# ON THE CONJUGACY OF INJECTORS IN LOCALLY FINITE GROUPS

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Abstract. We characterize the largest subgroup-closed classes of locally finite groups where injectors behave as in finite groups. The present paper is the second part of [1].

## 1. Results

The notations are the ones of [1], we add just the following definition.

**Definition 1.1.** Let $G$ be a locally finite group. If $\mathfrak{X}$ is a set of subgroups of $G$, a subgroup $V$ of $G$ is an $\mathfrak{X}$-*sn-injector* of $G$ if, for every subnormal subgroup $A$ of $G$, $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of $A$.

**Proposition 1.2.** *Let $G$ be a locally finite group, and $\mathfrak{X}$ a normal Fitting set of $G$ such that*

- *any $\mathfrak{X}$-group is radical;*
- *for any normal $\mathfrak{X}$-subgroup $K$ of a subgroup $H$ of $G$, if $H/K$ is abelian, then $H \in \mathfrak{X}$;*

*Moreover, we assume that any subgroup $H$ of $G$ satisfies one of the following conditions:*

- *$H$ has exactly one conjugacy class of $\mathfrak{X}$-sn-injectors and its $\mathfrak{X}$-sn-injectors are pronormal in $H$;*
- *$H$ has exactly one conjugacy class of $\mathfrak{X}$-a-injectors and its $\mathfrak{X}$-a-injectors are pronormal in $H$;*
- *$H$ has exactly one conjugacy class of $\mathfrak{X}$-injectors.*

*Then $G$ is an $\mathfrak{X}$-group.*

*Proof.* We prove the result in the case of "$\mathfrak{X}$-sn-injectors", but the same proof works in the other cases, except the last paragraph in the case of "$\mathfrak{X}$-injectors".

We show that $G$ is locally soluble. Let $H$ be a finite subgroup of minimal order for the condition that $H$ is not soluble. Then every proper normal subgroup of $H$ is soluble, and the subgroup $N$ generated by them is soluble too. Moreover, the $\mathfrak{X}$-sn-injectors of $H$ are the maximal $\mathfrak{X}$-subgroups of $H$ containing $N$. Let $U$ be an $\mathfrak{X}$-sn-injector of $H$, $p$ a prime, and $P/N$ a Sylow $p$-subgroup of $H/N$. Then $P$ is an $\mathfrak{X}$-subgroup, so $P$ is contained in a maximal $\mathfrak{X}$-subgroup of $H$, which is an $\mathfrak{X}$-sn-injector. By the conjugacy of $\mathfrak{X}$-sn-injectors of $H$, $U/N$ contains a Sylow $p$-subgroup of $H/N$ for every prime $p$. Hence we have $H = U$ and $H$ is soluble, so $G$ is locally soluble.

*Date*: November 11, 2013.
1991 *Mathematics Subject Classification.* Primary 20F17; Secondary 20D10.
*Key words and phrases.* locally finite groups, Fitting sets, Injectors.

We show that each $\mathfrak{X}$-sn-injector $V$ of $G$ covers all the chief factors $H/K$ of $G$. Since $V \cap K$ is a maximal $\mathfrak{X}$-subgroup of $K$, it is self-normalizing in $K$. Since $V \cap K$ is an $\mathfrak{X}$-sn-injector of $K$, a Frattini Argument applied with $V \cap K$ gives $H = KN_H(V \cap K)$. Since $G$ is locally soluble, the quotient $H/K \simeq N_H(V \cap K)/(V \cap K)$ is abelian, and $N_H(V \cap K)$ is an $\mathfrak{X}$-group. But $V \cap H$ is a maximal $\mathfrak{X}$-subgroup of $H$, so $V \cap H = N_H(V \cap K)$. Hence $V$ covers $H/K$.

We assume toward a contradiction that $G$ is not an $\mathfrak{X}$-group. Let $V$ be an $\mathfrak{X}$-sn-injector of $G$, and let $L = G_{\mathfrak{X}n}$. Then we have $L \leq V < G$. Moreover, for each $g \in N_G(V)$, the subgroup $V\langle g\rangle$ is an $\mathfrak{X}$-group, so $N_G(V) = V$ by the maximality of $V$, and we obtain $L < V$. Let $p$ be a prime such that $\rho(V/L)$ has a nontrivial Sylow $p$-subgroup $P/L$. Let $C/L$ be the intersection of the centralizers of all $p'$-chief factors of $G/L$. If $(H/L)/(K/L)$ is a $p'$-chief factor of $G/L$, then $V$ covers $H/K$ by the previous paragraph and $P$ avoids $H/K$. But $V$ normalizes $P$, so $[P, H]K = [P, V \cap H]K$ is contained in $(P \cap H)K = K$ and $P$ centralizes $H/K$. Therefore $C$ contains $P$. By [2, Theorem 3.8], $C/L$ has a normal Sylow $p$-subgroup $S/L$. Thus we obtain $P/L \leq S/L \leq \rho(G/L)$, and $V/L \cap \rho(G/L)$ is nontrivial.

Let $R/L = \rho(G/L)$. Then $V \cap R$ is pronormal in $R$, and since $V \cap R$ contains $L$, the subgroup $V \cap R$ is both pronormal and serial in $R$, so it is normal in $R$. But $V \cap R$ is a maximal $\mathfrak{X}$-subgroup of $R$, so it is self-normalizing in $R$, and we obtain $R \leq V$. Now $R$ is a normal $\mathfrak{X}$-subgroup of $G$, so it is contained in $L$, contradicting that $\rho(G/L)$ is nontrivial. Thus $G$ is an $\mathfrak{X}$-group as desired.

In the case of "$\mathfrak{X}$-injectors", the previous paragraph does not work. However, for each $r \in R$ the subgroup $\langle r\rangle L/L$ is serial in $R/L = \rho(G/L)$ so $\langle r\rangle L$ is serial in $G$. But since $L$ is an $\mathfrak{X}$-group, it is an $\mathfrak{X}$-group too, hence we find $r \in V$ and $V$ contains $R$. Thus $R$ is an $\mathfrak{X}$-group, contradicting $\mathfrak{X}n = L < R$. □

**Lemma 1.3.** *it Let $G$ be a locally finite group, and $\pi$ be a set of primes. We assume that any two $\mathfrak{X}_\pi$-injectors of a subgroup $H$ of $G$ are conjugate in $H$. If $G = S\rho(G)$ for a Sylow $\pi$-subgroup $S$ of $G$, then the Sylow $\pi$-subgroups of $G$ are conjugate and $S$ is an $\mathfrak{X}_\pi$-injector of $G$.*

*Proof.* Let $R$ be a Sylow $\pi$-subgroup of $G$, $P$ be the Sylow $\pi$-subgroup of $\rho(G)$ and $Q$ be the Sylow $\pi'$-subgroup of $\rho(G)$. In particular $S \cap R\rho(G)$ and $R$ contain $P$, and we have $R\rho(G) = (S \cap R\rho(G))\rho(G)$.

We show that, if $L$ is a $\pi$-subgroup of $G$ containing $P$ and if $L\rho(G) = R\rho(G)$, then $L$ is an $\mathfrak{X}_\pi$-injector of $R\rho(G)$. Let $A$ be a serial subgroup of $R\rho(G)$. Then $AP$ is a serial subgroup of $A\rho(G)$. Thus, if $A\rho(G)$ has a $\pi$-element $x \notin AP$, there exist $U$ and $V$ two subgroups of $A\rho(G)$ containing $AP$ such that $V \trianglelefteq U$ and $x \in U \setminus V$. Since $A\rho(G) = APQ$, $Q$ covers $U/V$ and $U/V$ has not a $\pi$-element, contradicting the choice of $x$. Therefore $AP$ contains all the $\pi$-elements of $A\rho(G)$ and we have $L \cap A\rho(G) = L \cap AP = (L \cap A)P$.

Let $T$ be a Sylow $\pi$-subgroup of $A$ containing $L \cap A$. We obtain

$$\begin{aligned} A &= A \cap (L \cap A\rho(G))\rho(G) = A \cap (L \cap A)\rho(G) = (L \cap A)(A \cap \rho(G)) \\ T &= (L \cap A)(T \cap A \cap \rho(G)) = (L \cap A)(T \cap \rho(G)) \leq (L \cap A)P \leq L. \end{aligned}$$

Thus $L \cap A$ is a Sylow $\pi$-subgroup of $A$ and $L$ is an $\mathfrak{X}_\pi$-injector of $R\rho(G)$.

By the paragraph above, $S \cap R\rho(G)$ and $R$ are two $\mathfrak{X}_\pi$-injectors of $R\rho(G)$, so they are conjugate in $R\rho(G)$. Thus, there exists $g \in R\rho(G)$ such that $S^g$ contains $R$, hence $R = S^g$ and $G = R\rho(G)$. This finishes the proof. □

*Notation* 1.4. For each set $\pi$ of primes, we denote by $\mathfrak{X}_\pi$ the set of $\pi$-subgroups of locally finite groups.

Let $\mathfrak{V}sn$ (resp. $\mathfrak{V}a$, $\mathfrak{V}$) be the largest subgroup-closed class of locally finite subsoluble groups (resp. $SN^*$-groups, radical groups) $G$ such that, for any set $\pi$ of primes, $G$ has a unique conjugacy class of $\mathfrak{X}_\pi$-sn-injectors (resp. $\mathfrak{X}_\pi$-a-injectors, $\mathfrak{X}_\pi$-injectors).

We recall that if a group $G$ has a normal $\mathfrak{U}$-subgroup $H$ such that $G/H$ is finite and soluble, then $G$ is a $\mathfrak{U}$-group too [3, Lemma 6.6].

**Lemma 1.5.** *Let $G$ be a $\mathfrak{V}$-sn-group (resp. $\mathfrak{V}$-a-group, $\mathfrak{V}$-group). If $G/\rho(G)$ is a Baer(resp. Gruenberg, locally nilpotent) group, then $G$ is a $\mathfrak{U}$-group.*

*Proof.* Let $\pi$ be a set of primes, and $V$ an $\mathfrak{X}_\pi$-sn-injector (resp. $\mathfrak{X}_\pi$-a-injector, $\mathfrak{X}_\pi$-injector) of $G$. Since $G/\rho(G)$ is locally nilpotent, $G/\rho(G)$ has a unique Sylow $\pi$-subgroup $G_\pi/\rho(G)$, and we have $V \leq G_\pi$.

By Lemma 1.3, we have just to prove that $G_\pi = V\rho(G)$. For each $g \in G_\pi$, $\rho(G)\langle g\rangle$ is a finite extension of $\rho(G)$ and, by [3, Lemma 6.6], $\rho(G)\langle g\rangle$ is a $\mathfrak{U}$-group. But $G/\rho(G)$ is a Baer group (resp. an Gruenberg group, a locally nilpotent group), so $\rho(G)\langle g\rangle$ is subnormal (resp. ascendant, serial) in $G$. Thus $V \cap \rho(G)\langle g\rangle$ is a Sylow $\pi$-subgroup of $\rho(G)\langle g\rangle$, and we obtain $\rho(G)\langle g\rangle = (V \cap \rho(G)\langle g\rangle)\rho(G)$ [2, Lemma 2.1 (ii)]. In particular, $V\rho(G)$ contains $g$ and $G_\pi$, so $G_\pi = V\rho(G)$, as desired. □

*Fact* 1.6. ([3, Theorem E] and [4, Lemma 1.1]) If $G$ is a $\mathfrak{U}$-group, then $G/\rho(G)$ is (2-step soluble)-by-finite. In particular $G/\rho(G)$ is soluble. Moreover, $G/\rho(G)$ is characteristically hyperfinite.

**Theorem 1.7.** *The classes $\mathfrak{V}sn$, $\mathfrak{V}a$ and $\mathfrak{V}$ are contained in $\mathfrak{U}$.*

*Proof.* We give the proof just for $\mathfrak{V}sn$, since the same proof works for $\mathfrak{V}a$ and $\mathfrak{V}$. Let $G$ be a $\mathfrak{V}sn$-group. Let $\alpha$ be the least ordinal such that, if we define $B_0(G) = \rho(G)$, $B_{i+1}(G)/B_i(G) = \beta(G/B_i(G))$ for each ordinal $i$ and $B_\gamma(G) = \cup_{i<\gamma} B_i(G)$ for limit ordinals $\gamma$, then $G = B_\alpha(G)$. We may assume that for each $\mathfrak{V}sn$-group $H$, if $H = B_i(H)$ for $i < \alpha$, then $H$ is a $\mathfrak{U}$-group. By Lemma 1.5, we may assume $\alpha \geq 2$.

(1) *$G/\beta(G)$ is hyperfinite.* If $\alpha > \omega$, then $B_\omega(G)$ is a $\mathfrak{U}$-group containing $B_n(B_\omega(G)) = B_n(G)$ for each integer $n$, contradicting Fact 1.6. So we have $\alpha \leq \omega$.

Suppose $\alpha = \omega$. Then $B_i(G)$ is a $\mathfrak{U}$-group for each $i \in \mathbb{N}$, and $B_i(G)/\rho(G)$ is characteristically hyperfinite for each $i \in \mathbb{N}$ (Fact 1.6), so $G/\rho(G)$ is hyperfinite.

Suppose $\alpha < \omega$. By the minimality of $\alpha$, $B_1(G)$ is a $\mathfrak{U}$-group. So $B_1(G)$ has a Carter subgroup $C$ [2, Theorem 5.4], $B_1(G) = \rho(G)C$ and the Carter subgroups of $B_1(G)$ are conjugate. Then we have $N_G(C) \cap B_1(G) = C$ and a Frattini Argument gives $G = \rho(G)N_G(C)$. We obtain $G/B_1(G) \simeq N_G(C)/C$, and since $C$ is locally nilpotent, $N_G(C) = B_{\alpha-1}(N_G(C))$. Therefore, by the minimality of $\alpha$, $N_G(C)$ is a $\mathfrak{U}$-group. Moreover, since $G = \rho(G)N_G(C)$, we have $\rho(N_G(C)) \leq N_G(C) \cap B_1(G) = C$, and $G/B_1(G) \simeq N_G(C)/C$ is hyperfinite (Fact 1.6). As $B_1(G)$ is a $\mathfrak{U}$-group, $B_1(G)/\rho(G)$ is characteristically hyperfinite (Fact 1.6), hence $G/\rho(G)$ is hyperfinite.

(2) *$G$ is a $\mathfrak{U}$-group.* By (1), there exists an ordinal $\gamma$ such that $G/\rho(G)$ has a normal series $(U_i/\rho(G))_{0\leq i\leq\gamma}$ with $U_{i+1}/U_i$ finite abelian for every $i < \gamma$, $U_0 = \rho(G)$ and $U_\gamma = G$. We may assume that $G$ is not a $\mathfrak{U}$-group. Then there exists a set $\pi$ of primes such that Sylow $\pi$-subgroups of $G$ are not conjugate. Since $G$ is an $\mathfrak{V}sn$-group, $G$ has a Sylow $\pi$-subgroup $S$ such that $S$ is not an $\mathfrak{X}_\pi$-sn-injector.

Let $\beta$ be the least ordinal such that there exists a Sylow $\pi$-subgroup $R$ of $RU_\beta$ which is not an $\mathfrak{X}_\pi$-sn-injector of $RU_\beta$. By Lemma 1.3, we have $\beta > 0$.

Assume that $\beta$ is a limit ordinal. Let $A$ be a normal subgroup of $RU_\beta$ and $T$ be a Sylow $\pi$-subgroup of $A$ containing $R \cap A$. By the minimality of $\beta$, $R$ is an $\mathfrak{X}_\pi$-sn-injector of $RU_i$ for every $i < \beta$, therefore $R \cap A$ is a Sylow $\pi$-subgroup of $A \cap RU_i$. We obtain $R \cap A = T \cap RU_i$ for every $i < \beta$, so $R \cap A = T \cap RU_\beta = T$. Thus $R$ is an $\mathfrak{X}_\pi$-sn-injector of $RU_\beta$, contradicting the choice of $R$.

Assume there is an ordinal $\delta$ such that $\beta = \delta + 1$. Let $C = C_{RU_\beta}(U_\beta/U_\delta)$. Then $C$ is a normal subgroup of finite index of $RU_\beta$. Since $U_\beta/U_\delta$ is abelian, we have $U_\beta \leq C$ and $C = (R \cap C)U_\beta$. By the minimality of $\beta$, $R$ is an $\mathfrak{X}_\pi$-sn-injector of $RU_\delta$, so $R \cap C = R \cap (C \cap RU_\delta)$ is an $\mathfrak{X}_\pi$-sn-injector of $C \cap RU_\delta$. Let $M = N_{RU_\beta}(R \cap C)$. Then $R/(R \cap C)$ is a finite Sylow $\pi$-subgroup of $M/(R \cap C)$. Thus, since $M/(R \cap C)$ is locally soluble, the Sylow $\pi$-subgroups of $M/(R \cap C)$ are finite, and conjugate. As $R \cap C$ is a $\pi$-subgroup, we obtain the conjugacy of the Sylow $\pi$-subgroups of $M$.

Let $V$ be an $\mathfrak{X}_\pi$-sn-injector of $RU_\beta$. As $C \cap RU_\delta$ is normal in $RU_\beta$, the subgroup $V \cap (C \cap RU_\delta)$ is an $\mathfrak{X}_\pi$-sn-injector of $C \cap RU_\delta$ and there is $x \in C \cap RU_\delta$ such that $(V \cap C \cap RU_\delta)^x = R \cap C$. Moreover $V$ normalizes $V \cap C \cap RU_\delta$ and we obtain $V^x \leq M$, so $V^x$ is a Sylow $\pi$-subgroup of $M$. Now $V^x$ and $R$ are conjugate in $M$ and $R$ is an $\mathfrak{X}_\pi$-sn-injector of $RU_\beta$, contradicting the choice of $R$. □

**Corollary 1.8.** *Let $G$ be a locally finite group. We assume that for $\mathfrak{X} = \mathscr{S}$, the set of subsoluble subgroups of $G$, and $\mathfrak{X} = \mathfrak{X}_\pi$ for any set $\pi$ of primes, any subgroup $H$ of $G$ satisfies:*

- *$H$ has exactly one conjugacy class of $\mathfrak{X}$-n-injectors;*
- *for every normal subgroup $A$ of $H$ and every $\mathfrak{X}$-n-injector $V$ of $H$, the subgroup $V \cap A$ is an $\mathfrak{X}$-n-injector of $A$;*
- *any $\mathfrak{X}$-n-injector of $H$ contained in $K \leq H$ is an $\mathfrak{X}$-n-injector of $K$,*

*Then $G$ is a subsoluble $\mathfrak{U}$-group.*

*Proof.* By our hypotheses, we may apply Proposition 1.2 with $\mathfrak{X} = \mathscr{S}$, so $G$ is subsoluble. Now Theorem 1.7 says that $G$ is a $\mathfrak{U}$-group. □

*Remark* 1.9. By [1, Theorem 1.4, Proposition 1.14 and Corollary 1.15], for any subsoluble $\mathfrak{U}$-group $G$, the hypotheses of Corollary 1.8 are satisfied for any normal Fitting set $\mathfrak{X}$ of $G$.

**Conclusion 1.10.** *The largest subgroup-closed class of locally finite groups $G$ such that, for any normal Fitting set $\mathfrak{X}$ of $G$,*

- *$G$ has exactly one conjugacy class of $\mathfrak{X}$-n-injectors;*
- *for every normal subgroup $A$ of $G$ and every $\mathfrak{X}$-n-injector $V$ of $G$, the subgroup $V \cap A$ is an $\mathfrak{X}$-n-injector of $G$;*
- *any $\mathfrak{X}$-n-injector of $H$ contained in $K \leq H$ is an $\mathfrak{X}$-n-injector of $K$,*

*is the class of subsoluble $\mathfrak{U}$-groups.*

Equivalently:

**Conclusion 1.11.** *The largest subgroup-closed class of locally finite groups $G$ such that, for any normal Fitting set $\mathfrak{X}$ of $G$,*

- *$G$ has exactly one conjugacy class of $\mathfrak{X}$-sn-injectors;*
- *any $\mathfrak{X}$-sn-injector of $H$ contained in $K \leq H$ is an $\mathfrak{X}$-sn-injector of $K$,*

*is the class of subsoluble $\mathfrak{U}$-groups.*

If we apply Proposition 1.2 with $\mathfrak{X}$ the class of locally finite $SN^*$-groups, we obtain:

**Conclusion 1.12.** *The largest subgroup-closed class of locally finite groups $G$ such that, for any ascendant Fitting set $\mathfrak{X}$ of $G$,*

- *$G$ has exactly one conjugacy class of $\mathfrak{X}$-a-injectors;*
- *any $\mathfrak{X}$-a-injector of $H$ contained in $K \leq H$ is an $\mathfrak{X}$-a-injector of $K$,*

*is the class $SN^* \cap \mathfrak{U}$.*

Now if we apply Proposition 1.2 with $\mathfrak{X}$ the class of radical locally finite groups, we obtain:

**Conclusion 1.13.** *The largest subgroup-closed class of locally finite groups $G$ such that, for any serial Fitting set $\mathfrak{X}$ of $G$,*

- *$G$ has exactly one conjugacy class of $\mathfrak{X}$-injectors,*

*is the class of $\mathfrak{U}$-groups.*

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