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CHAOTIC SOLUTION FOR THE BLACK-SCHOLES EQUATION

HASSAN EMAMIRAD, GISÈLE RUIZ GOLDSTEIN, AND JEROME A. GOLDSTEIN

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ABSTRACT. The Black-Scholes semigroup is studied on spaces of continuous functions on $(0,\infty)$ which may grow at both 0 and at ∞, which is important since the standard initial value is an unbounded function. We prove that in the Banach spaces

$$Y^{s,\tau} := \{u \in C((0,\infty)) : \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0,\ \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0\}$$

with norm $\|u\|_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right| < \infty$, the Black-Scholes semigroup is strongly continuous and chaotic for $s > 1$, $\tau \geq 0$ with $s\nu > 1$, where $\sqrt{2}\nu$ is the volatility. The proof relies on the Godefroy-Shapiro hypercyclicity criterion.

1. Introduction

In [B-S], F. Black and M. Scholes proved that under certain assumptions about the market, the value of a stock option, as a function of the current value of the underlying asset $x \in \mathbb{R}^+ = [0,+\infty)$ and time, $u(x,t)$, satisfies the final value problem

$$(\mathbf{BS}) \qquad \begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0,T]; \\ u(0,t) = 0 & \text{for } t \in [0,T]; \\ u(x,T) = (x-p)^+ & \text{for } x \in \mathbb{R}^+, \end{cases}$$

where $p > 0$ represents a given strike price, $\sigma > 0$ is the volatility and $r > 0$ is the interest rate.

Let $v(x,t) = u(x,T-t)$. Then v satisfies the forward Black-Scholes equation, which is a parabolic problem, defined for all time $t \in \mathbb{R}^+$ by

$$(\mathbf{FBS}) \qquad \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+; \\ v(0,t) = 0 & \text{for } t \in \mathbb{R}^+; \\ v(x,0) = f(x) & \text{for } x \in \mathbb{R}^+. \end{cases}$$

Strictly speaking, the condition $t \in \mathbb{R}^+$ should have been written as $0 \leq t \leq T$. But once one notes that, there is no problem considering all nonnegative values of time.

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In **(FBS)** we have

$$f(x) = (x-p)^+ = \begin{cases} x-p & \text{if } x > p, \\ 0 & \text{if } x \le p, \end{cases} \tag{1.1}$$

but for the time being we prefer to consider f merely as an arbitrary given function. Later we shall deal with (1.1). In order to put the **(FBS)** problem in an abstract form, let us denote by $D_\nu = \nu x \frac{d}{dx}$, where $\nu = \sigma/\sqrt{2}$, and let

$$\mathcal{B} = D_\nu^2 + \gamma D_\nu - rI = \nu^2 C_1 + C_2, \tag{1.2}$$

with $\gamma = r/\nu - \nu, C_1 := x^2 \frac{d^2}{dx^2} = D_1^2 - D_1$ and $C_2 := rD_1 - rI$. Then **(FBS)** can be written as

$$\text{(\textbf{AFBS})} \qquad \begin{cases} dv/dt = \mathcal{B}v, \\ v(0,t) = 0, \\ v(x,0) = f(x) \qquad \text{for } x \in \mathbb{R}^+. \end{cases}$$

For European call options, Cruz-Báez and González-Rodríguez [C-G1] and Arendt and de Pagter [AdP] showed that **(FBS)** is governed by a C_0-semigroup on a suitable Banach space. In [C-G2] the authors have generalized [C-G1] to American call options, a topic of interest in mathematical finance. But by working in the context of a contraction semigroup, these authors could not consider the issue of chaos. Recently, [GMR] gave a simple explicit representation of the solution of **(FBS)**, and this representation holds in the spaces $Y^{s,\tau}$ considered here.

2. Multiplicative (C_0) semigroups on the weighted space

For representing the Black-Scholes semigroup, we begin by introducing the translation on the multiplication group of positive numbers, $G = ((0,\infty), \cdot)$. We do this now and we postpone the Black-Scholes semigroup to Section 3.

Let μ be the Haar measure on G and suppose $\tau = \{\tau_t \ : \ t \in \mathbb{R}\}$ is the group of translations on G. Thus $d\mu = \frac{dx}{x}$, and $\tau_t(x) = \mathrm{e}^t x$, for $x > 0, t \in \mathbb{R}$.

Let $s, \tau \ge 0$ and let $C(0,\infty)$ be the space of all complex continuous functions on $(0,\infty)$. Define

$$Y^{s,\tau} := \{u \in C(0,\infty) : \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0\},$$

with norm

$$\|u\|_{s,\tau} = \sup_{x>0} \left| \frac{u(x)}{(1+x^{-\tau})(1+x^s)} \right| < \infty.$$

These are Banach spaces.

Fix $\nu \in \mathbb{R}\backslash\{0\}$. Define the translation group with parameter ν, $S_\nu := \{S_\nu(t) : t \in \mathbb{R}\}$ on $Y^{s,\tau}$, by

$$(S_\nu(t)f)(x) = \mathrm{e}^{tD_\nu} f(x) = f(\tau_{\nu t}(x))$$

for $f \in Y^{s,\tau}$, $x \in G$ and $t \in \mathbb{R}$. Since $\tau_{t+s} = \tau_t \tau_s$ for all $t, s \in \mathbb{R}$, S_ν forms a one-parameter group on each $Y^{s,\tau}$. Let D_ν be its infinitesimal operator in the sense of Hille, that is,

$$D_\nu f = \frac{d}{dt} S_\nu(t) f \ |_{t=0}$$

for all f for which this limit exists in $Y^{s,\tau}$; call this set $\mathcal{D}(D_\nu)$. Then $f \in \mathcal{D}(D_\nu)$ requires that $x \to f(x)$ and $x \to xf(x)$ are both in $Y^{s,\tau}$. Below we will establish the strong continuity and characterize $\mathcal{D}(D_\nu) = \mathcal{D}(D_1)$ in $Y^{s,\tau}$ for $s \geq 1$, $\tau \geq 0$. The spaces $Y^{s,\tau}$ are Banach spaces, for all $s \geq 0$, $\tau \geq 0$.

Let $\mathcal{M}(0,\infty)$ be the set of all finite complex Borel measures on $(0,\infty)$. Any $\psi \in \mathcal{M}(0,\infty)$ can be written as

$$\psi = \mathrm{Re}(\psi) + \mathrm{i}\,\mathrm{Im}(\psi) = \sum_{j=1}^{4} c_j P_j \tag{2.1}$$

where each P_j is a probability measure on $(0,\infty)$ and the scalars c_j satisfy $\mathrm{Re}(\psi) = (\mathrm{Re}\,\psi)_+ - (\mathrm{Re}\,\psi)_- = \psi_1 - \psi_2$, with $\psi_1 = c_1 P_1$ and $\psi_2 = -c_2 P_2$, and $c_1, c_2 \geq 0$. In the same way $\mathrm{Im}(\psi) = (\mathrm{Im}\,\psi)_+ - (\mathrm{Im}\,\psi)_- = \psi_3 - \psi_4$, with $\mathrm{i}\psi_3 = c_3 P_3$ and $\mathrm{i}\psi_4 = -c_4 P_4$, and $-\mathrm{i}c_3, -\mathrm{i}c_4 \geq 0$, and P_j is uniquely determined for each j for which $c_j \neq 0$. We also define $\psi \in \mathcal{M}_{loc}(0,\infty)$ to mean that for any $n \in \mathbb{N}$, the restriction of ψ to Borel subsets of $[\frac{1}{n}, n]$ is a finite complex Borel measure ψ_n satisfying

$$\begin{aligned}\psi_n &= \mathrm{Re}\,\psi_n + \mathrm{i}\,\mathrm{Im}\,\psi_n \\ &= (\mathrm{Re}\,\psi_n)_+ - (\mathrm{Re}\,\psi_n)_- + \mathrm{i}[(\mathrm{Im}\,\psi_n)_+ - (\mathrm{Im}\,\psi_n)_-].\end{aligned}$$

Let ξ_n denote any one of $(\mathrm{Re}\,\psi_n)_\pm, (\mathrm{Im}\,\psi_n)_\pm$. Then each such ξ_n determines uniquely a σ-finite Borel measure on $(0,\infty)$ via

$$\xi(A) = \lim_{n\to\infty} \xi_n(A \cap [\frac{1}{n}, n])$$

for all Borel sets $A \subset [0,\infty]$. In this sense we can view $(\mathrm{Re}\,\psi)_\pm, (\mathrm{Im}\,\psi)_\pm$ as measures in a certain sense. Note that the set functions $\psi = (\mathrm{Re}\,\psi)_+ - (\mathrm{Re}\,\psi)_- + \mathrm{i}[(\mathrm{Im}\,\psi)_+ - (\mathrm{Im}\,\psi)_-] \in \mathcal{M}_{loc}(0,\infty)$ are not in general complex measures, but nevertheless we can treat them locally (away from 0 and ∞) as if they were complex measures by using ψ_n for $n \in \mathbb{N}$.

We begin our study of $Y^{s,\tau}$ with the case of $s = 0$, $\tau = 0$. Note that

$$Y^{0,0} = C_0(0,\infty),$$

the continuous complex functions on $(0,\infty)$, which vanish at both 0 and ∞, with the norm

$$\|u\|_{0,0} = \frac{1}{4}\,\|u\|_\infty$$

for $u \in Y^{0,0}$. Note that the constant function $\mathbf{1}$ is in $Y^{s,\tau}$ if and only if $s > 0$, $\tau > 0$. By the Riesz Representation Theorem, the dual space of $Y^{0,0} = ((C_0(0,\infty), \|\cdot\|_{0,0})$ can be identified with $\mathcal{M}(0,\infty)$ with the norm

$$\|\psi\| = 4TV(\psi) = 4\sum_{j=1}^{4} |c_j|$$

when c_j is as in (2.1), and TV means total variation. The identification is made by mapping $u \in Y^{0,0}$ and $\psi \in \mathcal{M}(0,\infty)$ to

$$\langle u, \psi \rangle = \int_{(0,\infty)} u(x)\psi(dx).$$

We shall write $\int_0^\infty$ in place of $\int_{(0,\infty)}$.

One may view $Y^{0,0}$ as $\{u \in C[0,\infty] : u(0) = u(\infty) = 0\}$, the continuous functions on the compact interval $[0,\infty]$, which vanish at both 0 and ∞. Similarly,

$\mathcal{M}(0,\infty)$ may be viewed as the finite complex Borel measures ψ on $[0,\infty]$ satisfying $\psi(\{0,\infty\})=0$.

We next recall a well-known fact.

Lemma 2.1. *Let $U : X \to Y$ be an isometric isomorphism between Banach spaces. Then $U^* : Y^* \to X^*$ is also an isometric isomorphism between their dual spaces.*

Let

$$\mathcal{C}_c := \mathcal{C}_c(0,\infty) = \{u \in C(0,\infty) : u \text{ has compact support in } (0,\infty)\}.$$

Then $\mathcal{C}_c$ is dense in $Y^{s,\tau}$ for all $s,\tau \geq 0$. Let $u \in \mathcal{C}_c$. Then $u \in \mathcal{C}_0(\varepsilon, 1/\varepsilon)$ for some $\varepsilon > 0$. Let $\varphi \in \mathcal{M}([\varepsilon, 1/\varepsilon])$ be a finite complex measure on $[\varepsilon, 1/\varepsilon]$, which is the dual space of $\mathcal{C}[\varepsilon,1/\varepsilon]$. Then if $\psi \in (Y^{s,\tau})^*$, we have $\langle u,\psi\rangle = \int_0^\infty u(x)\varphi(dx)$ for some φ as above. We extend φ by requiring that $\varphi(A)=0$ for all Borel subsets A of $[0,\varepsilon]\cup[1/\varepsilon,\infty]$. For this φ, define χ by

$$\chi(dx) = (1+x^s)(1+x^{-\tau})\varphi(dx).$$

Then for $u \in Y^{s,\tau}$, $u = U_{s,\tau}v$ for a unique $v \in Y^{0,0}$, and

$$\begin{aligned}\langle u,\psi\rangle &:= \int_0^\infty u(x)\psi(dx) = \int_0^\infty \left(\frac{u(x)}{(1+x^s)(1+x^{-\tau})}\right)\big((1+x^s)(1+x^{-\tau})\psi(dx)\big)\\ &= \langle 4U_{s,\tau}u, \frac{1}{4}\chi\rangle = \langle U_{s,\tau}u,\chi\rangle = \langle v,\chi\rangle = \langle u, U^*_{s,\tau}\chi\rangle,\end{aligned}$$

since $u \in \mathcal{C}_c$, which is dense in $Y^{a,b}$ for all $a,b>0$. Here $U_{s,\tau}$ is the U of Lemma 2.1 corresponding to $X = Y_{s,\tau}$. Let

$$Z_{s,\tau} := \{\psi \in \mathcal{M}_{loc}(0,\infty) : \chi(dx) := (1+x^s)(1+x^{-\tau})\psi(dx) \;\text{ defines } \chi \in \mathcal{M}(0,\infty)\}$$

for $s,\tau \geq 0$. Then (2.2) below holds for $\psi \in Z_{s,\tau}$ and $\psi = U^*_{s,\tau}\chi$ for a unique $\chi \in \mathcal{M}(0,\infty)$, that is, $\psi \in Y^*_{s,\tau}$, and conversely. Thus we have proved that $Z_{s,\tau}$ can be identified with $(Y^{s,\tau})^*$ for all $s,\tau \geq 0$. We restate this now proved result as follows.

Lemma 2.2. *For $s,\tau \geq 0$, the dual space of $Y^{s,\tau}$ is*

$$(Y^{s,\tau})^* = \{\varphi \in \mathcal{M}_{loc}((0,\infty)) : \eta(dx) := (1+x^s)^{-1}(1+x^{-\tau})^{-1}\varphi(dx) \in \mathcal{M}((0,\infty))\}. \tag{2.2}$$

Let us define the space $\mathscr{S}_{s,\tau} := \{f \in C^1(0,\infty)\cap Y^{s,\tau} \;:\; f' \in L^\infty(0,\infty)\}$. In order to prove that S_ν is a (C_0) group on $Y^{s,\tau}$, we need the following lemma.

Lemma 2.3. *The space $\mathscr{S}_{s,\tau}$ is dense in $Y^{s,\tau}$ for all $s,\tau \geq 0$.*

Proof. Note that the map

$$\frac{f(x)}{4} \longrightarrow \frac{f(x)}{(1+x^s)(1+x^{-\tau})}$$

is an isometric isomorphism from $Y^{0,0}$ onto $Y^{s,\tau}$ which leaves invariant $C_c^\infty(0,\infty)$, the smooth functions with compact support in $(0,\infty)$. Therefore $C_c^\infty(0,\infty)$ and $\mathscr{S}_{s,\tau}$ are both dense in $Y^{s,\tau}$ since $C_c^\infty(0,\infty)$ is dense in $Y^{0,0}$. □

Theorem 2.4. *The family S_ν forms a (C_0) group on $Y^{s,\tau}$ for each $s \geq 1$ and $\tau \geq 0$.*

Proof. First we note that the constant function **1** belongs to $Y^{s,\tau}$ if and only if s, τ are both positive. Next, we observe that for $f \in Y^{s,\tau}$ and $t \in \mathbb{R}$,

$$\begin{aligned}\|S_\nu(t)f\|_{s,\tau} &= \sup_{x>0} \frac{|f(\mathrm{e}^{\nu t}x)|}{(1+x^s)(1+x^{-\tau})} \\ &= \sup_{y>0} \left| \frac{|f(y)|}{(1+[\mathrm{e}^{-\nu t}y]^s)(1+[\mathrm{e}^{-\nu t}y]^{-\tau})} \right|.\end{aligned}$$

Suppose $t\nu > 0$. Then

$$\|S_\nu(t)f\|_{s,\tau} \le \mathrm{e}^{\nu ts} \sup_{y>0} \frac{|f(y)|}{(1+y^s)(1+y^{-\tau})} = \mathrm{e}^{\nu ts}\|f\|_{s,\tau}.$$

For $t\nu \le 0$, we have

$$\begin{aligned}\|S_\nu(t)f\|_{s,\tau} &= \sup_{y>0} \left| \frac{|f(y)|}{(1+[\mathrm{e}^{-\nu t}y]^s)(1+[\mathrm{e}^{-\nu t}y]^{-\tau})} \right| \\ &\le \mathrm{e}^{|\nu t|\tau} \sup_{y>0} \frac{|f(y)|}{(1+y^s)(1+y^{-\tau})} = \mathrm{e}^{|\nu t|\tau}\|f\|_{s,\tau}.\end{aligned}$$

Thus $S_\nu(t) : Y^{s,\tau} \mapsto Y^{s,\tau}$ and $\|S_\nu(t)\| \le \mathrm{e}^{\omega|t|}$, $\omega = |\nu| \max\{s, \tau\}$.

Thanks to Lemma 2.3, it is enough to show the strong continuity on $\mathscr{S}_{s,\tau}$. In fact, for any $f \in \mathscr{S}_{s,\tau}$, choose $\chi \in C^\infty(0,\infty)$ such that $\chi(x) = 0$ for $0 \le x \le 1$, χ is increasing on $(1,2)$, and $\chi(x) = 1$ for $x \ge 2$.

Let $f_1 = f\chi$, $f_2 = f(1-\chi)$. Then $f_1, f_2 \in \mathscr{S}_{s,\tau}$, *supp*$f_1 \subset (1,\infty)$, *supp*$f_2 \subset (0,2)$, and $f_1 + f_2 = f$. Now for f_1,

$$\begin{aligned}\|S_\nu(t)f_1 - f_1\|_{s,\tau} &= \sup_{x\ge1} \frac{|f_1(\mathrm{e}^{\nu t}x) - f_1(x)|}{(1+x^s)(1+x^{-\tau})} \\ &\le \|f_1'\|_\infty \sup_{x\ge1} \frac{|\mathrm{e}^{\nu t}x - x|}{1+x^s} \\ &\le \|f_1'\|_\infty |\mathrm{e}^{\nu t} - 1| \to 0, \quad \text{as} \quad t \to 0^+ \text{ since } s \ge 1.\end{aligned}$$

For f_2, we have

$$\|S_\nu(t)f_2 - f_2\|_{s,\tau} \le \|f_2'\|_\infty \sup_{0<x<2} \frac{|\mathrm{e}^{\nu t}x - x|}{1+x^{-\tau}} \le \frac{2^{\tau+1}}{1+2^\tau}\|f_2'\|_\infty|\mathrm{e}^{\nu t} - 1| \to 0,$$

as $t \to 0^+$, and this proves the theorem. □

In the sequel we will need the following result, which is proved in [G1] and [deL, Theorem 11].

Lemma 2.5. *Suppose* iA *generates a strongly continuous group. Let* $p(t) = t^{2n} + q(t)$*, where* q *is a polynomial of degree less than* $2n$*. Then* $-p(A)$ *generates a holomorphic* (C_0) *semigroup of angle* $\pi/2$*.*

Take $A = -\mathrm{i}D_\nu$, so that iA generates a strongly continuous group on $X = Y^{s,\tau}$ and take $p(t) = t^2 - \mathrm{i}\gamma t + r$. Hence we have the following result.

Theorem 2.6. *The operator* $\mathcal{B}$ *defined in* (1.2) *generates a holomorphic* (C_0) *semigroup of angle* $\pi/2$ *on any* $Y^{s,\tau}$*, where* $s \ge 1, \tau \ge 0$*.*

3. The chaotic character of the Black-Scholes semigroup

Let X be a separable complex Banach space.

Definition 3.1. A strongly continuous semigroup (or (C_0) semigroup) $T = \{T(t) : t \geq 0\}$ of bounded linear operators on X is called *hypercyclic* if there exists a vector $x \in X$ such that its orbit $\{T(t)x : t \geq 0\}$ is dense in X, and T is called *chaotic* if in addition the set of periodic points of T,

$$\mathcal{P}_{per} := \{x \in X \ : \ \text{there exists } t_0 > 0 \text{ such that } T(t_0)x = x\},$$

is dense in X.

The notion of chaotic (C_0) semigroups was introduced independently by MacCluer [McC] and Protopopescu and Azmy [P-A]; the first systematic study of this concept is due to Desch, Schappacher and Webb [DSW]. So far, several specific examples of hypercyclic (C_0) semigroups have come up in the literature (see [GE1, GE2] for complete citations).

The following lemma is proved by G. Godefroy and J. Shapiro in [G-S, Corollary 1.5].

Lemma 3.2. *Suppose A is a linear bounded operator on a Banach space X, Q_1, Q_2 are dense subsets of X and $Z : Q_1 \mapsto Q_1$ such that*

(1) $AZy = y$, for all $y \in Q_1$,
(2) $\lim_{n\to\infty} Z^n y = 0$, for all $y \in Q_1$ and
(3) $\lim_{n\to\infty} A^n w = 0$, for all $w \in Q_2$.

Then A is hypercyclic.

Let $s > 1/\nu$, where $\nu > 0$ is given. Denote by

$$S_s = \{\lambda \in \mathbb{C} \ : \ 0 < \operatorname{Re} \lambda < \nu s\} \tag{3.1}$$

the open strip in $\mathbb{C}$ and let $h_\lambda(x) = x^\lambda$. This function is well-defined in $\mathbb{R}^+$ for any $\lambda \in S_s$.

Lemma 3.3. *The function $\lambda \mapsto h_\lambda(x)$ is analytic from S_s into $Y^{s,\tau}$ for each $s\nu > 1$ and $\tau \geq 0$.*

Proof. Note that when $\tau = 0$, any $\psi \in \mathcal{M}(0,\infty)$ cannot have an atom at 0 and $\mathbf{1} \notin Y^{s,0}$. Now, since weak analyticity is equivalent to analyticity, we have only to prove that

$$\lambda \mapsto \int_{(0,\infty)} h(x,\lambda)\psi(dx)$$

is analytic for any $\psi \in F$, where F is a norm-determining subset of $(Y^{s,\tau})^*$. The norm-determining set we use is

$$F := \{c\delta_x \ : \ c \in \mathbb{C}, \ x \in (0,\infty)\},$$

where δ_x denotes the Dirac point mass measure at x. Note that

$$\|f\|_{s,\tau} = \sup\{|cf(x)| = |\langle f, \psi\rangle| \ : \ \psi = c\delta_x, \ c \in \mathbb{C}, \ x \in (0,\infty), \ \|\psi\|_{(Y^{s,\tau})^*} = 1\},$$

and a choice of c that works above is $c = [(1+x^s)(1+x^{-\tau})]^{-1}$, when the supremum defining the norm of f is a maximum attained at x. Furthermore

$$\lambda \mapsto x^\lambda = \mathrm{e}^{(\ln x)\lambda} = \langle h_\lambda(x), \delta_x \rangle$$

is an entire function of $\lambda \in \mathbb{C}$ for all $x > 0$; and for $\lambda \in S_s$ and $x > 0$, $\lim_{x\to\infty} x^\lambda / (1 + x^s) = 0$. Hence $x^\lambda \in Y^{s,\tau}$ for all $\tau > 0$. □

If a linear operator L generates a (C_0) group on a Banach space X, then some polynomials in L (such as $L^2 + \alpha L + \beta I$ for arbitrary scalars α, β) generate (C_0) semigroups on X. For the operator $\mathcal{B}$, defined in (1.2), the Black-Scholes semigroup can be represented by $T(t) := f(D_\nu)$, where

$$f(z) = \mathrm{e}^{tg(z)} \quad \text{with } g(z) = z^2 + \gamma z - r. \tag{3.2}$$

According to Theorem 2.6 this (C_0) semigroup is well defined for each $t \in \mathbb{C}$, with $\mathrm{Re}(t) > 0$. These operators will be shown to be chaotic on $X = Y^{s,\tau}$ for $s > 1, \tau \geq 0$ when $s\nu > 1$. We begin by recalling the following lemma, which was proved in [DSW] and [dL-E], and we reproduce this proof in our case.

Lemma 3.4. *Suppose that there exists a set $\Omega \subset S_s$ which has an accumulation point in S_s. Then*

$$Q := \mathrm{Span}\{h_\lambda \ : \ \lambda \in \Omega\}$$

is dense in $Y^{s,\tau}$ for $s > 1$, $\tau \geq 0$.

Proof. Suppose $\psi \in Q^\perp$. Since ψ belongs to the dual of $Y^{s,\tau}$ and $h_\lambda = x^\lambda \in Y^{s,\tau}$, Lemma 3.3 asserts that $p(\lambda) = \langle \psi, x^\lambda \rangle$ is well defined and $p(\lambda)$ is analytic in S_s. Since $p(\lambda) = 0$ for all $\lambda \in \Omega$, which is a set with an accumulation point, then $p = 0$ in all of S_s and so $\psi = 0$, as desired. □

We continue to work in the spaces $Y^{s,\tau}$, $s > 1$, $\tau \geq 0$ with $s\nu > 1$.

Lemma 3.5. *Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and $\mathbb{T}$, the unit circle, be its boundary. The set $f(S_s) \cap \mathbb{T}$ is nonempty and possesses infinitely many accumulation points in the strip S_s, where f is as in* (3.2).

Proof. For $f(z) = \mathrm{e}^{tg(z)}$ with $t > 0$, in order to have $f(S_s) \cap \mathbb{T} \neq \emptyset$ we must find $z \in S_s$ such that

$$\mathrm{Re}\, g(z) = \mathrm{Re}(\nu^2 z^2 + (r - \nu^2)z - r) = \nu^2(x^2 - y_0^2 - x) + rx - r = 0$$

with $z = x + \mathrm{i}y_0$. Equivalently, we must find (x, y_0) with $0 < x < \nu s, y_0 \in \mathbb{R}$ such that

$$x^2 + (\frac{r}{\nu^2} - 1)x - \frac{r}{\nu^2} = y_0^2. \tag{3.3}$$

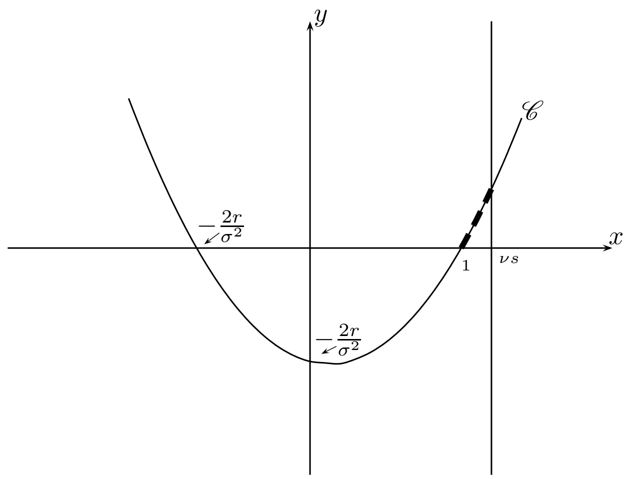

FIGURE 1

Call $\mathscr{C}$ the curve represented by the graph of the quadratic function $y = x^2 + (\frac{r}{\nu^2} - 1)x - \frac{r}{\nu^2}$. As Figure 1 shows, for $1 < x < \nu s$, there are uncountably many points (x, y) on the dashed portion of $\mathscr{C}$ with $y > 0$. For each such point let $y_0 = \sqrt{y}$. Then this gives uncountably many solutions of (3.3). $\square$

Now we can prove our main theorem.

Theorem 3.6. *The Black-Scholes* (C_0) *semigroup* T *is chaotic in* $Y^{s,\tau}$ *for each* $s > 1, \tau \geq 0$ *with* $s\nu > 1$.

Proof. First let us prove that the (C_0) semigroup $T = \{T(t) = f(D_\nu) = e^{t\mathcal{B}} \ : \ t \geq 0\}$ is hypercyclic. For this we will use Lemma 3.2, taking

$$\Omega_1 = \{\lambda \in \frac{1}{\nu}S_s \ : \ |f(\nu\lambda)| > 1\}, \quad \Omega_2 = \{\lambda \in \frac{1}{\nu}S_s \ : \ |f(\nu\lambda)| < 1\}$$

and

$$Q_j := \text{Span}\{h_\lambda \ : \ \lambda \in \Omega_j\} \quad \text{for } j = 1, 2.$$

Now, let $z_0 \in f(S_s) \cap \mathbb{T}$, since f is holomorphic and nonconstant, $f(S_s)$ is an open set, and $\Omega_1 = f(S_s) \cap \{z \in \mathbb{C} \ : \ |z| > 1\}$ and $\Omega_2 = f(S_s) \cap \{z \in \mathbb{C} \ : \ |z| < 1\}$ are also open, and any point in Ω_j is an accumulation point. So according to Lemma 3.4, Q_j is dense in $Y^{s,\tau}$ for $j = 1, 2$.

Let $A = f(D_\nu)$ and define $Z = (f(D_\nu))^{-1}$ on Q_1 so that

$$Z\left(\sum_{k=1}^{N} \alpha_k h_{\lambda_k}\right) = \sum_{k=1}^{N} \alpha_k \left(f(\nu\lambda_k)\right)^{-1} h_{\lambda_k}$$

for $\lambda_k \in \Omega_1, \alpha_k \in \mathbb{C}$ and $N \in \mathbb{N}$. It is clear that for any $y = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_1$, we have $AZy = y$. Furthermore for $\lambda_k \in \Omega_1, |f(\nu\lambda_k)| > 1$, and consequently

$$\lim_{n\to\infty} Z^n y = \lim_{n\to\infty} \sum_{k=1}^{N} \alpha_k \left(f(\nu\lambda_k)\right)^{-n} h_{\lambda_k} = 0.$$

Finally, for $w = \sum_{k=1}^{N} \alpha_k h_{\lambda_k} \in Q_2$ with $|f(\nu\lambda_k)| < 1$ for each k,

$$\lim_{n\to\infty} A^n y = \lim_{n\to\infty} \sum_{k=1}^{N} \alpha_k f(\nu\lambda_k)^n h_{\lambda_k}(x) = 0.$$

These imply that the hypotheses of the Godefroy-Shapiro Lemma 3.2 are satisfied and A is hypercyclic.

To see that $T(t) = f(D_\nu)$ is chaotic, we define $\Omega_3 = \{\lambda \in \frac{1}{\nu} S_s \ : \ f(\nu\lambda) \in \mathrm{e}^{2\pi \mathrm{i}\mathbb{Q}}\}$ and $Q_3 := \mathrm{Span}\{h_\lambda \ : \ \lambda \in \Omega_3\}$. Q_3 is contained in the set of all periodic points of $A = f(D_\nu)$. Suppose $f(\nu\lambda_k) = \mathrm{e}^{2\pi \mathrm{i} n_k/m_k}$. Then for $y = \sum_{k=1}^{N} \alpha_k h_{\lambda_k}$ and $m = \prod_{k=1}^{N} m_k$, one has $f(D_\nu)^m y = y$. So the set of all periodic points $\mathcal{P}_{per}$ of $f(D_\nu)$ is dense, and consequently $T(t)$ is chaotic. □

The real-world applications of (FBS) require nonnegative initial data and nonnegative solutions. The above proof that the Black-Scholes semigroup T is chaotic uses holomorphic functions and thus requires the use of spaces of complex-valued functions. Theorem 3.6 would be more satisfying from an applied standpoint if it were valid for real functions. This is precisely the content of the next result.

Let $Y^{s,\tau}_{\mathbb{R}}$ be the real functions in $Y^{s,\tau}$. This is a real Banach space. If $f \in Y^{s,\tau}$, then by [GMR, eq. (17)], the solution of (FBS) is given by

$$v(x,t) = (T(t)f)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2/(4t)} f\left(x \mathrm{e}^{(r-\sigma^2/2)t - (\sigma/\sqrt{2})y}\right) dy.$$

Thus $T(t)f$ is real (resp., nonnegative) for each $t \geq 0$ if and only if f is real (resp. nonnegative). Let S_T be the restriction of T to $Y^{s,\tau}_{\mathbb{R}}$. Then $S_T = \{S_T(t) : t \geq 0\}$ is a (C_0) semigroup on $Y^{s,\tau}_{\mathbb{R}}$ for $s \geq 1$, $\tau \geq 0$.

Theorem 3.7. *The semigroup S_T on $Y^{s,\tau}_{\mathbb{R}}$ is chaotic if $s > 1$ and $\tau \geq 0$ when $s\nu > 1$.*

Proof. Let $f \in Y^{s,\tau}$ be given, where $s > 1$ with $s\nu > 1$, and $\tau \geq 0$. Let $g \in Y^{s,\tau}$ have a dense $T-$orbit. Then there is a sequence of times $t_n \to \infty$ such that $\|T(t_n)g - f\|_{s,\tau} \to 0$ as $n \to \infty$. Consequently, since $\mathrm{Re}(T(t)h) = T(t)(\mathrm{Re}(h))$ for all $h \in Y^{s,\tau}$,

$$\begin{aligned} \|S_T(t_n)(\mathrm{Re}(g)) - f)\|_{s,\tau} &\leq \left\| \sqrt{[S_T(t_n)(\mathrm{Re}(g)) - f]^2 + [S_T(t_n)(\mathrm{Im}(g))]^2} \right\|_{s,\tau} \\ &= \|[\mathrm{Re}(T(t_n)g - f] + \mathrm{i}[\mathrm{Im}(T(t_n)g)]\|_{s,\tau} \\ &= \|T(t_n)g - f\|_{s,\tau} \to 0 \end{aligned}$$

as $n \to \infty$. It follows that S_T is hypercyclic.

Next, if f is periodic of period p, then so are $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. Thus S_T has a dense set of periodic points since T does. The theorem follows. □

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References

[AdP] W. Arendt and B. de Pagter, Spectrum and asymptotics of the Black-Scholes partial differential equation (L^1, L^∞)-interpolation spaces. *Pacific J. Math.* **202** (2002), 1-36. MR1883968 (2002m:35087)

[B-S] F. Black and M. Scholes, The pricing of options and corporate liabilities. *J. Polit. Econ.* **81** (1973), 637-654.

[C-G1] D. I. Cruz-Báez and J.M. González-Rodríguez, Semigroup theory applied to options. *J. Appl. Math.* **2** (2002), 131-139. MR1915662 (2003i:47042)

[C-G2] D. I. Cruz-Báez and J.M. González-Rodríguez, A semigroup approach to American options. *J. Math. Anal. Appl.* **302** (2005), 157-165. MR2107354 (2006b:91066)

[deL] R. deLaubenfels, Polynomials of generators of integrated semigroups. *Proc. Amer. Math. Soc.* **107** (1989), 197-204. MR975637 (90a:47100)

[dL-E] R. deLaubenfels and H. Emamirad, Chaos for functions of discrete and continuous weighted shift operators. *Ergodic Theory Dynam. Systems.* **21** (2001), 1411-1427. MR1855839 (2002j:47030)

[DSW] W. Desch, W. Schappacher and G. F. Webb, Hypercyclic and chaotic semigroups of linear operators. *Ergodic Theory Dynam. Systems* **17** (1997), 793-819. MR1468101 (98j:47083)

[G-S] G. Godefroy and J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds. *J. Funct. Anal.* **98** (1991), 229-269. MR1111569 (92d:47029)

[G1] J. A. Goldstein, Some remarks on infinitesimal generators of analytic semigroups. *Proc. Amer. Math. Soc.* **22** (1969), 91-93. MR0243384 (39:4706)

[G2] J. A. Goldstein, Abstract evolution equations. *Trans. Amer. Math. Soc.* **141** (1969), 159-185. MR0247524 (40:789)

[G3] J. A. Goldstein, *Semigroups of Linear Operators and Applications.* Oxford University Press, New York, 1985. MR790497 (87c:47056)

[GMR] J. A. Goldstein, R. M. Mininni and S. Romanelli, A new explicit formula for the solution of the Black-Merton-Scholes equation, in *Infinite Dimensional Stochastic Analysis* (ed. by A. Sengupta and P. Sundar), World Series Publ., 2008, 226-235. MR2412891 (2009k:60132)

[GE1] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators. *Bull. Amer. Math. Soc. (N.S.)* **36** (1999), 345-381. MR1685272 (2000c:47001)

[GE2] K.-G. Grosse-Erdmann, Recent developments in hypercyclicity. *RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.* **97** (2003), 273-286. MR2068180 (2005c:47010)

[McC] C. R. MacCluer, Chaos in linear distributed systems. *J. Dynam. Systems Measurement Control* **114** (1992), 322-324.

[M-T] M. Matsui and F. Takeo, Chaotic semigroups generated by certain differential operators of order 1. *SUT J. Math.* **37** (2001), 51-67. MR1849367 (2002f:47087)

[P-A] V. Protopopescu and Y. Y. Azmy, Topological chaos for a class of linear models. *Math. Models Methods Appl. Sci.* **2** (1992), 79-90. MR1159477 (93c:58126)

[Tak] F. Takeo, Chaos and hypercyclicity for solution semigroups to some partial differential equations. *Nonlinear Analysis* **63** (2005), 1943-1953.

Laboratoire de Mathématiques, Université de Poitiers, teleport 2, BP 179, 86960 Chasseneuil du Poitou, Cedex, France

E-mail address: emamirad@math.univ-poitiers.fr

Department of Mathematical Sciences, The University of Memphis, Memphis, Tennessee 38152

E-mail address: ggoldste@memphis.edu

Department of Mathematical Sciences, The University of Memphis, Memphis, Tennessee 38152

E-mail address: jgoldste@memphis.edu