CHAOTIC TENSOR PRODUCT SEMIGROUPS

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Abstract. *Suppose $\{G_1(t)\}_{t\geq 0}$ and $\{G_2(t)\}_{t\geq 0}$ be two semigroups on an infinite dimensional separable reflexive Banach space $\mathcal{X}$. In this paper we give the sufficient conditions for that the tensor product semigroup $\mathcal{G}(t): X \longrightarrow G_2(t)XG_1(t)$ became chaotic in $\mathcal{L}(\mathcal{X})$ with the strong operator topology and chaotic in the ideal of compact operators on $\mathcal{X}$ with the norm operator topology.*

1. Introduction

Let $\mathcal{X}$ be an infinite dimensional separable Banach space. It is formally clear that if the C_0–semigroups $G_1(t)$ and $G_2(t)$ are generated by the operators A_1 and A_2, then the application $\mathcal{G}(t)$ defined by

$$\mathcal{L}(\mathcal{X}) \ni \ X \longrightarrow G_2(t)XG_1(t) \in \mathcal{L}(\mathcal{X})$$

forms a family of semigroup on $\mathcal{L}(\mathcal{X})$ in the following sense

(1) $\mathcal{G}(0)X = X$;

(2) $$\begin{aligned}\mathcal{G}(t)\mathcal{G}(s)X &= \mathcal{G}(t)[G_2(s)XG_1(s)] \\ &= G_2(t)G_2(s)XG_1(s)G_1(t) \\ &= G_(t+s)XG_1(t+s) \\ &= \mathcal{G}(t+s)X, \quad s,t \geq 0;\end{aligned}$$

(3) $\mathcal{G}(t)X$ converges strongly to X as $t \to 0^+$, for any $X \in \mathcal{L}(\mathcal{X})$.

In general the convergence is not in the sense of uniform topology of $\mathcal{L}(\mathcal{X})$, but if $\mathcal{X}$ is a finite dimensional space or $\mathcal{G}(t)$ acts on the subalgebra $\mathcal{K}(\mathcal{X})$, the ideal of compact operators on $\mathcal{X}$, then it has the uniform convergence.

In the sequel we denote by so-lim [resp. uo-lim] the strong [resp. uniform] operator limit.

The semigroup $\mathcal{G}(t)$ has Δ_{A_2,A_1} as its infinitesimal generator where

$$\Delta_{A_2,A_1}X = A_2X + XA_1. \tag{1.1}$$

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This is strictly true when the generators A_1 and A_2 are bounded. This semigroup intervenes for studying the operator equation $A_2X + XA_1 = Q$ in the scattering theory and also in the theory of remediability (see [13] and [11]).

We denote T^* the adjoint operator of T.

It is clear that if $T(t)$ is a C_0-semigroup, then $T(t)^*$ satisfies the semigroup property. However, the adjoint semigroup $T(t)^*$ need not be a C_0-semigroup in $\mathcal{X}^*$, since it is not conserve the strong continuity of $T(t)$. In the special case where $\mathcal{X}$ is a reflexive Banach space we have that the adjoint semigroup $T(t)^*$ is a C_0-semigroup on $\mathcal{X}^*$ whose infinitesimal generator is A^* the adjoint of the infinitesimal generator of $T(t)$, [16, Corollary 10.6].

By applying $G_2(t) \otimes G_1(t)^*$ to operators of rank one, $u \otimes v^* = \langle ., v^* \rangle u$ $(u \in \mathcal{X}, v^* \in \mathcal{X}^*)$, it comes out that

$$\begin{aligned}
[G_2(t) \otimes G_1^*(t)]\,(u \otimes v^*) &= G_2(t)u \otimes G_1^*(t)v^* \\
&= \langle \cdot, G_1^*(t)v^* \rangle G_2(t)u \\
&= G_2(t)(u \otimes v^*)G_1(t).
\end{aligned}$$

Hence, $\mathcal{G}(t)$ is an extension to $\mathcal{L}(\mathcal{X})$ of the tensor product $G_2(t) \otimes G_1(t)^*$ acting on the space of operators of finite rank, $\mathcal{X} \otimes \mathcal{X}^*$. Therefore $\mathcal{G}(t)$ is referred to *tensor product semigroup*. This type of semigroup have been also studied in the literature (see [1] [2], [8], [12], [21], [22], [23] and [24]). In the following proposition we gather together the essential properties of the tensor product semigroup $\mathcal{G}(t)$.

Proposition 1.1. (see [12])

(1) *If $G_1(t)$ and $G_2(t)$ are two C_0–semigroups on a Banach space $\mathcal{X}$, then for any $X \in \mathcal{L}(\mathcal{X})$, $\mathcal{G}(t)X = G_2(t)XG_1(t)$ is continuous in t $(0 \le t < \infty)$ relative to the strong operator topology;*

(2) *Let $\mathcal{D}(\Delta_{A_2,A_1}) := \{X \in \mathcal{L}(\mathcal{X})$ such that $X\mathcal{D}(A_1) \subset \mathcal{D}(A_2)$ and the operator Δ_{A_2,A_1} has a bounded extension on $\mathcal{X}$ defined by* (1.1)$\}$. *Then Δ_{A_2,A_1} is closed for weak operator topology and*

$$\Delta_{A_2,A_1}X = \textit{so-lim}_{t\to 0}\frac{\mathcal{G}(t)X - X}{t}.$$

A bounded linear linear operator T in a Banach space $\mathcal{X}$ is *hypercyclic* if the orbit $\{T^n x \,:\, n \in \mathbb{N}\}$ is dense in $\mathcal{X}$. T is *chaotic* if it is hypercyclic and the set of periodic points is dense in $\mathcal{X}$. Similarly a strongly continuous semigroup of bounded linear operator $T(t)$ is hypercyclic provided that there exists $x \in \mathcal{X}$ such that $\{T(t)x\}_{t\ge 0}$ is dense in $\mathcal{X}$. Furthermore, it is chaotic if it is hypercyclic and the set $\{x \in \mathcal{X} \;:\; \exists t > 0,\; T(t)x = x\}$ is dense in $\mathcal{X}$.

In this paper we give some sufficient conditions on A_1 and A_2 in order that $\mathcal{G}(t)$ as an operator in $\mathcal{L}(\mathcal{X})$, be chaotic with the strong operator topology. In fact, the Banach algebra $\mathcal{L}(\mathcal{X})$ endowed with the norm operator topology is not separable and so is not suitable for studying the hypercyclicity or chaoticity. If $\mathcal{X}$ is a Banach space with *approximation property,* i.e. the identity operator can be approximated uniformly on compact subset of $\mathcal{X}$ by linear operators of finite rank, then we prove the hypercyclicity and chaoticity in the space $\mathcal{K}(\mathcal{X})$ of the compact operators on $\mathcal{X}$ endowed with the norm operator topology. This time the space is separable, since the set of finite rank operator are dense in it.

2. Preliminaries

Let us establish the necessary notation and preliminaries on tensor products.

Given G and H Banach spaces, the *projective norm* $\|\cdot\|_\pi$ on tensor product $G \otimes H$ is defined as follows

$$\|z\|_\pi := \inf_{n\in\mathbb{N}} \left\{ \sum_{j=1}^{n} \|x_j\|_G \|y_j\|_H : z = \sum_{j=1}^{n} x_j \otimes y_j \right\}, \quad z \in G \otimes H.$$

With this norm the space is denoted by $G \otimes_\pi H$, and its completion is $G\widetilde{\otimes}_\pi H$.

This space can be equipped with a weaker topology called *injective tensor norm* $\|\cdot\|_\varepsilon$, defined by

$$\|z\|_\varepsilon := \sup_{n\in\mathbb{N},\, \varphi\in G^*,\, \phi\in H^*,\, \|\varphi\|\le 1,\, \|\phi\|\le 1} \left\{ \left|\sum_{j=1}^{n} \varphi(x_j)\phi(y_j)\right| : z = \sum_{j=1}^{n} x_j \otimes y_j, \right\},$$

where $z \in G\otimes H$. So, $\|z\|_\varepsilon \le \|z\|_\pi$. With this norm the space is denoted by $G\otimes_\varepsilon H$, and its completion by $G\widetilde{\otimes}_\varepsilon H$. Let $\mathcal{X}$ be a Banach space endowed with the norm $\|\cdot\|$. We recall that $\mathcal{X}\otimes\mathcal{X}^*$ can be viewed as a subspace of $\mathcal{L}(\mathcal{X})$. To any $a = \sum e_i \otimes e_i^* \in \mathcal{X}\otimes\mathcal{X}^*$ one can correspond an operator A given by $Ax := \sum\langle x, e_i^*\rangle e_i$. Notice that if $\mathcal{X}_n$ is a finite dimensional subspace of $\mathcal{X}$, then the projection $P_n : \mathcal{X} \mapsto \mathcal{X}_n$, $P_n(x) = \sum_{1\le i\le n} \alpha_i^*(x) e_i$ belongs to $\mathcal{X}\otimes\mathcal{X}^*$, since $P_n = \sum_{1\le i\le n} e_i \otimes \alpha_i^*$.

Lemma 2.1. *Let $\mathcal{X}$ be a separable Banach space. The injection $a \hookrightarrow A$ is continuous and has dense image if we consider $\mathcal{X}\otimes\mathcal{X}^*$ with the injective tensor norm and $\mathcal{L}(\mathcal{X})$ with the strong operator topology.*

Proof. Let us denote $\{x_n\}$ the denumerable dense subset of $\mathcal{X}$. If $S \in \mathcal{L}(\mathcal{X})$ and $\mathcal{X}_n = \langle x_1, \cdots x_n\rangle$, we fix a projection $P_n : \mathcal{X} \to \mathcal{X}_n$ we get $S \circ P_n \in \mathcal{X}\otimes\mathcal{X}^*$, and $S \circ P_n x_i = S x_i$, for all $i = 1, \cdots, n$. This implies that $S\circ P_n$ converges strongly to S.□

The following Lemma is well-known.

Lemma 2.2. *If $\mathcal{X}$ is a separable Banach space, then*

(i) *If $\mathcal{X}$ is reflexive, then $\mathcal{X}^*$ is separable for its natural topology.*
(ii) *If $\mathcal{X}$ and $\mathcal{Y}$ are two separable Banach spaces, then $\mathcal{X}\widetilde{\otimes}_\epsilon\mathcal{Y}$ is also separable.*

Let E and F be two Fréchet spaces, $T_1 \in \mathcal{L}(E)$ and $T_2 \in \mathcal{L}(F)$. We can define the tensor product $T_1 \otimes T_2 : E \otimes F \to E \otimes F$ by $T_1\otimes T_2(x \otimes y) := T_1 x \otimes T_2 y$.

3. Chaotic tensor product semigroup in $\mathcal{L}(\mathcal{X})$

In this section we assume that $\mathcal{X}$ is an infinite dimensional separable reflexive Banach space. We prove the chaoticity of $\mathcal{G}(t)$ in $\mathcal{L}(\mathcal{X})$ with the strong operator topology. Our technique is similar to those used by Martínez-Giménez and Peris in [15] and Bonet-Martínez-Peris in [6].

Theorem 3.1. *Let $\mathcal{X}$ be an infinite dimensional separable reflexive Banach space, $G_2(t)$ and $G_1(t)$ C_0-semigroups in $\mathcal{X}$ and suppose that there exists t_0 such that the operator $G_2(t_0)$ is chaotic and $G_1^*(t_0)$ has a dense set of periodic points. Then the tensor product semigroup $\mathcal{G}(t)$ is a chaotic on $\mathcal{L}(\mathcal{X})$ with the strong operator topology.*

Before proving the theorem, we need the following lemmas:

Lemma 3.2. [15, Lemma 1.3] *Let $T_i : E_i \to E_i$, $i = 1, 2$ operators on separable Fréchet spaces E_i and let $\Psi : E_1 \to E_2$ be continuous map with dense range such that $T_2 \circ \Psi = \Psi \circ T_1$. That is, the following diagram is commutative*

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad T_1 \quad} & E_1 \\ \Psi \big\downarrow & & \big\downarrow \Psi \\ E_2 & \xrightarrow{\quad T_2 \quad} & E_2 \end{array}$$

(1) If T_1 is hypercyclic, then T_2 is also hypercyclic;
(2) If T_1 is chaotic, then T_2 is also chaotic.

Lemma 3.3. [15, Corollary 1.12] *If E and F are separable Fréchet spaces, the operator $T_1 : E \to E$ is chaotic, and $T_2 : F \to F$ has a dense set of periodic points, then*

$$T_1 \otimes T_2 : E \tilde{\otimes}_a F \to E \tilde{\otimes}_a F$$

is chaotic for projective $(a = \pi)$ and injective $(a = \varepsilon)$ tensor norm. In particular $T_1 \otimes T_2$ is chaotic if T_1 and T_2 are so.

Proof of Theorem 3.1. Since $\mathcal{X}$ is reflexive separable Banach space, according to Lemma 2.2, $\mathcal{X}^*$ with its natural topology and $\mathcal{X} \widetilde{\otimes}_\epsilon \mathcal{X}^*$ are also separable. Let us consider the following diagram

$$\begin{array}{ccc} \mathcal{X} \widetilde{\otimes}_\varepsilon \mathcal{X}^* & \xrightarrow{\; G_2(t_0) \otimes G_1^*(t_0) \;} & \mathcal{X} \widetilde{\otimes}_\varepsilon \mathcal{X}^* \\ \Psi \big\downarrow & & \big\downarrow \Psi \\ \mathcal{L}(\mathcal{X}) & \xrightarrow{\quad \mathcal{G}(t_0) \quad} & \mathcal{L}(\mathcal{X}) \end{array}$$

According to Lemma 2.2, $G_2(t_0)\otimes G_1^*(t_0)$ is chaotic and according to Lemma 2.1 the injection Ψ has a dense image in $\mathcal{L}(\mathcal{X})$ with strong operator topology. Hence in order to achieve the proof we shall use Lemma 2.1 and prove the commutative relation

$$\mathcal{G}(t_0)\circ\Psi = \Psi\circ G_2(t_0)\otimes G_1^*(t_0). \tag{3.1}$$

Let $x\otimes y^*\in\mathcal{X}\otimes\mathcal{X}^*$, by applying the second member of (3.1) to $x\otimes y^*$ we get

$$\begin{aligned}\Psi\circ G_2(t_0)\otimes G_1^*(t_0)(x\otimes y^*)(\cdot) &= \Psi\circ[G_2(t_0)x\otimes G_1^*(t_0)y^*](\cdot)\\ &= \langle\cdot, G_1^*(t_0)y^*\rangle G_2(t_0)x\\ &= \langle G_1(t_0)(\cdot), y^*\rangle G_2(t_0)x\\ &= G_2(t_0)\Psi(x\otimes y^*)G_1(t_0)(\cdot)\\ &= \mathcal{G}(t_0)\circ\Psi(x\otimes y^*)(\cdot)\end{aligned}$$

which implies (3.1) in view of Lemma 2.1. □

Example 3.4. Let $\mathcal{X}=\ell^p(\mathbb{N},\alpha)(1<p<\infty)$ be the Banach space of complex-valued sequences $\mathbf{x}:=\{x_n\}$ such that $\|\mathbf{x}\|^p:=\sum_{n=1}^{\infty}|x_n|^p\alpha_n$ is finite and the weight sequence $\alpha=(\alpha_n)$ satisfies

$$\sup_{n\geq 1}\frac{\alpha_n}{\alpha_{n+1}}\leq M.$$

Let B be the backward shift $((B\mathbf{x})_n = x_{n+1})$. Now if Q is a polynomial, D is the open unit disc and $R:=\limsup_{n\to\infty}\alpha_n^{-1/n}$ is the convergence radius of $\sum_{n=1}^{\infty}\alpha_n z^n$. If $Q(R^{1/p}D)\cap i\mathbb{R}$ is nonempty, then $\mathcal{G}(t)$ defined by $\mathcal{G}(t)X := e^{tQ(B)}X$, is a chaotic semigroup on $\mathcal{L}(\mathcal{X})$ with the strong operator topology. In fact in this example $\mathcal{G}(t)$ is a tensor product semigroup associated to $e^{tQ(B)}$ and identity and according to [9, Theorem 2.14], $e^{tQ(B)}$ is chaotic and I has a dense set of periodic points.

In the previous Theorem we have proved the chaoticity of the operator $\mathcal{G}(t_0)$. The question arises, if $G_2(t)$ and $G_1^*(t)$ are chaotic semigroups, is $\mathcal{G}(t)$ chaotic for all $t>0$? Our previous technique fails to give a positive answer to this question. However we can use the technique of Desch, Schappacher and Webb [10] to give this answer.

In the following lemma we have slightly modified Theorem 3.1 of [10]. In fact, this modification abridged the proof of their Theorem. For the sake of completeness we include the proof.

Lemma 3.5. *Let $\mathcal{X}$ be a separable Banach space and let A be the infinitesimal generator of a C_0–semigroup $T(t)$ on $\mathcal{X}$. Assume that there exists an open subset U of the point spectrum $\sigma_p(A)$ of A (i.e. for each $\lambda\in U$ there exists a nonzero element $x_\lambda\in\mathcal{X}$ such that $Ax_\lambda=\lambda x_\lambda$), which intersects the imaginary axis. Furthermore we assume that for every subset V of U which admits an accumulation point in U, $D_V:=\operatorname{span}\{x_\lambda \mid \lambda\in V\}$ is dense in $\mathcal{X}$. Then $T(t)$ is a chaotic semigroup. Moreover, $T(t)$ is a chaotic operator for all t.*

Proof. Let t_0 be an arbitrary real number. We will prove that $T(t_0)$ is a chaotic operator. Let U_+ and U_- be open subsets of U such that U_+ is contained in the right half plane and U_- in the left plane. By hypothesis D_{U_+} and D_{U_-} are dense sets in $\mathcal{X}$. Define

$$S(t_0)x_\mu := e^{-\mu t_0}x_\mu$$

for all $\mu \in U_+$. Therefore $\lim_{n\to\infty} T(t_0)^n x_\lambda = 0$ for all $\lambda \in U_-$, $\lim_{n\to\infty} S(t_0)^n x_\mu = 0$ and $T(t_0)S(t_0)x_\mu = x_\mu$ for all $\mu \in U_+$. Hence the operator $T(t_0)$ satisfies the Hypercyclicity Criterion (see [**?**, Theorem 4]). It remains to show that $T(t_0)$ has a dense set of periodic points. Let U_0 be a subset of U contained in the imaginary axis. Suppose without lost of generality that U_0 is a subset of $\frac{2\pi i}{t_0}\mathbb{Q}$, where $\mathbb{Q}$ is the rational numbers. If $\lambda = \frac{2\pi i}{t_0}\frac{p}{q} \in U_0$, then

$$T(t_0)^q x_\lambda = e^{\frac{2\pi i}{t_0}\frac{p}{q}t_0 q}x_\lambda = e^{2\pi i p}x_\lambda = x_\lambda.$$

So, D_{U_0} is dense and every point of D_{U_0} is a periodic point of $T(t_0)$. □

Corollary 3.6. *Let $\mathcal{X}$ be an infinite dimensional separable reflexive Banach space and suppose that A_2 and A_1^* are the generators of C_0-semigroup $G_2(t)$ and $G_1^*(t)$. If A_2 and A_1^* satisfy the hypotheses of Lemma 3.5 in $\mathcal{X}$ and $\mathcal{X}^*$, then the tensor product semigroup $\mathcal{G}(\cdot)$ is chaotic on $\mathcal{L}(\mathcal{X})$ with the strong operator topology.*

Proof. By Lemma 3.5, $G_2(t)$ and $G_1(t)^*$ are chaotic operators for all $t > 0$. Hence Theorem 3.1 shows that the tensor product semigroup $\mathcal{G}(t)$ is a chaotic on $\mathcal{L}(\mathcal{X})$ with the strong operator topology. □

Lemma 3.7. *Let $\mathcal{X}$ be a separable Banach space. If x is a periodic point of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then the orbit $\{T(t_n)x\}_{n\in\mathbb{N}}$ is precompact, for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers.*

Proof. Let t_0 be a positive real number such that $T(t_0)x = x$. Therefore

$$\{T(t_n)x \;:\; n \in \mathbb{N}\} \subset \{T(t)x \;:\; t \geq 0\} = \{T(t)x \;:\; t \in [0, t_0]\}$$

for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers. Since $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup, then $\{T(t)x \;:\; t \in [0, t_0]\}$ is a compact subset of $\mathcal{X}$. Hence $\{T(t_n)x \;:\; n \in \mathbb{N}\}$ is precompact. □

Theorem 3.8. *Let $\mathcal{X}$ be a separable reflexive Banach space. If $\{G_1(t)^*\}_{t\geq 0}$ and $\{G_2(t)\}_{t\geq 0}$ are chaotic C_0-semigroups, then the tensor product semigroup $\mathcal{G}(\cdot)$ is hypercyclic on $\mathcal{L}(\mathcal{X})$ with the strong operator topology.*

Proof. Since $\{G_2(t)\}_{t\geq 0}$ is a hypercyclic semigroup and $\mathcal{X}$ is separable , then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers and an element $x \in_c alx$ such that the sequence $\{G_2(t_n)x\}_{n\in\mathbb{N}}$ is dense in $\mathcal{X}$. Moreover, by Lemma 3.7, $\{G_2(t_n)x\}_{n\in\mathbb{N}}$ is precompact for every periodic point x of $\{G_2(t)\}_{t\geq 0}$. The same argument gives that $\{G_1^*(t)x^*\}_{t\geq 0}$ is precompact for every periodic point x^* of $\{G_1^*(t)\}_{t\geq 0}$. Using [4, Theorem 4.1] and [3, Corollary 2.3] the sequence $\{G_2(t_n)\}_{n\in\mathbb{N}}$ satisfies the Hypercyclicity Criterion. Define $S_1^*(t_k)x^* := G_1^*(n_k s_0 - t_k)x^*$ where s_0 is the smallest real number such that $G_1^*(s_0)x^* = x^*$ and n_k is the smallest integer such that $n_k s_0 \geq t_k$. Hence $\{G_1^*(t_n)x^*\}_{n\in\mathbb{N}}$ and $\{S_1^*(t_n)x^*\}_{n\in\mathbb{N}}$ are bounded and $G_1^*(t_n)S_1^*(t_n)x^* = x^*$ for every periodic point x^* of $\{G_1^*(t)\}_{t\geq 0}$. By [15, Theorem 1.8] $G_2(t_n)\otimes G_1^*(t_n)$ is hypercyclic on $\mathcal{X}\widetilde{\otimes}_\varepsilon\mathcal{X}^*$. The same proof of Theorem 3.1 gives the result. □

Example 3.9. Under the assumption of Example 3.4, Let $A_2 := -\alpha I+\beta B$, where B is the backward shift in $\ell^p(\mathbb{N},\alpha)$ with $1 < p < \infty$. In [9, Remark 2.15], it is proved that the bounded operator $A_2 := -\alpha_2 I + \beta_2 B$ in $\ell^p(\mathbb{N},\alpha)$ generates a semigroup $G_2(t) = e^{tA_2}$, which satisfies the assumptions of Lemma 3.5 provided that $\beta_2 > \alpha_2 \geq 0$ (see [9] and also [17] for the case $p = 1$). Hence this semigroup generated by A_2 is chaotic. Now, since $\ell^p(\mathbb{N},\alpha)$ is reflexive, it is well-known that $G_2(t)^*$ is also a C_0–semigroup in $\ell^q(\mathbb{N},\alpha)$, when $1/p + 1/q = 1$. But it is shown in [10] that the adjoint of $G_2(t)$ is no more chaotic and for this reason we take $G_1(t)$ the semigroup generated by $-\alpha_1 I+\beta_1 F$, when α_1 and β_1 verifying that $\beta_1 > \alpha_1 \geq 0$ and F is the forward shift in $\ell^p(\mathbb{N},\alpha)$. This time $G_1(t)$ is not chaotic but its adjoint $G_1(t)^*$ is chaotic in $\ell^q(\mathbb{N},\alpha)$. All these facts imply that we are in position to apply Corollary 3.6 and conclude that the tensor product semigroup $\mathcal{G}(t) : X \mapsto G_2(t)XG_1(t)$ is also chaotic in $\mathcal{L}(\ell^p(\mathbb{N},\alpha))$ with the strong operator topology.

Proposition 3.10. *If $\mathcal{G}(t)$ has a hypercyclic vector T in $\mathcal{L}(\mathcal{X})$ with the strong operator topology and x is any nonzero vector in $\mathcal{X}$, then x is a hypercyclic vector for the family of operators $\mathcal{G}(t)T$ on $\mathcal{X}$*

The proof of the above result is similar to [7, Proposition 7].

4. Chaotic tensor product semigroup in $\mathcal{K}(\mathcal{X})$

Let $\mathcal{K}(\mathcal{X})$ be the ideal of compact operators on $\mathcal{X}$. It is well-known that if the sequence $\{A_n\}$ of $\mathcal{L}(\mathcal{X})$ the Banach algebra of bounded operators on a Banach space $\mathcal{X}$, converges strongly to a bounded operator A, for any $C \in \mathcal{K}(\mathcal{X})$, the sequence $\{A_nC\}$ [resp. $\{CA_n\}$] converges to AC [resp. CA] relative to the norm operator topology. From this fact, for each fixed $C \in \mathcal{K}(\mathcal{X})$, $\mathcal{G}(t)C = G_2(t)CG_1(t)$ is continuous in $t \in [0,\infty)$ relative to the norm topology and hence $\mathcal{G}(t)$ defined by

$$\mathcal{K}(\mathcal{X}) \ni\ C \longrightarrow G_2(t)CG_1(t) \in \mathcal{K}(\mathcal{X})$$

forms a family of C_0–semigroup on $\mathcal{K}(\mathcal{X})$. As usual, in the theory of C_0-semigroup the generator is given by

$$\hat{\Delta}_{A_2,A_1}C := \text{uo-lim}_{t\to 0}\frac{1}{t}\left(\mathcal{G}(t)C - C\right) \tag{4.1}$$

with

$$\mathcal{D}(\hat{\Delta}_{A_2,A_1}) := \{C \in \mathcal{K}(\mathcal{X}) \mid \text{the uo-lim exists and is in } \mathcal{K}(\mathcal{X})\}.$$

The following lemma makes precise the sense in which $\hat{\Delta}_{A_2,A_1}$ generates $\mathcal{G}(t)$.

Lemma 4.1. *The domain $\mathcal{D}(\hat{\Delta}_{A_2,A_1})$, consists of those operators $C \in \mathcal{K}(\mathcal{X})$ such that $C\mathcal{D}(A_1) \subset \mathcal{D}(A_2)$ and the operator $A_2C + CA_1$ has an unique compact extension $K \in \mathcal{K}(\mathcal{X})$.*

Proof. Let $x \in \mathcal{D}(A_1)$. The proof can be carried out on the following identity

$$\frac{1}{t}(\mathcal{G}(t)C - C)x = \left[\frac{1}{t}(G_2(t) - I)\right] Cx + G_2(t)C\left[\frac{1}{t}(G_1(t) - I)\right] x.$$

The second term on the right hand side converges to $G_2(t)CA_1$ and the left hand side converges to Kx, where

$$K := \text{uo-lim}_{t\to 0}\frac{1}{t}(\mathcal{G}(t)C - C) = \text{uo-lim}_{t\to 0}\frac{1}{t}(G_2(t)CG_1(t) - C).$$

Hence the first term of the right hand side has a limit and $Cx \in \mathcal{D}(A_2)$. Furthermore, since the uniform limit of compact operators is compact, $K \in \mathcal{K}(\mathcal{X})$ and $Kx = A_2Cx + CA_1x$ for all $x \in \mathcal{D}(A_1)$. This proves that K is a compact extension of $\hat{\Delta}_{A_2,A_1}C$ and this extension is unique since A_1 is densely defined. □

Next, we give conditions in order $\mathcal{G}(t)$ to be chaotic on $\mathcal{K}(\mathcal{X})$.

Theorem 4.2. *Let $\mathcal{X}$ be an infinite dimensional separable reflexive Banach space with the approximation property and $G_2(t)$ and $G_1(t)$ C_0-semigroups in $\mathcal{X}$. Suppose that there exists t_0 such that the operator $G_2(t_0)$ is chaotic and $G_1^*(t_0)$ has a dense set of periodic points or vice versa. Then the tensor product semigroup $\mathcal{G}(t)$ is chaotic on $\mathcal{K}(\mathcal{X})$ with the norm operator topology.*

Proof. Since $\mathcal{X}$ is separable and reflexive, then $\mathcal{X}^*$ is separable. Moreover, $\mathcal{X}\otimes_\pi\mathcal{X}^*$ is a separable normed space and the extension of the continuous inclusion gives an operator $\Psi : \mathcal{X}\widetilde{\otimes}_\pi\mathcal{X}^* \to \mathcal{K}(\mathcal{X})$ continuous with dense range since $\mathcal{X}$ has the approximation property.

The theorem follows by applying Lemma 3.2 to the following commutative diagram

$$\begin{array}{ccc}
\mathcal{X}\widetilde{\otimes}_\pi\mathcal{X}^* & \xrightarrow{G_2(t_0)\otimes G_1^*(t_0)} & \mathcal{X}\widetilde{\otimes}_\pi\mathcal{X}^* \\
\Psi\downarrow & & \downarrow\Psi \\
\mathcal{K}(\mathcal{X}) & \xrightarrow{\mathcal{G}(t_0)} & \mathcal{K}(\mathcal{X})
\end{array}$$

since by Lemma 3.3, the operator $G_2(t_0)\otimes G_1^*(t_0)$ is chaotic in $\mathcal{X}\widetilde{\otimes}_\pi\mathcal{X}^*$. □

The following lemma relates $\sigma_p(A_1{}^*)$ and $\sigma_p(A_2)$ to $\sigma_p(\hat{\Delta}_{A_2,A_1})$

Lemma 4.3. *Let $\hat{\Delta}_{A_2,A_1}$ define by (1.1) be the infinitesimal generator of a tensor product semigroup $\mathcal{G}(t)$. Then*

$$\sigma_p(A_1{}^*) + \sigma_p(A_2) \subset \sigma_p(\hat{\Delta}_{A_2,A_1}).$$

Proof. Let $\lambda_1^* \in \sigma_p(A_1^*)$ and $\lambda_2 \in \sigma_p(A_2)$. If x_1^* is such that $A^*x_1^* = \lambda_1^*x_1^*$ and x_2 is such that $A_2x_2 = \lambda_2x_2$, then for any $x \in D(A_1)$ we have

$$\begin{aligned}
A_2(x_2\otimes x_1^*)x + (x_2\otimes x_1^*)A_1x &= A_2(\langle x, x_1^*\rangle x_2) + \langle A_1x, x_1^*\rangle x_2 \\
&= \lambda_2\langle x, x_1^*\rangle x_2 + \langle x, \lambda_1^*x_1^*\rangle x_2 \\
&= (\lambda_2 + \lambda_1^*)(x_2\otimes x_1^*)x.
\end{aligned}$$

Since $x_2 \otimes x_1^*$ is of rank one operator and $x_2 \in D(A_2)$, by Lemma 4.1 it is clear that $\lambda_2 + \lambda_1^*$ is an eigenvalue of $\hat{\Delta}_{A_2,A_1}$. □

The following corollary gives conditions on A_1 and A_2 in order that $\mathcal{G}(t)$ is an operator chaotic on $\mathcal{K}(\mathcal{X})$ with the norm operator topology.

Corollary 4.4. *Let $\mathcal{X}$ be an infinite dimensional separable reflexive Banach space with the approximation property. Suppose that A_2 and A_1^* are the generators of the C_0-semigroup $G_2(t)$ and $G_1^*(t)$ respectively. If A_2 and A_1^* satisfy the hypotheses of Lemma 3.5 in $\mathcal{X}$ and $\mathcal{X}^*$,respectively, then $\mathcal{G}(t)$ the tensor product semigroup is chaotic on $\mathcal{K}(\mathcal{X})$ with the norm operator topology.*

The above corollary is an immediate consequence of Theorem 4.2. However, we will give a different proof.

Lemma 4.5. [10, Theorem 2.3] *Let $\mathcal{X}$ be a separable Banach space and let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{X}$. Suppose that the sets*

$$\mathcal{X}_0 := \{x \in \mathcal{X} \mid \lim_{t\to\infty} T(t)x = 0\},$$

$$\mathcal{X}_\infty := \{x \in \mathcal{X} \mid \forall \epsilon > 0,\ \exists y \in \mathcal{X},\ t > 0\ \|y\| < \epsilon,\ T(t)y = x\}$$

and

$$\mathcal{X}_{per} := \{x \in \mathcal{X} \mid \text{there exists some } t > 0 \text{ such that } T(t)x = x\}$$

are all dense in $\mathcal{X}$. Then $T(t)$ is chaotic.

Proof of Corollary 4.4. Take U and U^* open subsets of the point spectrum of the generators A_2 and A_1^*, respectively, which intersects the imaginary axis. Let $V_- \subset U$ and $V_-^* \subset U^*$ such that each subset is contained in the left half plane and has an accumulation point in the left half plane. By Lemma 3.5 we have that D_{V_-} and $D_{V_-^*}$ are dense in $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Define

$$\mathcal{F}_{V_-,V_-^*} = \{\langle x_\lambda\rangle \otimes \langle x_\mu^*\rangle : x_\lambda \in D_{V_-},\ x_\mu^* \in D_{V_-^*}\}.$$

It is clear that $\mathcal{F}_{V_-,V_-^*}$ is dense in $\mathcal{X} \otimes \mathcal{X}^*$. Moreover, $\mathcal{F}_{V_-,V_-^*}$ is a subset of $\mathcal{X}_0$, where

$$\mathcal{X}_0 := \{C \in \mathcal{K}(\mathcal{X})\ : G_2(t)CG_1(t) \to 0 \text{ uniformly }\}$$

since $G_2(t)(x_\lambda \otimes x_\mu^*)G_1(t) = e^{(\lambda+\mu)t}(x_\lambda \otimes x_\mu^*) \to 0$ as $t \to \infty$ for all $\lambda \in V_-$ and $\mu \in V_-^*$.

Let $V_+ \subset U$ and $V_+^* \subset U^*$, analogous at above but now each subset is contained in the right half plane and has an accumulation point in the right half plane. By Lemma 3.5 we have that D_{V_+} and $D_{V_+^*}$ are dense in $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Define

$$\mathcal{F}_{V_+,V_+^*} = \{\langle x_\lambda\rangle \otimes \langle x_\mu^*\rangle : x_\lambda \in D_{V_+},\ x_\mu^* \in D_{V_+^*}\}.$$

It is clear that $\mathcal{F}_{V_+,V_+^*}$ is dense in $\mathcal{X} \otimes \mathcal{X}^*$, and $\mathcal{F}_{V_+,V_+^*}$ is a subset of $\mathcal{X}_\infty$, where

$$\mathcal{X}_\infty := \{C \in \mathcal{K}(\mathcal{X}) | \forall \varepsilon > 0,\ \exists K \in \mathcal{K}(\mathcal{X}),\ t > 0 :\ \|K\| < \varepsilon,\ \mathcal{G}(t)K = C\}$$

since $G_2(t)(x_\lambda \otimes x_\mu^*)G_1(t) = e^{(\lambda+\mu)t}(x_\lambda \otimes x_\mu^*)$.

Finally, it remains to show that $\mathcal{X}_{per}$ is dense. Let $V_0 \subset U$ and $V_0^* \subset U^*$ as before but with an accumulation point on the imaginary axis. Suppose without lost of generality that V_0 and V_0^* are subsets of $2\pi\mathrm{i}\mathbb{Q}$. It follows that $\mathcal{F}_{V_0,V_0^*}$ is a subset of $\mathcal{X}_{per}$. Moreover, if $\lambda = 2\pi\mathrm{i}\frac{p}{q} \in V_0$ and $\mu = 2\pi\mathrm{i}\frac{p'}{q'} \in V_0^*$, then

$$\mathcal{G}(t)(x_\lambda \otimes x_\mu^*) = G_2(t)(x_\lambda \otimes x_\mu^*)G_1(t) = e^{2\pi\mathrm{i}\left(\frac{p}{q}+\frac{p'}{q'}\right)t}(x_\lambda \otimes x_\mu^*).$$

Hence, taking $t = qq'$ we obtain that $\mathcal{G}(t)(x_\lambda \otimes x_\mu^*) = x_\lambda \otimes x_\mu^*$. By hypotheses of Lemma 3.5 we have that $\mathcal{F}_{V_0,V_0^*}$ is dense in $\mathcal{X} \otimes \mathcal{X}^*$. So, we get the desired result using Lemma 4.5. □

Example 4.6. We consider the Banach space $\ell^p(\mathbb{N},\alpha)$ as is defined in Example 3.9. This space has the approximation property (see [20, Example 4.5]). As we have seen in Example 3.9 A_1 and A_2^* satisfy the assumptions of Lemma 3.5. Hence we can use Corollary 4.4 and conclude that the tensor product semigroup $\mathcal{G}(t) : C \mapsto G_2(t)CG_1(t)$ is also chaotic in $\mathcal{K}(\ell^p(\mathbb{N},\alpha))$ with the norm operator topology.

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