CONSERVATION OF ULTRASPHERICAL CONVOLUTION

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ABSTRACT. The space of pseudomeasures $\mathcal{F}L^\infty$ lies between the space of Radon measures and distributions; $\mathcal{M} \subset \mathcal{F}L^\infty \subset (C_c^\infty)'$. The theory of strongly continuous semigroups id generalized by J.L. Lions [9] to distribution semigroups (DSG) $\mathscr{G} \in \mathscr{L}(C_c^\infty, \mathscr{L}(X))$, where $\mathscr{L}(X)$ is the algebra of linear continuous operators on a Banach space X, which satisfies the conservation of convolution $\mathscr{G}(\varphi * \psi) = \mathscr{G}(\varphi)\mathscr{G}(\psi)$ for all $\varphi, \psi \in C_c^\infty(\mathbb{R})_+$. In this paper we define the subspace Π_ν of $\mathscr{L}(L^2([-1,1], \mu_\nu))$ of pseudomeasure operators T, which satisfy $T(g *_\nu h) = (Tg) *_\nu h$ for any $g, h \in L^2(I, \mu_\nu)$. We prove that T^2 conserves the generalized convolution $T^2(f *_\nu g) = T(f) *_\nu T(g)$ and finally we study the different properties of DSG with different type of endomorphism $\mathscr{G} = T^2$.

1. INTRODUCTION

This paper concerns with the homomorphisms which map from a convolution algebra $\mathfrak{a}$ into an algebra $\mathcal{A}$. Well-known examples of such a situation are given by J.L. Lions in [9] and J. Faraut in [5]. In [9], Lions takes $\mathfrak{a} := C_c^\infty(\mathbb{R})$, $\mathcal{A} := \mathscr{L}(X)$ the algebra of bounded linear operators on the Banach space X and the homomorphism $\mathscr{G} : \mathfrak{a} \mapsto \mathcal{A}$ satisfies $\mathscr{G}(\varphi) = 0$ whenever φ has its support in $(-\infty, 0]$ and satisfies

$$\mathscr{G}(\varphi * \psi) = \mathscr{G}(\varphi)\mathscr{G}(\psi) \quad \text{for all } \ \varphi, \psi \in C_c^\infty(\mathbb{R})_+. \tag{1.1}$$

By imposing the extra conditions on $\mathscr{G}$, he defines $\mathscr{G}(T)$, for any distribution $T \in \mathfrak{a}'$ as an unbounded operator. In [5], Faraut takes $\mathfrak{a} := \mathcal{M}(\mathbb{R}_+)$, the space of Radon measures on $\mathbb{R}_+$ and $\mathcal{A} := \mathscr{L}(X)$. Here $\mathscr{G}(\mu_t)$ is defined for a family $\{\mu_t\}_{t\geq 0}$ of the measures semigroup on $\mathbb{R}_+$. In this case $\mathscr{G}(\mu_t)$ satisfies

$$\mathscr{G}(\mu_t * \mu_s) = \mathscr{G}(\mu_t)\mathscr{G}(\mu_s) \quad \text{for all } \ \mu_t, \mu_s \ \text{ satisfying } \ \mu_t * \mu_s = \mu_{t+s}. \tag{1.2}$$

Now the following problem arises. Is it possible to carry out the same formalism for the space of pseudomeasures $\mathcal{P}$ which is a subset of distributions $(C_c^\infty(\mathbb{R}))'$ and bigger than $\mathcal{M}(\mathbb{R}_+)$. In fact if we denote $\mathcal{F}L^1$ the space of distribution on $\mathbb{R}$ which are Fourier transform of functions in $L^1(\mathbb{R})$. According to Riemann-Lebesgue Theorem if $f \in L^1(\mathbb{R})$, then its Fourier transform $\widehat{f}$ is a continuous

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function vanishing at infinity. Hence $C_c^\infty(\mathbb{R}) \subset \mathcal{F}L^1 \subset C_0(\mathbb{R})$. Since the dual of $C_0(\mathbb{R})$ is $\mathcal{M}(\mathbb{R})$, the dual of $\mathcal{F}L^1$ (which can be identified with $\mathcal{F}L^\infty$) is referred to as the space of pseudomeasures (see [11] for more information on this space).

The same formalism is carried out for Fourier series or orthogonal systems (see [10] or [13] for details). In the next section we will define the ultraspherical polynomials $Q_n^\nu(x)$. Let $\mathfrak{a}_\nu$ be the space of the continuous functions on I := [-1, 1] of the form

$$f = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu \tag{1.3}$$

where $\widehat{f}(n) = \langle f, Q_n^\nu \rangle := \int_I f Q_n^\nu d\mu_\nu$ and

$$\omega_n^\nu = \frac{(n+\nu)\Gamma(n+2\nu)}{2^{2\nu-1}\Gamma(\nu+1/2)^2\Gamma(n+1)}. \tag{1.4}$$

We will see in Lemma 2.2 that $\mathfrak{a}_\nu$ equipped with the norm

$$\|f\|_\nu = \sum_{n\geq 0} \omega_n^\nu |\widehat{f}(n)| \tag{1.5}$$

is a commutative, semisimple Banach algebra for the simple product of functions.

Definition 1.1. *By a pseudomeasure is meant a distribution $g \in C^\infty(I)'$ such that $\widehat{g}(n) \in \ell^\infty(\mathbb{N})$. We denote by $\mathfrak{p}_\nu$ the space of all pseudomeasures.*

As in the Fourier transform case, we can identify the space of pseudomeasures with $\mathfrak{a}_\nu'$, the dual of $\mathfrak{a}_\nu$. As in [8], on $\mathfrak{a}_\nu$ we introduce the generalized convolution product denoted $*_\nu$, by mean of a kernel called *Gegenbauer's kernel.*

In the section 3, we will define a pseudomeasure operator T as an element of the Banach algebra $\mathscr{L}(L^2(I, \mu_\nu))$ such that $T(g *_\nu h) = (Tg) *_\nu h$ for any $g, h \in L^2(I, \mu_\nu)$. Then we prove the consistency of this definition and we justify this designation. Then we prove in order that the homomorphism $\mathscr{G} \in \mathscr{L}(\mathfrak{a}_\nu)$ satisfies the identity

$$\mathscr{G}(f *_\nu g) = Tf *_\nu Tg \quad \text{for all} \quad f, g \in \mathfrak{a}_\nu, \tag{1.6}$$

it is necessary and sufficient that $\mathscr{G} = T^2$.

In the last section we compare the different properties of T^2 in the framework of the theory of distribution semigroups (DSG). In the theory of DSG, if $\mathscr{G} \in \mathscr{L}(C_c^\infty(\mathbb{R})_+, \mathscr{L}(X))$ and satisfies what Lions called the assumptions of regular distribution semigroup, that is, for any $\varphi \in C_c^\infty(\mathbb{R})_+$ satisfying $\int_0^\infty \varphi(t)dt = 1$ we have

$$s - \lim_{s\to 0} \mathscr{G}(\varphi_s) = I, \quad \text{where} \quad \varphi_s(t) = \frac{1}{s}\varphi\left(\frac{t}{s}\right). \tag{1.7}$$

From this property he deduced that

(a)

$$\bigcup_{\varphi \in C_c^\infty(\mathbb{R})_+} \mathrm{Im}\mathscr{G}(\varphi) \text{ is dense in } X \tag{1.8}$$

(b)

$$\bigcap_{\varphi \in C_c^\infty(\mathbb{R})_+} \mathrm{Ker}\,\mathscr{G}(\varphi) = \{0\} \tag{1.9}$$

In [1, 2] the authors obtained the assertions (4.3) and (4.4) without assuming the regular conditions of Lions, only by introducing the notion of smooth distribution semigroup. This notion is redefined by P. J. Miana [12] by introducing the Weyl's fractional calculus. In the last section we prove as (1.7) the approximation of identity for admissible pseudomeasure operator T, where $\Phi_T(Q_n^\nu)$ has only a finite number of zeros. This gives readily (4.3) and (4.4) type assertions

2. Preliminaries

It is well-known that the orthogonal polynomials satisfying a second order differential equation of the form $Lu+\lambda_n u = 0$ where the operator L is a second order differential operator and λ_n's are the correspondent eigenvalues. For example for the Jacobi polynomials $P_n^{\alpha,\beta}(x)$, $L := (1-x^2)\frac{d^2}{dx^2} + [(\beta-\alpha) - (\alpha+\beta+2)x]\frac{d}{dx}$ and $\lambda_n = n(n+\alpha+\beta+1)$ in the Hilbert space $L^2([-1,1],(1-x)^\alpha(1+x)^\beta dx)$. These polynomials satisfy the Rodrigues' formula

$$(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{2.1}$$

Throughout this paper, for the sake of simplicity, we take $\beta = \alpha = \nu - \dfrac{1}{2}$, which corresponds to the ultraspherical (or Gegenbauer) polynomials Q_n^ν, but the same analysis can be readily generalized to the Jacobi polynomials. In order to respect the normalization we will use the customary notation for

$$\frac{\Gamma(n+\nu+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})\Gamma(n+1)} Q_n^\nu(x) = P_n^{\alpha,\beta}(x). \tag{2.2}$$

For $\nu > -1/2$, in $L^2(I,\mu_\alpha)$ the set of ultraspherical polynomials

$$Q_n^\nu(x) = \frac{(-1)^n}{2^n(\nu-1/2+1)...(\nu-1/2+n)}(1-x^2)^{-\nu+1/2}\frac{d^n}{dx^n}(1-x^2)^{\nu-1/2+n}$$

forms an orthogonal system on I for the measure $d\mu_\nu(x) = (1-x^2)^{\nu-1/2}dx$ (cf. [4]), with the inner product relation

$$\langle Q_m^\nu, Q_n^\nu\rangle_\nu = \int_I Q_m^\nu(x) Q_n^\nu(x) d\mu_\nu(x) = \begin{cases} 0, & if \quad n \neq m \\ (\omega_n^\nu)^{-1}, & if \quad n = m. \end{cases}$$

The well-known properties of these polynomials are:

(i) $Q_n^\nu(1) = 1$; (the normalization condition)
(ii) $Q_0^\nu(x) = 1, Q_1^\nu(x) = x$;
(iii) $Q_n^\nu(-x) = (-1)^n Q_n^\nu(x)$;
(iv) $|Q_n^\nu(x)| < 1$ if $\nu > -\frac{1}{2}, |x| < 1$ and $n \geq 1$;
(v) $xQ_n^\nu(x) = \frac{n}{2n+2\nu} Q_{n-1}^\nu(x) + \frac{n+2\nu}{2n+2\nu} Q_{n+1}^\nu(x)$ for $n \geq 1$.

Now, let us define the space $L^p(I, d\mu_\nu)$ of the real valued functions f, such that

$$\|f\|_p := \left(\int_I |f(x)|^p d\mu_\nu(x) \right)^{1/p} < \infty. \tag{2.3}$$

For $p = 2$ the inner product of $L^2(I, d\mu_\nu)$ is designated by

$$\langle f, g \rangle := \int_I f(x) g(x) d\mu_\nu(x).$$

For any $f, g \in L^1(I, d\mu_\nu)$, we can define the generalized convolution product of f by g by

$$(f *_\nu g)(x) = \int_I \int_I G_\nu(x, y, z) f(y) g(z) d\mu_\nu(y) d\mu_\nu(z), \tag{2.4}$$

where the Gegenbauer kernel $G_\nu(x, y, z)$ is defined by

$$G_\nu(x, y, z) = \frac{2^{1-2\nu}(1 - x^2 - y^2 - z^2 + 2xyz)_+^{\nu-1}}{\Gamma(\nu)^2 (1 - z^2)^{\nu-1/2} (1 - x^2)^{\nu-1/2} (1 - y^2)^{\nu-1/2}} \quad \text{for } \nu > 0, \tag{2.5}$$

where by f_+ we designate the positive part of the function f.

It is worthwhile to mention that this kernel produces also the following multiplication formula

$$Q_n^\nu(x) Q_n^\nu(y) = \int_I G_\nu(x, y, z) Q_n^\nu(z) d\mu_\nu(z), \quad (-1 < x, y < 1) \tag{2.6}$$

(see [8]). As a consequence of the above formula we get

$$(Q_n^\nu *_\alpha Q_m^\nu)(x) = \begin{cases} (\omega_n^\nu)^{-1} Q_m^\nu(x), & if \quad n = m \\ 0, & if \quad n \neq m. \end{cases} \tag{2.7}$$

(see [3] for the proof). The following Lemma is proved in [3].

Lemma 2.1. *Suppose $f, g \in L^2(I, \mu_\nu)$, then*

$$\widehat{f *_\nu g}(n) = \widehat{f}(n) \widehat{g}(n). \tag{2.8}$$

In [6] and [7], G. Gasper discusses this construction not only for the ultraspherical polynomials, but also for the Jacobi polynomials, where he shows the following Lemma.

Lemma 2.2. *(a) $L^1(I,\mu_\nu)$ endowed with the convolution product $*_\nu$ is a semisimple commutative Banach algebra.*

(b) For $1 \le p,q,r \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, if $f \in L^p(I,\mu_\nu)$ and $g \in L^q(I,\mu_\nu)$, then $f *_\nu g \in L^r(I,\mu_\nu)$, and one has $\|f *_\nu g\|_r \le \|f\|_p \|g\|_q$.

3. Pseudomeasure operators and the main result.

Definition 3.1. Let Π_ν be the subspace of the Banach algebra $\mathscr{L}(L^2(I,\mu_\nu))$ we say that $T \in \Pi_\nu$ is a *pseudomeasure operator* iff

$$T(g *_\nu h) = (Tg) *_\nu h \quad \text{for any } \ g,h \in L^2(I,\mu_\nu). \tag{3.1}$$

In order to prove the consistency of the above formula we need the following Lemma which is proved in [3].

Lemma 3.2. (a) *Let $g,h \in L^2(I,\mu_\nu)$, then $g *_\nu h \in L^2(I,\mu_\nu) \cap \mathfrak{a}_\nu$;*
(b) *For any $f \in \mathfrak{a}_\nu$ there exist $g,h \in L^2(I,\mu_\nu)$, such that $f = g *_\nu h$.*

The elements of Π_ν are called pseudomeasure operators; the following Lemma justifies this designation.

Lemma 3.3. *For any $T \in \Pi_\nu$ there exists a unique pseudomeasure $\Phi_T \in \mathfrak{a}'_\nu$ such that*

$$\Phi_T(g *_\nu h) = \langle Tg, h\rangle \quad \textit{for any } \ g,h \in L^2(I,\mu_\nu). \tag{3.2}$$

and this correspondence is an isometric isomorphism from Π_ν onto $\mathfrak{a}'_\nu$.

Proof. Let $T \in \Pi_\nu$ and $f \in \mathfrak{a}_\nu$. From Lemma 3.2 it follows that there exist $g,h \in L^2(I,\mu_\nu)$ such that $f = g *_\nu h$ and so $Tf = Tg *_\nu h \in \mathfrak{a}_\nu$. Consequently Tf is a continuous function on $[-1,1]$. Thus, one can define the linear form Φ_T by

$$\Phi_T(f) = [Tf](1). \tag{3.3}$$

Since $G_\nu(1,y,z)$ is a reproducing kernel on $L^2(I,\mu_\nu)$, i.e. $\int_I G_\nu(1,y,z)h(z)d\mu_\nu(z) = h(y)$, for all $h \in L^2(I,\mu_\nu)$, we have

$$[g *_\nu h](1) = \int_I \left(\int_I G_\nu(1,y,z)h(z)d\mu_\nu(z) \right) g(y)d\mu_\nu(y) =< g,h >,$$

therefore

$$\Phi_T(g *_\nu h) = [T(g *_\nu h)](1) = [Tg *_\nu h](1) =< Tg,h >,$$

regarding Lemma 3.2 (b);

$$|\Phi_T(f)| = |\Phi_T(g *_\nu h)| = | < Tg,h > | \le \|Tg\|_2 \|h\|_2 \le \|T\|_{\Pi_\nu} \|g\|_2 \|h\|_2 = \|T\|_{\Pi_\nu} \|f\|_\nu$$

which implies the boundedness of Φ_T by $\|T\|_{\Pi_\nu}$.

Conversely, if $\Phi \in \mathfrak{a}'_\nu$, then for any $g,h \in L^2(I,\mu_\nu)$, one has

$$|\Phi(g *_\nu h)| \le \|\Phi\|_{\mathfrak{a}'_\nu} \|g *_\nu h\|_\nu \le \|\Phi\|_{\mathfrak{a}'_\nu} \|g\|_2 \|h\|_2.$$

Thus, $(g,h) \mapsto \Phi(g *_\nu h)$ is a bilinear form on $L^2(I,\mu_\nu)$, therefore there exists $T_\Phi \in \mathscr{L}(L^2(I,\mu_\nu))$ such that $\Phi(g *_\nu h) = < h, T_\Phi g >$ and $\|T\|_\Phi \le \|\Phi\|_{\mathfrak{a}'_\nu}$.

It is not hard to see that, for any f, g and $h \in L^2(I,\mu_\nu)$ we have $< f, g *_\nu h > = < f *_\nu g, h >$ and consequently

$$\begin{aligned} < h, T_\Phi(g *_\nu h) > &= \Phi(h *_\nu (f *_\nu g)) = \Phi((h *_\nu f) *_\nu g) \\ &= < h *_\nu g, T_\Phi f > = < h, (T_\Phi f) *_\nu g >, \end{aligned}$$

which implies that T_Φ commutes with the generalized convolution products from left, and accordingly $T_\Phi \in \Pi_\nu$.

The fact that

$$\Phi_{T_\Phi}(g *_\nu h) = [(T_\Phi g) *_\nu h](1) = < T_\Phi g, h > = \Phi(g *_\nu h)$$

for any $g, h \in L^2(I,\mu_\nu)$ implies that $\Phi_{T_\Phi} = \Phi$. Hence by $\|\Phi_T\|_{\mathfrak{a}'_\nu} \le \|T\|_{\Pi_\Phi}$ and $\|T\|_\Phi \le \|\Phi\|_{\mathfrak{a}'_\nu}$ the mapping $T \mapsto \Phi_T$ is an isometric isomorphism. □

The following corollary will be useful in the sequel.

Corollary 3.4. *Suppose $T \in \Pi_\nu$. For any $f \in \mathfrak{a}_\nu, Tf \in \mathfrak{a}_\nu$ and $\widehat{Tf}(n) = \Phi_T(Q_n^\nu)\widehat{f}(n)$.*

Proof. In the proof of the Lemma 3.3 we have shown that if $f \in \mathfrak{a}_\nu$, then $Tf \in \mathfrak{a}_\nu$. From 2.7 and 3.2 it follows that

$$< T(Q_n^\nu), Q_m^\nu > = \Phi_T(Q_n^\nu *_\nu Q_m^\nu) = \begin{cases} (\omega_n^\nu)^{-1}\Phi_T(Q_n^\nu), & if \quad n = m \\ 0, & if \quad n \neq m, \end{cases}$$

and consequently $T(Q_n^\nu) = \Phi_T(Q_n^\nu)Q_n^\nu$. This shows that

$$\begin{aligned} \widehat{Tf}(n) &= \int_I (Tf)Q_n^\nu d\mu_\nu = \int_I \sum_{m\ge 0} \omega_m^\nu \widehat{f}(m) T(Q_m^\nu) Q_n^\nu d\mu_\nu \\ &= \omega_m^\nu \int_I \sum_{m\ge 0} \widehat{f}(m)\Phi_T(Q_m^\nu)Q_m^\nu Q_n^\nu d\mu_\nu = \Phi_T(Q_n^\nu)\widehat{f}(n). \end{aligned}$$

□

Example 3.5. Let us denote by P_j the projection on $\mathfrak{a}_\nu$. For any $f = \sum_{n\ge 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu \in \mathfrak{a}_\nu$, $P_j(f) = \omega_j^\nu \widehat{f}(j) Q_j^\nu$ belongs to Π_ν. In fact for $f = g *_\nu h$, we have

$$P_j(f) = \omega_j^\nu \widehat{f}(j) = \omega_j^\nu \widehat{g}(j)\widehat{h}(j) \qquad \text{(according to (2.8))} \tag{3.4}$$

$$= \omega_j^\nu \widehat{g}(j) Q_j^\nu *_\nu \sum_{n\ge 0} \omega_n^\nu \widehat{h}(n) Q_n^\nu(z) \qquad \text{(according to (2.7))} \tag{3.5}$$

$$= (P_j g) *_\nu h. \tag{3.6}$$

By using (3.3), we have also by normalization condition

$$\Phi_{P_j}(f) = P_j(f)(1) = \omega_j^\nu \widehat{f}(j) Q_j^\nu(1) = \omega_j^\nu \widehat{f}(j). \tag{3.7}$$

and

$$\Phi_{P_j}(Q_j^\nu) = \omega_j^\nu \widehat{Q}_j^\nu(j) = 1. \tag{3.8}$$

and for $n \neq j$, $\Phi_{P_j}(Q_n^\nu) = 0$.

Example 3.6. The identity is also a pseudomeasure operator since $I(f *_\nu g) = (If) *_\nu g$ and $\Phi_I(Q_n^\nu) = 1$ for all $n \in \mathbb{N}$

Concerning the pseudomeasure operators we have also the following characterization.

Lemma 3.7. *For any $\{\beta_n\} \in \ell^\infty(\mathbb{N})$, the operator T defined by*

$$Tf = \sum_{n \geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu(x),$$

where $f = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu(x)$, belongs to Π_ν. Conversely, if $T \in \Pi_\nu$, there exists a unique sequence $\{\beta_n\} \in \ell^\infty(\mathbb{N})$, such that $Tf = \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu(x)$, for any $f = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu(x)$, and we have $\|T\|_{\Pi_\nu} = \|\{\beta_n\}\|_\infty$.

Proof. It suffices to prove that T satisfies

$$T[f *_\nu g] = [Tf] *_\nu g \qquad \forall f, g \in L^2(I, \mu_\nu)$$

From the definition of T it follows that

$$\begin{aligned}
T[f *_\nu g](x) &= \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f *_\nu g}(n) Q_n^\nu(x) \\
&= \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) \widehat{g}(n) Q_n^\nu(x) \\
&= \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu(x) \int_I Q_n^\nu(z) g(z) d\mu_\nu(z) \\
&= \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) \int_I \int_I G_\nu(x,y,z) Q_n^\nu(y) g(z) d\mu_\nu(z) d\mu_\nu(y) \quad \textit{(by (2.6))} \\
&= \left[\sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu\right] *_\nu g(x) \\
&= [Tf] *_\nu g(x).
\end{aligned}$$

Conversely, let $T \in \Pi_\nu$ and $\Phi_T \in \mathfrak{a}'_\nu$ be the corresponding pseudomeasure given by Lemma 3.3. By designating $\beta_n := \Phi_T(Q_n^\nu)$ for any $f = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu \in \mathfrak{a}_\nu$, we have $\Phi_T(f) = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n)\Phi_T(Q_n^\nu) = \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n)$ and consequently

$$\|\Phi_T\|_{\mathfrak{a}'_\nu} = \sup_{\|f\|_{\mathfrak{a}_\nu}\leq 1} |\Phi_T(f)| = \sup_{\sum_{n\geq 0}\omega_n^\nu|\widehat{f}(n)|\leq 1} |\sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n)| = \sup_{n\geq 0} |\beta_n|.$$

Furthermore it follows from Corollary 3.4 that for any $f \in \mathfrak{a}_\nu$, $Tf \in \mathfrak{a}_\nu$ and

$$Tf = \sum_{n\geq 0} \omega_n^\nu \widehat{Tf}(n) Q_n^\nu = \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu.$$

□

Now, we can announce our main result.

Theorem 3.8. *Let $\mathscr{G} \in \mathscr{L}(\mathfrak{a}_\nu)$: There exist a pseudomeasure operator $T \in \Pi_\nu$, such that one of the following equivalent assertions is fulfilled*

(a)

$$\mathscr{G}(g *_\nu h) = (Tg) *_\nu (Th) \quad \text{for any } g, h \in L^2(I, \mu_\nu); \tag{3.9}$$

(b)

$$\mathscr{G} = T^2; \tag{3.10}$$

(c)

$$\widehat{\mathscr{G}(f)}(n) = \Phi_T^2(Q_n^\nu)\widehat{f}(n). \tag{3.11}$$

Proof. Let $\mathscr{G} \in \mathscr{L}(\mathfrak{a}_\nu)$, such that there exists a pseudomeasure operator $T \in \Pi_\nu$, satisfying (3.9). Then according to Lemma 3.7, there exists $\{\beta_n\} \in \ell^\infty(\mathbb{N})$, such that $Tf = \sum_{n\geq 0} \beta_n \omega_n^\nu \widehat{f}(n) Q_n^\nu(x)$, for any $f = \sum_{n\geq 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu$, and consequently for any $g, h \in L^2(I, \mu_\nu)$ we have

$$\begin{aligned}
\mathscr{G}(g *_\nu h)(z) &= \int_I \int_I G_\nu(x,y,z) Tg(x) Th(y) d\mu_\nu(x) d\mu_\nu(y),\\
&= \sum_{n\geq 0}\sum_{m\geq 0} \int_I \int_I G_\nu(x,y,z) \beta_n\beta_m \omega_n^\nu \omega_m^\nu \widehat{g}(n)\widehat{h}(m) Q_n^\nu(x) Q_m^\nu(y) d\mu_\nu(x) d\mu_\nu(y)\\
&= \sum_{n\geq 0}\sum_{m\geq 0} \beta_n\beta_m \omega_n^\nu \omega_m^\nu \widehat{g}(n)\widehat{h}(m) [Q_n^\nu *_\nu Q_m^\nu](z)\\
&= \sum_{n\geq 0} \beta_n^2 \omega_n^\nu \widehat{g}(n)\widehat{h}(n) Q_n^\nu(z) \qquad \text{(according to (2.7))}\\
&= \sum_{n\geq 0} \beta_n^2 \omega_n^\nu \widehat{g *_\nu h}(n) Q_n^\nu(z) \qquad \text{(according to (2.4))}
\end{aligned} \tag{3.12}$$

Now, Lemma 3.2(b) asserts that for any $f \in \mathfrak{a}_\nu$ there exist $g, h \in L^2(I, \mu_\nu)$ such that $f = g *_\nu h$, which according to (3.12) implies that $\widehat{(\mathscr{G}f)}(n) = \beta_n^2 \widehat{f}(n)$.

Now, since $\beta_n = \Phi_T(Q_n^\nu)$ and by Corollary 3.4 $\widehat{(Tf)}(n) = \Phi_T(Q_n^\nu)\widehat{f}(n)$, we get

$$\begin{aligned}\widehat{T^2 f}(n) &= \Phi_T^2(Q_n^\nu)\widehat{f}(n)\\ &= \widehat{(\mathscr{G}f)}(n),\end{aligned}$$

which is equivalent to (3.10).

Here we have also proved that (3.10) implies (3.11). Finally the fact that (3.9) follows from (3.11). In fact, according to Lemma 3.7, $Tf = \sum_{n\geq 0}\beta_n\omega_n^\nu\widehat{f}(n)Q_n^\nu \in \Pi_\nu$, and for $\mathscr{G}(f) = \sum_{n\geq 0}\beta_n^2\omega_n^\nu\widehat{f}(n)Q_n^\nu$, we follow (3.12) from bottom to top and we retrieve (3.9). □

4. Comparison with the theory of distribution semigroups

Definition 4.1. In the sequel we say a pseudomeasure operator is *admissible* if

(i) the set of $\{\Phi_T(Q_n^\nu), n \in \mathbb{N}\}$ has only a finite set of zeros;
(ii) the sequence $\{1/\Phi_T(Q_n^\nu)^2 \ \in \ell^\infty$ for $, n \geq M\}$, where $M := \max\{n \in \mathbb{N} : \Phi_T(Q_n^\nu) = 0\}$.

In the Examples 3.5 and 3.6, according to (3.8), P_j is not admissible, while the identity operator I is admissible, since $\Phi_I(Q_n^\nu) = 1$ for all $n \in \mathbb{N}$.

In the theory of DSG, if $\mathscr{G} \in \mathscr{L}(C_c^\infty(\mathbb{R})_+, \mathscr{L}(X))$ and satisfies the conditions of regular distribution semigroup, then for any $\varphi \in C_c^\infty(\mathbb{R})_+$ satisfying $\int_0^\infty \varphi(t)dt = 1$ we have (1.7) and consequently (4.3) and (4.4). Here we will prove a similar result for a endomorphism $\mathscr{G} = T^2$, where T is an admissible pseudomeasure operator.

Theorem 4.2. *Let T be an admissible pseudomeasure operator and $\mathscr{G} = T^2$, then there exist a sequence $g_m \in \mathfrak{a}_\nu$, such that*

$$\lim_{m\to\infty} \|\mathscr{G}(g_m *_\nu f) - f\|_\nu = 0 \quad \textit{for any } f \in \mathfrak{a}_\nu. \tag{4.1}$$

Proof. Let T be the admissible pseudomeasure operator and Φ_T the correspondent element in $\mathfrak{a}'_\nu$. For any integer $m \leq M$, we take $g_m = 0$ and for any $m > M$ we define

$$g_m(x) = \frac{\omega_{M+1}^\nu}{(\Phi_T({Q_{M+1}}^\nu))^2}Q_{M+1}^\nu(x) + \cdots + \frac{\omega_m^\nu}{(\Phi_T({Q_m}^\nu))^2}Q_m^\nu(x).$$

Now for any $n \in \mathbb{N}$ we take m so large that $m \geqslant \max\{n, M\}$, hence we have

$$\begin{aligned}\widehat{g_m}(n) = < g_m, Q_n^\nu > = \int_I g_m Q_n^\nu d\mu_\nu = \frac{\omega_n^\nu}{(\Phi_T({Q_n}^\nu))^2} < Q_n^\nu, Q_n^\nu > \\ = \frac{1}{(\Phi_T({Q_n}^\nu))^2}\end{aligned} \tag{4.2}$$

which is exactly what we were looking for. Since according to (2.8) and (ii) of Definition 4.1 $\|g_m *_\nu f\|_\nu \le C\|f\|_\nu$ and for any $f = \sum_{n\ge 0} \omega_n^\nu \widehat{f}(n) Q_n^\nu \in \mathfrak{a}_\nu$, we have

$$\mathscr{G}(g_m *_\nu f)(x) = \sum_{n\ge 0} (\Phi_T(Q_n^\nu))^2 \omega_n^\nu \widehat{g}_m(n) \widehat{f}(n) Q_n^\nu(x).$$

Hence we get

$$\|\mathscr{G}(g_m *_\nu f) - f\|_\nu = \sum_{n\ge 0} \omega_n^\nu |((\Phi_T({Q_n}^\nu))^2 \widehat{g}_m(n) - 1)\widehat{f}(n)|,$$

which according to (4.2) goes to zero as $m \to \infty$. □

Corollary 4.3. *For* $\mathscr{G} = T^2$

(a)

$$\mathscr{R} := \bigcup_{f\in\, \mathfrak{a}_\nu} \mathrm{Im}\mathscr{G}(f) \ \ \textit{is dense in } \mathfrak{a}_\nu \tag{4.3}$$

(b)

$$\mathscr{N} := \bigcap_{g\in\, \mathfrak{p}_\nu} \mathrm{Ker}\, \mathscr{G}(g *_\nu \ .) = \{0\}, \tag{4.4}$$

where $\mathfrak{p}_\nu$ *is the space of pseudomeasures (see Definition 1.1).*

Proof. (a) For any $f \in \mathfrak{a}_n u$, $g_m *_\nu f \in \mathfrak{a}_\nu$ and according to Corollary 3.4 for any m, Im $\mathscr{G}(g_m *_\nu f) \in \mathscr{R}$. Finally Theorem 4.2 implies the assertion.
(b) Since for any $g \in \mathfrak{p}_\nu$, $f \mapsto \mathscr{G}(g *_\nu f)$ belongs to $\mathscr{L}(\mathfrak{a}_\nu)$, if $f \in \mathscr{N}$, then $\mathscr{G}(g_m *_\nu f) = 0$ for all $g_m \in \mathfrak{p}_\nu$. According to assumption (ii) of Definition 4.1, g_m constructed in the proof of Theorem 4.2 belongs to $\mathfrak{p}_\nu$, hence $\mathscr{G}(g_m *_\nu f) = 0$ which goes to f by Theorem 4.2. This implies that $f = 0$. □

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