

A survey on the singularities of spherical varieties

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We list combinatorial criteria of some singularities, which appear in the Minimal Model Program or in the study of (singular) Fano varieties, for spherical varieties. Most of the results of this paper are already known or are quite easy corollary of known results. We collect these results, we precise some proofs and add few results to get a coherent and complete survey.

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1. Introduction

Spherical varieties form a big family of rational and normal complex varieties, including toric varieties and flag varieties. Here, we list some types of singularities of these varieties and we give combinatorial criteria of these singularities. Most of the results of this paper are already known or are quite easy corollary of known results.

We will only consider types of singularities that appear in birational geometry: in the Minimal Model Program or in the study of (singular) Fano varieties.

The paper is organized as follows. In section 2, we describe the theory of spherical varieties and fix notations. In section 3, we deal with smooth, locally factorial and \mathbb{Q} -factorial spherical varieties. In section 4, we quickly give the criteria of Gorenstein and \mathbb{Q} -Gorenstein singularities for spherical varieties. In section 5, we characterize terminal, canonical and kawamata log terminal (klt)

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singularities for spherical varieties. And in section 6, we conclude by an overview diagram.

We illustrate the theory of spherical varieties and their singularities with a common example all along section 2 and in section 6.

2. Notations

In all the paper, G denotes a connected reductive algebraic group over \mathbb{C} and varieties are algebraic varieties over \mathbb{C} .

A spherical variety X is combinatorially associated to:

- a lattice M of characters of a Borel subgroup B of G (and its dual N);
- a cone \mathcal{V} in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ (called the valuation cone);
- the finite set \mathcal{D} of B -stable, but not G -stable, irreducible divisors of X ;
- an injective map σ from the set of G -invariant valuations of $\mathbb{C}(X)$ to $\mathcal{V} \subset N_{\mathbb{Q}}$ (and a map, still called σ from \mathcal{D} to N);
- and a colored fan \mathbb{F}_X in $N_{\mathbb{Q}}$.

In this section, we describe all these objects and their connections to spherical varieties, and we recall the results we will need in the next sections. The main references of the theory summarized here are [LV83] and [Kno91] for the classification of spherical varieties, [Bri89] for properties of divisors of spherical varieties, [Bri97] and [Lun97] for the description of anticanonical divisors of spherical varieties.

2.1. Spherical embeddings.

Definition 2.1. A G -variety X is an algebraic variety over \mathbb{C} equipped with an algebraic action of G on X .

A spherical G -variety is a normal G -variety X such that there exists $x \in X$ and a Borel subgroup B of G satisfying that the B -orbit of x is open in X .

Remark that, since the Borel subgroups of G are all conjugated, we can fix a Borel subgroup B of G and give the following equivalent definition.

Definition 2.2. A spherical G -variety is a normal G -variety with an open B -orbit.

In the rest of the paper, we also fix a maximal torus in B , so that we can work with the root system of (G, B, T) .

When there is no possible confusion about the group acting on varieties, we write spherical varieties instead of spherical G -varieties.

We can easily remark that spherical varieties have an open G -orbit, and have finitely many B -orbits and G -orbits. The open G -orbit of a spherical variety X is isomorphic to an homogeneous space G/H that is spherical as a G -variety. We say that G/H is a spherical homogeneous space. Note that we can always choose H such that BH is open in G .

Among spherical varieties, we can distinguish several subfamilies of varieties: flag varieties (when H is a parabolic subgroup of G and then $X = G/H$), toric varieties (when $G = (\mathbb{C}^*)^n$ and $H = \{1\}$), horospherical varieties (when H contains a maximal unipotent subgroup of G) and symmetric varieties (when H is an open subgroup of the fixed points set of an involution σ in the automorphism group of G).

The classification of spherical varieties is divided into two parts : the classification of spherical homogeneous spaces and the classification of spherical varieties with some fixed open G -orbit (isomorphic to some fixed G/H). In this paper, we explicitly use the second classification called the Luna-Vust theory of spherical embeddings, and which generalizes the classification of toric varieties in terms of fans.

Definition 2.3. Let G/H be an homogeneous space. A G/H -embedding is a pair (X, x) , where X is a normal G -variety and x is a point of X such that $G \cdot x$ is open in X and H is the stabilizer of x in G .

Two G/H -embeddings (X, x) and (X', x') are isomorphic if there exists a G -equivariant isomorphism from X to X' that sends x to x' .

Note that, if G/H is a spherical (resp. toric, horospherical, symmetric) homogeneous space and (X, x) is a G/H -embedding, then X is a spherical (resp. toric, horospherical, symmetric) variety. Conversely, if X is a spherical (resp. toric, horospherical, symmetric) G -variety, let x be a point in the open G -orbit of X and let H be the stabilizer in G of x ; then G/H is a spherical (resp. toric, horospherical, symmetric) homogeneous space and (X, x) is a G/H -embedding.

By abuse, we often forget the point x and say that X is a G/H -embedding if X is a normal variety with an open G -orbit isomorphic to G/H .

Example 2.4. We consider the homogenous space G/H where $G = \mathrm{SL}_3(\mathbb{C})$ and H is the maximal unipotent subgroup of G consisting of lower triangular matrices with ones on the diagonal. We fix T to be the set of diagonal matrices in G and B to be the upper triangular matrices in G . Then it is easy to check that BH/H is open in G/H so that G/H is spherical. In fact, since H is the unipotent radical of the Borel subgroup B^- of G consisting of lower triangular matrices in G , G/H is a horospherical. Also note that the projection $\pi : G/H \rightarrow G/B^-$ is a torus fibration.

To construct a G/H -embedding, consider the trivial G -module \mathbb{C} and the fundamental G -modules \mathbb{C}^3 and $\bigwedge \mathbb{C}^3$. Denote by (e_1, e_2, e_3) the canonical basis of \mathbb{C}^3 . Let

$$x_0 := [1, e_3, e_3 \wedge e_2] \in \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^3 \oplus \bigwedge \mathbb{C}^3) \simeq \mathbb{P}^6.$$

Then the stabilizer of x_0 in G is H , and the closure X of $G \cdot x_0$ in \mathbb{P}^6 is a cone of vertex $[1, 0, \dots, 0]$ over the Grassmannian of planes in \mathbb{C}^4 . Indeed, if (e_0, e_1, e_2, e_3) denote the canonical basis of $\mathbb{C} \oplus \mathbb{C}^3$, then the map defined by $x + y \wedge z \mapsto x \wedge e_0 + y \wedge z$ gives an isomorphism of G -modules from $\mathbb{C}^3 \oplus \bigwedge \mathbb{C}^3$ to $\bigwedge^2 \mathbb{C}^4$. And

then, by an argument on dimensions, X is the cone of vertex $[1, 0, \dots, 0]$ over the Grassmannian $\mathrm{SL}_4(\mathbb{C}) \cdot e_3 \wedge (e_0 + e_2)$ in $\mathbb{P}(\wedge^2 \mathbb{C}^4)$.

In particular, (X, x_0) (or by abuse X) is a G/H -embedding.

As an exercise, the reader can compute the seven G -orbits of X : one is open and three are closed.

2.2. The lattices M , N , and the colors of G/H . The field $\mathbb{C}(X)$ of rational functions on a spherical variety X can be described by a lattice M by noticing the following multiplicity free property.

Remark 2.5. Let X be a spherical G -variety. The field $\mathbb{C}(X)$ of rational functions of X is naturally a G -module. Let χ be a character of the Borel subgroup B and f_1, f_2 be in $\mathbb{C}(X) \setminus \{0\}$ such that, for any $b \in B$, $b \cdot f_1 = \chi(b)f_1$ and $b \cdot f_2 = \chi(b)f_2$. Then $\frac{f_1}{f_2}$ is a rational function fixed by B . Since B acts with an open orbit on X , we deduce that $\frac{f_1}{f_2}$ is a constant.

Definition 2.6. Let G/H be a spherical homogeneous space.

- (1) We denote by M the lattice of weights χ of B such that there exists $f_\chi \in \mathbb{C}(G/H) \setminus \{0\}$ satisfying, for any $b \in B$, $b \cdot f_\chi = \chi(b)f_\chi$. Note that, by Remark 2.5, for any $\chi \in M$ the rational function f_χ is unique up to a scalar. We say that f_χ is a rational function of X of weight χ .
- (2) The dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M is denoted by N .
- (3) We denote by $M_{\mathbb{Q}}$ (resp. $N_{\mathbb{Q}}$) the \mathbb{Q} -vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $N \otimes_{\mathbb{Z}} \mathbb{Q}$).
- (4) The colors of the spherical homogeneous space G/H are the B -stable irreducible divisors of G/H .
- (5) We denote by \mathcal{D} the set of colors of G/H .

The rank of the lattice M (which is also the rank of N) is called the rank of G/H (or the rank of X if X is any G/H -embedding).

Example 2.7. Let G/H be as in Example 2.4. Denote by ϖ_1 and ϖ_2 the fundamental weights of (G, B, T) (ie $\varpi_1((a_{ij}) \in B) = a_{11}$ and $\varpi_2((a_{ij}) \in B) = a_{11}a_{22}$).

Then, there are two particular functions f_χ as in Definition 2.6 (1): $f_1 := f_{\varpi_1}$ and $f_2 := f_{\varpi_2}$ defined by the H -invariant maps $(a_{ij}) \in \mathrm{SL}_3(\mathbb{C}) \mapsto a_{22}a_{33} - a_{23}a_{32}$ and $(a_{ij}) \in \mathrm{SL}_3(\mathbb{C}) \mapsto a_{33}$ respectively.

Hence, the lattice M equals the lattice of weights of B (ie $\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$). In particular, N is the lattice generated by the two simple coroots α_1^\vee and α_2^\vee of (G, B, T) . And the rank of G/H is 2.

Moreover, G/H has two colors: the two B -stable irreducible divisors D_1 and D_2 of G/H defined as the zero sets of f_1 and f_2 respectively. In fact, D_1 and D_2 can also be defined as the inverse image by the fibration π of Schubert divisors of G/B^- .

2.3. The valuation cone and the maps σ . Let first recall the definition of a valuation in our context.

Definition 2.8. Let X be a normal variety. A valuation of X is a map ν from $\mathbb{C}(X) \setminus \{0\}$ to \mathbb{Q} satisfying:

- for any f_1 and f_2 in $\mathbb{C}(X) \setminus \{0\}$ such that $f_1 + f_2 \neq 0$, $\nu(f_1 + f_2) \geq \min(\nu(f_1), \nu(f_2))$;
- for any f_1 and f_2 in $\mathbb{C}(X) \setminus \{0\}$, $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$;
- ν vanishes on constant functions.

Moreover, if X is a G -variety, the valuation ν is said to be G -invariant if for any $g \in G$ and for any $f \in \mathbb{C}(X) \setminus \{0\}$ we have $\nu(g \cdot f) = \nu(f)$.

Definition 2.9. Let G/H be a spherical homogeneous space.

- (1) The set of G -invariant valuations of G/H is a cone \mathcal{V} that can be identified to a cone in $N_{\mathbb{Q}}$ by the injective map $\sigma : \mathcal{V} \rightarrow N_{\mathbb{Q}}$ defined by, for any $\nu \in \mathcal{V}$ and any $\chi \in M$, $\sigma(\nu)(\chi) = \nu(f_{\chi})$ where f_{χ} is as in Definition 2.6 (1). The image of \mathcal{V} in $N_{\mathbb{Q}}$ is called the valuation cone of G/H .
- (2) Any color D of the spherical homogeneous space G/H defines a B -invariant valuation on $\mathbb{C}(G/H) \setminus \{0\}$, and then it defines (similarly to the definition of σ) a point in $N_{\mathbb{Q}}$ that we also denote by $\sigma(D)$. It is called the image of the color D in $N_{\mathbb{Q}}$. (In fact, $\sigma(D) \in N$.)

Remark 2.10. • The valuation cone of G/H equals $N_{\mathbb{Q}}$ if and only if G/H is horospherical [BP87, Corollaire 5.4].

- Since the colors of a spherical homogeneous space G/H are not G -stable, it could happen that two colors of G/H have the same image in $N_{\mathbb{Q}}$.
- The opposite $-\mathcal{V}^{\vee}$ of the dual in $M_{\mathbb{Q}}$ of the valuation cone is simplicial, and generated by positive roots of (G, B, T) and sums of two strongly orthogonal positive roots of (G, B, T) [Bri90].

Definition 2.11. The primitive elements of the rays of the simplicial cone $-\mathcal{V}^{\vee}$ are called the spherical roots of G/H .

Example 2.12. We still consider the same horospherical homogeneous space G/H as in Examples 2.4 and 2.7. Then, by Remark 2.10, the valuation cone of G/H equals $N_{\mathbb{Q}}$.

Moreover, since the colors D_1 and D_2 are defined as the zero sets of f_1 and f_2 respectively, it is easy to see that $\sigma(D_i)(f_j)$ is zero if $i \neq j$ and one if $i = j$. In other words, for any $i \in \{1, 2\}$, $\sigma(D_i)$ equals the coroots α_i^{\vee} .

2.4. Colored fans.

Definition 2.13. Let G/H be a spherical homogeneous space, $N_{\mathbb{Q}}$, \mathcal{D} , \mathcal{V} and σ defined as above.

- (1) A colored cone in $N_{\mathbb{Q}}$ is a pair $(\mathcal{C}, \mathcal{F})$ such that
 - \mathcal{F} is a subset of \mathcal{D} ;

- \mathcal{C} is a cone in $N_{\mathbb{Q}}$ generated by finitely many elements of $N \cap \mathcal{V}$ and the $\sigma(D)$ with D in \mathcal{F} ;
 - the relative interior of \mathcal{C} intersects \mathcal{V} ;
 - $\sigma(D) \neq 0$ for any $D \in \mathcal{F}$ and \mathcal{C} contains no line.
- (2) A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ where \mathcal{C}' is a face of the cone \mathcal{C} , whose relative interior intersects \mathcal{V} , and $\mathcal{F}' = \{D \in \mathcal{F} \mid \sigma(D) \in \mathcal{C}'\}$. It is in particular a colored cone.
- (3) A colored fan in $N_{\mathbb{Q}}$ is a finite set \mathbb{F} of colored cones such that: any colored face of a colored cone of \mathbb{F} is in \mathbb{F} , and for any $u \in N_{\mathbb{Q}} \cap \mathcal{V}$ there exists at most one colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F} such that u is in the relative interior of \mathcal{C} .
- (4) A fan is complete if $\bigcup_{(\mathcal{C}, \mathcal{F}) \in \mathbb{F}} \mathcal{C} \supset \mathcal{V}$.

Note that if $G/H = (\mathbb{C}^*)^n$, then \mathcal{D} is empty (ie there is no color), $\mathcal{V} = N_{\mathbb{Q}}$, and the definitions of colored cones and colored fans are equivalent to the definitions of cones and fans in toric geometry.

See Figure 2, at the end of the paper, to get several examples of complete colored fans in the case where G/H is as in Example 2.4. Note that, in that case, the images of the colors are all distinct. Then, to represent a colored fan, we only draw the edges of the fan and we represent a color of the fan by bordering in grey the white circle corresponding to the image of the color of G/H .

2.5. The classification of G/H -embeddings. To any G/H -embedding (X, x) , we associate a colored fan as follows.

For any G -orbit Y of X , denote by X_Y the G -stable subset $\{x \in X \mid \overline{G \cdot x} \supset Y\}$. Denote by $\mathcal{D}_{G,Y}$ the set of G -stable irreducible divisors in X_Y and denote by \mathcal{F}_Y the set of $D \in \mathcal{F}$ such that the closure of D in X_Y contains Y . For any $D \in \mathcal{D}_{G,Y}$, denote by $\sigma(D)$ the image by σ of the G -invariant valuation on $\mathbb{C}(G/H) \setminus \{0\}$ associated to D . Denote by \mathcal{C}_Y the cone in $N_{\mathbb{Q}}$ generated by the $\sigma(D)$ with $D \in \mathcal{D}_{G,Y}$ and the $\sigma(D)$ with $D \in \mathcal{F}_Y$.

We can now state the classification of G/H -embeddings when G/H is spherical.

Theorem 2.14 ([Kno91]). *Let G/H be a spherical homogeneous space.*

Let (X, x) be a G/H -embedding. Then, for any G -orbit Y of X , $(\mathcal{C}_Y, \mathcal{F}_Y)$ is a colored cone in $N_{\mathbb{Q}}$ and the set of $(\mathcal{C}_Y, \mathcal{F}_Y)$ with Y in the set of G -orbits of X is a colored fan in $N_{\mathbb{Q}}$. It is called the colored fan of X and denoted by \mathbb{F}_X .

The map from the set of isomorphic classes of G/H -embeddings to the set of colored fans in $N_{\mathbb{Q}}$ that sends the class of (X, x) to \mathbb{F}_X is well-defined and bijective.

Moreover, X is complete if and only if \mathbb{F}_X is complete.

The set of colors of X (or of \mathbb{F}_X) is the union $\mathcal{F}_X := \bigcup_{(\mathcal{C}, \mathcal{F}) \in \mathbb{F}} \mathcal{F}$. It is a subset of \mathcal{D} .

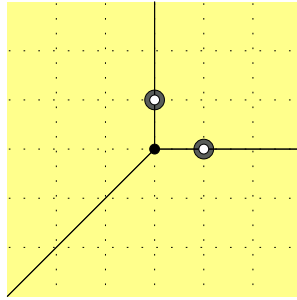
Example 2.15. We consider the horospherical homogeneous space G/H and the G/H -embedding X of Example 2.4. We can compute that there are three closed

G -orbits: the vertex of the cone X , $\mathbb{P}(\mathbb{C}^3)$ and $\mathbb{P}(\wedge^2 \mathbb{C}^3)$. Also, there is only one G -stable irreducible divisor X_1 in X , which is the intersection of X with $\mathbb{P}(\mathbb{C}^3 \oplus \wedge^2 \mathbb{C}^3)$ and is isomorphic to the Grassmannian of planes in \mathbb{C}^4 .

It is easy to check that X_1 contains the two 2-dimensional closed G -orbits, and that the closures in X of the two colors of G/H contains the vertex of the cone X and exactly one of the other closed G -orbits.

In fact, if we denote by $W_0, W_1, W_2, W_3, W_{12}, W_{13}, W_{23}$ the coordinates in $\mathbb{P}(\mathbb{C} \oplus \mathbb{C}^3 \oplus \wedge^2 \mathbb{C}^3)$, then the functions f_1 and f_2 (defining the colors D_1 and D_2 of G/H) correspond to the functions $\frac{W_{23}}{W_0}$ and $\frac{W_3}{W_0}$ respectively. Also, the G -stable divisor X_1 is defined by $W_0 = 0$. Hence, we also deduce that $\sigma(X_1)$ is $-\alpha_1^\vee - \alpha_2^\vee$.

Combining all we said above, we can prove that the colored fan of X is the following one.



2.6. G -equivariant morphisms between G/H -embeddings. The G -equivariant morphisms between spherical G -varieties are very well understood. We state here the description of birational G -equivariant morphisms (ie between two embeddings of the same spherical homogeneous space).

Note that if $f : X \rightarrow Y$ is a proper morphism between algebraic varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and if G acts on X , by a result of Blanchard (see also [BSU13, Prop. 4.2.1]), then there exists a unique action of G on Y such that f is G -equivariant. In particular, it is not so restrictive to only consider G -equivariant morphisms.

Proposition 2.16. *Let G/H be a spherical homogeneous space.*

Let $(X, x), (Y, y)$ be two G/H -embeddings. Then, there exists a G -equivariant morphism $f : X \rightarrow Y$ with $f(x) = y$ if and only if for any colored cone $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, there exists a colored cone $(\mathcal{C}', \mathcal{F}') \in \mathbb{F}_Y$ such that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{F} \subset \mathcal{F}'$.

See examples of proper G -equivariant morphisms between G/H -embeddings in Figure 2, with G/H as in Example 2.4.

2.7. Divisors, Cartier divisors. Let G/H be a spherical homogeneous space.

Let (X, x) be a G/H -embedding associated to the colored fan \mathbb{F}_X . Denote by X_1, \dots, X_m the irreducible G -stable divisors of X ($m \geq 0$). For any $i \in \{1, \dots, m\}$, we denote by x_i the image by σ of the valuation associated to X_i . Recall that it is

a primitive element of an edge of \mathbb{F}_X that has no color (ie contains no $\sigma(D)$ with $D \in \mathcal{F}_X$).

Proposition 2.17. (1) *Any Weil divisor of X is linearly equivalent to a B -stable divisor, ie of the form $\delta = \sum_{i=1}^m \delta_i X_i + \sum_{D \in \mathcal{D}} \delta_D D$.*

(2) *Such a divisor is Cartier if and only if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ there exists $\chi \in M$ such that for any $i \in \{1, \dots, m\}$ such that $x_i \in \mathcal{C}$, $\langle \chi, x_i \rangle = \delta_i$ and for any $D \in \mathcal{F}$, $\langle \chi, \sigma(D) \rangle = \delta_D$.*

Then any Cartier divisor $\delta = \sum_{i=1}^m \delta_i X_i + \sum_{D \in \mathcal{D}} \delta_D D$ of a complete spherical variety X is associated to a unique piecewise linear function h_δ on \mathbb{F}_X (well-defined function on \mathcal{V}), linear on each cone in \mathbb{F}_X , such that $\forall i \in \{1, \dots, m\}$, $h_\delta(x_i) = \delta_i$ and $\forall D \in \mathcal{F}_X$, $h_\delta(\sigma(D)) = \delta_D$. In that case, for any maximal cone \mathcal{C} of \mathbb{F}_X (ie for any maximal colored cone $(\mathcal{C}, \mathcal{F})$), we denote by $\chi_{\mathcal{C}, \delta}$ the element of M associated to the linear function defining h_δ on \mathcal{C} .

Definition 2.18. A piecewise linear function h_δ on \mathbb{F}_X is convex if:

- for any maximal cone \mathcal{C} of \mathbb{F}_X and any $x \in N_{\mathbb{Q}}$, we have $h_\delta(x) \geq \langle \chi_{\mathcal{C}, \delta}, x \rangle$;
- and for any $D \in \mathcal{D} \setminus \mathcal{F}_X$, $h_\delta(\sigma(D)) \leq \delta_D$.

We say that it is strictly convex if:

- for any maximal cone \mathcal{C} of \mathbb{F}_X and any $x \in N_{\mathbb{Q}} \setminus \mathcal{C}$, we have $h_\delta(x) > \langle \chi_{\mathcal{C}, \delta}, x \rangle$;
- and for any $D \in \mathcal{D} \setminus \mathcal{F}_X$, $h_\delta(\sigma(D)) < \delta_D$.

Proposition 2.19. *A Cartier divisor $\delta = \sum_{i=1}^m \delta_i X_i + \sum_{D \in \mathcal{D}} \delta_D D$ of a complete spherical variety X is globally generated (respectively ample) if and only if h_δ is convex (respectively strictly convex).*

Anticanonical divisors of spherical varieties are described in [Bri97, Proposition 4.1] and in [Lun97, Section 3.6], by dividing the colors of G/H into three types. The description of these three types was recently reconsidered in [Kno14] and in [GH15] in a simpler way by using that B -orbits of G/H correspond to H -orbits in $B \setminus G$. In the following theorem, we gather together their results that we will need later.

For any simple root α of (G, B, T) , denote by P_α the minimal parabolic subgroup of G containing B such that $-\alpha$ is a weight of the Lie algebra of P_α .

Let G/H be a spherical homogeneous space.

Denote by P the stabilizer in G of the open B -orbit of G/H (it is a parabolic subgroup of G containing B), and denote by \mathcal{S}_P the set of simple roots α of (G, B, T) such that $P_\alpha \not\subset P$.

Theorem 2.20. *With the notation above, let $D \in \mathcal{D}$. Choose a simple root α of (G, B, T) , such that $P_\alpha \cdot D \neq D$ (in particular, $\alpha \in \mathcal{S}_P$).*

Recall that the spherical roots of G/H are the primitive elements of $-\mathcal{V}^\vee$. Then one and only one of the following case occurs:

- (a) α is a spherical root of G/H ;

- (2a) 2α is a spherical root of G/H ;
 (b) neither α nor 2α is a spherical root of G/H .

Moreover, the case occurring does not depend on the choice of α . Then, we say that D is of type (a), (2a) and (b) respectively.

Denote by α_M^\vee the restriction to M of the coroot α^\vee ; it is an element of N . Then, the images $\sigma(D)$ of D in $N_{\mathbb{Q}}$ satisfy, respectively in each case:

- (a) $\langle \alpha, \sigma(D) \rangle = 1$;
 (2a) $\sigma(D) = \frac{1}{2}\alpha_M^\vee$, in particular $\langle \alpha, \sigma(D) \rangle = 1$;
 (b) $\sigma(D) = \alpha_M^\vee$.

For any $D \in \mathcal{D}$, we define an integer a_D as follows: if D is of type (a) or (2a), $a_D = 1$; and if D is of type (b), $a_D = \langle \sum_{\alpha \in \mathcal{R}_P^+} \alpha, \alpha^\vee \rangle$ (which is greater or equal to 2), where \mathcal{R}_P^+ is the set of positive roots with at least one non-zero coefficient for a simple root of \mathcal{S}_P .

Let (X, x) be a G/H -embedding. Denote by \mathcal{D}_X the set of irreducible G -stable divisors of X . Then, an anticanonical divisor of a G -spherical embedding (X, x) associated to a colored fan \mathbb{F}_X is

$$-K_X = \sum_{D \in \mathcal{D}_X} D + \sum_{D \in \mathcal{D}} a_D D.$$

Corollary 2.21. *For any B -stable irreducible divisor D of X , the coefficient attached to D in $-K_X$ is a positive integer. Moreover, if D is not G -stable and $a_D = 1$, then $\sigma(D)$ is not in the valuation cone \mathcal{V} .*

Proof. If $a_D = 1$, then D is of type (a) or (2a). In particular, α or 2α is in $-\mathcal{V}^\vee$. Hence, for any $v \in \mathcal{V} \subset N$, we get $\langle \alpha, v \rangle \leq 0$. But we also have that $\langle \alpha, \sigma(D) \rangle = 1$. Thus $\sigma(D)$ is not in \mathcal{V} . \square

Remark 2.22. If $\sigma(D)$ is not in \mathcal{V} , a_D is not necessary 1. For example, if $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and H is the diagonal in G , then G/H has only one spherical root that is the strongly orthogonal sum of the two simple roots; in particular, the unique B -stable divisor D of G/H is of type (b) and an easy computation gives $a_D = 2$.

Example 2.23. We still consider the horospherical homogeneous space G/H of the previous examples. Since the valuation cone of G/H is $N_{\mathbb{Q}}$, the cone $-\mathcal{V}^\vee$ is reduced to 0. Then G/H has no spherical root and all colors of G/H is of type (b).

In that case, $P = B$, \mathcal{S}_P is the set $\{\alpha_1, \alpha_2\}$ of simple roots of (G, B, T) and \mathcal{R}_P^+ is the set of positive roots of (G, B, T) . In particular for any $i \in \{1, 2\}$, $a_{D_i} = 2$.

If X is the G/H -embedding defined in Example 2.4, then $-K_X = X_1 + 2D_1 + 2D_2$.

In [GH15, Section 6], there are more examples of spherical homogeneous spaces with their spherical data: SL_2/T , which admits two colors of type (a) , where T is a maximal torus; or SL_2/N , which admits one color of type $(2a)$, where N is the normalizer of a maximal torus; etc.

3. Smooth, locally factorial and \mathbb{Q} -factorial varieties

Definition 3.1. A variety X is locally factorial if all Weil divisors of X are Cartier.

A variety X is \mathbb{Q} -factorial if all Weil divisors of X are \mathbb{Q} -Cartier.

The smoothness of spherical varieties is the type of singularities that is the most complicated to characterize for spherical varieties. For toric varieties, it is not difficult and well-known because any locally factorial toric variety is smooth (see for example [Ful93]). For horospherical varieties, there is a more complicated criterion, simultaneously obtained in [Pas06] and [Tim11], which mixes the combinatorial aspects of colored fans and root systems. For general spherical varieties, a smoothness criterion was first given in [Bri91], and a more practical one was recently given in [Gag15]. And if we admit a conjecture (satisfied by horospherical varieties and symmetric varieties), then we get a very simple smoothness criterion [GH17]. Moreover, there is another smooth criterion for horospherical varieties in [BM13], which is expected to hold also for arbitrary spherical varieties.

In this paper, we will not write these smoothness criteria, but we have to note that the main tool of their proofs is the local structure theorem of spherical varieties [BP87, Proposition 3.4]. This tool easily permits to prove the following useful (and well-known) result.

Proposition 3.2. *Let X be a locally factorial spherical variety such that \mathcal{F}_X is empty. Then X is smooth.*

Proof. Let G/H be a spherical homogeneous space, and (X, x) be a G/H -embedding. Recall that P is the stabilizer in G of the open B -orbit of G/H .

We apply the local structure theorem when \mathcal{F}_X is empty and we get that a P -stable open set \mathcal{U} of X is isomorphic to $R_u(P) \times Z$, where $R_u(P)$ is the unipotent radical of P and Z is a toric variety under the action of the neutral component of a Levi of P . Moreover, this open set \mathcal{U} equals $X \setminus \bigcup_{D \in \mathcal{D}} \overline{D}$ (where \overline{D} is the closure of D in X). If \mathcal{F}_X is empty, then \mathcal{U} intersects every closed G -orbit. If X is locally factorial, then Z is a locally factorial toric variety and so Z is smooth. We conclude that, if X is locally factorial with \mathcal{F}_X empty, X is smooth along all its closed G -orbits. Hence X is smooth. \square

We now write and prove locally factorial and \mathbb{Q} -factorial criteria.

Using the criterion of Cartier divisors of spherical varieties (Proposition 2.17), we get the following result.

Proposition 3.3. *Let X be a spherical variety associated to a colored fan \mathbb{F}_X . Then X is locally factorial (respectively \mathbb{Q} -factorial) if and only if for any $(\mathcal{C}, \mathcal{F}) \in$*

\mathbb{F}_X , the colors of \mathcal{F} have distinct images in $N_{\mathbb{Q}}$ and there exists a basis $(u_1, \dots, u_k) \cup (\sigma(D))_{D \in \mathcal{F}}$ of the lattice N (respectively of the vector space $N_{\mathbb{Q}}$) such that \mathcal{C} is generated by the family $(u_1, \dots, u_{k'}) \cup (\sigma(D))_{D \in \mathcal{F}}$ (where $k' \leq k$ are non-negative integers).

Proof. The “if” part is quite easy, so we prove only the “only if” part.

Suppose that X is \mathbb{Q} -factorial. Let $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, and D_1, D_2 be two colors in \mathcal{F} . The Weil divisor $\delta = D_1$ is \mathbb{Q} -Cartier, so that there exists $\chi_{\mathcal{C}, \delta} \in M_{\mathbb{Q}}$ such that $\langle \chi_{\mathcal{C}, \delta}, \sigma(D_1) \rangle = 1$ and $\langle \chi_{\mathcal{C}, \delta}, \sigma(D_2) \rangle = 0$. In particular, $\sigma(D_1) \neq \sigma(D_2)$. Now, denote by $u_1, \dots, u_{k'}$ the primitive elements of the edges of \mathcal{C} that are not generated by some $\sigma(D)$ with $D \in \mathcal{F}$. We can suppose that $u_1, \dots, u_{k'}$ correspond to the G -stable irreducible divisors $X_1, \dots, X_{k'}$ respectively.

For any $i \in \{1, \dots, k'\}$, X_i is \mathbb{Q} -Cartier, so there exists $\chi_{\mathcal{C}, X_i} \in M_{\mathbb{Q}}$ such that $\langle \chi_{\mathcal{C}, X_i}, u_i \rangle = 1$, $\langle \chi_{\mathcal{C}, X_i}, u_j \rangle = 0$, for any $j \in \{1, \dots, k'\} \setminus \{i\}$, and $\langle \chi_{\mathcal{C}, X_i}, \sigma(D) \rangle = 0$, for any $D \in \mathcal{F}$. Similarly, for any $D' \in \mathcal{F}$, there exists $\chi_{\mathcal{C}, D'} \in M_{\mathbb{Q}}$ such that $\langle \chi_{\mathcal{C}, D'}, \sigma(D') \rangle = 1$, $\langle \chi_{\mathcal{C}, D'}, u_j \rangle = 0$, for any $j \in \{1, \dots, k'\}$, and $\langle \chi_{\mathcal{C}, D'}, \sigma(D) \rangle = 0$, for any $D \in \mathcal{F} \setminus \{D'\}$.

Let $b_1, \dots, b_{k'}$ and, for any $D \in \mathcal{F}$, b_D be rational numbers such that $b_1 u_1 + \dots + b_{k'} u_{k'} + \sum_{D \in \mathcal{F}} b_D \sigma(D) = 0$ (*). Applying $\chi_{\mathcal{C}, X_i}$ for $i \in \{1, \dots, k'\}$ and $\chi_{\mathcal{C}, D'}$ for $D' \in \mathcal{F}$ to (*), we get that $b_1 = \dots = b_{k'} = 0$ and $b_D = 0$ for any $D \in \mathcal{F}$. The family $(u_1, \dots, u_{k'}) \cup (\sigma(D))_{D \in \mathcal{F}}$ is linearly independent (and generates \mathcal{C}), in particular we can complete it to get a basis $(u_1, \dots, u_k) \cup (\sigma(D))_{D \in \mathcal{F}}$ of $N_{\mathbb{Q}}$.

Suppose moreover that X is locally factorial. Then, we can choose the elements $\chi_{\mathcal{C}, X_i}$ for $i \in \{1, \dots, k'\}$, and $\chi_{\mathcal{C}, D'}$ for $D' \in \mathcal{F}$, in the lattice M . With the same proof as above, we can prove that the family $(\chi_{\mathcal{C}, X_1}, \dots, \chi_{\mathcal{C}, X_{k'}}) \cup (\chi_{\mathcal{C}, D})_{D \in \mathcal{F}}$ is linearly independent. Also, for any element u in the intersection of N with the \mathbb{Q} -vector space generated by $(u_1, \dots, u_{k'}) \cup (\sigma(D))_{D \in \mathcal{F}}$, we have $u = \langle \chi_{\mathcal{C}, X_1}, u \rangle u_1 + \dots + \langle \chi_{\mathcal{C}, X_{k'}}, u \rangle u_{k'} + \sum_{D \in \mathcal{F}} \langle \chi_{\mathcal{C}, D}, u \rangle \sigma(D)$ and then u is in the sublattice generated by $(\chi_{\mathcal{C}, X_1}, \dots, \chi_{\mathcal{C}, X_{k'}}) \cup (\chi_{\mathcal{C}, D})_{D \in \mathcal{F}}$. We conclude, by Lemma 3.4, that we can find $u_{k'+1}, \dots, u_k$ in N such that $(u_1, \dots, u_k) \cup (\sigma(D))_{D \in \mathcal{F}}$ is a basis of N . \square

A particular case of a classical theorem on free modules over a principal rings [Lan02, III, Theorem 7.8] gives the following result.

Lemma 3.4. *Let L be a lattice. Let \mathcal{E} be a linearly independent family of elements of L . Denote by L' the sublattice generated by \mathcal{E} and suppose that L' equals the intersection of L with the \mathbb{Q} -vector space generated by \mathcal{E} .*

Then we can complete \mathcal{E} into a basis of L .

Remark 3.5. A consequence of Propositions 2.16, 3.2 and 3.3 is that, for any spherical variety X , a G -equivariant resolution $f : V \rightarrow X$ is given by erasing all colors in \mathbb{F}_X and by sufficiently subdividing the cones in \mathbb{F}_X . (Such a subdivision exists for example by [Cox00, Theorem 1.5].)

Moreover, the exceptional locus of f is G -stable. But, still by Propositions 3.2 and 3.3, G -stable irreducible subvarieties of V are smooth. Then, by blowing-up the irreducible components of the exceptional locus of f of codimension at least 2, we obtain a G -equivariant resolution $\tilde{f} : \tilde{V} \rightarrow X$ such that the exceptional

locus of \tilde{f} is of pure codimension one. Remark also that, since $\mathcal{F}_{\tilde{V}}$ is empty, the exceptional divisors of \tilde{V} , which are G -stable, are smooth.

By the local structure theorem (already used in Proposition 3.2), we can also prove that the exceptional locus of \tilde{f} is a simple normal crossing divisor.

4. Gorenstein and \mathbb{Q} -Gorenstein varieties

Definition 4.1. A normal variety X is Gorenstein (respectively \mathbb{Q} -Gorenstein) if the anticanonical divisor $-K_X$ is Cartier (respectively \mathbb{Q} -Cartier).

The criterion of these types of singularities for spherical varieties is an easy consequence of the criterion of Cartier divisors of spherical varieties (Proposition 2.17) and the description of anticanonical divisor (Theorem 2.20).

Proposition 4.2. *Let G/H be a spherical homogeneous space. For any color $D \in \mathcal{D}$, we define a_D as in Section 2.7. Let (X, x) be a G/H -embedding associated to a colored fan \mathbb{F}_X . Then X is Gorenstein (respectively \mathbb{Q} -Gorenstein) if and only if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, there exists $m_{\mathcal{C}} \in M$ (respectively $m_{\mathcal{C}} \in M_{\mathbb{Q}}$) such that, for any primitive element x of an edge of \mathcal{C} that is not generated by some $\sigma(D)$ with $D \in \mathcal{D}$, $\langle m_{\mathcal{C}}, x \rangle = 1$, and for any $D \in \mathcal{F}$, $\langle m_{\mathcal{C}}, \sigma(D) \rangle = a_D$.*

5. (log) terminal and canonical singularities

Definition 5.1. Let X be a normal \mathbb{Q} -Gorenstein variety. Let $f : V \rightarrow X$ be a resolution of X (ie f is a proper birational morphism and V is smooth). Then $K_V - f^*(K_X) = \sum_{i \in \mathcal{I}} c_i E_i$ where $\{E_i \mid i \in \mathcal{I}\}$ is the set of exceptional divisors of f .

We say that X has

- canonical singularities if, for any $i \in \mathcal{I}$, $c_i \geq 0$;
- terminal singularities if, for any $i \in \mathcal{I}$, $c_i > 0$.

Note that the definition does not depend on the choice of the resolution. Moreover, if X is spherical, recall that, by Remark 3.5, we can construct a resolution by deleting the colors of X and by taking subdivision of the cones of \mathbb{F}_X . Then, still with the criterion of Cartier divisors of spherical varieties, we get the following characterizations of canonical and terminal singularities.

Proposition 5.2. *Let G/H be a spherical homogeneous space. Let (X, x) be a \mathbb{Q} -Gorenstein G/H -embedding associated to a colored fan \mathbb{F}_X .*

For any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , denote by $h_{\mathcal{C}}$ the linear function such that for any $D \in \mathcal{F}$, $h_{\mathcal{C}}(\sigma(D)) = a_D$ and, for any primitive element u of an edge of \mathcal{C} that is not generated by some $\sigma(D)$ with $D \in \mathcal{F}$, $h_{\mathcal{C}}(u) = 1$.

- X has canonical singularities if and only if for any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , for any $x \in \mathcal{C} \cap N \cap \mathcal{V}$, $h_{\mathcal{C}}(x) \geq 1$.

- X has terminal singularities if and only if for any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , for any $x \in \mathcal{C} \cap N \cap \mathcal{V}$, $h_{\mathcal{C}}(x) \leq 1$ implies that x is the primitive element of an edge of \mathcal{C} that is not generated by some $\sigma(D)$ with $D \in \mathcal{F}$ (ie $x = x_i$ with our notation).

We begin by proving the following result.

Lemma 5.3. *Let G/H be a spherical homogeneous space. Let (X, x) and (V, v) be two G/H -embeddings respectively associated to colored fans \mathbb{F}_X and \mathbb{F}_V . Suppose that there exists a G -equivariant dominant morphism $f : V \rightarrow X$. And let δ be a Cartier divisor of X .*

Recall that δ is associated to a piecewise linear function h_{δ} on \mathbb{F}_X . Then, for any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , we denote by $h_{\mathcal{C}, \delta}$ the restriction of h_{δ} on $(\mathcal{C}, \mathcal{F})$. Moreover, there exists an element $\chi_{\mathcal{C}, \delta}$ of M such that $h_{\mathcal{C}, \delta}$ is the restriction of the linear form $\chi_{\mathcal{C}, \delta}$.

Let E be an exceptional divisor of f . Since f is birational and G -equivariant, it is G -stable.

Then, for any $(\mathcal{C}, \mathcal{F})$ in \mathbb{F}_X such that $\sigma(E) \in \mathcal{C}$, the coefficient attached to E in $f^(\delta)$ is $h_{\mathcal{C}, \delta}(\sigma(E))$.*

Proof of Lemma 5.3. To describe $f^*(\delta)$ we need to look deeper at the Cartier criterion.

Since δ is a B -stable Cartier divisor of X , $\mathcal{O}(\delta)$ is given by $(\mathcal{U}_{\mathcal{C}}, f_{-\chi_{\mathcal{C}, \delta}})$, where $(\mathcal{C}, \mathcal{F})$ runs through the set of maximal colored cones in \mathbb{F}_X , where $\mathcal{U}_{\mathcal{C}}$ is the open set of X associated to the colored cone $(\mathcal{C}, \mathcal{F})$ and $f_{-\chi_{\mathcal{C}, \delta}}$ is a non-zero rational function on X associated to the weight $-\chi_{\mathcal{C}, \delta}$ (unique up to a scalar). Moreover, for any maximal colored cones $(\mathcal{C}, \mathcal{F})$ in \mathbb{F}_X , $\text{div}(f_{-\chi_{\mathcal{C}, \delta}})$ equals $-\delta$ on the open set $\mathcal{U}_{\mathcal{C}}$. Then $f^*\mathcal{O}(\delta)$ is given by $(f^{-1}(\mathcal{U}_{\mathcal{C}}), f_{-\chi_{\mathcal{C}, \delta}} \circ f)$.

Now, we remark that the map from $\mathbb{C}(X)$ to $\mathbb{C}(V)$ that sends g to $g \circ f$ is a G -equivariant isomorphism, so that $f_{-\chi_{\mathcal{C}, \delta}} \circ f$ is a rational function on V associated to the weight $-\chi_{\mathcal{C}, \delta}$. And we also notice that $f^{-1}(\mathcal{U}_{\mathcal{C}})$ is the union of the open set of V associated to the colored cones $(\mathcal{C}', \mathcal{F}')$ of \mathbb{F}_V such that $\mathcal{C}' \subset \mathcal{C}$ (and $\mathcal{F}' \subset \mathcal{F}$). Then, for any irreducible G -stable divisor E of V , the coefficient attached to E in $f^*(\delta)$ is $\langle \chi_{\mathcal{C}, \delta}, \sigma(E) \rangle$ for any maximal $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ such that $\sigma(E) \in \mathcal{C}$.

Hence, for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ such that $\sigma(E) \in \mathcal{C}$, the coefficient attached to E in $f^*(\delta)$ is $h_{\mathcal{C}, \delta}(\sigma(E))$. \square

Proof of Proposition 5.2. Let $f : V \rightarrow X$ be a G -equivariant resolution of X . We can apply Lemma 5.3 to the Cartier divisor $\delta = -kK_X$ for a large enough positive integer k . In that case, for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, $h_{\mathcal{C}, \delta} = kh_{\mathcal{C}}$. Then, for any $i \in \mathcal{I}$ and for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ such that $\sigma(E_i) \in \mathcal{C}$, the coefficient attached to E_i in $k(K_V - f^*(K_X))$ is $kh_{\mathcal{C}}(\sigma(E_i)) - k$, so that the coefficient c_i attached to E_i in $K_V - f^*(K_X)$ is $h_{\mathcal{C}}(\sigma(E_i)) - 1$.

- Suppose that X has canonical singularities. Let $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ and $x \in \mathcal{C} \cap N \cap \mathcal{V}$. We can suppose that x is primitive in N . By Remark 3.5, there exists a resolution V of X such that x is the primitive element of an edge without

color of the colored fan of V . Then x is the image by σ of an exceptional (G -stable and irreducible) divisor E_i of V . Since $c_i \geq 0$ and $c_i = h_{\mathcal{C}}(x) - 1$, we get $h_{\mathcal{C}}(x) \geq 1$.

The “if” part proof works with the same arguments.

- The proof is almost the same as above with “ $>$ ” instead of “ \geq ”.

□

Definition 5.4. Let X be a normal variety and let D be an effective \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier. The pair (X, D) is said to be klt (Kawamata log terminal) if for any resolution $f : V \rightarrow X$ of X such that $K_V = f^*(K_X + D) + \sum_{i \in \mathcal{I}} c_i E_i$, we have $c_i > -1$ for any $i \in \mathcal{I}$.

We say that X has log terminal singularities if X is \mathbb{Q} -Gorenstein and $(X, 0)$ is klt.

Remark 5.5. (1) In fact, it is enough to check the above property for one log-resolution to say that a pair (X, D) is klt.

- (2) The condition “ $c_i > -1$ for any $i \in \mathcal{I}$ ” can be replaced by: $[D] = 0$ and for any $i \in \mathcal{I}$ such that E_i is exceptional for f , $c_i > -1$.

Still with the criterion of Cartier divisors of spherical varieties (Proposition 2.17), we get the following result.

Proposition 5.6. [AB04] *Let X be a spherical variety. Then X has log terminal singularities.*

In fact, in [AB04], V. Alexeev and M. Brion proved that, if X is a spherical G -variety and D be an effective \mathbb{Q} -divisor of X such that $D + K_X$ is \mathbb{Q} -Cartier, $[D] = 0$ and $D = D_G + D_B$ where D_G is G -stable and D_B is stable under the action of a Borel subgroup B of G , then $(X, D_G + D'_B)$ has klt singularities for general D'_B in the linear system $|D_B|$.

We can give a short proof of the proposition.

Proof. Let $f : V \rightarrow X$ be G -equivariant log-resolution of $(X, 0)$, ie such that the exceptional locus of f is a simple normal crossing divisor (see Remark 3.5). With the same arguments as in the first part of the proof of Proposition 5.2, we can prove that, for any exceptional divisor E_i of f , we have $c_i = h_{\mathcal{C}, \delta}(\sigma(E_i)) - 1$ where $\sigma(E_i) \in \mathcal{C}$ and $\delta = -K_X$.

By Proposition 2.20, we notice that $h_{\mathcal{C}, \delta}(x) > 0$ for any $x \in \mathcal{C} \setminus \{0\}$. In particular $h_{\mathcal{C}, \delta}(\sigma(E_i)) > 0$ and then $c_i > -1$. □

To complete what we know on klt singularities of spherical varieties, we note that the author prove in [Pas16] the following result, by using Bott-Samelson resolutions of flag varieties.

Theorem 5.7. *Let X be a horospherical variety and let D be an effective \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier. The pair (X, D) is klt if and only if $[D] = 0$.*

The author does not know if this result could be generalized to spherical varieties.

6. Conclusion and example

To complete the natural connections between these singularities we can also state the following result.

Proposition 6.1. *Any locally factorial spherical variety has terminal singularities.*

And any Gorenstein spherical variety has canonical singularities.

Remark 6.2. By [KM98, Corollary 5.24], any variety with rational and Gorenstein singularities have canonical singularities. But, spherical varieties are rational, then the second assertion of Proposition 6.1 is already known.

Remark 6.3. Remark 6.2 cannot be made for the first assertion of Proposition 6.1. Indeed, C. Casagrande points the author the following example of a locally factorial variety with canonical but not terminal singularities, which is the variety X' described in [JPR05, Table A.4, No.25] with $r = 2$, $r' = 1$ and $d = 6$. It is constructed as follows. Let Y be a smooth cubic threefold in \mathbb{P}^4 and let S be a general linear section of Y , then S is smooth cubic surface. Let C be a general smooth curve in $|(-K_Y)|_S|$ and let $\phi : X \rightarrow Y$ be the blow-up of C in Y , then $-K_X$ is globally generated and the anticanonical morphism $\phi : X \rightarrow X'$ contracts the strict transform D of S by ϕ to a point. By construction, X' has canonical but not terminal singularities, and is a \mathbb{Q} -factorial Fano variety. Moreover, there exists a curve C' in S such that $-K_S \cdot C' = 1$, then the strict transform C'' of C' by ϕ satisfies $-K_X \cdot C'' = 0$ and we have $D \cdot C'' = -1$. Hence, we deduce that X' is locally factorial.

Proof. Fix a spherical homogeneous space G/H .

Let X be a locally factorial G/H -embedding and let $f : V \rightarrow X$ be a G -equivariant resolution of X . Let E_i be an exceptional divisor of f . And let $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ such that $\sigma(E) \in \mathcal{C}$. By Proposition 3.3, the elements of \mathcal{F} have distinct images in $N_{\mathbb{Q}}$ and there exists a basis $(u_1, \dots, u_k) \cup (\sigma(D))_{D \in \mathcal{F}}$ of the lattice N such that \mathcal{C} is generated by the family $(u_1, \dots, u_{k'}) \cup (\sigma(D))_{D \in \mathcal{F}}$ (where $k' \leq k$ are non-negative integers). In particular, $\sigma(E_i) = \sum_{j=1}^{k'} \lambda_j u_j + \sum_{D \in \mathcal{F}} \lambda_D \sigma(D)$, where the λ_j 's and the λ_D 's are non-negative integers. Moreover, either at least two of these integers are not zero, or only one λ_D is not zero, because $\sigma(E_i)$ is a primitive element different from the u_i 's. In the second case, we must have $\sigma(D)$ in \mathcal{V} so that, by Corollary 2.21, $a_D \geq 2$.

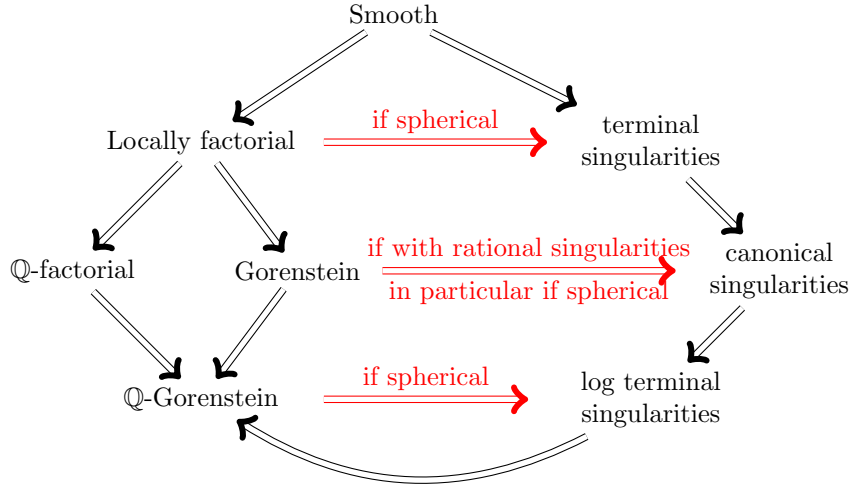
Hence, $h_{\mathcal{C}}(\sigma(E_i)) = \sum_{j=1}^{k'} \lambda_j + \sum_{D \in \mathcal{F}} \lambda_D a_D$ is at least two. We conclude by Proposition 5.2.

The proof of the second assertion is easier.

Let X be a Gorenstein G/H -embedding and let $f : V \rightarrow X$ be a G -equivariant resolution of X . Let E_i be an exceptional divisor of f . And let $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ such that $\sigma(E_i) \in \mathcal{C}$ so that $h_{\mathcal{C}}(\sigma(E_i))$ is positive. Then, since X is Gorenstein, $h_{\mathcal{C}}(\sigma(E_i))$ is a positive integer. Hence, $c_i = h_{\mathcal{C}}(\sigma(E_i)) - 1$ is a non-negative integer. It implies that X has canonical singularities. \square

We can now conclude the paper by the following diagram and example.

Figure 1. Relations between the singularities considered in this paper



Example 6.4. We consider the horospherical homogeneous space G/H that we already considered in section 2 (Examples 2.4, 2.7, 2.12, 2.15 and 2.23).

Then we give in Figure 2 a list a colored fans (corresponding to projective G/H -embeddings), by pointing those who are smooth, locally factorial, \mathbb{Q} -factorial or not \mathbb{Q} -factorial, Gorenstein, \mathbb{Q} -Gorenstein or not \mathbb{Q} -Gorenstein, with terminal or canonical singularities, or only with log terminal singularities. When the variety X is not \mathbb{Q} -Gorenstein, we can also precise if there exists, or not, a \mathbb{Q} -divisor D such that the pair (X, D) is klt.

We only write the optimal singularities.

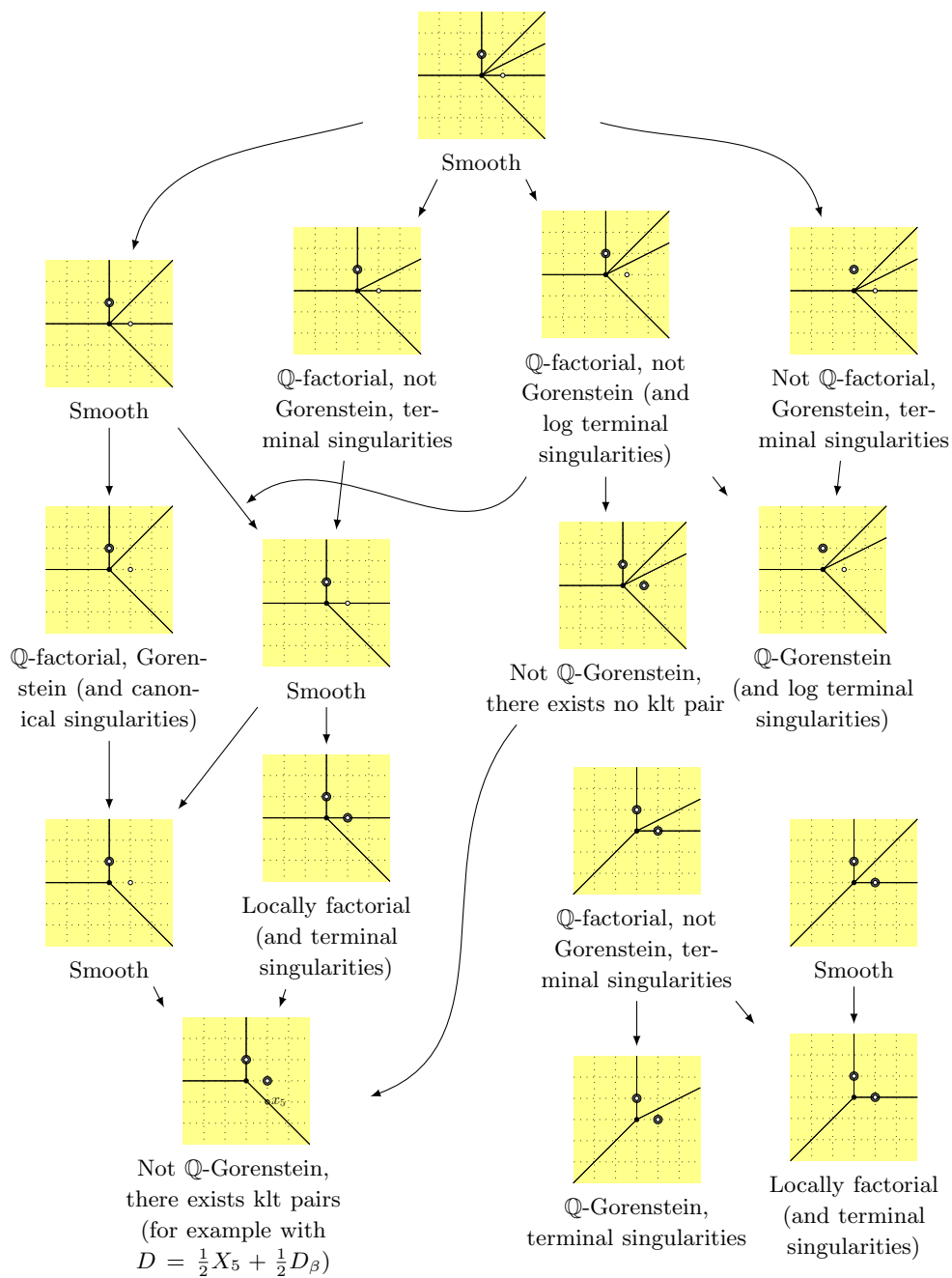
We also represent by arrows all G -equivariant morphisms between these G/H -embeddings.

In this example, we see in particular that there exist horospherical varieties with terminal singularities that are either not Gorenstein or not \mathbb{Q} -factorial. It means that we list all possible implications in Figure 1.

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Figure 2. Singularities of the G/H -embeddings of Example 6.4



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