma} \leq Ch^2(|u_t|_2 + |u_t|_{2,\Gamma}), \\ |\rho_{tt}^u|_0 + |\rho_{tt}^u|_{0,\Gamma} + h|\rho_{tt}^u|_1 + h|\rho_{tt}^u|_{1,\Gamma} \leq Ch^2(|u_{tt}|_2 + |u_{tt}|_{2,\Gamma}), \\ |\rho_t^w|_0 + h|\rho_t^w|_1 \leq Ch^2|w_t|_2. \end{gathered} \tag{4.41}$$

The regularity required on u, u_t implies that $u \in C^1([0,T]; H_p^2(\Omega))$ and by the Sobolev continuous injection $H_p^2(\Omega) \subset C^0(\overline{\Omega})$, we see that u and u_t belong to $C^0([0,T]; C^0(\overline{\Omega}))$. Thus,

$$\sup_{t\in[0,T]} \|u(t)\|_{C^0(\overline{\Omega})} < R \text{ and } \sup_{t\in[0,T]} \|u_t(t)\|_{C^0(\overline{\Omega})} < R,$$

for some $R > 0$. We also have

$$\begin{aligned} \|u_0^h - u_0\|_{C^0(\overline{\Omega})} &\leq \|u_0^h - I^h u_0\|_{C^0(\overline{\Omega})} + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \\ &\leq Ch^{-d/2}|u_0^h - I^h u_0|_0 + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \text{ (using (3.6))} \\ &\leq Ch^{-d/2}(|u_0^h - u_0|_0 + |u_0 - I^h u_0|_0) + \|I^h u_0 - u_0\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{4.42}$$

Using the embedding that $H_p^2(\Omega) \subset C^{0,\gamma}(\Omega)$, where $\gamma \in (0,1)$, we find

$$\|I^h u_0 - u_0\|_{C^0(\overline{\Omega})} \leq C' h^\gamma |u_0|_2. \tag{4.43}$$

Due to estimates (3.4), (4.13) and assumptions (4.40), we have

$$|u_0^h - u_0|_0 + |u_0 - I^h u_0|_0 \leq Ch^2(|u_0|_2 + |u_0|_{2,\Gamma}). \tag{4.44}$$

Thus by (4.43) and (4.44), we deduce that

$$\|u_0^h - u_0\|_{C^0(\overline{\Omega})} \leq (Ch^{2-d/2} + C'h^\gamma)|u_0|_2 + Ch^{2-d/2}|u_0|_{2,\Gamma}. \tag{4.45}$$

For h small enough, we obtain

$$\|u_0^h\|_{C^0(\overline{\Omega})} < R, \tag{4.46}$$

and we may apply Lemma (4.3).
We claim that $\mathcal{N}(0) \leq Ch^4$, with $\mathcal{N}(t) = |\theta^u|_1^2 + \lambda|\theta^u|_{0,\Gamma}^2 + |\nabla_\Gamma \theta^u|_{0,\Gamma}^2 + |\theta_t^u|_{-1,h}^2 + |\theta_t^u|_{0,\Gamma}^2$. Indeed, by assumptions (4.40), we have

$$\mathcal{N}(0) = |\theta_t^u(0)|_{-1,h}^2 + |\theta_t^u(0)|_{0,\Gamma}^2. \tag{4.47}$$

Using the fact that $\theta^w(0) = 0$ and estimate (4.24) at $t = 0$, we see that $u^h(0)$ satisfies

$$(\theta_t^u(0), \varphi)_\Omega = -(\rho_t^u(0), \varphi)_\Omega, \ \forall \varphi \in V^h. \tag{4.48}$$

Choosing $\varphi = G^h\theta_t^u(0)$ in (4.48) and using (3.10) and (4.41), we obtain

$$\begin{aligned}
|\theta_t^u(0)|_{-1,h}^2 =& - (\rho_t^u(0), G^h\theta_t^u(0))_\Omega \\
\leq& |\rho_t^u(0)|_{-1,h}|\theta_t^u(0)|_{-1,h} \\
\leq& c|\rho_t^u(0)|_0|\theta_t^u(0)|_{-1,h} \\
\leq& Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma})|\theta_t^u(0)|_{-1,h}.
\end{aligned} \tag{4.49}$$

For $\varphi = \theta_t^u(0)$ in (4.48), we find

$$|\theta_t^u(0)|_0 \leq |\rho_t^u(0)|_0 \leq Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}). \tag{4.50}$$

Using assumptions (4.40) and choosing $\chi = \theta_t^u(0)$ in (4.26) considered at $t = 0$, we obtain

$$\begin{aligned}
|\theta_t^u(0)|_{0,\Gamma}^2 =& (\rho^w(0), \theta_t^u(0))_\Omega - (f(u_0^h) - f(u_0), \theta_t^u(0))_\Omega \\
&- (g(u_0^h) - g(u_0), \theta_t^u(0))_\Gamma - (\rho_t^u(0), \theta_t^u(0))_\Gamma.
\end{aligned} \tag{4.51}$$

Using the fact that $\|u_0\|_{C^0(\overline{\Omega})} < R$ and the fact that $u_0^h - u_0 = \rho^u(0)$, (4.51) gives

$$\begin{aligned}
|\theta_t^u(0)|_{0,\Gamma}^2 \leq& (|\rho^w(0)|_0 + L_f|\rho^u(0)|_0)|\theta_t^u(0)|_0 \\
&+ (L_g|\rho^u(0)|_{0,\Gamma} + |\rho_t^u(0)|_{0,\Gamma})|\theta_t^u(0)|_{0,\Gamma},
\end{aligned} \tag{4.52}$$

where L_f and L_g are respectively the Lipschitz constants of f and g on $[-R, R]$. By (4.5), (4.13), (4.41) and (4.50), estimate (4.52) yields

$$\begin{aligned}
|\theta_t^u(0)|_{0,\Gamma}^2 \leq& Ch^2(|w(0)|_2 + |u_0|_2 + |u_0|_{2,\Gamma})Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}) \\
&+ (L_gCh^2(|u_0|_2 + |u_0|_{2,\Gamma}) + Ch^2(|u_t(0)|_2 + |u_t(0)|_{2,\Gamma}))|\theta_t^u(0)|_{0,\Gamma},
\end{aligned} \tag{4.53}$$

and

$$|\theta_t^u(0)|_{0,\Gamma}^2 \leq Ch^4 + Ch^2|\theta_t^u(0)|_{0,\Gamma}. \tag{4.54}$$

In particular, (4.54) implies $|\theta_t^u(0)|_{0,\Gamma} \leq Ch^2$. Thus,

$$\mathcal{N}(0) \leq Ch^4. \tag{4.55}$$

The regularity assumptions on u and w and estimates (4.5), (4.13) and (4.41), imply that

$$\int_0^t |\rho_t^u|_0^2 + |\rho^w|_0^2 + |\rho^u|_0^2 + |\rho_{tt}^u|_0^2 \mathrm{d}s \leq Ch^4. \tag{4.56}$$

Using estimates (4.55) and (4.56) we deduce from (4.22) that

$$\mathcal{N}(t) \leq Ch^4, \quad \forall t \in [0, T^h]. \tag{4.57}$$

Estimate (4.57) implies in particular that

$$|\theta^u(t)|_0^2 \leq Ch^2, \quad \forall t \in [0, T^h]. \tag{4.58}$$

Arguing as in (4.42), we deduce that

$$\sup_{t\in[0,T^h]} \|u^h(t) - u(t)\|_{C^0(\overline{\Omega})} \to 0 \text{ as } h \to 0. \tag{4.59}$$

We conclude by noticing that for h small enough, $T^h = T$. □

Remark 1. We remark here that the regularity required in (4.38) and (4.39) is a strong one, this is due to the fact that we need strong regularity results in order to estimate the term θ_t^u.

5 Stability of the backward Euler scheme

In what follows, we denote by $\delta t = T/N$ the time step with $N \in \mathbb{N}^*$. We study the following backward in time Euler scheme:
Let $u_h^0 \in V^h$ and for $n = 1, 2, ...$, find $(u_h^n, w_h^n) \in V^h \times V^h$ such that

$$\begin{cases} (\overline{\partial} u_h^n, \varphi)_\Omega = -(\nabla w_h^n, \nabla\varphi)_\Omega - (w_h^n, \varphi)_\Omega, \\ (w_h^n, \chi)_\Omega = (\nabla u_h^n, \nabla\chi)_\Omega + (f(u_h^n), \chi)_\Omega + (\nabla_\Gamma u_h^n, \nabla_\Gamma\chi)_\Gamma + (\tilde{g}(u_h^n), \chi)_\Gamma + (\overline{\partial} u_h^n, \chi)_\Gamma, \end{cases} \tag{5.1}$$

for all $\varphi, \chi \in V^h$, where we denote by $\overline{\partial}$ the operator which to a sequence $(v^n)_{n\geq 0}$ associates the sequence defined by

$$\overline{\partial} v^n = \frac{v^n - v^{n-1}}{\delta t}, \; n \geq 1, \tag{5.2}$$

and the function $\tilde{g}$ is given by

$$\tilde{g}(\sigma) = \lambda\sigma + g(\sigma), \quad \forall \sigma \in \mathbb{R}.$$

Note that the dissipativity assumptions (2.1) imply that

$$f'(v) \geq -C_f \text{ and } \tilde{g}'(v) \geq -C_g, \quad \forall v \in \mathbb{R}, \tag{5.3}$$

where C_f and C_g are positive constants.

Theorem 5.1. *For every $u_h^0 \in V^h$, there exists a sequence $(u_h^n, w_h^n)_{n\geq 1}$ generated by (5.1) and which satisfies*

$$\mathcal{E}(u_h^n) + \frac{1}{2\delta t}|u_h^n - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\delta t}|u_h^n - u_h^{n-1}|_{0,\Gamma}^2 \leq \mathcal{E}(u_h^{n-1}),\ \forall n \geq 1. \tag{5.4}$$

Furthermore, if $\delta t < \delta t^*$, where $\delta t^* = \min\left\{\dfrac{4}{C_f^2 + C_f^2 C}, \dfrac{1}{C_g}\right\}$, then this sequence is uniquely defined.

Proof. Consider the variational problem:

$$J^h(u) = \inf_{v\in V^h} J^h(v), \tag{5.5}$$

where

$$J^h(v) = \mathcal{E}(v) + \frac{1}{2\delta t}|v - u_h^{n-1}|_{-1,h}^2 + \frac{1}{2\delta t}|v - u_h^{n-1}|_{0,\Gamma}^2. \tag{5.6}$$

Using (2.3), we have

$$J^h(v) \geq \frac{1}{2}(|v|_1^2 + |v|_{1,\Gamma}^2 + \lambda|v|_{0,\Gamma}^2) - c_2(|\Omega| + |\Gamma|), \quad \forall v \in V^h. \tag{5.7}$$

Since J^h is continuous, there exists a solution to (5.5). Such a solution u satisfies

$$\begin{aligned} 0 =&(\nabla u, \nabla\chi)_\Omega + (f(u), \chi)_\Omega + (\nabla_\Gamma u, \nabla_\Gamma \chi)_\Gamma + (\tilde{g}(u), \chi)_\Gamma \\ &+ \frac{1}{\delta t}(G^h(u - u_h^{n-1}), \chi)_\Omega + \frac{1}{\delta t}(u - u_h^{n-1}, \chi)_\Gamma,\ \forall \chi \in V^h. \end{aligned} \tag{5.8}$$

Setting $u_h^n = u$, $w_h^n = -\frac{1}{\delta t}G^h(u - u_h^{n-1})$, we find that (u_h^n, w_h^n) solves problem (5.1). Thanks to (5.5), $J^h(u_h^n) \leq J^h(u_h^{n-1})$, which implies (5.4).
In order to prove uniqueness, set $\xi = (u_h^n)^1 - (u_h^n)^2$ and $\eta = (w_h^n)^1 - (w_h^n)^2$ to be the difference of two possible solutions $((u_h^n)^i, (w_h^n)^i)$ $(i = 1, 2)$ of (5.1) for a given u_h^{n-1}. Then, (ξ, η) satisfies

$$\left\{\begin{aligned} (\xi, \varphi)_\Omega =& -\delta t(\nabla\eta, \nabla\varphi)_\Omega - \delta t(\eta, \varphi)_\Omega, \\ (\eta, \chi)_\Omega =&(\nabla\xi, \nabla\chi)_\Omega + (f((u_h^n)^1) - f((u_h^n)^2), \chi)_\Omega + (\nabla_\Gamma \xi, \nabla_\Gamma \chi)_\Gamma \\ &+ \big(\tilde{g}((u_h^n)^1) - \tilde{g}((u_h^n)^2), \chi\big)_\Gamma + (\xi/\delta t, \chi)_\Gamma, \end{aligned}\right. \tag{5.9}$$

for all $\varphi, \chi \in V^h$. Taking $\varphi = \eta$ in the first equation of (5.9) and $\chi = \xi$ in the second equation of (5.9) and subtracting the resulting equations, we obtain

$$\delta t|\eta|_1^2 + \delta t|\eta|_0^2 + |\xi|_1^2 + |\xi|_{1,\Gamma}^2 + \frac{1}{\delta t}|\xi|_{0,\Gamma}^2 \leq C_f|\xi|_0^2 + C_g|\xi|_{0,\Gamma}^2, \tag{5.10}$$

where we have used the inequalities

$$(f((u_h^n)^1) - f((u_h^n)^2), \xi)_\Omega \geq -C_f|\xi|_0^2,$$

$$(\tilde{g}((u_h^n)^1) - \tilde{g}((u_h^n)^2), \xi)_\Gamma \geq -C_g|\xi|_{0,\Gamma}^2.$$

By choosing $\chi = \xi$ in the first equation of (5.9), we find

$$\begin{aligned} C_f|\xi|_0^2 =& - C_f\delta t(\nabla\eta, \nabla\xi)_\Omega - C_f\delta t(\eta, \xi)_\Omega \\ \leq& \delta t|\eta|_1^2 + \frac{C_f^2\delta t}{4}|\xi|_1^2 + \delta t|\eta|_0^2 + \frac{C_f^2\delta t}{4}|\xi|_0^2, \\ \leq& \delta t|\eta|_1^2 + \frac{C_f^2\delta t}{4}|\xi|_1^2 + \delta t|\eta|_0^2 + \frac{C_f^2 C\delta t}{4}|\xi|_1^2, \end{aligned} \tag{5.11}$$

which leads to

$$\left(1 - \delta t(\frac{C_f^2}{4} + \frac{C_f^2 C}{4})\right)|\xi|_1^2 + |\xi|_{1,\Gamma}^2 + \left(\frac{1}{\delta t} - C_g\right)|\xi|_{0,\Gamma}^2 \leq 0. \tag{5.12}$$

The smallness assumption on δt implies $\xi = 0$ and by (5.9), we deduce that $\eta = 0$. $\square$

Corollary 1. *If f and g are analytic, then, for all $u_h^0 \in V^h$, any sequence $(u_h^n, w_h^n)_{n\geq 1}$ generated by* (5.1) *and which satisfies the energy estimate* (5.4) *converges to a steady state $(\overline{u}_h, \overline{w}_h)$ as $n \to +\infty$.*

Proof. The proof of this result is based on the Lojasiewicz gradient inequality (see [1] and [14]). Let $u_h^0 \in V^h$. By (5.4), the sequence $(\mathcal{E}(u_h^n))_n$ is non-increasing and since it is bounded from below by 0, we have $\mathcal{E}(u_h^n) \to \mathcal{E}^*$. We assume without loss of generality that $\mathcal{E}^* = 0$. By (3.2), $\mathcal{E}(v) \to +\infty$ as $\|v\|_V \to +\infty$ and $(u_h^n)_n$ is bounded: there exist $u_h^\infty \in V^h$ and a subsequence $(u_h^{n_k})_k$ such that $u_h^{n_k} \to u_h^\infty$ in V^h as $k \to +\infty$.
Using the same matrix notation as introduced in the proof of Proposition 1, problem (5.1) reads

$$\begin{pmatrix} (A+I) & I \\ -I & M_\Gamma \end{pmatrix} \begin{pmatrix} W^n \\ (U^n - U^{n-1})/\delta t \end{pmatrix} = - \begin{pmatrix} 0 \\ AU^n + F^h(U^n) + A_\Gamma U^n + G_\Gamma^h(U^n) \end{pmatrix},$$

where U^n (resp. W^n) is the vector of the coordinates of u_h^n (resp. w_h^n). The matrix $A + I$ is invertible. Thus, eliminating W^n, we obtain

$$((A+I)^{-1} + M_\Gamma)\frac{U^n - U^{n-1}}{\delta t} = -(AU^n + F^h(U^n) + A_\Gamma U^n + G_\Gamma^h(U^n)) = -\nabla E^h(U^n), \tag{5.13}$$

where

$$E^h(V) = \mathcal{E}(\sum_{i=1}^M v_i\varphi_i), \quad \forall V = (v_1, \cdots, v_M) \in \mathrm{I\!R}^M.$$

If we take the Euclidean norm of (5.13), we see that

$$\lambda_1\frac{\|U^n - U^{n-1}\|}{\delta t} \leq \|\nabla E^h(U^n)\| \leq \lambda_M\frac{\|U^n - U^{n-1}\|}{\delta t}, \tag{5.14}$$

where $0 \leq \lambda_1 < \lambda_M < +\infty$ are respectively the smallest and the largest eigenvalues of $((A+I)^{-1} + M_\Gamma)$ since $((A+I)^{-1} + M_\Gamma)$ is a symmetric positive definite matrix. On the other hand, since f and g are real analytic, the function E^h is real analytic on $\mathrm{I\!R}^M$ and it satisfies the Lojasiewicz inequality; more precisely, there exist $\sigma, \gamma > 0$ and $\nu \in (0, 1/2]$ such that

$$\forall V \in \mathrm{I\!R}^M, \|V - U^\infty\| < \sigma \Rightarrow |E^h(V)|^{1-\nu} \leq \gamma\|\nabla E^h(V)\|, \tag{5.15}$$

where we have used the fact that $E^h(U^\infty) = \mathcal{E}(u_h^\infty) = \mathcal{E}^* = 0$ and where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^M$. Now let n be such that $\|U^n - U^\infty\| \le \sigma$. We recall that E^h satisfies the following inequality:

$$E^h(U^n) + \frac{1}{2\delta t}|U^n - U^{n-1}|^2_{-1,h} + \frac{1}{2\delta t}|U^n - U^{n-1}|^2_{0,\Gamma} \le E^h(U^{n-1}),\ \forall n \ge 1. \tag{5.16}$$

We consider the following two cases:
Case 1: $E^h(U^n) > E^h(U^{n-1})/2$. In what follows, we will use the fact that all norms are equivalent on V^h. Since $x \mapsto x^{\nu-1}$ is non-increasing, we have

$$2E^h(U^n) > E^h(U^{n-1}) \Rightarrow E^h(U^{n-1})^{\nu-1} > 2^{\nu-1}E^h(U^n)^{\nu-1}. \tag{5.17}$$

We also know that

$$|v^h|^2_{-1,h} + |v^h|^2_{0,\Gamma} \ge c_h|v^h|^2_0, \quad \forall v^h \in V^h \tag{5.18}$$

for some positive constant $c_h > 0$ since all norms are equivalent on V^h. Then

$$\begin{aligned}
E^h(U^{n-1})^\nu - E^h(U^n)^\nu &= \int_{E^h(U^n)}^{E^h(U^{n-1})} \nu x^{\nu-1}\mathrm{d}x \\
&\ge \int_{E^h(U^n)}^{E^h(U^{n-1})} \nu(E^h(U^{n-1}))^{\nu-1}\mathrm{d}x \text{ (using (5.17))} \\
&\ge 2^{\nu-1}\nu E^h(U^n)^{\nu-1}[E^h(U^{n-1}) - E^h(U^n)] \text{ (using (5.16))} \\
&\ge 2^{\nu-1}\nu E^h(U^n)^{\nu-1}\frac{1}{2\delta t}(|U^n - U^{n-1}|^2_{-1,h} + |U^n - U^{n-1}|^2_{0,\Gamma}) \\
&= 2^{\nu-2}\nu E^h(U^n)^{\nu-1}\frac{1}{\delta t}(|U^n - U^{n-1}|^2_{-1,h} + |U^n - U^{n-1}|^2_{0,\Gamma}) \\
&\ge 2^{\nu-2}\nu c_h\frac{\|U^n - U^{n-1}\|^2}{\delta t E^h(U^n)^{1-\nu}}.
\end{aligned} \tag{5.19}$$

Using (5.15) and (5.14), we obtain

$$\begin{aligned}
E^h(U^{n-1})^\nu - E^h(U^n)^\nu \ge& \frac{2^{\nu-2}\nu c_h}{\gamma\delta t}\frac{\|U^n - U^{n-1}\|^2}{\|\nabla E^h(U^n)\|} \\
\ge& \frac{2^{\nu-2}\nu c_h}{\lambda_M\gamma}\|U^n - U^{n-1}\|.
\end{aligned} \tag{5.20}$$

Case 2: $E^h(U^n) \le E^h(U^{n-1})/2$. We have

$$\begin{aligned}
& E^h(U^n) \le \frac{E^h(U^{n-1})}{2} \\
\Longleftrightarrow & (E^h(U^n))^{1/2} \le \frac{(E^h(U^{n-1}))^{1/2}}{\sqrt{2}} \\
\Longleftrightarrow & (E^h(U^{n-1}))^{1/2} - \frac{(E^h(U^{n-1}))^{1/2}}{\sqrt{2}} \le (E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2} \\
\Longleftrightarrow & \left(1 - \frac{1}{\sqrt{2}}\right)(E^h(U^{n-1}))^{1/2} \le (E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2} \\
\Longleftrightarrow & (E^h(U^{n-1}))^{1/2} \le \left(1 - \frac{1}{\sqrt{2}}\right)^{-1}\left((E^h(U^{n-1}))^{1/2} - (E^h(U^n))^{1/2}\right).
\end{aligned} \tag{5.21}$$

Using (5.16), (5.18) and (5.21), we obtain

$$\begin{aligned}\|U^n - U^{n-1}\| &\leq \frac{1}{\sqrt{c_h}}(|U^n - U^{n-1}|_{-1,h} + |U^n - U^{n-1}|_{0,\Gamma})\\ &\leq 2\left(\frac{\delta t}{c_h}\right)^{1/2}\left(E^h(U^{n-1}) - E^h(U^n)\right)^{1/2}\\ &\leq 2\left(\frac{\delta t}{c_h}\right)^{1/2} E^h(U^{n-1})^{1/2}\\ &\leq 2\left(1-\frac{1}{\sqrt{2}}\right)^{-1}\left(\frac{\delta t}{c_h}\right)^{1/2}\left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right).\end{aligned} \tag{5.22}$$

Thus, in both cases, we have

$$\begin{aligned}\|U^n - U^{n-1}\| \leq &\frac{2^{2-\nu}\lambda_M\gamma}{\nu c_h}\left(E^h(U^{n-1})^\nu - E^h(U^n)^\nu\right)\\ &+ 2\left(1-\frac{1}{\sqrt{2}}\right)^{-1}\left(\frac{\delta t}{c_h}\right)^{1/2}\left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right)\\ \leq &\frac{2^{2-\nu}\lambda_M\gamma}{\nu c_h}\left(E^h(U^{n-1})^\nu - E^h(U^n)^\nu\right)\\ &+ \left(\frac{\delta t}{c_h}\right)^{1/2}\left(E^h(U^{n-1})^{1/2} - E^h(U^n)^{1/2}\right).\end{aligned} \tag{5.23}$$

Now, let $\tilde{E} > 0$ be small enough so that

$$\frac{2^{2-\nu}\lambda_M\gamma}{\nu c_h}\tilde{E}^\nu + \left(\frac{\delta t}{c_h}\right)^{1/2}\tilde{E}^{1/2} \leq \sigma/3. \tag{5.24}$$

We choose $\overline{n}$ large enough such that $\|U^{\overline{n}} - U^\infty\| \leq \frac{1}{\sqrt{c_h}}(|U^{\overline{n}} - U^\infty|_{-1,h} + |U^{\overline{n}} - U^\infty|_{0,\Gamma}) < \sigma/3$ and $E^h(U^{\overline{n}}) \leq \tilde{E}$. Let $N-1 \geq \overline{n}$ be the largest integer (including $+\infty$) such that

$$\|U^n - U^\infty\| \leq \frac{1}{\sqrt{c_h}}(|U^n - U^\infty|_{-1,h} + |U^n - U^\infty|_{0,\Gamma}) < 2\sigma/3,$$

for all n with $\overline{n} \leq n \leq N-1$. Assume by contradiction that N is finite. We deduce from (5.16) that

$$\begin{aligned}\|U^N - U^\infty\| \leq &\frac{1}{\sqrt{c_h}}(|U^N - U^\infty|_{-1,h} + |U^N - U^\infty|_{0,\Gamma})\\ \leq &\frac{1}{\sqrt{c_h}}\left(|U^N - U^{N-1}|_{-1,h} + |U^N - U^{N-1}|_{0,\Gamma}\right)\\ &+ \frac{1}{\sqrt{c_h}}\left(|U^{N-1} - U^\infty|_{-1,h} + |U^{N-1} - U^\infty|_{0,\Gamma}\right)\\ \leq &\frac{1}{\sqrt{c_h}}\left(\sqrt{2\delta t E^h(U^{N-1})} + |U^{N-1} - U^\infty|_{-1,h} + |U^{N-1} - U^\infty|_{0,\Gamma}\right)\\ \leq &\sigma/3 + 2\sigma/3 = \sigma.\end{aligned} \tag{5.25}$$

So we may apply (5.23) to every $\overline{n} \le n \le N-1$ and since $(E^h(U^n))_n$ is non-increasing, we obtain

$$\sum_{n=\overline{n}}^{N} \|U^n - U^{n-1}\| \le \frac{2^{2-\nu}\lambda_M \gamma}{\nu c_h} E^h(U^{N-1})^\nu + 5\left(\frac{\delta t}{c_h}\right)^{1/2} E^h(U^{N-1})^{1/2} \le \sigma/3. \tag{5.26}$$

Thus,

$$\begin{aligned} \|U^N - U^\infty\| &\le \|U^N - U^{\overline{n}}\| + \|U^{\overline{n}} - U^\infty\| \\ &\le \sum_{n=\overline{n}}^{N} \|U^n - U^{n-1}\| + \|U^{\overline{n}} - U^\infty\| \\ &\le \sigma/3 + \sigma/3 = 2\sigma/3, \end{aligned} \tag{5.27}$$

which is in contradiction with the definition of $N-1$. So $N = +\infty$ and the whole sequence (U^n) converges to U_∞. Since w_h^n, defined by (5.1), is a continuous function of u_h^n, w_h^n also has a limit w_h^∞ as $n \to +\infty$. We see that (u_h^∞, w_h^∞) is necessarily a steady state by passing to the limit in (5.1). □

6 Numerical simulations

In this section, we illustrate some numerical simulations in two space dimensions. The fully discrete scheme (5.1) requires at each time step the resolution of a nonlinear system and for the numerical computation of solutions of the space semi-discrete scheme (3.7), we propose instead of (5.1) a semi-implicit time discretizations which is the semi-implicit Euler (SIE) scheme, i.e. (3.7) but the implicit nonlinear terms $f(u_h^n)$ and $\tilde{g}(u_h^n)$ are respectively replaced by the explicit terms $f(u_h^{n-1})$ and $\tilde{g}(u_h^{n-1})$. Using the same arguments as in the proof of Proposition 1 we see that the matrix of the SIE scheme is positive semidefinite and invertible, so that the SIE scheme is well-posed.

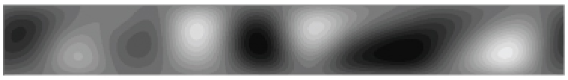

Figure 1: $t = 5$

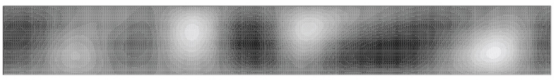

Figure 2: $t = 10$

Figure 3: $t = 25$

Figure 4: $t = 50$

In Figures 1-4, we see the result of the SIE scheme on the slab $L_x \times L_y = 80 \times 10$. The triangulation Ω^h was obtained by dividing the slab into 256×50 rectangles and by dividing every rectangle along the same diagonal into two triangles. The nonlinearities are

$$f(v) = \frac{1}{2}(v^3 - v) \quad \text{and} \quad \tilde{g}(v) = (\lambda + k)v - h, \quad v \in \mathbb{R}, \tag{6.1}$$

with $\lambda + k = 1$ ($\lambda = 0.5$, for instance), $h = 0$ and $\delta t = 0.1$. In each picture, the maximum and minimum values of u are colored in white and black and values of u in between correspond to different shades of grey. In these numerical simulations, we chose the same parameters as in [1] and [13]. Since $h = 0$, none of the components is preferably attracted by the walls, which is visible on the fact that both white and black zones appear at the boundary.

Figure 5: $t = 2$ ($h = 0$)

Figure 6: $t = 2$ ($h = 0.7$)

In Figures 5 and 6, we consider the nonlinearity $f(v) = v^3 - \dfrac{v}{2}$, $v \in \mathbb{R}$, and the same nonlinearity $\tilde{g}(v) = (\lambda + k)v - h, v \in \mathbb{R}$. This time, $\delta t = 0.01$ and the geometry is different;

the domain Ω is a disk of radius 80 centered at $(0,0)$ from which we have cut off a disk of radius 40 and centered at $(20,0)$. The exterior boundary is divided into 600 intervals and the internal boundary into 400 intervals, yielding a triangulation Ω^h of Ω with 59 048 triangles and 30 024 vertices. In these figures, we see the difference between the case $h = 0$, where no phase is preferentially attracted by the walls, and the case $h = 0.7$, where one of the components is preferentially attracted by the walls. We also remark that away from the boundary, Figures 5 and 6 present the same patterns.

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E-mail address: **haydiisrael@hotmail.com**
E-mail address: **alain.miranville@math.univ-poitiers.fr**
E-mail address: **madalina.petcu@math.univ-poitiers.fr**.