

CHAOTIC WEIGHTED SHIFTS IN BARGMANN SPACE

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ABSTRACT. This paper deals with the unilateral backward shift operator T on a Bargmann space $F(\mathbb{C})$. This space can be identified with the sequence space $\ell^2(\mathbb{N})$. We use the hypercyclicity Criterion of Bès, Chan and Seubert and the program of K.-G. Grosse-Erdmann to give a necessary and sufficient condition in order that T be a chaotic operator. The chaoticity of differentiation which correspond to the annihilation operator in quantum radiation field theory is in view, since the Bargmann space is an infinite dimensional separable complex Hilbert space.

1. INTRODUCTION

In this paper the Bargmann space is denoted by $F(\mathbb{C})$. This space has been studied by many authors. Its roots can be found in mathematical problems of relativistic physics (see [21]) or in quantum optics (see [16]). In physics the Bargmann space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field (see [15] and for other application see [11]). The space $F(\mathbb{C})$ has also been discovered in the theory of the wavelets. In fact, the Bargmann transform is a unitary map from $L^2(\mathbb{R})$ onto $F(\mathbb{C})$ which transforms the family of evaluation functionals at a point into canonical coherent state which are noting but the Gabor wavelets. Since this family is not complete in $L^2(\mathbb{R})$, in [5] Daubechies and Grossman have characterized its frames via Bargmann space.

The Hilbert space $F(\mathbb{C})$ introduced by V. Bargmann [1, 2] is the space of entire functions equipped with the inner product

$$(f, g) = \int_{\mathbb{C}} f(z) \overline{g(z)} d\mu(z),$$

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and the norm $\|f\| = \sqrt{(f, f)}$, where $d\mu(z) = \pi^{-1}e^{-\bar{z}z}dxdy$, ($z = x + iy$), is the Gaussian measure on \mathbb{C} . A straightforward computation leads that

$$(z^k, z^{k'}) = \begin{cases} 0, & \text{for } k \neq k' \\ k!, & \text{for } k = k'. \end{cases}$$

This implies that $e_k = z^k/\sqrt{k!}$ establishes an orthonormal basis in $F(\mathbb{C})$. Related to this basis we can say that a function f belongs to $F(\mathbb{C})$ iff $f(z) = \sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}}$ with $\sum_{k \geq 0} |c_k|^2 < \infty$ and the Parseval identity implies that

$$\|f\|^2 = \sum_{k \geq 0} |c_k|^2. \quad (1.1)$$

In the last decade it has been observed that chaotic behaviour in sense of Devaney [7] can occur in some infinite-dimensional space for a linear operator. A continuous linear operator T on a topological vector space X is called *hypercyclic* if there exists a vector $x \in X$ whose orbit $\{T^n x \mid n = 0, 1, \dots\}$ is dense in X . A *periodic point* for T is a vector $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$. Finally, T is said to be *chaotic* if it is hypercyclic and its set of periodic points is dense in X .

Examples of hypercyclic operators have been previously reported in [4, 12, 13, 17, 19, 20]. In [4], Birkhoff showed the hypercyclicity of the operator of translation on the space $H(\mathbb{C})$ of entire functions equipped with topology of local uniform convergence, while MacLane's result was on the hypercyclicity of the differentiation operator $Df = f'$ on $H(\mathbb{C})$. Although $F(\mathbb{C}) \subset H(\mathbb{C})$, the differentiation operator D is not continuous on the Bargmann space $F(\mathbb{C})$. The same is true on the Hardy space H^2 . Motivated by this result, Bès, Chan and Seubert have defined in [3] the notion of hypercyclicity and chaos for an unbounded operator as follows:

Definition 1.1. *Let X be a separable infinite dimensional Banach space. An unbounded densely defined operator $A : X \mapsto X$ is called hypercyclic if there is a vector $f \in X$ whose orbit under A ; $\text{orb}(A, f) := \{f, Af, A^2f, \dots\}$, is densely defined in X . Every such vector f is called hypercyclic vector for A . If there exist an integer $N \in \mathbb{N}$ and a vector $f \in D(A^N)$ such that $A^N f = f$, such vector is called periodic and the operator A is said to be chaotic if it has both sets of periodic points and hypercyclic vectors.*

It is clear that if the unbounded operator A is hypercyclic then $D(A^\infty) := \bigcap_{n=0}^\infty D(A^n)$ which contains $\text{orb}(A, f)$ is also dense. In [6], one can find a similar definition, where there are two different Banach spaces X and Y and $A : X \mapsto Y$.

The identity (1.1) identifies the Bargmann space $F(\mathbb{C})$ with the sequence space $\ell^2(\mathbb{N})$. Since any unilateral weighted backward shifts operator $T : F(\mathbb{C}) \mapsto F(\mathbb{C})$,

$$T \left(\sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}} \right) = \sum_{k \geq 0} \omega_k c_{k+1} \frac{z^k}{\sqrt{k!}} \quad (1.2)$$

can be identified with

$$\tilde{T}(c_k)_{k \in \mathbb{N}} = (\omega_k c_{k+1})_{k \in \mathbb{N}}, \quad (1.3)$$

It is clear that by assuming the following condition on the weight sequence $\{\omega_k\}$:

$$\sup_{k \in \mathbb{N}} |\omega_k| \leq M, \quad (1.4)$$

the operator T (resp. \tilde{T}) becomes continuous in $F(\mathbb{C})$ (resp. in $\ell^2(\mathbb{N})$). In [20], H. Salas has shown the following result:

Lemma 1.2. *Let \tilde{T} be a positively weighted backward shift on $\ell^2(\mathbb{N})$ defined by (1.3) with (1.4). Then \tilde{T} is hypercyclic if and only if there is an increasing sequence (n_k) of positive integers such that the weight sequence $\{\omega_k\}$ satisfies*

$$\prod_{n=1}^{n_k} \omega_n \rightarrow \infty \quad (1.5)$$

as $n_k \rightarrow \infty$.

This result is generalized by K.-G. Grosse-Erdmann in [13] in the following terms:

Lemma 1.3. *Let \tilde{T} be a weighted backward shift on a Fréchet-sequence space X in which $(e_n)_{n \in \mathbb{N}}$ is an unconditional basis. Then the following assertions are equivalent:*

- (i) \tilde{T} is chaotic;
- (ii) \tilde{T} has a non-trivial periodic point;
- (iii) the series

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n \omega_k \right)^{-1} e_n \quad (1.6)$$

converges in X .

One of the aims of [13] was to apply the above lemma to the differentiation operator $Df = f'$ on $H(\mathbb{C})$ the space of entire functions, endowed with the topology of uniform convergence on compact subsets and retrieve the result of G. Godefroy and J. H. Shapiro [12], which asserts that D is a chaotic operator on $H(\mathbb{C})$. Our aim in this paper is to extend the above result to the unbounded weighted shifts for establishing a necessary and sufficient condition in order that a such operator to be hypercyclic or chaotic on the Bargmann space. As a corollary one can retrieve the chaoticity of the differentiation operator on $F(\mathbb{C})$.

For achieving this end we have to use a Hypercyclicity Criterion for unbounded operators. Such a criterion has been already provided by Bès, Chan and Seubert [3] in the following terms.

Theorem 1.4. *Let X be a separable infinite dimensional Banach space and let A be a densely defined linear operator on X for which A^n is closed for any $n \in \mathbb{N}$. Then A is hypercyclic if there exist a dense subset $\mathcal{D} \subset D(A)$ and a mapping $B : \mathcal{D} \rightarrow \mathcal{D}$ such that*

- (i) $AB = I_{\mathcal{D}}$ the identity map of \mathcal{D} , and
- (ii) $A^n, B^n \rightarrow 0$ pointwise on \mathcal{D} .

By using the above Criterion we will prove in the section 3 the following theorem.

Theorem 1.5. *Let T be a linear unbounded backward shift operator on $F(\mathbb{C})$. Then the following assertions are equivalent:*

- (i) T is chaotic;
- (ii) T has a non-trivial periodic point;
- (iii) the positive series

$$\sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{1}{|\omega_j|^2}, \quad (1.7)$$

converges.

This theorem implies that the differentiation is a chaotic unbounded operator in $F(\mathbb{C})$.

2. HYPERCYCLICITY OF UNBOUNDED BACKWARD SHIFT IN BARGMANN SPACE

Let $\{\omega_n\}_{n \in \mathbb{N}}$ be an arbitrary weight sequence, we define the iterated unbounded backward shift T^n in Bargmann space by

$$T^n \left(\sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}} \right) = \sum_{k \geq 0} \left(\prod_{j=k}^{n+k-1} \omega_j \right) c_{n+k} \frac{z^k}{\sqrt{k!}} \quad (2.1)$$

with its domain in $F(\mathbb{C})$.

$$D(T^n) := \left\{ f(z) = \sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}} ; \sum_{k \geq 0} |c_k|^2 < \infty \text{ and } \sum_{k \geq 0} \left| \prod_{j=k}^{m+k-1} \omega_j \right|^2 |c_{k+m}|^2 < \infty \right\}, \quad (2.2)$$

for all $m \in \mathbb{N}$, $1 \leq m \leq n$.

Lemma 2.1. *The subspace $D(T^\infty) := \bigcap_{n \geq 0} D(T^n)$ is dense in $F(\mathbb{C})$.*

Proof. Since $e_k(z) := z^k/\sqrt{k!}$ is the complete orthonormal basis in $F(\mathbb{C})$ and each $e_k \in D(T^\infty)$, this implies that $D(T^\infty)$ is dense in $F(\mathbb{C})$. \square

This Lemma legitimates the research of hypercyclicity of the unbounded operator T in the framework of Definition 1.1.

Lemma 2.2. *Assume $|\omega_k|$ is a nondecreasing sequence which tends to infinity as $k \rightarrow \infty$. Then the spectrum of T is the whole complex plane.*

Proof. Let λ be an arbitrary complex number and

$$\varphi_\lambda(z) = \sum_{k \geq 0} \left(\prod_{j=0}^{k-1} \frac{\lambda}{\omega_j} \right) \frac{z^k}{\sqrt{k!}}.$$

For $1 < \alpha < 1$, choose $n \in \mathbb{N}$ enough large such that $|\lambda| \leq \alpha |\omega_N|$ and write

$$\begin{aligned} \sum_{k \geq 0} \prod_{j=0}^{k-1} \left| \frac{\lambda}{\omega_j} \right|^2 &= \sum_{k=0}^N \prod_{j=0}^{k-1} \left| \frac{\lambda}{\omega_j} \right|^2 + \sum_{k=N+1}^{\infty} \prod_{j=0}^{k-1} \left| \frac{\lambda}{\omega_j} \right|^2 \\ &\leq \sum_{k=0}^N \prod_{j=0}^{k-1} \left| \frac{\lambda}{\omega_j} \right|^2 + \left(\frac{1}{1-\alpha} \right) \prod_{j=0}^N \left| \frac{\lambda}{\omega_j} \right|^2 < \infty. \end{aligned}$$

Hence $\varphi_\lambda \in F(\mathbb{C})$ and

$$\begin{aligned} T\varphi_\lambda(z) &= \sum_{k \geq 0} \omega_k \left(\prod_{j=0}^k \frac{\lambda}{\omega_j} \right) \frac{z^k}{\sqrt{k!}} \\ &= \lambda \sum_{k \geq 0} \left(\prod_{j=0}^{k-1} \frac{\lambda}{\omega_j} \right) \frac{z^k}{\sqrt{k!}} = \lambda \varphi_\lambda(z) \end{aligned}$$

shows that λ is an eigenvalue for T . \square

In order to use the hypercyclicity criterion (Theorem 1.4) we have to ensure that for each $n \in \mathbb{N}$, the operator T^n is closed. The above Lemma show that the resolvent set of the operator T can be empty and we cannot use the fact that every power of a closed unbounded operator with a nonempty resolvent set must be closed (see [10, p. 602]). So to prove this assertion we have to proceed in a direct manner.

Lemma 2.3. *For each $n \in \mathbb{N}$, the operator T^n defined by (2.1) and (2.2) is closed.*

Proof. Let $\{(f_j, T^n f_j)\}$ be a sequence in the graph of T^n which converges to a point (f^*, g^*) in $F(\mathbb{C}) \times F(\mathbb{C})$. Since $F(\mathbb{C})$ is a Hilbert space, each coefficients $c_{k,j}$ of $f_j(z) = \sum_{k \geq 0} c_{k,j} \frac{z^k}{\sqrt{k!}}$ converges to the coefficient c_k^* of $f^*(z) = \sum_{k \geq 0} c_k^* \frac{z^k}{\sqrt{k!}}$, as $j \rightarrow \infty$. Hence

$$\left(\prod_{j=k}^{n+k-1} \omega_j \right) c_{n+k,j} \rightarrow \left(\prod_{j=k}^{n+k-1} \omega_j \right) c_{n+k}^*.$$

On the other hand, since $T^n f_j$ converges to g^* ,

$$\left(\prod_{j=k}^{n+k-1} \omega_j \right) c_{n+k,j} \rightarrow d_k^*$$

where $g^*(z) = \sum_{k \geq 0} d_k^* \frac{z^k}{\sqrt{k!}}$, we can conclude that $d_k = \left(\prod_{j=k}^{n+k-1} \omega_j \right) c_{n+k}^*$. This proves that $f^* \in D(T^n)$ and $T^n f^* = g^*$. \square

Theorem 2.4. *A linear unbounded backward shift operator $T : F(\mathbb{C}) \mapsto F(\mathbb{C})$ is hypercyclic if and only if there is an increasing sequence (n_k) of positive integers such that*

$$\prod_{j=0}^{n_k} |\omega_j| \rightarrow \infty \quad \text{as } n_k \rightarrow \infty. \quad (2.3)$$

Proof. For the necessary part of the proof we use a simplified version of Salas' argument. Assume that T is hypercyclic. Since the set of hypercyclic vectors for T is dense, then for a δ_k enough small there is a hypercyclic vector $f \in D(T^\infty)$ for T such that

$$\|f - 1\| < \delta_k \quad (2.4)$$

and there is also an arbitrarily large $n_k \in \mathbb{N}$, such that

$$\|T^{n_k} f - 1\| < \delta_k \quad (2.5)$$

Now if $f(z) = \sum_{j \geq 0} c_j \frac{z^j}{\sqrt{j!}}$, the inequality (2.4) implies that $|c_j| < \delta_k$ for all $j \geq 1$ and by taking the inner product of (2.1) with 1, (2.5) implies that $|\left(\prod_{j=0}^{n_k-1} \omega_j\right) c_{n_k} - 1| < \delta_k$. This in turn implies

$$\prod_{j=0}^{n_k-1} |\omega_j| > (1 - \delta_k)/|c_{n_k}| > (1 - \delta_k)/\delta_k$$

By choosing δ_k enough small we see that the condition (2.3) is necessary if T is hypercyclic.

For the sufficient part of the proof we will use the Hypercyclicity Criterion stated in Theorem 1.4. The closedness of T^n is already proved in Lemma 2.3. Take as \mathcal{D} the linear subspace generated by finite combinations of the canonical basis $\{z^k/\sqrt{k!}\}$. As Lemma 2.1 \mathcal{D} is dense in $F(\mathbb{C})$ and we can define the pseudo-inverse of T on \mathcal{D} by

$$S \left(\sum_{0 \leq k \leq n} c_k \frac{z^k}{\sqrt{k!}} \right) = \sum_{1 \leq k \leq n+1} \frac{c_{k-1}}{\omega_{k-1}} \frac{z^k}{\sqrt{k!}}.$$

Since $T(z^k/\sqrt{k!}) = \omega_{k-1} z^{k-1}/\sqrt{(k-1)!}$, for $n > k$, $T^n(z^k/\sqrt{k!}) = 0$ and any element of \mathcal{D} can be annihilated by a finite power of T . On the other hand, according to (2.3),

$$S^{n_k}(z^k/\sqrt{k!}) = \left(\prod_{j=k}^{n_k+k} \omega_j \right)^{-1} z^{k+n_k}/\sqrt{(k+n_k)!} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in $F(\mathbb{C})$. In fact, as it is noted in [3], Theorem 1.4 actually holds when the entire sequence of positive intergers in Hypothesis (ii) is replaced by a subsequence of positive integers. With this we have proved all the assumptions of Theorem 1.4. \square

Corollary 2.5. *The operator of differentiation $D : f \mapsto f'$ defined on $D(D) := \{f \in F(\mathbb{C}) : f' \in F(\mathbb{C})\}$ is hypercyclic on $F(\mathbb{C})$.*

Proof. It is enough to remark that

$$D \left(\sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}} \right) = \sum_{k \geq 0} \sqrt{k+1} c_{k+1} \frac{z^k}{\sqrt{k!}}$$

and $\omega_k = \sqrt{k+1}$ satisfies the assumption (2.3). \square

Remark 2.6. This Corollary is already established by Bès, Chan and Seubert in the Hardy space H^2 (see [3, Corollary 2.3]). In the next section we go further and we prove that the operator of differentiation is even chaotic in $F(\mathbb{C})$.

3. THE PROOF OF THEOREM 1.5

(i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii). Since

$$T^N \left(\sum_{k \geq 0} c_k \frac{z^k}{\sqrt{k!}} \right) = \sum_{k \geq 0} \left(\prod_{j=k}^{N+k-1} \omega_j \right) c_{N+k} \frac{z^k}{\sqrt{k!}}$$

if there exists $N \in \mathbb{N}$ such that $T^N f = f$, then

$$\left(\prod_{j=k}^{N+k-1} \omega_j \right) c_{N+k} = c_k, \quad \text{for any } k \geq 0.$$

This implies that for any $l = 0, 1, \dots, N-1$ and any $k \geq 1$

$$c_{kN+l} = \left(\prod_{j=l}^{kN+l-1} \frac{1}{\omega_j} \right) c_l. \quad (3.1)$$

Now let us decompose the series $\sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{1}{|\omega_j|^2}$ as follows

$$\sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{1}{|\omega_j|^2} = \sum_{l=1}^N \frac{1}{\prod_{j=0}^{l-1} |\omega_j|^2} \left(1 + \sum_{k=1}^{\infty} \prod_{j=l}^{kN-1+l} \frac{1}{|\omega_j|^2} \right). \quad (3.2)$$

Since for any $l = 0, 1, \dots, N-1$, the series $\sum_{k=1}^{\infty} c_{kN+l}^2$ converges, the series (3.2) also converges.

(iii) \Rightarrow (i). If the series (1.7) converges, the product $\prod_{j=0}^{n-1} \frac{1}{|\omega_j|^2}$ has to tend to zero, which is equivalent to (2.3). Thus T is hypercyclic. We shall now show that T has a dense periodic points.

First we remark that with $c_l = \prod_{j=0}^{l-1} \frac{1}{\omega_j}$ the identity (3.1) holds. Hence for any integers $\nu \geq 0$ and $N \geq \nu$,

$$g_{\nu, N}(z) := \frac{z^\nu}{\sqrt{\nu!}} + \sum_{k=1}^{\infty} \left(\prod_{j=\nu}^{kN+\nu-1} \frac{1}{\omega_j} \right) \frac{z^{kN+\nu}}{\sqrt{(kN+\nu)!}} \quad (3.3)$$

is a N -periodic point for T . Due to (1.7) $g_{\nu, N} \in F(\mathbb{C})$. We have also to show that $g_{\nu, N} \in D(T^N)$. In (3.3), we have only the terms $kN + \nu$, thus from (2.2) the series

$$\sum_{k=0}^{\infty} \prod_{j=kN+\nu}^{(k+1)N+\nu-1} |\omega_j|^2 |c_{(k+1)N+\nu}|^2$$

should be finite. Since $c_{kN+\nu} = \left(\prod_{j=\nu}^{kN+\nu-1} \frac{1}{\omega_j} \right)$ this series is equal

$$\sum_{k=0}^{\infty} \prod_{j=\nu}^{kN+\nu-1} |\omega_j|^{-2}$$

which under condition (iii) converges.

We shall now show that T has a dense set of periodic points. To see this, it suffices to show that for every element f in the dense subspace \mathcal{D} , defined in the proof of Theorem 2.4, there is a periodic point g arbitrarily close to it. Let $f(z) := \sum_{\nu=0}^m c_\nu z^\nu / \sqrt{\nu!}$ and $\varepsilon > 0$. We can assume without loss of generality that

$$\left| c_\nu \prod_{j=0}^{\nu-1} \omega_j \right| \leq 1 \quad \text{for } \nu = 0, 1, \dots, m. \quad (3.4)$$

Furthermore the condition (1.7) implies that the series $\sum_{k \geq 0} \left(\prod_{j=0}^k \frac{1}{\omega_j} \right) \frac{z^k}{\sqrt{k!}}$ converges, which in turn implies the existence of an $N \geq m$ such that

$$\left\| \sum_{k \geq N+1} \varepsilon_k \left(\prod_{j=0}^k \frac{1}{\omega_j} \right) \frac{z^k}{\sqrt{k!}} \right\| < \frac{\varepsilon}{m+1} \quad (3.5)$$

for every sequence (ε_k) taking values 0 or 1. One can choose the periodic point for T as $g(z) := \sum_{\nu=0}^m c_\nu g_{\nu,N}(z)$. With this choice

$$\begin{aligned} \|g - f\| &= \left\| \sum_{\nu=0}^m c_\nu \left(g_{\nu,N}(z) - \frac{z^\nu}{\sqrt{\nu!}} \right) \right\| \\ &= \left\| \sum_{\nu=0}^m \left(c_\nu \prod_{j=0}^{\nu-1} \omega_j \right) \sum_{k=1}^{\infty} \left(\prod_{j=0}^{kN+\nu-1} \frac{1}{\omega_j} \right) \frac{z^{kN+\nu}}{\sqrt{(kN+\nu)!}} \right\| \\ &\leq \sum_{\nu=0}^m \left\| \sum_{k=1}^{\infty} \left(\prod_{j=0}^{kN+\nu-1} \frac{1}{\omega_j} \right) \frac{z^{kN+\nu}}{\sqrt{(kN+\nu)!}} \right\| \quad \text{by (3.4)} \\ &< \varepsilon \quad \text{by (3.5)}. \end{aligned}$$

This achieves the proof of the Theorem.

4. CHAOTICITY OF THE ANNIHILATION OPERATOR

In bosonic coherent state theory of radiation field, if the physical coordinates and momentum are measured in standard units ($\sqrt{\hbar/m\omega} \equiv 1$ for the position variable x and $\sqrt{\hbar m\omega} \equiv 1$ for the momentum $\frac{d}{dx}$), the one-dimensional oscillator hamiltonian can be expressed by $H = 2(A^\dagger A + AA^\dagger)$, where A^\dagger and A given by

$$A^\dagger = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right) \quad \text{and} \quad A = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right)$$

are the creation and annihilation operators. The particularities of these operators lie in the fact that for the orthogonal basis of the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, $\phi_n(x) = \pi^{-1/4} (e^n n!)^{-1/2} e^{-x^2/2} H_n(x)$, where $H_n(x)$ is the n -th order Hermite polynomial, one has

$$A^\dagger \phi_n = \sqrt{n+1} \phi_{n+1} \quad \text{and} \quad A \phi_n = \sqrt{n} \phi_{n-1}. \quad (4.1)$$

In [14], Gulisashvili and MacCluer have shown that the annihilation operator A is chaotic in the Fréchet space

$$\mathcal{E} := \left\{ \phi \in \mathcal{H} \mid \phi = \sum_{n=0}^{\infty} c_n \phi_n, \text{ with } \sum_{n=0}^{\infty} |c_n|^2 (n+1)^\ell < \infty \text{ for all } \ell \in \mathbb{N}^* \right\}.$$

The choice of such space comes from the fact that the operator A is not bounded in \mathcal{H} , but as it is mentioned in [1] by complexification, the natural space for creation and annihilation operators would be the Bargmann space $F(\mathbb{C})$. In fact, in this space the orthogonal basis is $\phi_n(z) := z^n / \sqrt{n!}$ and the creation and annihilation operators are multiplication and differentiation

$$A^\dagger : \varphi(z) \mapsto z\varphi(z) \quad \text{and} \quad A : \varphi(z) \mapsto d\varphi(z)/dz.$$

It is clear that for these operators the relations (4.1) are fulfilled.

Corollary 4.1. *The annihilation operator $A : f \mapsto f' = df/dz$ defined on $D(A) := \{f \in F(\mathbb{C}) : f' \in F(\mathbb{C})\}$ is chaotic on $F(\mathbb{C})$.*

Proof. As we have seen in Corollary 2.5, for the operator $A = D$ we have $\omega_k = \sqrt{k+1}$ which satisfies the assumption (1.7). \square

5. CONCLUSION

In [14] Gulisashvili and MacCluer have given a nice interpretation of the chaoticity of annihilation operator in the performance of nanomachinery (see [9]). In fact, the interconnection of a nanosystem can be conceived as the iteration of the annihilation operator and by the laws of thermodynamics, the closed system will gradually decay from order to chaos, tending toward maximum entropy. Here we want to give another interpretation: The relationship between chaos and irreversibility occurs in many domains of sciences. In nonequilibrium statistical mechanics this is discussed in chapters 7 and 8 of [8]. In the chapter 11 of [18] the emergence of irreversibility and quantum chaos is elucidated. In $L^2(\mathbb{R})$ space the irreversibility of annihilation operator can be considered as the loss of informations on the states of particles, but since in the Bargmann space this operator can be identified with the differentiation, hence the irreversibility is much more apparent. In fact, the differentiation is not invertible or is invertible modulo a constant.

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