AN APPROXIMATING FAMILY FOR THE DIRICHLET-TO-NEUMANN SEMIGROUP

HASSAN EMAMIRAD AND IDRISS LAADNANI

ABSTRACT. *In this paper we prove that the Dirichlet-to-Neumann semigroup $S(t)$ is an analytic compact Markov irreducible semigroup in $C(\partial\Omega)$ in any bounded smooth domain Ω. By a generalization of the Lax semigroup, we construct an approximating family for $S(t)$. We prove some regularizing characters and compactness of this family. By using the ergodic properties of $S(t)$, we deduce its asymptotic behavior. At the end we conjecture some open problems.*

1. INTRODUCTION.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let γ be a C^∞ matrix- valued function on $\overline{\Omega}$ which represents the electrical conductivity of Ω. We suppose that $\gamma(x) = [\gamma_{i,j}(x)]$ satisfies the following hypotheses:

(H1) The coefficients are symmetric and $\gamma_{i,j}(x) = \gamma_{j,i}(x) \in \mathcal{C}^\infty(\overline{\Omega})$; and there exist two constants $0 < c_1 \leq c_2 < \infty$ such that for all $\xi \in \mathbb{R}^n$, we have

$$c_1\|\xi\|^2 \leq \sum_{i,j=1}^{n} \xi_i\xi_j\gamma_{i,j}(x) \leq c_2\|\xi\|^2.$$

The Dirichlet-to-Neumann operator Λ_γ is defined by

$$\Lambda_\gamma f = -\partial v/\partial\nu_\gamma = -\nu \cdot \gamma\nabla v \mid_{\partial\Omega}, \tag{1.1}$$

where ν is the outer normal vector at $\overline{x} \in \partial\Omega$ and v is a solution to the electrical conductivity equation

$$\textbf{(P1)} \qquad \begin{cases} A_\gamma v := \operatorname{div}(\gamma\nabla v) = 0 & \text{in } \Omega, \\ v = f & \text{on } \partial\Omega. \end{cases}$$

Such a function is called γ-harmonic lifting of f.

Let X be the Banach space $C(\overline{\Omega})$ and H the Hilbert space $L^2(\Omega)$. On X or H, the operator A_γ is defined on its maximal domain

$$D(A_\gamma) := \{u \in X \text{ or } H \mid A_\gamma u \in X \text{ or } H\}.$$

Date: February 28, 2006.

1991 *Mathematics Subject Classification.* 47A40, 35P25.

Key words and phrases. Dirichlet-to-Neumann semigroup.

We introduce the *boundary spaces* $\partial X := C(\partial\Omega)$ and $\partial H := L^2(\partial\Omega)$. On ∂X we define Λ_γ as un unbounded operator with domain

$$D(\Lambda_\gamma) := \{f \in \partial X \mid \Lambda_\gamma f \in \partial X\}.$$

In [Esc], J. Escher defines the Dirichlet-to-Neumann semigroup in a more general context. He proves that, for any bounded smooth domain Ω in $\mathbb{R}^n$, the following elliptic system with dynamical boundary condition

$$\textbf{(P2)} \qquad \begin{cases} -\operatorname{div}(\gamma\nabla u(t,.)) + \boldsymbol{a}\cdot\nabla u(t,.) + a_0 u(t,.) = 0 & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial}{\partial t}u(t,.) + \frac{\partial}{\partial\nu_\gamma}u(t,.) + b_0 u(t,.) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,.) = f & \text{on } \partial\Omega. \end{cases}$$

where $(\boldsymbol{a}, a_0, b_0) \in [C^\infty(\overline{\Omega})]^{n+2}$, has a unique solution $u(t,f)$, for any f in the Besov space $B^s_{p,p}(\partial\Omega)$ for $s\in\mathbb{R}$, $p\in(0,\infty)$. Since $C(\partial\Omega) \hookrightarrow B^s_{p,p}(\partial\Omega)$ for $s<0$, Escher defines the Dirichlet-to-Neumann semigroup on $C(\partial\Omega)$ by taking the trace of this solution on $C(\partial\Omega)$, i.e.,

$$S(t)f := u(t,f)\mid_{\partial\Omega} \quad \text{for any } f\in C(\partial\Omega). \tag{1.2}$$

In [Esc], we can find the following Theorem.

Theorem 1.1. *The family* $\{S(t)\}_{t\geq 0}$ *defined by* (1.2) *forms an analytic, compact and positive* C_0*-semigroup on* ∂X.

Before going into the heart of matter, let us recapitulate some known results in the Hilbertian framework. On ∂H, the definition of $D(\Lambda_\gamma)$ is more subtle. In order for problem **(P1)** to admit a unique solution in $H^1(\Omega)$, we have to assume that $f \in H^{1/2}(\partial\Omega)$. So, we define

$$D(\Lambda_\gamma) := \{f \in H^{1/2}(\partial\Omega) \mid \Lambda_\gamma f \in \partial H\}.$$

As in [Tay, sections 11 and 12 of Chapter 7], one can prove that Λ_γ is a nonpositive selfadjoint operator on ∂H. So, by using [Kat, page 491], Λ_γ generates an analytic C_0-semigroup of contractions in ∂H. The compactness of the imbedding $D(\Lambda_\gamma) \hookrightarrow \partial H$ implies that this semigroup is also compact. Consequently, the corresponding Cauchy problem which is problem **(P2)** with $(\boldsymbol{a}, a_0, b_0)$=0 has a unique solution $u \in C([0,\infty);\partial H)\cap C((0,\infty);D(\Lambda_\gamma))\cap C^1((0,\infty);\partial H)$ for any $f\in\partial H$. This procedure is exploited in [Vra, Theorem 6.4.1] for the harmonic lifting of $f\in\partial H$ and can be transcribed for the γ–harmonic lifting of f.

Finally, it is worthwhile to mention that recently various authors have studied in a direct manner the properties of the semigroup generated by Wentzel dynamical boundary condition which generalizes our C_0–semigroup (see [AMPR],[FGGR1] and [FGGR2]).

In this paper we begin by reciting the results which are proved by P. Lax in his book [Lax]. Then we will continue to show its other properties such as analyticity, irreducibility, Markovian character and asymptotic behavior.

In Section 3, we based ourself on the results of [Esc] and [Hin] for the generation of the semigroup $S(t)$ and we continue to use harmonic analysis rather than functional analysis to get more information on $S(t)$. For example we will use Hopf's Lemma for proving the positivity of $S(t)$ and we will prove the irreducibility, the markovian character and the asymptotic behavior of this semigroup.

In Section 4, we will construct an approximating family $\{V(t)\}_{t\geq 0}$ for this semigroup which satisfies the Chernoff's product formula. Our construction follows from a generalization of Lax semigroup [Lax, 36.2] for a γ-lifting of a continuous function on the boundary of Ω. As in Section 2, the regularizing characters and the compactness of $V(t)$ will be proved. The main result of Section 5 is that, for any $f \in \partial X$, $\lim_{t\to\infty} S(t)f = c_f$, where c_f is a constant which depends on f, furthermore there exists a probability measure μ such that

$$c_f = \int_{\partial\Omega} f(\sigma)d\mu(\sigma).$$

Finally, in the last section we bring some open problems on the geometry of Ω.

2. Lax semigroup.

In [Lax, 36.2], P. Lax has introduced a semigroup in ∂X and ∂H when Ω is the unit ball in $\mathbb{R}^n$ and $\gamma = I$ is the identity. Let v be the solution of the following system

$$\textbf{(P3)} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ Lv = f & \text{on } \partial\Omega. \end{cases}$$

where L is the trace operator on $\partial\Omega$. We define the lifting operator L_0 which associate to any $f \in \partial X$ or in ∂H the harmonic function v which solves problem **(P3)**. The Lax semigroup is defined by

$$S(t) := LT(t)L_0. \tag{2.1}$$

where $T(t)v(x) := v(\mathrm{e}^{-t}x)$. In other words, if $f \in \partial X$ or ∂H, then we have for any $\omega \in \partial\Omega$,

$$S(t)f(\omega) = v(\mathrm{e}^{-t}\omega).$$

The semigroup defined by (2.1) is a Dirichlet-to-Neumann semigroup. In fact

- It is clear that $S(0)f(\omega) = v(\omega) = f(\omega)$ on $\partial\Omega$.
- Since the function $x \mapsto v(\mathrm{e}^{-t}x)$ is harmonic, then we have

$$\begin{aligned} S(s)S(t)f(\omega) &= LT(s)v(\mathrm{e}^{-t}\omega) \\ &= Lv(\mathrm{e}^{-s}\mathrm{e}^{-t}\omega) \\ &= S(s+t)f(\omega). \end{aligned}$$

- The strong continuity in ∂X follows from the fact that a harmonic function is continuous in $\overline{\Omega}$, hence whose values on the boundary are continuous and the continuity in ∂H follows from the density of ∂X in ∂H.
- Finally, the generator of $S(t)$ in ∂X is $\Lambda f := -\dfrac{\partial v}{\partial \omega}$ with domain

$$D(\Lambda) := \{f \in \partial X \mid L_0 f \in C^1_\omega(\overline{\Omega})\}, \tag{2.2}$$

where $C^1_\omega(\overline{\Omega})$ consists of all $v \in C(\overline{\Omega})$ for which the outer normal derivative

$$\frac{\partial v}{\partial \omega}(\overline{x}) := -\lim_{t\to 0} \frac{v(\overline{x} - t\omega(\overline{x})) - v(\overline{x})}{t}$$

exists uniformly for $\overline{x} \in \partial\Omega$. In fact

$$\lim_{t\to 0}\sup_{\omega\in\partial\Omega} |\frac{1}{t}\left(S(t)f(\omega) - f(\omega)\right) - \Lambda f(\omega)|$$
$$= \lim_{t\to 0}\sup_{\omega\in\partial\Omega}\left|\frac{1}{t}\left(v(\mathrm{e}^{-t\omega}\omega) - v(\omega)\right) + \omega\cdot\nabla v(\omega)\right|.$$

By the mean value theorem, there exists $\tau \in (0,t)$ such that

$$\frac{1}{t}\left(v(\mathrm{e}^{-t}\omega) - v(\omega)\right) = \left(\frac{\mathrm{e}^{-t}-1}{t}\right)\omega\cdot\nabla v\left(\mathrm{e}^{-\tau}\omega\right))$$

and when $t\to 0$,

$$\frac{\mathrm{e}^{-t}-1}{t} \to -1 \quad \text{and} \quad \nabla v\left(\mathrm{e}^{-\tau}\omega\right) \to \nabla v(\omega)$$

uniformly. Hence, we have

$$\lim_{t\to 0}\sup_{\omega\in\partial\Omega} |\frac{1}{t}\left(S(t)f(\omega) - f(\omega)\right) - \Lambda f(\omega)| = 0.$$

In $\partial H := L^2(\partial\Omega)$, as we have mention in the Introduction, there is a nice discusion in [Vra, section 7.5.]. This discussion leads to the existence of a semigroup $S(t)$ generated by the following dynamical boundary problem

$$\textbf{(P4)} \qquad \begin{cases} \Delta u(t,.) = 0 & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial}{\partial t}u(t,.) + \frac{\partial}{\partial\nu}u(t,.) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,.) = f, & \text{on } \partial\Omega. \end{cases}$$

By the uniqueness of the semigroup generated by Λ, we have

$$S(t)f = v(\mathrm{e}^{-t}\omega) = u(t,\omega), \qquad \text{for every } f\in\partial H.$$

This shows that for any $f\in\partial H$, $S(t)f$ given by (2.1) is well-defined.

P. Lax proved the consistency of this definition by considering the Fourier expansion of $f\in\partial H$ and also the main properties of $S(t)$ which we gather together in the following Lemma.

Lemma 2.1. *In ∂X the family $\{S(t)\}_{t\geq 0}$ forms a compact C_0-semigroup of contractions and on ∂H it forms a compact C_0-semigroup of contractions with a self-adjoint generator.*

For the proof of the forthcoming Theorem we will use the following Lemmas.

Lemma 2.2. ([v-C, Proposition 5.2.]). *Let Λ be the generator of a C_0-semigroup $S(t)$. Then $S(t)$ is analytic if and only if*

$$\limsup_{t\to 0}\sup\{t\|\Lambda S(t)f\|_{\partial X} \mid f\in D(\Lambda),\ \|f\|_{\partial X}\leq 1\} < \infty. \tag{2.3}$$

Lemma 2.3. *Let v be a harmonic function on Ω with $u(\omega) = f(\omega)$ for any $\omega\in\partial\Omega$, then*

$$\sup_{x\in\Omega}(1-|x|)|\nabla v(x)| \leq \sqrt{n}(n+2)\|f\|_{\partial X}. \tag{2.4}$$

Proof. Note that if $|x|$ is the Euclidean norme in $\mathbb{R}^n$ and $|x|_1 = \sum_{i=1}^n |x_i|$, then the equivalence between $|\ .\ |$ and $|\ .\ |_1$ can be retrieved from $|\ x\ | \le |\ x\ |_1 \le \sqrt{n}|\ x\ |$. In order to have an explicit expression of u from its boundary values $f(\omega)$, we will use the Poisson integral

$$u(x) = \int_{\partial\Omega} f(\omega) \frac{1-|x|^2}{|x-\omega|^n} d\sigma(\omega),$$

which gives

$$\frac{\partial u(x)}{\partial x_i} = \int_{\partial\Omega} f(\omega) \frac{-2x_i|x-\omega|^n - n|x-\omega|^{n-2}(x_i-\omega_i)(1-|x|^2)}{|x-\omega|^{2n}} d\sigma(\omega), \quad 1 \le i \le n.$$

Hence,

$$\begin{aligned} |\nabla u(x)| &\le |\nabla u(x)|_1 \\ &\le \int_{\partial\Omega} |f(\omega)| \frac{2\sqrt{n}\ |x||x-\omega|^n + n\sqrt{n}|x-\omega|^{n-1}(1-|x|^2)}{|x-\omega|^{2n}} d\sigma(\omega). \end{aligned}$$

Now, it follows from the facts $\int_{\partial\Omega} \frac{1-|x|^2}{|x-\omega|^n} d\sigma(\omega) = 1$ and $1-|x| \le |\omega - x|$, that

$$(1-|x|)|\nabla u(x)| \le \left(\frac{2\sqrt{n}|x|}{1+|x|} + n\sqrt{n} \right) \|f\|_{\partial X},$$

which implies (2.4). □

Remark 2.4. The above Lemma was given as an exercise in [ABR, Exercise 2.10], where it was not clear that how the constant depends on the function f.

Theorem 2.5. *In ∂X the semigroup $S(t)$ is also analytic.*

Proof. In order to use Lemma 2.2, we take $f \in D(\Lambda)$ with $\|f\|_{\partial X} \le 1$ and we calulate for v solution of **(P3)**

$$\begin{aligned} |t\Lambda S(t)f(\omega)| &= |t\frac{d}{dt}v(\mathrm{e}^{-t}\omega)| \\ &= |t\mathrm{e}^{-t}\omega \cdot \nabla v(\mathrm{e}^{-t}\omega)| \\ &\le \frac{C_n t\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} \qquad \text{(according to Lemma 2.3)}, \end{aligned}$$

where $C_n = (2+n)\sqrt{n}$. Finally as $t \to 0$,

$$\frac{C_n t\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} = \frac{C_n t}{\mathrm{e}^t - 1} \to C_n.$$

□

The explicit expression of this semigroup permits us to conjecture some properties of a general Dirichlet-to-Neumann semigroup and try to prove them. This allows us also to bring some open problems at the end of the paper. Before listing some results which will generalize for a Dirichlet-to-Neumann semigroup, let us provide some standard definitions. A *Markov semigroup* on ∂X is defined to be a positive C_0–semigroup $T(t)$ such that $T(t)1 = 1$ for all $t \ge 0$. We say that an open set U of $\partial\Omega$ is *invariant* under the Markov semigroup $T(t)$ if $T(t)(\mathcal{J}_U) \subset \mathcal{J}_U$ for all $t \ge 0$, where

$\mathcal{J}_U := \{f \in C(\partial\Omega) \mid f(\overline{x}) = 0 \text{ if } \overline{x} \notin U\}$. Finally the semigroup $T(t)$ is called *irreducible* if $\emptyset$ and ∂X are the only invariant open sets.

Proposition 2.6. *In ∂X the semigroup $S(t)$ is a Markov irreducible semigroup.*

Proof. In fact, if $f \geq 0$ on $\partial\Omega$, according to Maximum Principle, $v(x) \geq 0$ for all $x \in \Omega$, hence $S(t)f(\omega) = v(\mathrm{e}^{-t}(\omega)) \geq 0$. Furthermore if $f(\omega) = 1$ for all $\omega \in S^{n-1}$, then $v(x) = 1$ and consequently $S(t)(1) = 1$. This proves that $S(t)$ is a Markov semigroup. For the irreducibility, one takes $f > 0$, hence $S(t)f > 0$ for all $t \in (0, \infty)$ and consequently $\langle S(t)f, \varphi^* \rangle > 0$ for all $0 < \varphi^* \in C^*(\partial\Omega)$. According to [Cle, Proposition 7.6.] this implies the irreducibility. □

Proposition 2.7. *The semigroup $S(t)$ has the following asymptotic behavior*

$$\lim_{t \to \infty} S(t)f = v(0).$$

Proof. Obvious. □

Remark 2.8. We can also generalize this procedure to multiple disconnected balls in $\mathbb{R}^n$. Let $\Omega = \bigcup_{1 \leq k \leq N} \Omega_k$, where $\{\Omega_k\}_{1 \leq k \leq N}$ are N mutually disjoint balls in $\mathbb{R}^n$, each of which centered at x_k with radius r_k. In this case f is an N–uple, $f_k := f\,|_{\partial\Omega_k} \in C(\partial\Omega_k)$ and

$$S(t) : \prod_{k=1}^{N} C(\partial\Omega_k) \mapsto \prod_{k=1}^{N} C(\partial\Omega_k) \tag{2.5}$$

with

$$[S(t)f(x)]_k := Lv_k(x_k + \mathrm{e}^{-\frac{t}{r}k}(r_k\omega)),$$

where v_k is the harmonic lifting of f_k.

3. Dirichlet-to-Neumann semigroup in ∂X.

In this section we assume that Ω is an arbitrary bounded domain in $\mathbb{R}^n$ (not necessarily a ball) with a smooth boundary $\partial\Omega$. We define the lifting operator

$$L_0 : \ C(\partial\Omega) \mapsto C^2(\Omega) \cap C(\overline{\Omega}). \tag{3.1}$$

In [G-T, Theorem 6.25], one can find the proof of the following Lemma, with less restrictive assumptions on γ and $\partial\Omega$.

Lemma 3.1. *The lifting L_0 associates to any $f \in C(\partial\Omega)$ a unique γ-harmonic function v solution of the problem* **(P1)**.

In [Eng] this operator is called *Dirichlet operator* and is given by $L_0 := \left(L|_{\operatorname{Ker} A_\gamma}\right)^{-1}$, where L is the trace operator on $\partial\Omega$. In other words $L_0 \in \mathcal{L}(\partial X, X)$ and for any $f \in \partial X$, $\quad L_0 f = v$, where $v \in D(A_\gamma)$ is the unique solution of

$$\textbf{(P5)} \qquad \begin{cases} A_\gamma v = 0 & \text{in } \Omega, \\ Lv = f & \text{on } \partial\Omega. \end{cases}$$

This representation shows that

$$LL_0 = I \quad \text{on} \quad \partial X \qquad \text{and} \qquad L_0 L = I \quad \text{on} \quad \operatorname{Ker} A_\gamma. \tag{3.2}$$

Now we define the Dirichlet-to-Neumann semigroup $S(t)$ on $\mathcal{C}(\overline{\Omega})$ by (1.2).

Lemma 3.2. *The semigroup $S(t)$ is a positive semigroup.*

Proof. In spite of the fact that this result is already shown in [Esc] by using the density of the Besov space $B_p^{1-1/p}(\partial\Omega)$ in $C(\partial\Omega)$, here we give a direct proof which uses the Hopf's Lemma. Let $0 \leq f \in \partial X$ and suppose $t_0 \geq 0$ is the first time that $x \mapsto u(t_0, x)$ solution of

$$\textbf{(P6)} \qquad \begin{cases} \operatorname{div}(\gamma \nabla u(t,.)) = 0 & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial}{\partial t} u(t,.) + \frac{\partial}{\partial \nu_\gamma} u(t,.) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0,.) = f & \text{on } \partial\Omega, \end{cases}$$

becomes zero at $y \in \partial\Omega$. Hence $u(t_0, y)$ is the minimum of $u(t_0, x)$ over Ω. The Hopf's Lemma implies that $\frac{\partial}{\partial \nu_\gamma} u(t,y) < 0$ (see [Eva]) and consequently $\frac{\partial}{\partial t} u(t,y) = -\frac{\partial}{\partial \nu_\gamma} u(t,y) > 0$. Since $S(t_0)f(y) = u(t_0, y)$, $S(t)f(y)$ becomes strictly positive immediately after time t_0, i.e. $t = t_0 + \epsilon$. So $S(t)f(\overline{x}) \geq 0$ for any $t > 0$ and any $\overline{x} \in \partial\Omega$. □

We can also use this Lemma in order to prove the following result.

Theorem 3.3. *For any $t \geq 0$ and any $y \in \partial\Omega$ there exists a level surface $\Gamma_{t,y} \subset \overline{\Omega}$ such that*

$$S(t)f(y) = v(x) \qquad \text{for any} \quad x \in \Gamma_{t,y},$$

where v is the solution of **(P1)**.

Proof. Take $t \geq 0$ and $u(t,x)$ the solution of **(P6)**. Let us denote

$$m_t := \inf_{y \in \partial\Omega} u(t,y) \qquad \text{and} \qquad M_t := \sup_{y \in \partial\Omega} u(t,y).$$

For $f(y) = u(0,y)$ we denote

$$m := \inf_{y \in \partial\Omega} f(y) \qquad \text{and} \qquad M := \sup_{y \in \partial\Omega} f(y).$$

The positivity of the semigroup $S(t)$ and the fact that $w(t,x) = u(t,x) - m$ is the solution of **(P6)** with $w(0,y) = f(y) - m \geq 0$ is nonnegative imply that $w(t,x) \geq 0$ and consequently $m \leq u(t,x)$ for all $x \in \overline{\Omega}$. By changing w with $w = -u + M$, we get also $u(t,x) \leq M$, hence we have $m \leq S(t)f(y) \leq M$ and $\Gamma_{t,y} = \{x \in \overline{\Omega} \mid v(x) = S(t)f(y)\}$ forms a level surface in $\overline{\Omega}$. □

This Theorem has the following Corollary.

Corollary 3.4. *The semigroup $S(t)$ is a contraction semigroup.*

Proof. In fact, according to maximum principle the solution of **(P5)** satisfies $\|v\|_{\partial X} \leq \|f\|_{\partial X}$. Since for any $t > 0$ and any $y \in \partial\Omega$ there exists $x \in \Gamma_{t,y} \subset \overline{\Omega}$ such that $S(t)f(y) = v(x)$ we have $\|S(t)f\|_{\partial X} \leq \|f\|_{\partial X}$. □

The next theorem shows the Markovian character of $S(t)$.

Theorem 3.5. *The semigroup $S(t)$ is a Markov irreducible semigroup.*

Proof. Since, for any $t > 0$ and any $y \in \partial\Omega$, there exists at least a point $x \in \overline{\Omega}$ such that if v is the γ–lifting of $f \in \partial X$, then $S(t)f(y) = v(x)$. For $f = 1$, $u(t, x) = 1$ for all $x \in \Omega$, hence $S(t)1 = 1$. This, together with the positivity of $S(t)$ imply that the semigroup $S(t)$ is a Markov semigroup. For the irreducibility one can use the same argument as in Proposition 2.6. □

Remark 3.6. In [Dav, Theorem 7.22], it is shown that the markovian character of $S(t)$ is equivalent to the fact that if $f(\overline{x})$ realizes the maximum of $f(x)$ on $\partial\Omega$, then $\Lambda_\gamma f(\overline{x}) \leq 0$. Hence we recover the Hopf's Lemma by this result.

4. An approximating family for the Dirichlet-to-Neumann semigroup

In order to define the approximating family for a semigroup, let us recall the following Theorem which is proved in [Che].

Theorem 4.1. Chernoff's product formula. *Let $\mathcal{X}$ be a Banach space and $\{V(t)\}_{t\geq 0}$ be a family of contractions on $\mathcal{X}$ with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and the closure Λ of $V'(0)\mid_{\mathcal{D}}$ generates a C_0–semigroup $S(t)$ of contractions. Then, for each $f \in \mathcal{X}$,*

$$\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f = S(t)f,$$

uniformly for t in compact subsets of $\mathbb{R}_+$.

In the sequel, we will call $\{V(t)\}_{t\geq 0}$ the approximating family of $S(t)$ and we will construct the approximating family for the Dirichlet-to-Neumann semigroup when Ω is a ball of radius r in $\mathbb{R}^n$. Let us denote $\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)$ the solution of the linear system

$$\begin{cases} \dot{\psi}(r\omega, t) + \frac{1}{r}\gamma(r\omega)\psi(r\omega, t) = 0 \\ \psi(r\omega, 0) = r\omega. \end{cases}$$

We define $V(t) \in \mathcal{L}(\partial X)$ by

$$V(t) := LT(t)L_0 : \ \partial X \mapsto \partial X, \tag{4.1}$$

where $T(t)$ always acts on the γ–harmonic functions $u \in \operatorname{Ker} A_\gamma$ as follows

$$T(t)v(x) = v(\exp(-\frac{t}{r}\gamma(x))x), \quad \text{for any } x \in \overline{\Omega}.$$

We can see that

- $V(0)f = f$ for all $f \in \partial X$;
- $V(t)f \to f$ strongly as $t \to 0$ in ∂X;
- For any $f \in D(\Lambda_\gamma)$, $\frac{d}{dt}V(t)f\mid_{t=0} = -\omega \cdot \gamma(r\omega)\nabla v(r\omega) = \Lambda_\gamma f$.

The family $V(t)$ is not a semigroup, for $v(\exp(-\frac{t}{r}\gamma(x))x)$ is not γ-harmonic and, consequently, if v_t is the γ-harmonic lifting of $v(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega))$,

$$\begin{aligned} V(s)V(t)f(r\omega) &= LT(s)v(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)) \\ &= Lv_t(\mathrm{e}^{-\frac{s}{r}\gamma(r\omega)}(r\omega)) \\ &\neq Lv(\mathrm{e}^{-\frac{s+t}{r}\gamma(r\omega)}(r\omega)). \end{aligned}$$

So we see that the fact that $S(t)f$ should be the trace of a γ–harmonic function is essential. But, fortunately, the three properties mentioned above show that the family $\{V(t)\}_{t\geq 0}$ can be considered as the approximating family for $S(t)$. Indeed, the fact that $V(t)$ is of contraction follows from the maximum principle and the third property implies that $V'(0)\mid_{D(\Lambda_\gamma)}$ is closed. So we have

$$\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f = S(t)f, \quad \text{for every } f \in \partial X. \tag{4.2}$$

Here will show that $V(t)$ satisfies some regularizing properties.

Lemma 4.2. *For any $t \geq 0$, $V(t)$ maps ∂X into $D(\Lambda_\gamma)$.*

Proof. According to Lemma 3.1, for any $f \in C(\partial\Omega)$, $v = L_0 f \in C^2(\Omega) \cap C(\overline{\Omega})$. Since γ is a matrix-valued $C^\infty(\overline{\Omega})$ function,

$$\varphi: \quad \partial\Omega \to \Omega, \qquad \varphi(x) = \exp\left(-\frac{t}{|x|}\gamma(x)\right)x,$$

belongs to $C^\infty(\partial\Omega)$. Consequently $V(t)f(r\omega) = v(\varphi(r\omega))$ belongs also to $C^2(\partial\Omega)$. Now let w be the γ-harmonique lifting of $v(\varphi(r\omega))$. This time $w \in C^2(\Omega)$ and its boundary values $w(r\omega) \in C^2(\partial\Omega)$. Hence $w \in C^2(\overline{\Omega})$ and $\Lambda_\gamma w \mid_{\partial\Omega} = -\omega\cdot\gamma(r\omega)\nabla w(r\omega) \in C^1(\partial\Omega) \subset \partial X$. $\square$

In Lemma 2.2, we saw that condition (2.3) is equivalent to the analyticity of the semigroup $S(t)$. Here we have some similar result.

Theorem 4.3. *The operator $V(t)$ verifies*

$$\limsup_{t\to 0} \sup\{t\|\frac{d}{dt}V(t)f\|_{\partial X}\ ;\ f\in\partial X;\ \ \|f\|_{\partial X}\leq 1\} < \infty. \tag{4.3}$$

Proof. Take $f \in \partial X$, with $\|f\|_{\partial X} \leq 1$, and $v(x)$ the solution of **(P1)**. As before

$$\begin{aligned} \frac{d}{dt}V(t)f(r\omega) &= \frac{d}{dt}v\left(\exp\left(-\frac{t}{r}\gamma(r\omega)\right)r\omega\right) \\ &= -\omega\cdot\exp\left(-\frac{t}{r}\gamma(r\omega)\right)\gamma(r\omega)\nabla v\left(\exp\left(-\frac{t}{r}\gamma(r\omega)\right)r\omega\right). \end{aligned}$$

Now, we can use Lemma 3.1 of [G-W], which asserts that there exists $K > 0$ such that, for any $x \in \Omega$, we have

$$|\nabla u(x)| \leq \frac{K}{\operatorname{dist}(x, \partial\Omega)} \|f\|_{\partial X}.$$

For $t \geq t_0$, $\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega) \in \Omega$, consequently,

$$\left|\nabla u\left(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega), \partial\Omega\right)\right| \leq \frac{K}{\operatorname{dist}\left(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega), \partial\Omega\right)}.$$

Since

$$\operatorname{dist}\left(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega), \partial\Omega\right) = r - |\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)|,$$

by the Mean Value Theorem, there exists $\tau \in [0, t]$ such that

$$r - |\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)| = (-t)\frac{d}{dt}\left(\left|\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)\right|\right)_{t=\tau}.$$

The symmetry of $\gamma(r\omega)$ implies that

$$\frac{d}{dt}\left(\left|\mathrm{e}^{-\frac{t}{r}\gamma(r\omega)}(r\omega)\right|\right)_{t=\tau} = \frac{\left(-\frac{1}{r}\gamma(r\omega)\mathrm{e}^{-\frac{\tau}{r}\gamma(r\omega)}(r\omega)\right) \cdot \left(\mathrm{e}^{-\frac{\tau}{r}\gamma(r\omega)}(r\omega)\right)}{\left|\mathrm{e}^{-\frac{\tau}{r}\gamma(r\omega)}(r\omega)\right|}.$$

As $t \to 0$, τ goes also to zero and the fact that $\exp\left(-\frac{\tau}{r}\gamma(r\omega)\right)(r\omega) \neq 0$ for all $\tau \in [0, t]$ yields

$$\lim_{\tau \to 0} \frac{d}{dt}\left(\left|\mathrm{e}^{-\frac{t}{r}\gamma r\omega)}(r\omega)\right|\right)_{t=\tau} = -\gamma(r\omega)\omega \cdot \omega.$$

Consequently, we have

$$\limsup_{t \to 0} \sup_{\omega \in S^{n-1}} \left|t\omega \cdot \nabla v\left(\mathrm{e}^{-\frac{t}{r}\gamma r\omega)}(r\omega)\right)\right| \leq \frac{K}{\gamma(r\omega)\omega \cdot \omega}$$

which implies (4.3). □

As in Lemma 2.1, we prove that:

Theorem 4.4. *The operator $V(t)$ is compact.*

Proof. In $\partial X = \mathcal{C}(\partial\Omega)$, in order for $V(t)$ to be compact, it suffices that for $\mathcal{B}$ the unit ball of ∂X, $V(t)\mathcal{B}$ be relatively compact in ∂X and this can be deduced from the Arzela-Ascoli criterion. In fact, we have to prove that $V(t)\mathcal{B}$ is an equicontinuous set in ∂X. Take ω_1, ω_2 two points of S^{n-1}. Since $u \in C^2(\Omega)$, the uniformly bounded first derivatives on a compact subset $K \subset \Omega$ such that $\mathrm{e}^{-\frac{t}{r}\gamma(r\omega_j)}(r\omega_j) \in K$ (for $j = 1, 2$) guarantees the existence of $M > 0$ such that

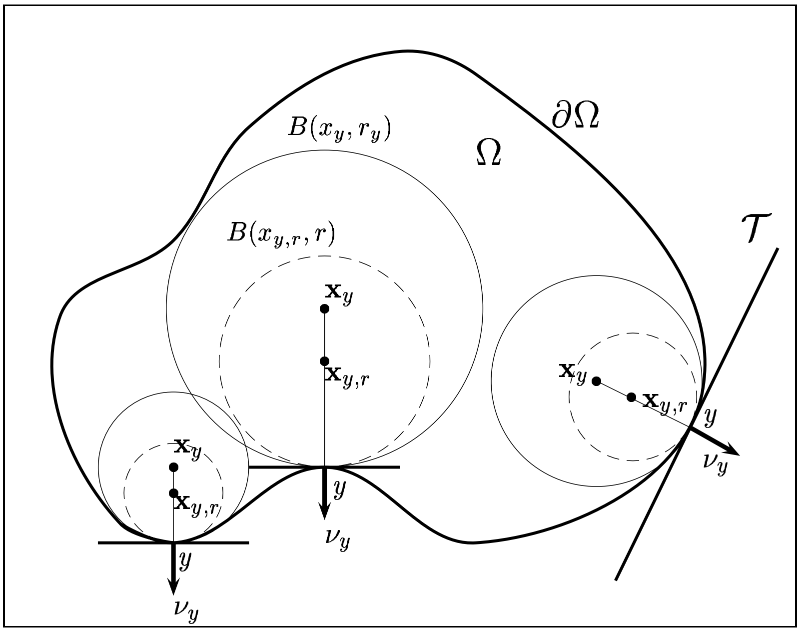
$$\begin{aligned} |V(t)f(r\omega_1) - V(t)f(r\omega_2)| &= |u\left(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega_1)}(r\omega_1)\right) - u\left(\mathrm{e}^{-\frac{t}{r}\gamma(r\omega_2)}(r\omega_2)\right)| \\ &\leq M|\mathrm{e}^{-\frac{t}{r}\gamma(r\omega_1)}(r\omega_1) - \mathrm{e}^{-\frac{t}{r}\gamma(r\omega_2)}(r\omega_2)|. \end{aligned}$$

Hence the equicontinuity of $V(t)\mathcal{B}$ follows from the continuous differentiability of the mapping $y \mapsto \mathrm{e}^{-\frac{t}{r}\gamma(y)}y$. □

Here, we construct a new family of approximating operator for a more general geometry of Ω, where Ω has the property of the interior ball, i.e., For any $y \in \partial\Omega$ there exists a tangent surface $\mathcal{T}$, which is tangent to $\partial\Omega$ at y, and a ball tangent to $\mathcal{T}$ at y which is totally included in Ω. If Ω has this property, then to any point $y \in \partial\Omega$, one can correspond a unique point x_y which is the center of $B(x_y, r_y)$ the biggest ball included in Ω which has this property and r_y its radius. For any $0 < r \le r_y$, we can construct $V_r(t)$ related to the ball $B(x_{r,y}, r) := \{x \in \Omega \mid |x - x_{r,y}| \le r\}$ of radius r centered at $x_{r,y} = \frac{r}{r_y} x_y + (1 - \frac{r}{r_y}) y$ by

$$V_r(t) f(x) := Lv \left(x_{r,y} + \mathrm{e}^{-\frac{t}{r}\gamma(y)} (r\nu_y) \right),$$

where v is the γ-harmonic lifting of $f(x)$, $y \in \partial\Omega$. As before,



- We see that $V_r(0)f(y) = v(x_{r,y} + r\nu_y) = v(y)$, because $\nu_y = (x_{r,y} - y)/r$;
- The derivative of $V(t)f(y)$ with respect to t is $V_r'(0)f(y) = -\gamma(y)\nu_y \cdot \nabla v(x_{r,y} + r\nu_y) = \Lambda_\gamma f(y)$ and, finally,
- as Corollary 3.4, $\|V_r(t)f\|_{\partial X} \le \|f\|_{\partial X}$.

These properties show that

Theorem 4.5. *Assume that Ω has the property of the interior ball and*

$$\inf_{y \in \partial\Omega} \{r > 0 \mid B(x_y, r_y) \subset \Omega\} > 0 \quad \textit{and} \quad \sup_{y \in \partial\Omega} \{r > 0 \mid B(x_y, r_y) \subset \Omega\} < \infty.$$

For each $s \in (0,1]$, we denote $V_s(t) := V_{sr_y}(t)$, i.e., $V_s(t)f(y) := v(x_s + \mathrm{e}^{-\frac{t}{sr_y}\gamma(y)}(sr_y\nu_y)$, where $x_s = sx_y + (1-s)y$. In this case we have for any $s \in (0,1]$

$$\lim_{n\to\infty} V_s\left(\frac{t}{n}\right)^n f = S(t)f, \quad \textit{for every } f \in \partial X. \tag{4.4}$$

The above Theorem shows that, for any arbitrary geometry of Ω, contrary to the case of the ball, one has to consider a family of approximating operators.

5. Asymptotic behavior of $S(t)$

Before announcing the main theorem of this section, let us give some results on the theory of semigroup in a general Banach spaces.

In [Gol, Exercise 8.24.17], one can find the following result as an exercise. For the sake of completeness we will give the proof of this result, since we will show, with the same lines of proof, that we can replace the reflexivity assumption of $\mathcal{X}$ with the compactness of the semigroup.

Lemma 5.1. *If A is the generator of an analytic semigroup $T(t)$ of type (α, M) on a reflexive Banach space $\mathcal{X}$ and P the projection of $\mathcal{X}$ onto $Ker(A)$, then $T(t)f$ goes to Pf, as $t \to \infty$, for any f in $\mathcal{X}$.*

Proof. For $f \in \mathcal{X}$ we consider the sequence $\{T(n)f\}$. Since $T(n)f \in B_f := \{T(t)f \mid t > 0\}$ and this set is bounded in $\mathcal{X}$, the reflexivity of $\mathcal{X}$ implies that there exists a subsequence $T(n_k)f$ which converges weakly to g. From the analyticity of $T(t)$ it follows that

$$\|AT(n_k)f\| \leq \frac{M}{n_k} \to 0,$$

as $k \to \infty$. Since the generator A is also weakly closed, we have $g \in D(A)$ and $Ag = 0$.

Now, the fact that $g \in \ker(A)$ implies that, for any $s \geq 0$, we have $T(s)g = g$. So, for any $t \geq 0$, there exists k such that $t \in [n_k, n_{k+1}]$. Finally, when $k \to \infty$, we get $t \to \infty$ and

$$\lim_{t\to\infty} T(t)f = \lim_{k\to\infty} T(t-n_k)T(n_k)f = g. \tag{5.1}$$

□

In the following Theorem we will see if we assume that $T(t)$ is compact for some $t > 0$, then we can remove the assumption of the reflexivity of $\mathcal{X}$.

Proposition 5.2. *If A is the generator of an eventually compact analytic semigroup $T(t)$ of type (α, M) on a Banach space $\mathcal{X}$ and P the projection of $\mathcal{X}$ onto $Ker(A)$, then $T(t)f$ goes to Pf, as $t \to \infty$, for any f in $\mathcal{X}$.*

Proof. As in the proof of the previous lemma, for $f \in \mathcal{X}$, the set $B_f := \{T(t)f\ ;\ t \geq 0\}$ is bounded in $\mathcal{X}$ and $T(t)B_f = B_f$ implies that B_f is relatively compact for some large $t > 0$. Hence there is a sequence t_n such that $T(t_n)f \to g$, as $t_n \to \infty$ and as in the proof of previous lemma, $AT(t_n)f \to 0$, which implies that $g \in D(A)$ and $Ag = 0$. □

Now we consider Ω a bounded domain with a smooth boundary $\partial\Omega$.

Lemma 5.3. *$f \in \operatorname{Ker} \Lambda_\gamma$ if and only if f is constant on $\partial\Omega$.*

Proof. It is clear that if $f(\overline{x}) = c$ for any $\overline{x} \in \partial\Omega$, then $u(x) = c$ for all $x \in \Omega$. Conversely, suppose that $f \in \operatorname{Ker} \Lambda_\gamma$, then $\nu \cdot \nabla u = 0$, hence $u(x)$ is constant for any $x \in \Omega$, in particular $u(\overline{x}) = f(\overline{x}) = c$. □

Now, we are in a position to give our main result of this section.

Theorem 5.4. *The semigroup $S(t)$ has the following asymptotic behavior*

$$\lim_{t\to\infty} S(t)f = c_f, \tag{5.2}$$

where c_f is a constant which depends on f. Furthermore there exists a probability measure μ such that

$$c_f = \int_{\partial\Omega} f(\sigma)d\mu(\sigma). \tag{5.3}$$

Proof. Proposition 5.2 and Lemma 5.3 achieve the proof of (5.2). For proving (5.3), we observe from Proposition 5.2 that

$$S(t)f \to Pf, \qquad \text{for every } \; f \in X, \tag{5.4}$$

where P is the projection onto $\operatorname{Ker} \Lambda_\gamma$ along $\overline{\operatorname{Ran} \Lambda_\gamma}$, hence $Pf = c$. Since from Lemma 3.2 the semigroup $S(t)$ is positive, P is a positive linear functional on ∂X. According to the Riesz Representation Theorem (see [Rud]), there exists a unique positive measure such that

$$Pf = \int_{\partial\Omega} f(\sigma)d\mu(\sigma). \tag{5.5}$$

Since $S(t)$ is a Markov semigroup $P1 = 1$, which proves that the measure μ is a probability measure. □

Remark 5.5. A similar result can be found in [Dav, Theorem 7.29].

Remark 5.6. For the Lax semigroup, property (5.2) is already given in Lemma 2.7 and property (5.3) is the well-known formula

$$u(0) = \frac{1}{meas(S^{n-1})} \int_{S^{n-1}} u(\omega)d\omega.$$

6. Open problems

Concerning the Lax semigroup, the first natural question which comes to mind is:

(OP1) $\begin{cases} \textit{Is it possible to generalize the Lax explicit representation for} \\ \textit{some other geometry } \Omega, \textit{ dif and only iferent from a ball?} \end{cases}$

The Lax semigroup gives us the precise location of the semigroup. In fact, by computing a harmonic lifting u of $f \in \partial X$, the value of $S(t)f(\omega)$ is at $u(\mathrm{e}^{-t}\omega)$. This property is very valuable for the calculation of the eigenvalues of Λ_γ as we will see in a forthcoming paper.

So, our second open question is the following

$$\textbf{(OP2)} \qquad \begin{cases} \textit{Can we precise the location of } S(t)f(y) \\ \textit{on the level surface } \ \Gamma_{t,y}? \end{cases}$$

We think that this problem would be very hard for an arbitrary geometry of Ω, since there is no unique approximating family for which the location is known.

For an general Dirichlet-to-Neumann semigroup, even in the case of a ball, we are not able to deduce the analyticity and the compactness of the semigroup from the analyticity and the compactness of the approximating family. The reason is that in the Chernoff's product formula we do not have a norm-operator convergence for this formula. But it seems that such type of convergence can be obtained by methods similar to those of [C-Z]. So our next open problems would be

$$\textbf{(OP3)} \qquad \begin{cases} \textit{Can we prove} \\ \lim_{n\to\infty} \left\| V\left(\frac{t}{n}\right)^n - S(t) \right\|_{\mathcal{L}(\partial X)} = 0 \ ? \end{cases}$$

$$\textbf{(OP4)} \qquad \begin{cases} \textit{What is the angle of the sector } S_\alpha \\ \textit{where } \ S(t) \textit{ is analytic } ? \end{cases}$$

Acknowledgment

We would like to express our gratitude to Professor Louis Nirenberg for the interesting and helpful discussion that we have had and for his nice proof of Lemma 3.2, which he provided to us.

References

[ABHN] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy Problems.* Birkhäuser. Basel. 2000.

[AMPR] W. Arendt, G. Metafune, D. Pallara and S. Romannelli, The Laplacian with Wenzell-Robin boudary conditions on spaces of continuous functions. *Semigroup Forum* **67**,(2003) 247-261.

[ABR] S. Axler, P. Boudon and W. Ramey, *Harmonic Function Theory.* Springer-Verlag, New-York. 2001.

[Bre] H. Brezis, *Analyse fonctionnelle* Masson, Paris, 1884.

[C-Z] V. Cachia and V. A. Zagrebnov, Operator-norm approximation of semigroups by quasi-sectorial contractions. *J. Funct. Analysis.* **180** (2001), 176–194.

[Che] P. R. Chernoff, Note on product formulas for operator semigroups. *J. Funct. Analysis.* **2** (1968), 238–242.

[Cle] Ph. Clément and al., *One-Parameter Semigroups* CWI Monographs, North-Holland, Amesterdam 1887.

[Dav] E. B. Davis, *One-Parameter Semigroups* Acad. Press, London, New-York, 1880.

[Eng] K.-J. Engel, The Laplacian on $C(\overline{\Omega})$ with generalized Wenzell boudary conditions., *Arch. Math.* **81** (2003), 548–558.

[E-N] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations.* Springer-Verlag, New-York. 2000.

[Esc] J. Escher, The Dirichlet-Neumann operator on continuous functions. *Annali della Scuola Norm. Sup. Pisa* **21** (1994), 235–266.

[Eva] L. Evans, *Partial Differential Equations* Amer. Math. Soc., Providence, 2002.

[FGGR1] A. Favini, G. Ruiz Goldstein, J. A. Goldstein and S. Romannelli, C_0-semigroups generated by second order differential operators with general Wentzell boundary conditions. *Proc. Amer. Math. Soc.* **128** (2000), 1981–1989.

[FGGR2] A. Favini, G. Ruiz Goldstein, J. A. Goldstein and S. Romannelli, The heat equation with generalized Wentzell boundary conditions. *J. Evol. Equs.* **2** (2002), 1–19.

[G-T] D. Gilbarg and N. S. Trudinger, *Partial Differential Equations of Second Order.* Springer-Verlag, New-York. 1983.

[Gol] J. A. Goldstein, *Semigroups of Operators and Applications.* Oxford University Press, London. 1985.

[G-W] M. Grüter and K.-O Widman, The Green function for uniformly elliptic equations. *Manus. Math.* **37**,(1982) 303-342.

[Hin] T. Hintermann, Evolution equations with dynamic boundary conditions. *Proc. R. Soc. Edinb., Sect. A* **113**,(1989) 43-60.

[Lax] P. D. Lax, *Functional Analysis* Wiley Inter-science, New-York. 2002.

[Kat] T. Kato, *Perturbation Theory for Linear Operators* Springer-Verlag, New-York. 1966.

[Rud] W. Rudin, *Real and complex analysis.* Third edition. McGraw-Hill Book Co., New York, 1987.

[Tay] M. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations.* Springer-Verlag, New-York. 1996.

[v-C] J. A. van Casteren, *Generators of Strongly Continuous Semigroups.* Pitman, Boston, London. 1985.

[Vra] I. I. Vrabie, *C_0–Semigroups and Applications.* North-Holland, Amsterdam, 2003.

Laboratoire de Mathématiques, Université de Poitiers. teleport 2, BP 179, 86960 Chasseneuil du Poitou, Cedex, France

E-mail address: emamirad@math.univ-poitiers.fr

E-mail address: Idriss.Laadnani@math.univ-poitiers.fr