SOBOLEV AND GEVREY REGULARITY RESULTS FOR THE PRIMITIVE EQUATIONS IN THREE SPACE DIMENSIONS

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Abstract. The aim of this paper is to present a qualitative study of the Primitive Equations in a three dimensional domain, with periodical boundary conditions. We start by recalling some already existing results regarding the existence locally in time of weak solutions and existence and uniqueness of strong solutions, and we prove the existence of very regular solutions, up to $\mathcal{C}^\infty$-regularity. In the second part of the paper we prove that the solution of the Primitive Equations belongs to a certain Gevrey class of functions.

1. Introduction

In this article we consider the Primitive Equations for the ocean or for the atmosphere in 3 space dimensions, with periodic boundary conditions. The general form of the equations governing the movement of the oceans and atmosphere is derived from the basic conservation laws, but the resulting equations are too difficult to handle. That is why, using scale analysis methods and physical observations, the equations are usually approximated by different models, having simpler forms (in principle), one of them being the Primitive Equations (for more details on the form of the Primitive Equations and their derivation, see e.g., [7], [8], [9]).

As we already mentioned, n this article we consider the 3D Primitive Equations with space periodicity and start by recalling the known results of existence, uniqueness and regularity of solutions, in the usual H^1 Sobolev space (see [7], [8], [15]). We then prove a regularity result in higher order Sobolev spaces; for a similar result for the Primitive Equations in space dimension 2, see [11]. We also study the Gevrey regularity for the PEs; in fact, we show that considering a forcing term which is analytic in time with values in some Gevrey space, the solutions of the PEs starting with initial data in the Sobolev space H^1 instantly become elements of a certain Gevrey class and remain there for a certain interval of time. The study of the Gevrey regularity for the solutions was inspired by the article of Foias and Temam [4] who proved this type of results for the Navier–Stokes equations in 2 and 3 space dimensions with periodic boundary conditions.

We also mention the works of Promislow [12], of Ferrari and Titi [3], who obtained Gevrey regularity results for a certain class of nonlinear parabolic equations; also, Cao, Rammaha and Titi [2] established the Gevrey regularity for a certain class of analytic nonlinear parabolic equations on the sphere. The Gevrey regularity of the Primitive Equations in 2 space dimensions was proven in [10].

The Primitive Equations in their dimensional form read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + v\frac{\partial u}{\partial x_2} + w\frac{\partial u}{\partial x_3} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x_1} = \nu\Delta u + F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} + v\frac{\partial v}{\partial x_2} + w\frac{\partial v}{\partial x_3} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial x_2} = \nu\Delta v + F_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial x_3} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x_1} + v\frac{\partial T}{\partial x_2} + w\frac{\partial T}{\partial x_3} = \mu\Delta T + F_T. \tag{1.1e}$$

In the system above, $(u,\, v,\, w)$ are the three components of the velocity vector and p, ρ and T are respectively the perturbations of the pressure, of the density and of the temperature from the reference (average) constant state p_0, ρ_0, and T_0. The relation between the temperature and the density is given by the equation of state, and we consider here a version of this equation linearized around the reference state ρ_0 and T_0,

$$\rho_{\text{full}} = \rho_0(1 - \beta_T(T - T_0)), \tag{1.2}$$

so that for the perturbations ρ and T:

$$\rho = -\beta_T\rho_0 T. \tag{1.3}$$

The constant g is the gravitational acceleration and f the Coriolis parameter, ν and μ are the eddy diffusivity coefficients, (F_u, F_v) represent body forces per unit of mass and F_T represents a heating source. In applications F_u, F_v vanish for the ocean but we consider here nonzero forces for mathematical generality. When required, we denote by F the vector (F_u, F_v, F_T).

We work in a limited domain:

$$\Omega = (0, L_1) \times (0, L_2) \times (-L_3/2, -L_3/2), \tag{1.4}$$

and we assume space periodicity with period Ω, meaning that all functions are taken to satisfy:

$$f(x_1, x_2, x_3, t) = f(x_1 + L_1, x_2, x_3, t) = f(x_1, x_2 + L_2, x_3, t) = f(x_1, x_2, x_3 + L_3, t), \tag{1.5}$$

when extended to $\mathbb{R}^3$.

All functions being periodic, they admit Fourier series, hence we can write:

$$f(x_1, x_2, x_3, t) = \sum_{k \in \mathbb{R}^3} f_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)}, \tag{1.6}$$

where, for notational conciseness, we set $k_j' = 2\pi k_j / L_j$ for $j = 1, 2, 3$.

Moreover, we assume as in [10], [11], that the following symmetries hold:

$$\begin{aligned} u(x_1, x_2, x_3, t) &= u(x_1, x_2, -x_3, t), & F_u(x_1, x_2, x_3, t) &= F_u(x_1, x_2, x_3, t), \\ v(x_1, x_2, x_3, t) &= v(x_1, x_2, -x_3, t), & F_v(x_1, x_2, x_3, t) &= F_v(x_1, x_2, -x_3, t), \\ T(x_1, x_2, x_3, t) &= -T(x_1, x_2, -x_3, t), & F_T(x_1, x_2, x_3, t) &= -F_T(x_1, x_2, -x_3, t), \\ w(x_1, x_2, x_3, t) &= -w(x_1, x_2, -x_3, t), & p(x_1, x_2, x_3, t) &= p(x_1, x_2, -x_3, t); \end{aligned} \tag{1.7}$$

in other words, u, v, p are even and w, T odd in x_3. These conditions are often used in numerical studies of rotating stratified turbulence (see e.g. [1]). Note that without these symmetry properties, space periodicity is not consistent with (1.1).

The following function spaces are used:

$$V = \{U = (u,\, v,\, T) \in (\dot{H}^1_{\mathrm{per}}(\Omega))^3,\ u,\, v \text{ even in } x_3,\ T \text{ odd in } x_3, \tag{1.8}$$

$$\int_{-L_3/2}^{L_3/2} (u_{x_1}(x_1, x_2, x_3') + v_{x_2}(x_1, x_2, x_3'))\, \mathrm{d}x_3' = 0\},$$

$$H = \text{closure of } V \text{ in } (\dot{L}^2(\Omega))^3. \tag{1.9}$$

Here the dots above $\dot{H}^1_{\mathrm{per}}$ and $\dot{L}^2$ denote the functions with zero average over Ω.

These spaces are endowed with the following scalar products: on H we consider the scalar product

$$(U,\, \tilde{U})_H = (u,\, \tilde{u})_{L^2} + (v,\, \tilde{v})_{L^2} + \kappa(T,\, \tilde{T})_{L^2}, \tag{1.10}$$

and on V the scalar product is

$$((U,\, \tilde{U}))_V = ((u,\, \tilde{u})) + ((v,\, \tilde{v})) + \kappa((T,\, \tilde{T})), \tag{1.11}$$

where we have written

$$((\Phi,\, \tilde{\Phi})) = \int_\Omega \nabla\Phi \cdot \nabla\tilde{\Phi}\, \mathrm{d}\Omega. \tag{1.12}$$

The positive constant κ will be chosen below. Since we assumed that all functions have zero average, a generalized Poincaré inequality holds, meaning that we have:

$$|U|_H \le c_0 \|U\|_V, \quad \forall\, U \in V, \tag{1.13}$$

which guarantees that $\|\cdot\|$ is indeed a norm on V equivalent to the usual H^1 norm.

In system (1.1), the unknown functions are regrouped in two sets: the prognostic variables u, v and T for which an initial value problem will be defined, and the diagnostic variables ρ, w and p which can be defined, at each instant of time, as functions of the prognostic variables, using the equations and the boundary conditions. The density ρ is already expressed in terms of the temperature T by the equation of state (1.3). Given the prognostic variable $U = (u, v, T) \in V$, we can uniquely determine the vertical velocity w from the conservation of mass equation as:

$$w(U) = w(x_1, x_2, x_3, t) = -\int_0^{x_3} (u_{x_1} + v_{x_2})\, \mathrm{d}x_3', \tag{1.14}$$

where we used $w(x_1, x_2, 0, t) = 0$, since w is odd in x_3. Using (1.1d), the fact that w is periodic gives the constraint

$$\int_{-L_3/2}^{L_3/2} (u_{x_1} + v_{x_2})\, \mathrm{d}x_3 = 0. \tag{1.15}$$

From equation (1.1c), the pressure can be determined uniquely in terms of T, up to its value p_s at $x_3 = 0$, namely,

$$p(x_1, x_2, x_3, t) = p_s(x_1, x_2, 0, t) + \beta_T \rho_0 \int_0^{x_3} T(x_1, x_2, x_3', t)\, \mathrm{d}x_3'. \tag{1.16}$$

In fact, we fully determine the Fourier coefficients p_k of the pressure p for $k_3 \neq 0$ but not for $k_3 = 0$. That means that the part of the pressure we can not determine is the average of the pressure in the vertical direction:

$$p_\star(x_1, x_2) = \frac{1}{L_3} \int_{-L_3/2}^{L_3/2} p(x_1, x_2, x_3, t)\, \mathrm{d}x_3 = \sum_{k,\, k_3 = 0} p_k(t) e^{i(k_1' x_1 + k_2' x_2)}. \tag{1.17}$$

We deduce below the relation between the average of the pressure in the vertical direction and the surface pressure:

(1.18)

$$\begin{aligned}
p(x_1, x_2, x_3, t) &= p_s(x_1, x_2, 0, t) + \beta_T \rho_0 \int_0^{x_3} \sum_k T_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3')}\, \mathrm{d}x_3 \\
&= p_s(x_1, x_2, t) + \beta_T \rho_0 \sum_{k,\, k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i(k_1' x_1 + k_2' x_2)} (e^{i k_3' x_3} - 1) \\
&= \sum_{(k_1, k_2)} (p_{s,\,(k_1,k_2)} - \beta_T \rho_0 \sum_{k_3 \neq 0} \frac{T_k(t)}{i k_3'}) e^{i(k_1' x_1 + k_2' x_2)} + \beta_T \rho_0 \sum_{k,\, k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i k' \cdot x} \\
&= \sum_{(k_1, k_2)} (p_{\star,\,(k_1,k_2)} e^{i(k_1' x_1 + k_2' x_2)} + \beta_T \rho_0 \sum_{k,\, k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i k' \cdot x},
\end{aligned}$$

where $p_\star$ is the average of p in the vertical direction. Then:

$$p_{\star,\,(k_1,k_2)} = p_{s,\,(k_1,k_2)} - \beta_T \rho_0 \sum_{k_3 \neq 0} \frac{T_k(t)}{i k_3'}. \tag{1.19}$$

The variational formulation of the problem

In order to obtain the variational formulation of this problem, we consider a test function $U^\flat = (u^\flat, v^\flat, T^\flat) \in V$, multiply (1.1a) by $u^\flat$, (1.1b) by $v^\flat$, and (1.1e) by $\kappa T^\flat$, and integrate over Ω. Using the integration by parts and the space periodicity, we find that system (1.1) is formally equivalent to the following problem:

To find $U : [0, t_0] \to V$, such that,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(U, U^\flat)_H + a(U, U^\flat) + b(U, U, U^\flat) + e(U, U^\flat) = (F, U^\flat)_H, \quad \forall\, U^\flat \in V, \\ &U(0) = U_0. \end{aligned} \tag{1.20}$$

In (1.20) we introduced the bilinear, continuous form $a : V \times V \to \mathbb{R}$ as:

$$a(U, U^\flat) = \nu((u, u^\flat)) + \nu((v, v^\flat)) + \kappa\mu((T, T^\flat)), \tag{1.21}$$

the trilinear form b as:

$$\begin{aligned} b(U, U^\sharp, U^\flat) = &\int_\Omega (u \frac{\partial u^\sharp}{\partial x} u^\flat + v \frac{\partial u^\sharp}{\partial y} u^\flat + w(U) \frac{\partial u^\sharp}{\partial z} u^\flat) \,\mathrm{d}\Omega \\ &+ \int_\Omega (u \frac{\partial v^\sharp}{\partial x} v^\flat + v \frac{\partial v^\sharp}{\partial y} v^\flat + w(U) \frac{\partial v^\sharp}{\partial z} v^\flat) \,\mathrm{d}\Omega \\ &+ \int_\Omega (u \frac{\partial T^\sharp}{\partial x} T^\flat + v \frac{\partial T^\sharp}{\partial y} T^\flat + w(U) \frac{\partial T^\sharp}{\partial z} \tilde{T}) \,\mathrm{d}\Omega, \end{aligned} \tag{1.22}$$

and the bilinear form e, $e : V \times V \to \mathbb{R}$ which is continuous:

$$e(U, U^\flat) = f \int_\Omega (uv^\flat - vu^\flat) \,\mathrm{d}\Omega - g\beta_T \int_\Omega T w(U^\flat) \,\mathrm{d}\Omega. \tag{1.23}$$

We note that

$$a(U, U) + e(U, U) = \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - g\beta_T \int_\Omega T w(U) \,\mathrm{d}\Omega. \tag{1.24}$$

We then estimate:

$$|g\beta_T \int_\Omega T w(U) \,\mathrm{d}\Omega| \le g\beta_T |T|_{L^2} |w(U)|_{L^2} \le cg\beta_T(\|u\| + \|v\|)\|T\|; \tag{1.25}$$

here we used (1.14) and the Poincaré inequality. We find:

$$a(U, U) + e(U, U) \ge \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - cg\beta_T\|u\|\|T\| - cg\beta_T\|v\|\|T\|. \tag{1.26}$$

From equation (1.26), we see that choosing κ large enough, more specifically $\kappa \ge 2(cg\beta_T)^2/(\nu\mu)$, the bilinear continuous form $a + e$ is coercive on V, and

$$a(U, U) + e(U, U) \ge \frac{\nu}{2}\|u\|^2 + \frac{\nu}{2}\|v\|^2 + \frac{\kappa\mu}{2}\|T\|^2 \ge c_1\|U\|_V^2,\ c_1 = \min(\nu, \mu). \tag{1.27}$$

In order to study the properties of the form b, we introduce the space V_2, defined as:

$$V_2 = \text{ the closure of } V \cap (\dot{H}^2_{\rm per}(\Omega))^3 \text{ in } (\dot{H}^2_{\rm per}(\Omega))^3. \tag{1.28}$$

We have the following result on b:

Lemma 1.1. *The form b is trilinear continuous from $V \times V_2 \times V$ into $\mathbb{R}$ and from $V \times V \times V_2$ into $\mathbb{R}$, and*

$$|b(U, U^\sharp, U^\flat)| \le c_2 \|U\| |U^\sharp|_H^{1/2} \|U^\sharp\|^{1/2} \|U^\flat\|_{V_2}, \quad \forall\, U, U^\sharp \in V,\ U^\flat \in V_2. \tag{1.29}$$

Furthermore,

$$b(U, U^\flat, U^\flat) = 0 \quad \forall\, U \in V,\ U^\flat \in V_2,$$

and

$$b(U, U^\flat, U^\sharp) = -b(U, U^\sharp, U^\flat), \quad \forall\, U, U^\flat, U^\sharp \in V \text{ with } U^\flat \text{ or } U^\sharp \in V_2.$$

Proof. The proof is based on appropriate estimates for the terms of $b(U, U^\sharp, U^\flat)$; Hölder, Sobolev and interpolation inequalities are used. For more details on how this type of results is derived, see [7], [11] or [15]. □

We can now write (1.20) as an evolution equation in the Hilbert space V_2', which is the dual space of V_2. For that purpose we observe that we can associate the following operators to the forms a, b and e above:

$$A \text{ linear continuous from } V \text{ into } V', \text{ defined by } \langle AU, U^\flat\rangle = a(U, U^\flat), \quad \forall U, U^\flat \in V,$$

$$B \text{ bilinear, continuous from } V \times V \text{ into } V_2', \text{ defined by}$$
$$\langle B(U, U^\flat),\, U^\sharp\rangle = b(U, U^\flat, U^\sharp) \quad \forall\, U, U^\flat \in V,\ \forall\, U^\sharp \in V_2,$$

$$E \text{ linear continuous from } V \text{ into } V', \text{ defined by } \langle EU, U^\flat\rangle = e(U, U^\flat), \quad \forall\, U, U^\flat \in V.$$

Then equation (1.20) is equivalent to the following operator evolution equation in V_2':

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U,U) + EU = F, \\ &U(0) = U_0. \end{aligned} \tag{1.30}$$

In the second section we present some existence, uniqueness and Sobolev regularity results for the Primitive Equations, that is (1.20) or (1.30). We start by recalling the existence of weak solutions (result already available thanks to [7]), the existence and uniqueness of strong solutions (result already available, see [15]) and we conclude by proving the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity. For these high regularity results we use periodic boundary conditions; for a similar result in two space dimensions, see [11].

In the third section we prove that the solutions of the Primitive Equations in space dimension three are real functions analytic in time with values in some Gevrey space.

2. Sobolev regularity results

As we mentioned before, we start by recalling some results already available and then we prove the existence of very regular solutions.

Theorem 2.1. *Given $U_0 \in H$ and $F \in L^\infty(\mathbb{R}_+; H)$, there exists at least one solution U of problem (1.20) such that:*

$$U \in L^\infty(\mathbb{R}_+; H) \cap L^2(0, T; V), \quad \forall T > 0. \tag{2.1}$$

Proof. The proof of this theorem is based on the a priori estimates given below which, by classical methods, lead to (2.1); we briefly recall these calculations needed below.

Taking $U^\flat = U(t)$ in equation (1.20), for an arbitrary fixed $t > 0$, we obtain after some basic computations:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_1\|U\|_H^2 \leq c_1'|F|_\infty^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + \frac{c_1}{c_0}|U|_H^2 \leq c_1'|F|_\infty^2, \tag{2.2}$$

where $|F|_\infty$ is the norm of F in $L^\infty(\mathbb{R}_+; H)$. Using Gronwall inequality, from (2.2) we find

$$|U(t)|_H^2 \leq |U(0)|_H^2 e^{-\frac{c_1 t}{c_0}} + \frac{c_1' c_0}{c_1}(1 - e^{-\frac{c_1 t}{c_0}})|F|_\infty^2. \tag{2.3}$$

Inequality (2.3) implies:

$$\limsup_{t\to\infty} |U(t)|_H^2 \leq \frac{c_1' c_0}{c_1}|F|_\infty^2 =: r_0^2. \tag{2.4}$$

After these a priori estimates of U in $L^\infty(\mathbb{R}_+; H)$, we integrate (2.3) and find:

$$\int_0^{t_1} \|U\|^2 \,\mathrm{d}t \leq K(U_0, F, t_1), \quad \forall t_1 > 0, \tag{2.5}$$

where $K(U_0, F, t_1)$ denotes a constant depending on the initial data U_0 and on the time $t_1 > 0$. These estimates are at the basis of the proof of existence in Theorem 2.1 (for more details, see [7]).

We also note that for a forcing independent on t, $F(t) \equiv F \in H$, inequality (2.4) implies that any ball $\mathcal{B}(0, r_0')$ in H, with $r_0' > r_0$ is an absorbing ball. □

The existence and uniqueness of a strong solution is given by the following theorem (see e.g., [5], [15]):

Theorem 2.2. *Given $U_0 \in V$ and $F \in L^2(0, T; H)$, there exists $t_\star > 0$, $t_\star = t_\star(\|U_0\|)$ and a unique solution $U = U(t)$ of (1.20) on $(0, t_\star$, such that:*

$$U \in \mathcal{C}(0, t_\star; V) \cap L^2(0, t_\star; (\dot{H}^2_{\mathrm{per}}(\Omega))^3).$$

Proof. The proof is based, as usual, on some a priori estimates for the solution U, obtained by taking $U^\flat = -\Delta U$ in (1.20). First of all, let us note that the "standard" treatment of the bilinear term gives the estimate,

$$|(B(U,U),\,\Delta U)_H| \le c\|U\|_V^{1/2}|\Delta U|_H^{5/2}, \tag{2.6}$$

The term $|\Delta U|_H^{5/2}$ is too strong to be dominated, meaning it cannot be majorized by $|\Delta U|^2$ on the left-hand side.

In order to overcome this difficulty, the idea is to use an anisotropic treatment for the terms in $b(U,\tilde U,U^\sharp)$ which contain $w(U)$. This gives the following result, which is proved in [15] (see also [5]):

Lemma 2.1. *The trilinear form b is continuous from $V_2\times V_2\times H$ into $\mathbb{R}$, and:*

$$|b(U,\tilde U,U^\sharp)| \le c_3(\|U\|_V\|\tilde U\|_V^{1/2}\|\tilde U\|_{V_2}^{1/2} + \|U\|_V^{1/2}\|U\|_{V_2}^{1/2}\|\tilde U\|_V^{1/2}\|\tilde U\|_{V_2}^{1/2})|U^\sharp|_H, \tag{2.7}$$

for all U, $\tilde U$ in V_2 and $U^\sharp$ in H.

We return to the proof of the theorem. Using Lemma 2.1, we can estimate the trilinear term as:

$$|(B(U,U),\Delta U)_H| \le c_4\|U\|_V|\Delta U|_H^2. \tag{2.8}$$

This estimate allows us to obtain some a priori estimates, but since the estimate is a weak one (the term $|\Delta U|_H$ has power 2), a direct estimate would force us to work with small initial data. In order to avoid imposing such a restriction, we split the solution U of equation (1.30) into $U = U^\star + \tilde U$, where $U^\star$ is the solution of the linear problem (as in [5], [15]):

$$\begin{aligned} &\frac{\mathrm{d}U^\star}{\mathrm{d}t} + AU^\star + EU^\star = F,\\ &U^\star(0) = U_0, \end{aligned} \tag{2.9}$$

and $\tilde U$ is the solution of the following nonlinear problem, in which $U^\star$ is now known:

$$\begin{aligned} &\frac{\mathrm{d}\tilde U}{\mathrm{d}t} + A\tilde U + B(\tilde U,\tilde U) + B(\tilde U,U^\star) + B(U^\star,\tilde U) + E\tilde U = -B(U^\star,U^\star),\\ &\tilde U(0) = 0. \end{aligned} \tag{2.10}$$

We start by deriving a priori estimates for $U^\star$. We take the scalar product of (2.9) with $-\Delta U^\star$ in H and we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^\star\|_V^2 + c_1|\Delta U^\star|_H^2 \le \frac{c_1}{2}|\Delta U^\star|_H^2 + \frac{1}{2c_1}|F|_H^2, \tag{2.11}$$

which leads to:

$$\sup_{0\le t\le t_1}\|U^\star(t)\|_V^2 \le \frac{1}{2c_1}|F|^2_{L^2(0,t_1;H)} + \|U_0\|_V^2, \tag{2.12}$$

and

$$\int_0^{t_1} |\Delta U^\star(t)|_H^2 \,\mathrm{d}t \le \frac{1}{2{c_1}^2}|F|^2_{L^2(0,t_1;H)} + \frac{1}{c_1}\|U_0\|_V^2. \tag{2.13}$$

Using the following a priori estimates and classical methods (e.g. Galerkin's method), we prove the existence of a solution of (2.10) on some interval $(0, t_\star)$, where $t_\star$ is also determined below. Assuming that $U^\star$ is known in $L^\infty(0, t_1; V) \cup L^2(0, t_1; H^2)$ for all $t_1 > 0$, we take the scalar product of (2.10) with $\Delta\tilde{U}$ in H and we use Lemma 2.1. We have the following estimates:

$$|(B(\tilde{U}, U^\star), \Delta\tilde{U})_H| = |b(\tilde{U}, U^\star, \Delta\tilde{U})| \le \frac{c_1}{8}|\Delta\tilde{U}|_H^2 + c\|\tilde{U}\|_V^2(1 + \|U^\star\|_{H^2}^2), \tag{2.14}$$

$$|(B(U^\star, \tilde{U}), \Delta\tilde{U})_H| = |b(U^\star, \tilde{U}, \Delta\tilde{U})| \le \frac{c_1}{8}|\Delta\tilde{U}|_H^2 + c\|U^\star\|_V^2\|U^\star\|_{H^2}^2\|\tilde{U}\|_V^2, \tag{2.15}$$

and

$$|(B(U^\star, U^\star), \Delta\tilde{U})_H| = |b(U^\star, U^\star, \Delta\tilde{U})| \le \frac{c_1}{8}|\Delta\tilde{U}|_H^2 + c\|U^\star\|_V^2\|U^\star\|_{H^2}^2. \tag{2.16}$$

Taking into account all these estimates, (2.10) leads to the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{U}\|_V^2 + (c_1 - c_4\|\tilde{U}\|_V)|\Delta\tilde{U}|_H^2 \le \gamma(t)\|\tilde{U}\|_V^2 + \eta(t), \tag{2.17}$$

with

$$\begin{aligned}\gamma(t) &= c(1 + \|U^\star\|_{H^2}^2 + \|U^\star\|_V^2\|U^\star\|_{H^2}^2,\\ \eta(t) &= c\|U^\star\|_V^2\|U^\star\|_{H^2}^2.\end{aligned}$$

Using (2.12) and (2.13), we see that the functions γ and η are integrable on any interval $(0, t_1)$. Since $\tilde{U}(0) = 0$, we may assume that:

$$\|\tilde{U}\|_V \le \frac{c_1}{2c_4}, \text{ on some finite interval of time } (0, t_0). \tag{2.18}$$

On that interval, we can write (2.17) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{U}\|_V^2 + \frac{c_1}{2}|\Delta\tilde{U}|_H^2 \le \gamma(t)\|\tilde{U}\|_V^2 + \eta(t). \tag{2.19}$$

Applying the Gronwall lemma to (2.19), we deduce the following estimate on $(0, t_0)$:

$$\|\tilde{U}\|_V^2 \le \int_0^t \eta(s)\exp\Big(\int_s^t \gamma(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{2.20}$$

Since the functions γ and η are integrable on $(0, T)$, we can define $t_\star = t_\star(F, U_0)$ as the first time for which

$$\int_0^{t_\star} \eta(s)\exp\Big(\int_s^{t_\star} \gamma(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = \Big(\frac{c_1}{2c_4}\Big)^2. \tag{2.21}$$

Then, on the interval $(0,t_\star)$ we find $\|\tilde{U}\|_V \leq c_1/2c_4$. Hence, on $(0,t_\star)$ the solution $\tilde{U}$ satisfies both (2.17) and (2.18).

We have then the necessary a priori estimates in order to deduce, using the Fourier–Galerkin method, the existence of a solution U of (1.20) such that:

$$U \in L^\infty(0,t_\star;V) \cap L^2(0,t_\star;(\dot{H}^2_{\mathrm{per}}(\Omega))^3). \tag{2.22}$$

The continuity of U from $[0,t_\star]$ into V is proved using an interpolation argument, see e.g. [6] or [14].

The uniqueness of the solution is easily obtained by classical methods, meaning we consider two solutions U_1, U_2 of (1.30) which satisfy (2.22) and estimate $U = U_1 - U_2$ in the H norm, we find that the solutions coincide. □

As we mentioned at the beginning of this section, we now prove the existence and uniqueness of the regular solution of the Primitive Equations, up to $\mathcal{C}^\infty$ regularity. We have the following result:

Theorem 2.3. *Given $m \in \mathbb{N}$, $m \geq 2$, $U_0 \in V \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^3$ and $F \in L^\infty(0,T;H \cap (\dot{H}^{m-1}_{\mathrm{per}}(\Omega))^3)$, there exists $t_{\star\star} = t_{\star\star}(F,U_0)$ and a unique solution U of equation (1.30) on $[0,t_{\star\star}]$ such that:*

$$U \in \mathcal{C}(0,t_{\star\star};(\dot{H}^m_{\mathrm{per}}(\Omega))^3) \cap L^2(0,t_{\star\star};(\dot{H}^{m+1}_{\mathrm{per}}(\Omega))^3). \tag{2.23}$$

Moreover, if $U_0 \in V$ and $F \in L^\infty(0,T;H\cap(\dot{H}^{m-1}_{\mathrm{per}}(\Omega))^3)$, then the solution U of equation (1.30) belongs to $\mathcal{C}((0,t_{\star\star}];\dot{H}^m_{\mathrm{per}}(\Omega))^3)$.

Proof. The proof is based on a priori estimates on the higher order derivatives.

We set $|U|_m = (\sum_{[\alpha]=m}|D^\alpha U|_H^2)^{1/2}$, where D^α is the differential operator $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$, $D_i = \partial/\partial x_i$ α is a multi-index, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_i \in \mathbb{N}$ and $[\alpha] = \alpha_1+\alpha_2+\alpha_3$. In equation (1.20) we take $\tilde{U} = (-\Delta)^m U(t)$, with $m \geq 2$ and t arbitrarily fixed and we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(U,(-\Delta)^m U)_H + a(U,(-\Delta)^m U) + b(U,U,(-\Delta)^m U) + e(U,(-\Delta)^m U)\\ = (F,(-\Delta)^m U)_H.\end{aligned} \tag{2.24}$$

We also note that:

$$a(U,(-\Delta)^m U) + e(U,(-\Delta)^m U) = (a+e)((-\Delta)^{m/2}U,(-\Delta)^{m/2}U) \geq c_1|U(t)|^2_{m+1},$$

where we used the coercivity of $a+e$.

Integrating by parts and using the periodicity, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U(t)|^2_m + c_1|U(t)|^2_{m+1} \leq |b(U,U,(-\Delta)^m U)| + |(F,(-\Delta)^m U)_H|. \tag{2.25}$$

We need to estimate the terms on the right hand side of (2.25). The last term can be easily estimated as:

$$|(F, (-\Delta)^m U)_H| \leq c|F|_{m-1}^2 + \frac{c_1}{2(2m+3)}|U|_{m+1}^2. \tag{2.26}$$

In order to estimate the term $b(U, U, (-\Delta)^m U)$, we note that the integrals we need to consider are of the types:

$$\begin{aligned}&\int_\Omega u\frac{\partial u}{\partial x}D_1^{2\alpha_1}D_2^{2\alpha_2}D_3^{2\alpha_3}u\,\mathrm{d}\Omega, \quad \int_\Omega v\frac{\partial u}{\partial y}D_1^{2\alpha_1}D_2^{2\alpha_2}D_3^{2\alpha_3}u\,\mathrm{d}\Omega,\\ &\int_\Omega w(U)\frac{\partial u}{\partial z}D_1^{2\alpha_1}D_2^{2\alpha_2}D_3^{2\alpha_3}u\,\mathrm{d}\Omega,\end{aligned} \tag{2.27}$$

where, as before, $\alpha_i \in \mathbb{N}$ with $[\alpha] = \alpha_1 + \alpha_2 + \alpha_3 = m$. Integrating by parts and using periodicity, the integrals become:

$$\int_\Omega D^\alpha\big(u\frac{\partial u}{\partial x}\big)D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega D^\alpha\big(v\frac{\partial u}{\partial y}\big)D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega D^\alpha\big(w(U)\frac{\partial u}{\partial z}\big)D^\alpha u\,\mathrm{d}\Omega. \tag{2.28}$$

Using Leibniz' formula, we see that the integrals can be written as sums of integrals of the form

$$\int_\Omega uD^\alpha\frac{\partial u}{\partial x}D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega vD^\alpha\frac{\partial u}{\partial y}D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega w(U)D^\alpha\frac{\partial u}{\partial z}D^\alpha u\,\mathrm{d}\Omega, \tag{2.29}$$

and of integrals of the form

$$\int_\Omega \delta^k u\delta^{m-k}\frac{\partial u}{\partial x}D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega \delta^k v\delta^{m-k}\frac{\partial u}{\partial y}D^\alpha u\,\mathrm{d}\Omega, \quad \int_\Omega \delta^k w(U)\delta^{m-k}\frac{\partial u}{\partial z}D^\alpha u\,\mathrm{d}\Omega, \tag{2.30}$$

where $k = 1, ..., m$ and δ^k is some differential operator D^α with $[\alpha] = k$.

Note that for each α, after integration by parts, the sum of the integrals of type (2.29) is zero because of the mass conservation equation (1.1d). It remains to estimate the integrals of type (2.30). The first two integrals in (2.30) lead to the same kind of estimates, so in fact we only need to estimate the first and last integrals, which we do using Sobolev and interpolation inequalities. For the first integral, we write:

$$\begin{aligned}\Big|\int_\Omega \delta^k u\delta^{m-k}\frac{\partial u}{\partial x}D^\alpha u\,\mathrm{d}\Omega\Big| &\leq |\delta^k u|_{L^3}\big|\delta^{m-k}\frac{\partial u}{\partial x}\big|_{L^6}|D^\alpha u|_{L^2}\\ &\leq c|U|_k^{1/2}|U|_{k+1}^{1/2}|U|_{m-k+2}|U|_m,\end{aligned} \tag{2.31}$$

where $k = 1, ..., m$.

For the last integral we write, when $k < m$:

$$\begin{aligned}\Big|\int_\Omega \delta^k w(U)\delta^{m-k}\frac{\partial u}{\partial z}D^\alpha u\,\mathrm{d}\Omega\Big| &\leq |\delta^k w(U)|_{L^2}\big|\delta^{m-k}\frac{\partial u}{\partial z}\big|_{L^3}|D^\alpha u|_{L^6}\\ &\leq c|U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_{m+1},\end{aligned} \tag{2.32}$$

and when $k = m$:

$$\Big|\int_\Omega \delta^m w(U)\frac{\partial u}{\partial z}D^\alpha u\,\mathrm{d}\Omega\Big| \le |\delta^m w(U)|_{L^2}\Big|\frac{\partial u}{\partial z}\Big|_{L^6}|D^\alpha u|_{L^3} \le c|U|_2|U|_m^{1/2}|U|_{m+1}^{3/2}. \tag{2.33}$$

Gathering relations (2.31), (2.32) and (2.33), we find:

$$\begin{aligned} |b(U,U,(-\Delta)^m U)| &\le c\sum_{k=1}^{m}|U|_{m-k+2}|U|_k^{1/2}|U|_{k+1}^{1/2}|U|_m \\ &\quad + c\sum_{k=1}^{m-1}|U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_{m+1} + c|U|_2|U|_m^{1/2}|U|_{m+1}^{3/2}. \end{aligned} \tag{2.34}$$

We now need to bound the terms from the right-hand side of (2.34):

For the case when $m > 2$, we notice that not all terms on the right hand side of (2.34) contain $|U|_{m+1}$. From the first sum, only terms corresponding to $k = 1$ and to $k = m$ contain $|U|_{m+1}$, and we estimate them as:

$$\begin{aligned} &|U|_{m+1}|U|_1^{1/2}|U|_2^{1/2}|U|_m \le \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_1'|U|_1|U|_2|U|_m^2; \\ &|U|_2|U|_m^{3/2}|U|_{m+1}^{1/2} \le \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_2'|U|_2^{4/3}|U|_m^2. \end{aligned}$$

Terms from the second sum corresponding to $k = 2, ..., m-1$, are estimated as:

$$c|U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_{m+1} \le \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_3'|U|_{k+1}^2|U|_{m-k+1}|U|_{m-k+2}, \tag{2.35}$$

while for the term for $k = 1$, as well as for the last term in (2.34), we have:

$$c|U|_2|U|_m^{1/2}|U|_{m+1}^{3/2} \le \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_4'|U|_2^4|U|_m^2. \tag{2.36}$$

Gathering all the estimates above, we obtain the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_m^2 + c_1|U|_{m+1}^2 \le \theta + \varphi|U|_m^2, \tag{2.37}$$

where the expressions of the functions $\theta = \theta(t)$ and $\varphi = \varphi(t)$ can be easily derived from the estimates above. The functions θ and φ are formed from sums involving the terms $|U|_k$, with $k \le m$.

We also note that for $m = 2$ we obtain, using the Young inequality, the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_2^2 + c_1|U|_3^2 \le c|F|_1^2 + c|U|_1|U|_2^3 + c|U|_1^4|U|_2^2 + c|U|_2^{10/3}. \tag{2.38}$$

Inequality (2.38) can be also written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(1 + |U|_2^2) \le K(|F|_{L^\infty(H^1)}^2, |U|_{L^\infty(0,t_\star;H^1)})(1 + |U|_2^2)^{5/3}, \tag{2.39}$$

and we obtain that there exists a time $t_{\star\star} \leq t_\star$ depending on F, $|U_0|_2$ and on $t_\star$ of Theorem 2.2, such that:

$$|U(t)|_2 \leq K(U_0), \quad \forall\, 0 \leq t \leq t_{\star\star}. \tag{2.40}$$

Using the Gronwall lemma, we find that for each $m \geq 2$, we have a bound for U in $L^\infty(0, t_{\star\star}; \dot{H}^m_{\text{per}}(\Omega))$ and $L^2(0, t_{\star\star}; \dot{H}^{m+1}_{\text{per}}(\Omega))$, where $t_{\star\star}$ was defined above. From this result, the first part of the theorem easily follows.

For the second part of the theorem we notice that since U_0 belongs to V, the solution U of problem (1.30) belongs, according to Theorem 2.2, to $L^2(0, t_\star; \dot{H}^2_{\text{per}})$. This means that $U(t) \in \dot{H}^2_{\text{per}}(\Omega)$ almost everywhere on $(0, t_\star)$, so there exists a t_1 arbitrarily small such that $U(t_1) \in \dot{H}^2_{\text{per}}(\Omega)$. Using now the first part of the theorem we obtain that the solution U is such that:

$$U \in \mathcal{C}([t_1, t_{\star\star}); \dot{H}^2_{\text{per}}(\Omega)) \cap L^2(t_1, t_{\star\star}; \dot{H}^3_{\text{per}}(\Omega)).$$

Using the same argument as before, we find a t_2 belonging to the interval $[t_1, t_{\star\star}]$, arbitrarily close to t_1, such that $U(t_2) \in \dot{H}^3_{\text{per}}(\Omega)$. Applying the result deduced before, in the first part of the theorem, we obtain that the solution U is such that:

$$U \in \mathcal{C}([t_2, t_{\star\star}]; \dot{H}^3_{\text{per}}(\Omega)) \cap L^2(t_2, t_{\star\star}; \dot{H}^4_{\text{per}}(\Omega)).$$

Recurrently we arrive at:

$$U \in \mathcal{C}([t_{m-1}, t_{\star\star}]; \dot{H}^m_{\text{per}}(\Omega)) \cap L^2(t_{m-1}, t_{\star\star}; \dot{H}^{m+1}_{\text{per}}(\Omega)).$$

where t_{m-1} is arbitrarily close to zero. From this relation, the result follows immediately:

$$U \in \mathcal{C}((0, t_{\star\star}]; \dot{H}^m_{\text{per}}(\Omega))$$

□

Remark 2.1. Note here that $t_{\star\star}$ is independent of m; in fact $t_{\star\star} = t(F, |U_0|_2, t_\star)$ is the time for which $|U|_2 \in L^\infty(0, t_{\star\star})$. Then, for each $m > 2$, the functions θ and φ from (2.37) are locally integrable on $(0, t_{\star\star})$ so, by the Gronwall lemma, we obtain a bound of $|U(t)|_m$ on the same interval $(0, t_{\star\star})$.

As a consequence of the above remark, we also deduce the following result:

Remark 2.2. Given $U_0 \in (\dot{\mathcal{C}}^\infty(\bar{\Omega}))^3$ and $F \in L^\infty(0, t, (\dot{\mathcal{C}}^\infty(\bar{\Omega}))^3)$, Theorem 2.3 gives also the existence of a solution U continuous from $(0, t_{\star\star})$ into $\cap_{m\geq 0} \dot{H}^m_{\text{per}}(\Omega) = \dot{\mathcal{C}}^\infty_{\text{per}}(\Omega)$.

If $F \in \mathcal{C}^\infty(\bar{\Omega} \times [0, t])$, estimates on the time derivatives of U can be also obtained as e.g. in [13] for the case of Navier-Stokes equations, so that U is finally $\mathcal{C}^\infty$ in space and time on $(0, t_{\star\star})$.

3. Gevrey regularity results

As mentioned in the introduction, the aim of this paper is also to prove that the solutions of the PEs are real functions analytic in time with values in a Gevrey space; in fact we prove that the solutions are the restriction to a positive real interval of some complex analytic function in time. We start this section by introducing some notations and defining the Gevrey spaces we are will consider.

We introduce the following notation:

$$[U_k]_\kappa^2 = |u_k|^2 + |v_k|^2 + \kappa |T_k|^2.$$

Considering the Laplacian Δ, we define the Gevrey class $D(e^{\tau(-\Delta)^{1/2}})$, $\tau > 0*$ as the set of functions U in H satisfying

$$|\Omega| \sum_{k\in\mathbb{Z}^3} e^{2\tau|k'|}[U_k]_\kappa^2 = |e^{\tau(-\Delta)^{1/2}}U|_H^2 < \infty. \tag{3.1}$$

The norm of the Hilbert space $D(e^{\tau(-\Delta)^{1/2}})$ is given by

$$|U|_\tau := |U|_{D(e^{\tau(-\Delta)^{1/2}})} = |e^{\tau(-\Delta)^{1/2}}U|_H, \text{ for } U \in D(e^{\tau(-\Delta)^{1/2}}), \tag{3.2}$$

and the associated scalar product is

$$(U,V)_\tau := (U,V)_{D(e^{\tau(-\Delta)^{1/2}})} = (e^{\tau(-\Delta)^{1/2}}U, e^{\tau(-\Delta)^{1/2}}V)_H, \text{ for } U,V \in D(e^{\tau(-\Delta)^{1/2}}). \tag{3.3}$$

Another Gevrey space that we will use is $D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})$, $m \geq 1$ integer, which is a Hilbert space when endowed with the inner product:

$$(U,V)_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} = ((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U, (-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}V)_H; \tag{3.4}$$

the norm of the space is given by

$$|U|^2_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} = |(-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U|_H^2 = |\Omega| \sum_{k\in\mathbb{Z}^3} |k'|^{2m} e^{2\tau|k'|}[U_k]_\kappa^2. \tag{3.5}$$

Estimate on b:

We start with the following estimate on b, following the idea of Foias and Temam for the Navier-Stokes equations [4]:

Lemma 3.1. *Let U, $U^\sharp$ and $U^\flat$ be given in $D((-\Delta)^{3/2}e^{\tau(-\Delta)^{1/2}})$, for $\tau \geq 0$. Then the following inequality holds:*

$$\begin{aligned} |((-\Delta)^{1/2}B(U,U^\sharp), (-\Delta)^{3/2}U^\flat)_\tau| &\leq c|\Delta U|_\tau |\Delta U^\sharp|_\tau^{1/2} |(-\Delta)^{3/2}U^\sharp|_\tau^{1/2} |(-\Delta)^{3/2}U^\flat|_\tau \\ &\quad + c|\Delta U|_\tau^{1/2}|(-\Delta)^{3/2}U|_\tau^{1/2}|\Delta U^\sharp|_\tau |(-\Delta)^{3/2}U^\flat|_\tau. \end{aligned} \tag{3.6}$$

Proof. We first write the trilinear form b in Fourier modes. In order to simplify the writing, we define, for each $j \in \mathbb{Z}^3$, $\delta_{j,n}$ as j'_n/j'_3 when $j'_3 \neq 0$ and as 0 when $j'_3 = 0$, for $n = 1, 2$. With obvious notations, the trilinear form is then written as:

$$
\begin{aligned}
b(U, U^\sharp, U^\flat) &= \sum_{j+l+k=0} i(l'_1 u_j + l'_2 v_j + l'_3 w_j) u^\sharp_l u^\flat_k \\
&\quad + \sum_{j+l+k=0} i(l'_1 u_j + l'_2 v_j + l'_3 w_j) v^\sharp_l v^\flat_k + \sum_{j+l+k=0} i(l'_1 u_j + l'_2 v_j + l'_3 w_j) T^\sharp_l T^\flat_k \\
&= \text{using the fact that, by definition, } w_j = 0 \text{ for } j_3 = 0 \text{ } (w \text{ is odd in } x_3) \\
&= \sum_{j+l+k=0} i[(l'_1 - \delta_{j,1} l'_3) u_j + (l'_2 - \delta_{j,2} l'_3) v_j](u^\sharp_l u^\flat_k + v^\sharp_l v^\flat_k + \kappa T^\sharp_l T^\flat_k).
\end{aligned}
\tag{3.7}
$$

We then compute:

$$
\begin{aligned}
&((-\Delta)^{1/2} B(U, U^\sharp), (-\Delta)^{3/2} U^\flat)_\tau \\
&\quad = \sum_{j+l+k=0} i[(l'_1 - \delta_{j,1} l'_3) u_j + (l'_2 - \delta_{j,2} l'_3) v_j] e^{2\tau|k'|} |k'|^4 (u^\sharp_l u^\flat_k + v^\sharp_l v^\flat_k + \kappa T^\sharp_l T^\flat_k).
\end{aligned}
\tag{3.8}
$$

We associate to each function u, a function $\check{u}$ defined by:

$$
\check{u} = \sum_{j \in \mathbb{Z}^3} \check{u}_j e^{i(j'_1 x + j'_2 y + j'_3 z)}, \text{ where } \check{u}_j = e^{\tau|j'|} |u_j|;
\tag{3.9}
$$

we also use similar notations for the other functions.

Since all the terms are similar, we need only to estimate the first sum from (3.8), denoted by I. We find:

$$
|I| \leq \sum_{j+k+l=0} |l'||j'||k'|^4 e^{2\tau|k'|} |u_j||u^\sharp_l||u^\flat_k|,
\tag{3.10}
$$

where we used the estimate $|l'_1 - \delta_{j',1} l'_3| \leq (L_3/2\pi)|j'||l'|$. Since $j + k + l = 0 \iff j' + k' + l' = 0$, we find $|k'| - |l'| - |j'| \leq 0$ and we have:

$$
\begin{aligned}
|I| &\leq \sum_{j+k+l=0} |l'||j'|(|l'| + |j'|)|k'|^3 \check{u}_j \check{u}^\sharp_l \check{u}^\flat_k \\
&\leq \sum_{j+k+l=0} |j'||l'|^2 |k'|^3 \check{u}_j \check{u}^\sharp_l \check{u}^\flat_k + \sum_{j+k+l=0} |j'|^2 |l'||k'|^3 \check{u}_j \check{u}^\sharp_l \check{u}^\flat_k \\
&= \frac{1}{|\Omega|} \int_\Omega q_1(x) q^\sharp_2(x) q^\flat_3(x) \, \mathrm{d}\Omega + \frac{1}{|\Omega|} \int_\Omega q_2(x) q^\sharp_1(x) q^\flat_3(x) \, \mathrm{d}\Omega,
\end{aligned}
\tag{3.11}
$$

where we wrote:

$$\begin{aligned} q_1(x) &= \sum_{j\in\mathbb{Z}^3} |j'|\check{u}_j e^{i(j_1'x+j_2'y+j_3'z)}, \quad q_2(x) = \sum_{j\in\mathbb{Z}^3} |j'|^2\check{u}_j e^{i(j_1'x+j_2'y+j_3'z)}, \\ q_3(x) &= \sum_{j\in\mathbb{Z}^3} |j'|^3\check{u}_j e^{i(j_1'x+j_2'y+j_3'z)}, \end{aligned} \tag{3.12}$$

and the definitions for $q_i^\sharp$ and $q_i^\flat$ for $i=1,2,3$ are the similar ones.

Using the Hölder and the imbedding inequalities, we find:

$$\begin{aligned} |I| &\le |q_1|_{L^6}|q_2^\sharp|_{L^3}|q_3^\flat|_{L^2} + |q_2|_{L^3}|q_1^\sharp|_{L^6}|q_3^\flat|_{L^2} \\ &\le c|q_1|_{H^1}|q_2^\sharp|_{L^2}^{1/2}|q_2^\sharp|_{H^1}^{1/2}|q_3^\flat|_{L^2} + c|q_2|_{L^2}^{1/2}|q_2|_{H^1}^{1/2}|q_1^\sharp|_{H^1}|q_3^\flat|_{L^2} \\ &\le c|\Delta U|_\tau|\Delta U^\sharp|_\tau^{1/2}|(-\Delta)^{3/2}U^\sharp|_\tau^{1/2}|(-\Delta)^{3/2}U^\flat|_\tau \\ &\qquad + c|\Delta U|_\tau^{1/2}|(-\Delta)^{3/2}U|_\tau^{1/2}|\Delta U^\sharp|_\tau|(-\Delta)^{3/2}U^\flat|_\tau. \end{aligned} \tag{3.13}$$

Analogue estimates for the other terms, yield Lemma 3.1. □

A priori estimates for the real case

We first derive some a priori estimates in the real-time case and then we consider the complex-time case. In all that follows we assume that the forcing term F is analytic in time with values in the Gevrey space $D(e^{\sigma_1(-\Delta)^{1/2}}(-\Delta)^{1/2})$, for some $\sigma_1 > 0$, and $U_0 \in D(-\Delta)$. Setting $\varphi(t) = \min(t,\sigma_1)$, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}\Delta$ to equation (1.30), then we take the scalar product in H with $e^{\varphi(t)(-\Delta)^{1/2}}\Delta U$.

Since $a+e$ is coercive, we have:

$$\begin{aligned} &(e^{\varphi(t)(-\Delta)^{1/2}}\Delta AU, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H + (e^{\varphi(t)(-\Delta)^{1/2}}\Delta EU, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H \\ &\quad = a(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U) + e(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U) \\ &\quad \ge c_1|e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H^2. \end{aligned}$$

For the bilinear term, we apply Lemma 3.1 and find:

$$\begin{aligned} |(e^{\varphi(t)(-\Delta)^{1/2}}\Delta B(U,U), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H| &\le c_2|\Delta U|_{\varphi(t)}^{3/2}|(-\Delta)^{3/2}U|_{\varphi(t)}^{3/2} \\ &\le \frac{c_1}{4}|(-\Delta)^{3/2}U|_{\varphi(t)}^2 + c_3|\Delta U|_{\varphi(t)}^6. \end{aligned} \tag{3.14}$$

For the term containing the time derivative of U, we have:

$$\begin{aligned}
&(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U'(t), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U(t))_H \\
&\quad = \big(\frac{\mathrm{d}}{\mathrm{d}t}(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U\big)_H \\
&\quad\quad - \varphi'(t)(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U)_H \\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|_H^2 - \varphi'(t)(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U)_H \\
&\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|_H^2 - |e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|_H \\
&\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|_{\varphi(t)}^2 - \frac{c_1}{4}|(-\Delta)^{3/2}U|_{\varphi(t)}^2 - \frac{1}{c_1}|\Delta U|_{\varphi(t)}^2.
\end{aligned}$$

The term containing the force F is estimated as:

$$\begin{aligned}
(e^{\varphi(t)(-\Delta)^{1/2}}\Delta F, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H &\leq |(-\Delta)^{1/2}F|_{\varphi(t)}|(-\Delta)^{3/2}U|_{\varphi(t)} \\
&\leq \frac{1}{c_1}|(-\Delta)^{1/2}F|_{\varphi(t)}^2 + \frac{c_1}{4}|(-\Delta)^{3/2}U|_{\varphi(t)}^2.
\end{aligned} \tag{3.15}$$

Gathering all these estimates, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|_{\varphi(t)}^2 + c_1|(-\Delta)^{3/2}U|_{\varphi(t)}^2 \leq \frac{2}{c_1}|\Delta U|_{\varphi(t)}^2 + c_2'|\Delta U|_{\varphi(t)}^6 + c_3'|(-\Delta)^{1/2}F|_{\varphi(t)}^2. \tag{3.16}$$

We consider the function $y(t) = 1 + |\Delta U|_{\varphi(t)}^2$. Since

$$|(-\Delta)^{1/2}F|_{\varphi(t)}^2 \leq |(-\Delta)^{1/2}F|_{\sigma_1},$$

we find, for any $t_1 > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leq c_4 y^3(t), \qquad 0 < t < t_1, \tag{3.17}$$

where c_4 is a constant depending on the norm of F in $L^\infty(0, t_1; D((-\Delta)^{1/2}e^{\sigma_1(-\Delta)^{1/2}}))$.

We easily deduce that there exists a time $t_\star'$, $0 < t_\star' \leq t_1$, $t_\star' = t_\star'(F, U_0) = 3/8y^2(0)c_4$, such that $y(t) \leq 2y(0)$ for all $0 \leq t \leq t_\star'(F, U_0)$. We then obtain the following a priori estimate:

$$|\Delta U(t)|_{\varphi(t)}^2 \leq 1 + 2|\Delta U_0|_H^2, \quad \forall\, t \leq t_\star'(F, U_0). \tag{3.18}$$

A priori estimates for the complex case

In order to prove that the solution is analytic in time and coincides with the restriction of a complex function in time to a real positive interval, we consider equation (1.30) with a complex time $\zeta \in \mathbb{C}$, and U a complex function. We take the complexified spaces H and V denoted $H_{\mathbb{C}}$ and $V_{\mathbb{C}}$ [1], so equation (1.30) is rewritten as:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}\zeta} + AU + B(U,U) + E(U) = F, \\ &U(0) = U_0, \end{aligned} \tag{3.19}$$

where $\zeta \in \mathbb{C}$ is the complex time.

We consider ζ of the form $\zeta = se^{i\theta}$, where $s > 0$ and $\cos\theta > 0$ so that the real part of ζ is positive. We apply $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta$ to equation (3.19) and take the scalar product in $H_{\mathbb{C}}$ with $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U$. We then multiply the resulting equation by $e^{i\theta}$ and take the real part. We find:

$$\begin{aligned} &\operatorname{Re} e^{i\theta}\big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\frac{\mathrm{d}U}{\mathrm{d}\zeta}, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U\big)_H \\ &\quad= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U|_H^2 \\ &\quad\quad+ \varphi'(s\cos\theta)\cos\theta\operatorname{Re} e^{i\theta}\big(\Delta e^{\varphi(s\cos\theta)(-\Delta)^{3/2}}U, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U\big)_H \\ &\quad\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\Delta U|_{\varphi(s\cos\theta)}^2 - \cos\theta|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}|\Delta U|_{\varphi(s\cos\theta)}. \end{aligned} \tag{3.20}$$

Since $a + e$ is coercive for our choice of κ, we also find:

$$\begin{aligned} &\operatorname{Re} e^{i\theta}\big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta AU, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U\big)_H \\ &\quad+ \operatorname{Re} e^{i\theta}\big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta EU, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U\big)_H \\ &\quad\geq c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H^2 = c_1\cos\theta|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}^2. \end{aligned} \tag{3.21}$$

For the forcing term, we write:

$$\begin{aligned} &|\operatorname{Re} e^{i\theta}\big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta F, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U\big)_H| \leq |(-\Delta)^{1/2}F|_{\varphi(s\cos\theta)}|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)} \\ &\quad\leq \frac{c_1}{6}\cos\theta|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}^2 + \frac{1}{c_1\cos\theta}|(-\Delta)^{1/2}F|_{\varphi(s\cos\theta)}^2. \end{aligned} \tag{3.22}$$

For the bilinear term B we use Lemma 3.1 and the Young inequality:

$$\begin{aligned} |\operatorname{Re} e^{i\theta}(\Delta B(U,U), \Delta U)_{\varphi(s\cos\theta)}| &\leq c_2|\Delta U|_{\varphi(s\cos\theta)}^{3/2}|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}^{3/2} \\ &\leq \frac{c_1}{6}\cos\theta|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}^2 + \frac{c_3}{(\cos\theta)^3}|\Delta U|_{\varphi(s\cos\theta)}^6. \end{aligned} \tag{3.23}$$

[1] For the scalar products and the norms we use the same notations as in the real case.

Gathering all the estimates above, we find the following differential inequality:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\Delta U|^2_{\varphi(s\cos\theta)} + \frac{c_1}{2}\cos\theta|(-\Delta)^{3/2}|^2_{\varphi(s\cos\theta)} &\leq \frac{1}{c_1\cos\theta}|(-\Delta)^{1/2}F|^2_{\varphi(s\cos\theta)} \\ &+ \frac{\cos\theta}{c_1}|\Delta U|^2_{\varphi(s\cos\theta)} + \frac{c_3}{(\cos\theta)^3}|\Delta U|^6_{\varphi(s\cos\theta)}.\end{aligned} \tag{3.24}$$

We restrict θ such that $\sqrt{2}/2 \leq \cos\theta \leq 1$ (in fact we can restrict θ to any domain such that $\cos\theta \geq c > 0$). Writing

$$y(s) = 1 + |\Delta U(s)|^2_{\varphi(s\cos\theta)},$$

the differential inequality (3.24) becomes:

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} \leq c(F)y^3(s), \quad 0 < s < t_1, \tag{3.25}$$

where $c(F)$ is a constant depending as before on the forcing term F. Therefore, by similar reasoning as for the real case, we find that there exists a time $t'_\star$, $0 < t'_\star \leq t_1$, $t'_\star = t'_\star(F, U_0)$ such that:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U(se^{i\theta})|^2_H \leq 1 + 2|\Delta U_0|^2_H, \quad \forall\, 0 \leq s \leq t'_\star(F, U_0). \tag{3.26}$$

Considering the complex region

$$\mathcal{D}(U_0, F, \sigma_1) = \{\zeta = se^{i\theta},\ |\theta| \leq \pi/4,\ 0 < s < t'_\star(F, U_0)\}, \tag{3.27}$$

estimate (3.26) gives us a bound for $U(\zeta)$, when $\zeta \in \mathcal{D}(U_0, F, \sigma_1)$.

We can now state the main result of this section:

Theorem 3.1. *Let U_0 be given in $\dot{H}^2_{\mathrm{per}}(\Omega)$ and let F be a given function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}}(-\Delta)^{1/2})$ for some $\sigma_1 > 0$. Then there exists $t'_\star > 0$ depending on the data, including U_0, and a unique solution U of (1.30) on $(0, t'_\star)$ such that the function*

$$t \to \Delta e^{\varphi(t)(-\Delta)^{1/2}}U(t),$$

is analytic from $(0, t'_\star)$ with values in H, where $\varphi(t) = \min(t, \sigma_1)$ and $t'_\star$ was defined above.

Proof. The proof is based on the a priori estimates obtained above and the use of the Galerkin–Fourier method; see e.g. [4]. The solutions of the Galerkin approximation satisfy rigorously the estimates formally derived above, and the bounds are independent of the order m of the Galerkin approximation. We can then pass to the limit $m \to \infty$, using classical results on the convergence of analytic functions. □

Remark 3.1. Taking into account the second part of Theorem 2.3, we see that the result of Theorem 3.1 still holds true while starting with initial data $U_0 \in V$, since at arbitrarily small time t the solution U satisfies $U(t) \in \dot{H}^2_{\mathrm{per}}(\Omega)$ and $U \in \mathcal{C}((0, t_{\star\star}]; \dot{H}^2_{\mathrm{per}}(\Omega))$.

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