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Géométrie birationnelle des variétés horosphériques
-
Birational geometry of horospherical varieties

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Au(x) futur(e)(s) à venir.

Contents

Introduction (en français)

Ce mémoire porte uniquement sur l'ensemble de mes travaux sur la géométrie birationnelle des variétés horosphériques ([Pas08], [Pas09], [Pas10a], [PP10], [Pas14a] and [Pas14b]). Je ne parlerai pas ici de mes travaux sur la cohomologie des fibrés en droites sur les variétés de Bott-Samelson (voir [Pas10b]), sur les courbes elliptiques de certains espaces homogènes (en collaboration avec N. Perrin, voir [PP13]), sur la conjecture PRV ou plus généralement sur des problèmes de restriction des représentations à un sous-groupe (en collaboration avec P.L. Montagard et N. Ressayre, voir [MPR11], [MPR15] et [PR13]).

La géométrie birationnelle complexe constitue un des principaux domaines de la géométrie algébrique. Elle comprend notamment l'étude des variétés de Fano et le Programme des Modèles Minimaux (ou théorie de Mori). Le principe de la géométrie birationnelle complexe est d'étudier des variétés algébriques complexes ayant en commun un même ouvert, et de comprendre les différents morphismes qui permettent de passer de l'une à l'autre. L'objectif, très ambitieux, des thérories développées dans ce domaine, est d'obtenir une classification des variétés algébriques complexes.

Pour construire et mieux comprendre les théories mais aussi pour tester des conjectures, on utilise très souvent des familles de variétés (birationnellement équivalentes) assez faciles à étudier.

En particulier, la plus connue et la plus utilisée est celle des variétés toriques de dimension fixée n (avec $n \geq 1$) : un variété algébrique complexe est appelée variété torique de dimension n si elle est normale et si elle est munie d'une action du tore $(\mathbb{C}^*)^n$ de sorte qu'elle ait une $(\mathbb{C}^*)^n$-orbite ouverte isomorphe à $(\mathbb{C}^*)^n$.

Ces variétés sont classifiées en termes d'éventails et leurs propriétés géométriques se caractérisent assez facilement de manière combinatoire sur les éventails associés. En effet, on peut décrire les singularités des variétés toriques et les morphismes $(\mathbb{C}^*)^n$-équivariants entre les variétés toriques. On peut aussi calculer des nombres d'intersections dans les variétés toriques, on peut décrire le cône des courbes effectives des variétés toriques,...

La famille des variétés toriques est encore très utilisée pour illustrer des résultats, des questions ouvertes, et aussi pour donner des exemples ou contre-exemples à certaines théories.

Si les variétés toriques sont assez faciles à étudier et ont une belle combinatoire, c'est simplement grâce au fait qu'elles sont munies de l'action d'un tore de telle sorte que leur corps de fonctions rationnelles (qui ici est aussi celui du tore) est décrit par le réseau des caractères du tore.

Il est donc naturel de se demander si on peut élargir cette famille en faisant agir un groupe non isomorphe à un tore tout en conservant la propriété que le corps des fonctions rationnelles des variétés soit décrit par un réseau de caractères d'un tore du groupe. On obtient ainsi la famille des variétés sphériques : une variété algébrique complexe X est une variété sphérique si elle est normale et si elle est munie d'une action d'un groupe algébrique complexe réductif et connexe G, tel que X ait une G-orbite ouverte isomorphe à G/H où H est un sous-groupe fermé de G tel que BH soit ouvert dans G pour un certain sous-groupe de Borel B de G. Une telle variété a un corps de fonctions rationnelles qui se décrit par un sous-réseau des caractères du sous-groupe de Borel B (ou de manière équivalente d'un tore maximal de G).

La famille des variétés sphériques contient celle des variétés toriques (lorsque $G = (\mathbb{C}^*)^n$ et $H = \{1\}$) et aussi celle des variétés de drapeaux (i.e. les variétés homogènes projectives).

Cette famille, moins utilisée que celle des variétés toriques car plus compliquée, peut pourtant être étudiée de manière analogue. Il existe une classification en termes d'éventails coloriés due à D. Luna et T. Vust, et leurs propriétés géométriques se caractérisent aussi de manière combinatoire sur les éventails coloriés associés grâce aux différents travaux de F. Knop puis de M. Brion. En particulier, on connaît les diviseurs de Cartier, ceux qui sont engendrés par leur sections globales et ceux qui sont amples, on comprend aussi assez bien les courbes des variétés sphériques, on sait y calculer des nombres d'intersections,...

Ces dix dernières années, je me suis particulièrement intéressé à une certaine sous-famille de variétés sphériques : la famille des variétés horosphériques. Cette famille contient aussi celles des variétés toriques et de drapeaux. Les avantages de se focaliser sur ces variétés, plutôt que de considérer toutes les variétés sphériques, sont :

- les espaces homogènes G/H horosphériques se classifient facilement ;
- la combinatoire est plus simple (par exemple le cône de valuation est l'espace tout entier, voir la définition 1.12) ;
- et les variétés horosphériques projectives se plongent ≪ facilement ≫ dans des espaces projectifs, comme adhérence de l'orbite d'une somme de vecteurs de plus haut poids (voir le corollaire 2.11 et la remarque 2.12).

Mes travaux sur les variétés horosphériques m'ont amené à étudier le Programme des Modèles Minimaux (MMP) en termes de polytopes moments. Ce point de vue, nouveau même pour les variétés toriques, permet de faire tourner le MMP de façon algorithmique à partir de toute variété horosphérique projective (pas trop singulière, voir la section 4.4).

Les grands principes du MMP sont les suivants. Soit X une variété algébrique projective normale et ≪ pas trop singulière ≫. Notons $-K_X$ un diviseur anticanonique de X (c'est le diviseur associé au déterminant du fibré tangent sur la partie lisse de X). L'idée première du MMP est de transformer X petit à petit en contractant des courbes strictement positives le long de $-K_X$. Le fait de contracter de telles courbes se traduit par la donnée d'un morphisme de X dans une autre variété normale Y, qui est notamment projectif et à fibres connexes ; un tel morphisme est appellé une contraction. Si au cours du processus on obtient une variété de même dimension que X (et ≪ pas trop singulière ≫) mais qui n'a plus de courbe strictement positive le long du diviseur anticanonique, on dit que cette variété est un modèle minimal. Si par contre, on obtient une variété de dimension plus petite que celle de X, la dernière contraction est alors appelée une fibration de Mori. Aussi, on distingue 2 types de contractions telles que $\dim(X) = \dim(Y)$. Ces contractions sont des morphismes birationnels (i.e. ils définissent des isomorphismes entre des ouverts non vides de X et Y). Le premier type de contractions, appelé type divisoriel, donne une variété Y ≪ pas trop singulière ≫, on peut alors continuer le processus avec Y. Le second type de contractions, appelé type flip, donne une variété Y ≪ trop singulière ≫ qu'on doit partiellement désingulariser en ajoutant des courbes strictement négatives sur le diviseur anticanonique de la nouvelle variété notée X^+. On continue alors le processus avec X^+.

L'existence des contractions de courbes strictement positives le long du diviseur anticanonique est donnée par les théorèmes du cône et de contraction (dus entre autres à Y. Kawamata, J. Kollàr, M. Reid et V.V. Shokurov). Il reste alors trois difficultés afin de pouvoir avoir un processus qui fonctionne et qui finit.

- Il faut d'abord définir convenablement les types de singularités de X et des variétés « pas trop singulières » rencontrées lors du processus. Selon le choix des singularités, quelques détails du MMP peuvent s'en trouver changés. On aura par exemple des différences majeures entre le MMP $\mathbb{Q}$-factoriel et le MMP $\mathbb{Q}$-Gorenstein (aussi appelé MMP non $\mathbb{Q}$-factoriel).
- La deuxième difficulté est l'existence des flips, i.e. l'existence de désingularisations partielles « assez petites » qui ne font apparaître que des courbes strictement négatives sur le diviseur anticanonique.
- La dernière difficulté est la terminaison en temps fini du processus. Pour cela, il ne doit pas exister de suite infinie de flips.

Pour compléter cette théorie, il est aussi nécessaire d'étudier les modèles minimaux et les fibrations de Mori. Je ne parlerai pas ici des modèles minimaux, car il n'y en a pas pour les variétés sphériques, puisque ces variétés sont rationnelles. Autrement dit, pour les variétés sphériques, le MMP se termine toujours par une fibration de Mori. Les fibres générales des fibrations de Mori sont des variétés de Fano (projectives et de nombre de Picard 1). L'étude des variétés de Fano est donc aussi une partie importante du MMP.

Le MMP a été décrit dans le cas des variétés toriques par M. Reid et dans le cas des variétés sphériques par M. Brion (sans la description des fibres générales des fibrations de Mori dans ce dernier cas). Il est utile de noter que, si la variété de départ est torique (respectivement horosphérique et sphérique), toutes les variétés obtenues tout au cours du processus du MMP seront aussi toriques (respectivement horosphériques et sphériques). Leur théorie repose sur une description du cône des courbes effectives de ces variétés. Malheureusement, ce cône n'est pas très facile à calculer en pratique. J'ai donc eu l'idée de proposer un point de vue « dual » pour le MMP dans le cas des variétés horosphériques projectives. Il suffit alors de connaître un diviseur ample pour se ramener à la simple étude d'une famille à un paramètre de polytopes [Pas14a].

Cette famille de polytopes est définie par 3 matrices A, B et C. La matrice A est donnée par la variété X (ou son éventail colorié), la matrice B est une matrice colonne donnée par le diviseur anticanonique de X, et la matrice C est une matrice colonne donnée par le diviseur ample de X choisi. La famille à un paramètre de polytopes $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ est alors définie par

$$Q^\epsilon := \{X \mid AX \geq B + \epsilon C\}.$$

Chaque ligne de l'égalité $AX = B + \epsilon C$ définit un hyperplan affine. On dit que 2 polytopes de cette famille sont dans la même classe si leurs faces sont en bijection de telle sorte que deux faces associées soient exactement dans les mêmes hyperplans affines définis ci-dessus. On montre alors que les classes de polytopes dans la famille $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ définissent une partition finie de $\mathbb{Q}_{\geq 0}$ en intervalles et singletons. Cette partition décrit complètement le MMP à partir de X jusqu'à la fibration de Mori finale. En effet, outre le dernier intervalle (ouvert et de longueur infinie) qui correspond à la classe des polytopes vides, chacun des intervalles et chacun des singletons correspond à une unique variété horosphérique projective. Chaque extrémité ouverte de ces intervalles correspond à une contraction ; en particulier chaque singleton, sauf le dernier, correspond à un flip, et le dernier singleton correspond à une fibration de Mori.

Ce point de vue permet aussi de décrire les fibres générales d'une fibration de Mori entre variétés horosphériques projectives [Pas14a]. Ces fibres générales sont des variétés horosphériques de Fano associées à une ≪ section générale ≫ du dernier polytope de dimension maximale de la famille $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ ci-dessus.

On remarque de plus que les singularités les plus naturelles pour le MMP via l'étude d'une famille de polytopes, sont les singularités $\mathbb{Q}$-Gorenstein, plus générales que les singularités $\mathbb{Q}$-factorielles considérées en général pour le MMP. Le MMP non-$\mathbb{Q}$-factoriel, bien que déjà plus ou moins connu des experts du domaine, n'a pas beaucoup été considéré car il a malheureusement l'inconvénient de faire augmenter le nombre de Picard lors de certains flips, contrairement au MMP $\mathbb{Q}$-factoriel. J'ai cependant trouvé intéressant de détailler et de démontrer les points essentiels de la stratégie du MMP non-$\mathbb{Q}$-factoriel dans le cas général [Pas14b]. Ceci m'a aussi permis de démontrer, dans le cas des variétés sphériques, l'existence des flips non-$\mathbb{Q}$-factoriels et la finitude des suites de flips non-$\mathbb{Q}$-factoriels.

Passons maintenant à l'étude des variétés horosphériques de Fano. Dans ma thèse, j'ai classifié les variétés horosphériques de Fano ($\mathbb{Q}$-Gorenstein et Gorenstein), puis j'ai majoré le nombre de Picard des variétés horosphériques $\mathbb{Q}$-factorielles, Gorenstein de Fano, ainsi que le degré des variétés horosphériques de Fano localement factorielles [Pas08]. Après ma thèse, j'ai complété ces résultats en démontrant une conjecture de L. Bonavero, C. Casagrande, O. Debarre et S. Druel sur le pseudo-indice des variétés lisses de Fano, aux variétés horosphériques $\mathbb{Q}$-factorielles, Gorenstein de Fano [Pas10a]. J'ai aussi décrit toutes les variétés horosphériques projectives lisses de nombre de Picard 1 (et donc de Fano) [Pas09], puis j'ai démontré (en collaboration avec N. Perrin) que toutes ces variétés sauf une sont localement rigides (i.e. leurs structures complexes n'admettent pas de déformation locale) [PP10].

Mes travaux sur les variétés horosphériques de Fano ont été complétés et étendus à d'autres famille de variétés.

A. Ruzzi a étudié les variétés symétriques lisses de nombre de Picard 1, et les variétés symétriques de Fano de petit rang [Ruz10], [Ruz11] and [Ruz12].

G. Gagliardi et J. Hofscheier ont récemment classifié les variétés sphériques de Fano et ont obtenu la même majoration du nombre de Picard pour les variétés sphériques $\mathbb{Q}$-factorielles, Gorenstein de Fano [GH15]. Ces mêmes auteurs ont aussi récemment demontré la conjecture sur le pseudo-indice dans le cas des variétés symétriques [GH14].

K. Langlois et R. Terpereau ont commencé à regarder certaines G-variétés de complexité 1 (non sphérique) en donnant notamment un critère de lissité proche de celui que j'ai donné pour les variétés horosphériques [LT14].

J. Hong a très récemment réalisé presque toutes les variétés horosphériques projectives lisses de nombre de Picard 1, comme sections linéaires de variétés de drapeaux.

Les variétés horosphériques ont aussi été utilisées dans d'autres théories, par exemple par V. Batyrev et A. Moreau [BM13], ou par Q. Li [Li15].

Il reste malgré tout encore beaucoup à faire autour des variétés horosphériques et sphériques en géométrie birationnelle.

On pense bien sûr aux diverses généralisations non encore connues des résultats ci-dessus aux variétés sphériques ou aux G-variétés de complexité 1.

Mais on peut aussi se demander s'il est possible de classifier et d'obtenir une description des variétés horosphériques projectives de nombre de Picard 1 non lisses ($\mathbb{Q}$-factorielles et à singularités terminales par exemple).

Voir la partie 6 pour plus de détail sur ces projets.

Une autre question, plus mystérieuse, peut aussi être posée : que peut nous apprendre le MMP non $\mathbb{Q}$-factoriel sur les variétés horosphériques ou sphériques ?

La description plus détaillée des mes résultats sur la géométrie birationnelle des variétés horosphériques est organisée comme suit.

Dans une première partie, on rappelle les définitions des variétés sphériques et horosphériques avec déjà quelques exemples. On se focalise ensuite sur les variétés horosphériques même si tous les résultats de cette section existent aussi dans le cas général des variétés sphériques. On donne la classification des variétés horosphériques en termes d'éventails coloriés puis on décrit les morphismes G-équivariants entre les variétés horosphériques en termes d'éventails coloriés.

Dans une deuxième partie, on donne une classification des variétés horosphériques projectives polarisées en termes de polytopes moments. Pour cela on utilise les critères dus à M. Brion qui permettent de déterminer si un diviseur est de Cartier, engendré par ses sections globales, ou ample.

Dans une troisième partie, on donne les définitions des différents types de singularités qu'on utilise dans les théories du MMP et des variétés de Fano, avant de donner des critères combinatoires de ces singularités dans le cas des variétés horosphériques.

Dans une quatrième partie, on rappelle les théories du MMP $\mathbb{Q}$-factoriel et du MMP non $\mathbb{Q}$-factoriel. On explique en détail comment faire fonctionner ces MMP pour les variétés horosphériques en utilisant la classification en termes de polytopes moments donnée en deuxième partie. Puis on introduit le programme en SAGE (disponible sur ma page personnelle) qui renvoie les différentes étapes du MMP à partir de n'importe quelle variété horosphérique projective et $\mathbb{Q}$-Gorenstein donnée.

Ces quatre parties sont illustrées par un même exemple (de plongements de SL_3/U).

Dans une cinquième partie, on décrit rapidement les résultats que j'ai obtenus sur les variétés horosphériques de Fano : classification, majorations du nombre de Picard et du degré, résultat sur le pseudo-indice, puis une classification, une description et quelques propriétés des variétés horosphériques projectives lisses de nombre de Picard 1.

Enfin, dans une sixième partie, on décrit quelques projets de recherche, prolongements naturels des résultats exposés dans ce manuscrit.

Introduction (in english)

This manuscript only deals with my works on the birational geometry of horospherical varieties ([Pas08], [Pas09], [Pas10a], [PP10], [Pas14a] and [Pas14b]). Here, I will not talk about my works on the cohomology of line bundles on Bott-Samelson varieties (see [Pas10b]), on elliptic curves of some homogeneous spaces (joint with N. Perrin, see [PP13]), on the PRV conjecture or more generally on branching rules problems (joint with P.L. Montagard and N. Ressayre, see [MPR11], [MPR15] and [PR13]).

Birational complex geometry forms one of the main domains of algebraic geometry. It particularly includes the study of Fano varieties and the Minimal Model Program (or Mori Theory). The principle of birational complex geometry is to study complex algebraic varieties that have a common open set, and to understand the different morphisms between these varieties. The very ambitious aim of the theories developed in this domain, is to obtain a classification of complex algebraic varieties.

To construct and to better understand theories, but also to test conjectures, we often use families of varieties that are quite easy to study.

In particular, the most famous and used one is the family of toric varieties of a given dimension n (with $n \geq 1$): a complex algebraic variety X is said to be toric of dimension n if X is normal and if there is a torus $(\mathbb{C}^*)^n$ acting on X with an open orbit isomorphic to $(\mathbb{C}^*)^n$.

These varieties are classified in terms of fans, and their geometric properties are easily characterized by the combinatorial properties of the associated fans. Indeed, we can describe combinatorially the singularities of toric varieties, and the $(\mathbb{C}^*)^n$-equivariant morphisms between them. We can also compute intersection numbers in toric varieties, we can describe the cone of effective curves of toric varieties,...

The family of toric varieties is still often used to illustrate results, open questions, and also to give examples or counterexamples of some theories.

If toric varieties are so easy to study and have a nice combinatorial setting, that is because there are endowed with a torus action, so that their field of rational functions (which also is the field of rational functions of the torus) is described by the lattice of characters of the torus.

Then, it is natural to wonder if we could expand this family by choosing a group (not necessarily a torus) acting on the varieties so that we still have the property that the field of rational functions of a variety is described by a lattice of characters of a torus of the group. Thus, we obtain the family of spherical varieties: a complex algebraic variety is said to be spherical if it is normal and if it is endowed with an action of a complex, connected and reductive algebraic group G, with an open orbit isomorphic to G/H where H is a closed subgroup of G such that BH is open in G for some Borel subgroup B of G. The field of rational functions of such a variety is described by a sublattice of the lattice of characters of the Borel subgroup B (or equivalently of a maximal torus of G).

The family of spherical varieties contains the family of toric varieties (when $G = (\mathbb{C}^*)^n$ and $H = \{1\}$) and also the family of flag varieties (i.e. projective homogeneous varieties).

This family is not used as soon as the family of toric varieties because of its complexity, but it can be studied in an analogue way. There exists a classification in terms of colored fans due to D. Luna and T. Vust, and several properties of spherical varieties can be characterized combinatorially on the associated colored fans, thanks to different works of F. Knop and M. Brion. In particular, we know Cartier divisors, those who are globally generated and those who are ample, we also understand quite well the curves of spherical varieties, we can compute intersection numbers,...

The last ten years, I was particularly interested in a special subfamily of spherical varieties: the family of horospherical varieties. This family also contains the families of toric varieties and flag varieties. The reasons to focus on these varieties, rather than all spherical varieties are:

- the horospherical homogeneous spaces G/H are very easy to classified;
- the combinatorial setting is simpler (for example the valuation cone is the total space, see Definition 1.12);
- and projective horospherical varieties can be "easily" embedded in projective spaces, as the closure of the orbit of a sum of highest weight vectors (see Corollary 2.11 and Remark 2.12).

My works on horospherical varieties lead me to study the Minimal Model Program (MMP) in terms of moment polytopes. This point of view, even new for toric varieties, enables us to run the MMP algorithmically from any (not too singular) projective horospherical variety (see Section 4.4).

Here are the key features of the MMP. Let X be a normal projective variety, we suppose that X is "not too singular". We denote by $-K_X$ the anticanonical divisor of X (it is the divisor associated to the determinant of the tangent bundle on the smooth part of X). The first idea of the MMP is to operate little changes on X by contracting curves of X that are positive along $-K_X$. By contracting such curves, we obtain a morphism from X to another normal variety Y, that is in particular projective and with connected fibres; such a morphism is called a contraction. If, during the process, we obtain a ("not too singular") variety with the same dimension as X, but that has no curve positive along its anticanonical divisor, then we say that this variety is a minimal model. If we obtain a variety with a smaller dimension, the last contraction is called a Mori fibration. Also, we distinguish two types of contractions such that $\dim(X) = \dim(Y)$. These contractions are birational morphisms (i.e. they define isomorphisms between non-empty open sets of X and Y). The first type of contraction, called divisorial type, give a variety Y that is "not too singular", and then we can continue the process with Y. The second type of contractions, called flipping type, give a "too singular" variety Y, and we need to find a partial desingularization that adds only curves that are negative along the anticanonical divisor of the new variety denoted by X^+. Then we continue the process with X^+.

The existence of contractions of curves that are positive along the anticanonical divisor is given by the Cone Theorem and the Contraction Theorem (due in particular to Y. Kawamata, J. Kollàr, M. Reid and V.V. Shokurov). To get a process that works and ends, we have to overcome three difficulties.

- First, we need to choose appropriately the types of singularities of X and of the "not too singular" varieties that we could meet during the process. There is not only one choice here. And few details could be different in the MMP, according to the chosen type of singularities. For example, we observe serious differences between the $\mathbb{Q}$-factorial MMP and the $\mathbb{Q}$-Gorenstein MMP (also called non-$\mathbb{Q}$-factorial MMP).
- The second difficulty is the existence of flips, i.e. the existence of partial desingularizations "small enough" that only add curves negative along the anticanonical divisor.
- The last one is to know if the process ends in finitely many steps. For this, there could not exist an infinite sequence of flips.

To complete this theory, it is also necessary to study minimal models and Mori fibrations. Here I will not deal with minimal models, because there are no minimal model that are spherical, since spherical varieties are rational. In other words, for spherical varieties, the MMP always ends with a Mori fibration. General fibres of Mori fibrations are Fano varieties (projective and with Picard number one). Hence, the study of Fano varieties also takes an important part of the MMP.

The MMP was described in the case of toric varieties by M. Reid and in the case of spherical varieties by M. Brion (without the description of general fibres in this latter case). It is useful to note that, if the variety we begin with is toric (respectively horospherical and spherical), then all the varieties that appear during the process are also toric (respectively horospherical and spherical). Their theory is based on a description of the cone of effective curves of these varieties. Unfortunately, this cone is not really easy to compute concretely. That is why, I proposed a "dual" point of view for the MMP in the case of projective horospherical varieties. Then, it becomes enough to know an ample divisor to reduce the MMP to a simple study of a one-parameter family of polytopes [Pas14a].

This family of polytopes is defined by three matrices A, B and C. The matrix A is given by the variety X (or the colored fan of X), the matrix B is a column matrix given by the anticanonical divisor of X, and the matrix C is a column matrix given by the chosen ample divisor of X. The one-parameter family of polytopes $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ is then defined by

$$Q^\epsilon := \{X \mid AX \geq B + \epsilon C\}.$$

Each line of the equality $AX = B + \epsilon C$ defines an affine hyperplane. We say that two polytopes of this family are in the same class, if their faces are in bijection such that two associated faces are exactly in the same affine hyperplanes defined above. Then we prove that the classes of polytopes in the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ define a finite partition of $\mathbb{Q}_{\geq 0}$ in segments and points. And this partition describes completely the MMP from X until the final Mori fibration. Indeed, except the last (open and of infinite length) segment which corresponds to the class of empty polytopes, each segment or point corresponds to a unique projective horospherical variety. Each open extremity of a segment corresponds to a contraction; in particular each point, except the last one, corresponds to a flip, and the last isolated point corresponds to the Mori fibration.

This point of view also enables us to describe the general fibres of a Mori fibration between projective horospherical varieties [Pas14a]. These general fibres are Fano horospherical varieties associated to some "general section" of the last polytope of maximal dimension in the above family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$.

Moreover, we notice that the most natural singularities for the MMP via the study of a family of polytopes, are the $\mathbb{Q}$-Gorenstein singularities (more general than the $\mathbb{Q}$-factorial singularities we usually consider in the MMP). Even if it is more or less already known by the specialists of the MMP, the non-$\mathbb{Q}$-factorial MMP has been rarely considered, because the Picard number can unfortunately increase in some flips, contrary to $\mathbb{Q}$-factorial MMP. Nevertheless, I found interesting to detail and prove the essential points of the strategy of the non-$\mathbb{Q}$-factorial MMP in the general context [Pas14b]. This enables me to prove, in the case of spherical varieties, the existence of non-$\mathbb{Q}$-factorial flips and the finiteness of sequences of non-$\mathbb{Q}$-factorial flips.

Consider now the study of Fano horospherical varieties. In my thesis, I classified ($\mathbb{Q}$-Gorenstein and Gorenstein) Fano horospherical varieties, and I gave upper bounds for the Picard number of $\mathbb{Q}$-factorial, Gorenstein Fano horospherical varieties and the degree of locally factorial Fano horospherical varieties [Pas08]. After my thesis, I completed these results, by proving a conjecture of L. Bonavero, C. Casagrande, O. Debarre and S. Druel on the pseudo-index of smooth Fano varieties, in the case of $\mathbb{Q}$-factorial, Gorenstein Fano horospherical varieties [Pas10a]. I also described all smooth, projective, horospherical varieties with Picard number one (and then Fano) [Pas09], and I proved (in a joint work with N. Perrin) that all these varieties, except one, are locally rigid (i.e. they admit no local deformation of their complex structures) [PP10].

My works on Fano horospherical varieties have been completed and extended to others families of varieties.

A. Ruzzi studied smooth symmetric varieties with Picard number one, and Fano symmetric varieties of small rank [Ruz10], [Ruz11] and [Ruz12].

G. Gagliardi and J. Hofscheier recently classified Fano spherical varieties and obtain the same upper bound for the Picard number of $\mathbb{Q}$-factorial, Gorenstein Fano spherical varieties [GH15]. They also recently proved the conjecture on the pseudo-index in the case of symmetric varieties [GH14].

K. Langlois and R. Terpereau began to look at some (not spherical) G-varieties of complexity one, and in particular, they gave a smooth criterion (similar to the one I gave for horospherical varieties) for these varieties [LT14].

J. Hong very recently realized, almost all smooth projective horospherical varieties with Picard number one, as linear sections of some flag varieties.

Horospherical varieties have been also considered, in other contexts, for example by V. Batyrev and A. Moreau in [BM13], or by Q. Li in [Li15].

In spite of all these results, there still are a lot of works to do around the birational geometry of horospherical or spherical varieties.

There are, of course, the different possible generalizations of all above results to spherical varieties or to G-variety of complexity one.

But we could also try to classify and describe not necessarily smooth and projective horospherical varieties with Picard number one ($\mathbb{Q}$-factorial ones with terminal singularities for example).

See Section 6 for more details.

Another more mysterious question can also be arisen: what can we learn from the non-$\mathbb{Q}$-factorial MMP applied to horospherical or spherical varieties?

The following more detailed description of my results on birational geometry of horospherical varieties is organized as follows.

In a first section, we recall the definitions of spherical and horospherical varieties, with first few examples. After that, we focus on horospherical varieties, even if the results of this section also exist in the general case of spherical varieties. We give the classification of horospherical varieties in terms of colored fans, and then we describe the G-equivariant morphisms between horospherical varieties in terms of colored fans.

In a second section, we classify polarized projective horospherical varieties in terms of moment polytopes. For this, we use the criteria due to M. Brion, that determine if a divisor is Cartier, globally generated or ample.

In a third section, we recall the definitions of the different types of singularities that we usually meet in the theories of MMP and Fano varieties, before to give combinatorial criteria to these types of singularities in the case of horospherical varieties.

In a forth section, we recall the theories of $\mathbb{Q}$-factorial MMP and non-$\mathbb{Q}$-factorial varieties. We explain in detail, how to run these MMP's for projective horospherical varieties, by using the classification in terms of moment polytopes given in the second section. Then we introduce the SAGE program (available on my Web Page), that sends the different steps of the MMP from any given $\mathbb{Q}$-Gorenstein projective horospherical variety.

These four sections are illustrated by a common example (of SL_3/U-embeddings).

In the fifth section, we list the results I obtain on Fano horospherical varieties: classification, upper bounds of the Picard number and the degree, result on the pseudo-index, and a classification, a description and some properties of smooth projective horospherical varieties with Picard number one.

We finish in a sixth section with some possible research projects, natural extensions of the results I describe in this manuscript.

1 Spherical and horospherical varieties: definitions, examples and first properties

1.1 Definitions

In all the text, G denotes a connected reductive algebraic group over $\mathbb{C}$.

We give here several equivalent definitions of spherical and horospherical varieties.

Definition 1.1. A G-variety X is an algebraic variety over $\mathbb{C}$ equipped with an action of G on X.

A spherical G-variety is a normal G-variety X such that there exist $x \in X$ and a Borel subgroup B of G satisfying that the B-orbit of x is open in X.

A horospherical G-variety is a normal G-variety such that there exist $x \in X$ and a maximal unipotent subgroup U of G satisfying that the orbit $G \cdot x$ is open in X and U fixes the point x.

Remark that, since the Borel subgroups of G (and the maximal unipotent subgroups of G) are all conjugated, we can fix a Borel subgroup B of G, denote by U the unipotent radical of B, and give the following equivalent definition.

Definition 1.2. A spherical G-variety is a normal G-variety with an open B-orbit.

A horospherical G-variety is a normal G-variety such that there exists $x \in X$ satisfying that the orbit $G \cdot x$ is open in X and U fixes x.

Note that horospherical G-varieties are spherical G-varieties. Indeed, if B^- denotes an opposite Borel of B (i.e. such that $B \cap B^-$ is a maximal torus), $B^- U$ is open in G, so $B^- \cdot x$ is open in $G \cdot x$ and also in X.

When there is no possible confusion about the group acting on varieties, we speak about spherical and horospherical varieties.

We can easily remark that spherical varieties have an open G-orbit, and have finitely many B-orbits and G-orbits. The open G-orbit of a spherical variety X is isomorphic to an homogeneous space G/H that is spherical as a G-variety. We say that G/H is a spherical homogeneous space. Note that, X is horospherical if and only if G/H is horospherical.

We still can give other equivalent definitions of spherical and horospherical varieties, as follows.

We say that two G-varieties X and X' are birationally isomorphic if there exist two non-empty open G-stable subsets U and U' of X and X', and a G-equivariant isomorphism from U to U'.

Proposition 1.3. *[Bri, Théorème 2.1] Let X be a normal G-variety, the following assertions are equivalent.*

1. *X is spherical.*
2. *Every G-variety birationally isomorphic to X has finitely many G-orbits.*
3. *For any G-linearized line bundle $\mathcal{L}$ on X, the G-module $H^0(X, \mathcal{L})$ has no multiplicity.*

The second item motivates to test birational results in the family of spherical varieties. And the third item explain why spherical varieties have a nice combinatorial setting.

Remark 1.4. We could also easily prove a multiplicity free result on the field $\mathbb{C}(X)$ of rational functions of X. It is naturally a G-module. Let χ be a character of B and f_1, f_2 be in $\mathbb{C}(X)\backslash\{0\}$ such that, for any $b \in B$, $b \cdot f_1 = \chi(b) f_1$ and $b \cdot f_2 = \chi(b) f_2$. Then $\frac{f_1}{f_2}$ is a rational function fixed by B. Since B acts with an open orbit on X, we deduce that $\frac{f_1}{f_2}$ is a constant. This means that the G-module $\mathbb{C}(X)$ has no multiplicity.

Proposition 1.5. *Let X be a normal G-variety, the following assertions are equivalent.*

1. *X is horospherical.*

2. *There exists $x \in X$ such that $G \cdot x$ is open X, and the stabilizer H of x in G is the kernel of finitely many characters of a same parabolic subgroup P of G containing B (in particular $U \subset H$).*

3. *There exists an open G-orbit in X, isomorphic to a torus bundle over a flag variety.*

Moreover, in (3), the flag variety is G/P with P given in (2), and the dimension of the torus is the minimal number of characters necessary to define H as in (2).

1.2 First examples of horospherical varieties

We begin by two easy and well-known examples, which are the two extremal cases of Proposition 1.5 (3).

Example 1.6. Flag varieties are horospherical varieties. Indeed, every flag variety is isomorphic to G/P where P is a parabolic subgroup G containing B, and then it also contains U. It is also obvious that, if $x = P/P \in G/P$, then $B^- \cdot x$ is open and U fixes x.

Example 1.7. Toric varieties are horospherical varieties. Here $G = P = B = B^- = T = (\mathbb{C}^*)^n$ and $H = U = \{1\}$. Then it is quite easy to check that any toric variety of dimension n is a horospherical $(\mathbb{C}^*)^n$-variety.

Example 1.8. In $\mathbb{P}^6$, consider the hypersurface X defined by the equation $W_1W_2 - W_3W_4 + W_5W_6 = 0$ (singular at the point $[1,0,0,0,0,0,0]$). Then, X is a projective cone over the Grassmannian $\mathrm{Gr}_{2,4}$ (embedded in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \simeq \mathbb{P}^5$ by the Plücker embedding). In other words, SL_4 acts naturally on $\mathbb{P}(\mathbb{C} \oplus \bigwedge^2 \mathbb{C}^4) \simeq \mathbb{P}^6$ so that X is isomorphic to the closure of the orbit $G \cdot [1, x \wedge y]$ for any linearly independent vectors x and y in $\mathbb{C}^4$. And the stabilizer of $[1, x \wedge y]$ in SL_4 is isomorphic to the kernel of the fundamental weight ϖ_2 in the maximal parabolic subgroup $P(\varpi_2)$ of SL_4.

A lot of other examples of horospherical varieties can be found in [Pas06]. Here we give the example of odd symplectic grassmannians, which belongs to the list (given in [Pas09]) of smooth projective horospherical varieties of Picard number one.

Example 1.9. (The odd symplectic grassmannian) Let $n \geq 2$ be an integer. Pick a skew-symmetric form ω on $\mathbb{C}^{2n+1}$ of maximal index. For any $i \in \{2, \dots, n\}$, denote by $\mathrm{Gr}_\omega(i, 2n+1)$ the variety of ω-isotropic i-dimensional vector subspaces of $\mathbb{C}^{2n+1}$. Then the varieties $\mathrm{Gr}_\omega(i, 2n+1)$ are smooth, projective, of Picard number one and called odd symplectic grassmannians. Moreover they have two orbits under the action of their (not reductive) automorphism group.

These varieties have been especially studied in [Mih07]. And in [Pas09], we noticed that they belongs to the list of smooth projective horospherical varieties of Picard number one. In particular, we can also define them as follows.

Fix a basis $(e_1,\dots,e_{2n+1})$ of $\mathbb{C}^{2n+1}$ such that the skew-symmetric form ω satisfies, for any $j\in\{1,\dots,n\}$, $\omega(e_j,e_{2n-j+1})=-\omega(e_{2n-j+1},e_j)=1$ and $\omega(e_k,e_l)=0$ in any other cases (with $k,\,l\in\{1,\dots,2n+1\}$).

Denote by V the $2n$-dimensional subspace generated by $e_1,\dots,e_{2n}$. The restriction of ω to V defines a complex symplectic group G acting naturally on $\bigwedge^i V$ for any $i\in\{1,\dots,n\}$. Denote by v_i the vector $e_1\wedge\cdots\wedge e_i$ in $\bigwedge^i V$. The stabilizer of the family of lines generated by the v_i is a Borel subgroup B of G. And for any $i\in\{1,\dots,n\}$, v_i is a highest weight vector for B in $\bigwedge^i V$, we denote by ϖ_i its weight, which is a fundamental weight.

Hence, for any $i\in\{2,\dots,n\}$, $\mathrm{Gr}_\omega(i,2n+1)$ is isomorphic to the closure in $\mathbb{P}(\bigwedge^i V\oplus\bigwedge^{i-1}V)(\simeq\mathbb{P}(\bigwedge^i\mathbb{C}^{2n+1}))$ of the G-orbit of $[v_i,v_{i-1}]$. Let us check that it is horospherical. The stabilizer H of $[v_i,v_{i-1}]$ in G is included in the stabilizer of both lines generated by v_i and v_{i-1}, which is a parabolic subgroup P of G that contains B. More precisely, H is the kernel of the character $\varpi_i-\varpi_{i-1}:P\longrightarrow\mathbb{C}^*$. By Proposition 1.5 (2), and assuming here that $\mathrm{Gr}_\omega(i,2n+1)$ is normal, we get that it is horospherical.

Remark 1.10. A horospherical G-variety with more than one G-orbit could be homogeneous under the action of a bigger reductive group. Consider for example $G=\mathrm{SL}_2$ acting on $\mathbb{C}$ trivially and on $\mathbb{C}^2$ as usual. Then these actions induce an action of G on $\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^2)\simeq\mathbb{P}^2$. We can check that we have three G-orbits in $\mathbb{P}^2$: the point $\mathbb{P}(\mathbb{C}\oplus\{0\})$, the projective line $\mathbb{P}(\{0\}\oplus\mathbb{C}^2)$ and the open set $\{[x,y]\mid x\in\mathbb{C}^*,\,y\in\mathbb{C}^2\backslash\{0\}\}$, which is isomorphic to SL_2/U.

This implies that the study of birational geometry of horospherical varieties depends on the choice of the group G. Indeed, $\mathbb{P}^2$ is the only SL_3-variety birationally isomorphic to $\mathbb{P}^2$. But the blow-up of a point in $\mathbb{P}^2$ is another (smooth) projective SL_2-variety birationally isomorphic to $\mathbb{P}^2$.

In [Pas09], two non-homogeneous smooth spherical varieties with two orbits are described, which are not horospherical. Other notable examples of (not horospherical) spherical varieties are symmetric varieties and wonderful varieties.

1.3 A classification in terms of colored fans

The first part of the classification of spherical varieties is originally due to D. Luna and T. Vust. They classify spherical varieties with a fixed open G-orbit, in terms of colored fans [LV83] (see also [Kno91]). In that text, we only use this classification, but it is notable that it has been recently completed by a classification of spherical homogeneous spaces [Los09].

Definition 1.11. Let G/H be an homogeneous space. A G/H-embedding is a couple (X,x), where X is a normal G-variety and x is a point of X such that $G\cdot x$ is open in X and H is the stabilizer of x in G.

Two G/H-embeddings (X,x) and (X',x') are isomorphic if there exists a G-equivariant isomorphism from X to X' that sends x to x'.

Note that, if G/H is a spherical (resp. horospherical) homogeneous space and (X,x) is a G/H-embedding, then X is a spherical (resp. horospherical) variety. Inversely, if X is a spherical (resp. horospherical) G-variety, let x be a point in the open G-orbit of X and let H be the stabilizer in G of x. Then, G/H is a spherical (resp. horospherical) homogeneous space and (X, x) is a G/H-embedding.

By abuse, we often forget the point x and say that X is a G/H-embedding if X is a normal variety with an open G-orbit isomorphic to G/H.

Definition 1.12. Let G/H be a spherical homogeneous space.

1. We denote by M the lattice of weights χ of B such that there exists $f_\chi \in \mathbb{C}(G/H)\backslash\{0\}$ satisfying, for any $b \in B$, $b \cdot f_\chi = \chi(b)f$. Note that, by Remark 1.4, for any $\chi \in M$ the rational function f_χ is unique up to a scalar.

2. The dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M is denoted by N.

3. We denote by $M_{\mathbb{Q}}$ (resp. $N_{\mathbb{Q}}$) the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $N \otimes_{\mathbb{Z}} \mathbb{Q}$).

4. The colors of the spherical homogeneous space G/H are the B-stable divisors of G/H.

5. The set of G-invariant valuations on $\mathbb{C}(G/H)\backslash\{0\}$ (over $\mathbb{Q}$) is a cone $\mathcal{V}$ and there is an injective map $\sigma : \mathcal{V} \longrightarrow N_{\mathbb{Q}}$ defined by, for any $\nu \in \mathcal{V}$ and any $\chi \in M$, $\sigma(\nu)(\chi) = \nu(f_\chi)$ where f_χ is as in (1). The image of $\mathcal{V}$ in $N_{\mathbb{Q}}$ is called the valuation cone of G/H.

6. Any color D of the spherical homogeneous space G/H defines a B-invariant valuation on $\mathbb{C}(G/H)\backslash\{0\}$, and then it defines (similarly to the definition of σ) a point in $N_{\mathbb{Q}}$ that we also denote by $\sigma(D)$. It is called the image of the color D in $N_{\mathbb{Q}}$. (In fact, $\sigma(D) \in N$.)

The rank of the lattice M (which is also the rank of N) is called the rank of G/H (or the rank of X if X is any G/H-embedding).

Remark 1.13.

- The valuation cone of G/H equals $N_{\mathbb{Q}}$ if and only if G/H is horospherical [BP87, Corollaire 5.4].

- If G/H is horospherical, there exists a parabolic subgroup P containing H such that the projection $\pi : G/H \longrightarrow G/P$ is a torus fibration over G/P, and the lattice M is the set of characters of P whose restrictions to H are trivial (see Proposition 1.5). Note also that P can be defined as the normalizer of H in G.

  In particular the colors of an horospherical homogeneous space G/H are the inverse image by π of the Schubert divisors of G/P, they are in bijection with the simple roots α of (G, B, T) such that $-\alpha$ is not a weight of the Lie algebra of P. Denote by D_α the color associated to α. Then, $\sigma(D_\alpha)$ is the restriction of the coroot $\alpha^\vee$ to M.

- Since the colors of a spherical homogeneous space G/H are not G-stable, it could happen that two colors of G/H have the same image in $N_{\mathbb{Q}}$.

From now on, we focus only on horospherical varieties.

Definition 1.14. Let H be a closed subgroup of G that contains U. Denote by P the normalizer of H in G. It is a parabolic subgroup of G that contains B. Let M be the lattice of characters of P whose restrictions to H are trivial. Recall that N is the dual of M, and $M_{\mathbb{Q}}$ (resp. $N_{\mathbb{Q}}$) is the $\mathbb{Q}$-vector space generated by M (resp. N). We denote by $\mathcal{S}_P$ the set of simple roots α of (G, B, T) such that $-\alpha$ is not a weight of the Lie algebra of P. Then for any $\alpha \in \mathcal{S}_P$, denote by $\alpha^\vee_M \in N$ the restriction $\alpha^\vee_M$ of the coroot $\alpha^\vee$.

1. A colored cone in $N_\mathbb{Q}$ is a couple $(\mathcal{C},\mathcal{F})$ where $\mathcal{F}$ is a subset of $\mathcal{S}_P$ and $\mathcal{C}$ is a cone in $N_\mathbb{Q}$ generated by finitely many elements of N and the $\alpha_M^\vee$ with α in $\mathcal{F}$ such that, $\alpha_M^\vee\neq 0$ for any $\alpha\in\mathcal{F}$ and $\mathcal{C}$ contains no line.

2. A colored face of a colored cone $(\mathcal{C},\mathcal{F})$ is a couple $(\mathcal{C}',\mathcal{F}')$ where $\mathcal{C}'$ is a face of the cone $\mathcal{C}$ and $\mathcal{F}'=\{\alpha\in\mathcal{F}\mid\alpha_M^\vee\in\mathcal{C}'\}$. It is in particular a colored cone.

3. A colored fan in $N_\mathbb{Q}$ is a finite set $\mathbb{F}$ of colored cones such that: any colored faces of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$, and for any $u\in N_\mathbb{Q}$ there exists at most one colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}$ such that u is in the relative interior of $\mathcal{C}$.

4. A fan is complete if $\bigcup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}}\mathcal{C}=N_\mathbb{Q}$.

Note that if $G/H=(\mathbb{C}^*)^n$, then $\mathcal{S}_P$ is empty (i.e. the set of colors of $(\mathbb{C}^*)^n$ is empty) and the definitions of colored cones and colored fans are equivalent to the definitions of cones and fans in toric geometry.

To any G/H-embedding (X,x), we associate a colored fan as follows.

For any G-orbit Y of X, denote by X_Y the G-stable subset $\{x\in X\mid\overline{G\cdot x}\supset Y\}$. Denote by $\mathcal{D}_{G,Y}$ the set of G-stable irreducible divisors in X_Y and denote by $\mathcal{F}_Y$ the set of $\alpha\in\mathcal{S}_P$ such that the closure of D_α in X_Y contains Y. For any $D\in\mathcal{D}_{G,Y}$, denote by $\sigma(D)$ the image by σ of the G-invariant valuation on $\mathbb{C}(G/H)^*$ associated to D. Denote by $\mathcal{C}_Y$ the cone in $N_\mathbb{Q}$ generated by the $\sigma(D)$ with $D\in\mathcal{D}_{G,Y}$ and the $\alpha_M^\vee$ with $\alpha\in\mathcal{F}_Y$.

We can now state the classification of G/H-embeddings when G/H is horospherical.

Theorem 1.15 ([Kno91]). *Let $U\subset H\subset G$. Let (X,x) be a G/H-embedding. Then, for any G-orbit Y of X, $(\mathcal{C}_Y,\mathcal{F}_Y)$ is a colored cone in $N_\mathbb{Q}$ and the set of $(\mathcal{C}_Y,\mathcal{F}_Y)$ with Y in the set of G-orbits of X is a colored fan in $N_\mathbb{Q}$. It is called the colored fan of X and denoted by $\mathbb{F}_X$.*

The map from the set of isomorphic classes of G/H-embeddings to the set of colored fans in $N_\mathbb{Q}$ that sends the class of (X,x) to $\mathbb{F}_X$ is well-defined and bijective.

Moreover, X is complete if and only if $\mathbb{F}_X$ is complete.

Since horospherical homogeneous spaces are easy to classify, we deduce the following general classification of horospherical varieties.

Definition 1.16. Assume that G and B are fixed.

A couple (X,x) is said to be a G-horospherical embedding if $G\cdot x$ is open in X and the stabilizer of x in G contains U. (In particular, X is horospherical.)

Two G-horospherical embeddings (X,x) and (X',x') are isomorphic if there exists a G-equivariant isomorphism from X to X' that sends x to x'. (In particular, $G\cdot x$ and $G\cdot x'$ are isomorphic homogeneous spaces.)

Corollary 1.17. *Let (X,x) be a G-horospherical embedding. Denote by P the normalizer in G of the stabilizer H of x in G. Let M be the lattice of characters of P whose restrictions to H are trivial. And denote by $\mathbb{F}_X$ the colored fan of X in $N_\mathbb{Q}$ (defined as above).*

Then this defines a bijective map from the set of isomorphic classes of G-horospherical embeddings to the set of triples $(P,M,\mathbb{F}_X)$, where P is a parabolic subgroup of G containing B, M is a sublattice of the lattice $\mathfrak{X}(P)$ of characters of P, and $\mathbb{F}_X$ is a colored fan of $N_\mathbb{Q}$.

In the rest of the text, we denote by $\mathcal{F}_X$ the set of colors of $\mathbb{F}_X$, i.e. the union $\bigcup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}_X}\mathcal{F}$.

Example 1.18. Consider $G=\mathrm{SL}_4$, the maximal parabolic subgroup P of G such that $\mathcal{S}_P=\{\alpha_2\}$ and $M=\mathfrak{X}(P)$. Then, (P,M) is associated to the subgroup H of G defined by the kernel of the fundamental weight ϖ_2 in P. Thus, $N\simeq\mathbb{Z}$ and G/H has one color D_{α_2} whose image in N is $\alpha_{2M}^\vee=1\in\mathbb{Z}\simeq N$. We exactly get two complete colored fans: $\mathbb{F}=\{(\{0\},\emptyset),(\mathbb{Q}_{\geq 0},\{\alpha_2\}),(\mathbb{Q}_{\leq 0},\emptyset)\}$ and $\mathbb{F}'=\{(\{0\},\emptyset),(\mathbb{Q}_{\geq 0},\emptyset),(\mathbb{Q}_{\leq 0},\emptyset)\}$.

Then the horospherical G-variety X associated to the triple $(P,M,\mathbb{F})$ is the projective cone over the grassmannian $\mathrm{Gr}(2,4)$ that we already consider in Example 1.8. Note that the closure of the divisor D_{α_2} in X is an hyperplane section of X that contains the vertex of the cone.

And the horospherical G-variety X' associated to the triple $(P,M,\mathbb{F}')$ is obtained by blowing up the vertex of the projective cone X. In that case, the closure of the divisor D_{α_2} in X' does not contains the exceptional divisor of the blow-up, which is a closed SL_4-orbit.

1.4 G-equivariant morphisms

To motivate our focus on G-equivariant morphisms, we recall a result due to A. Blanchard [Bla56, Prop. I.1] in the setting of complex geometry, whose proof can be adapted to the setting of algebraic geometry.

Proposition 1.19. *[BSU13, Prop. 4.2.1] Let $f:X\longrightarrow Y$ be a proper morphism between algebraic varieties such that $f_*(\mathcal{O}_X)=\mathcal{O}_Y$. If a connected algebraic group G acts on X, then there exists a unique action of G on Y such that f is G-equivariant.*

A consequence of this, is that every contraction (projective morphism $f:X\longrightarrow Y$ such that $\phi_*(\mathcal{O}_X)=\mathcal{O}_Y$) from a spherical (resp. horospherical) G-variety goes to a spherical (resp. horospherical) G-variety.

We now give the description of G-equivariant morphisms between horospherical G-varieties (due for spherical varieties to F. Knop [Kno91]).

Let $f:X\longrightarrow Y$ be a G-equivariant morphism between two G-horospherical embeddings (X,x) and (Y,y) that sends x to x'. Denote by H the stabilizer of x in G and by H' the stabilizer of y in G. In particular $H\subset H'$. Then f induces a G-equivariant projection $\pi:G/H\longrightarrow G/H'$. Moreover the normalizer P of H in G is contained in the normalizer P' of H' in G, and π induces an injective morphism $\pi^*:M'\longrightarrow M$ where M and M' are respectively the lattices of characters of P trivial on H, and of characters of P' trivial on H'. We denote by $\pi_*:N\longrightarrow N'$ the projection obtained by dualizing π^*. Note also that $\mathcal{S}_{P'}\subset\mathcal{S}_P$; $\forall\alpha\in\mathcal{S}_{P'}$, $\alpha_{M'}^\vee=\pi_*(\alpha_M^\vee)$; and $\forall\alpha\in\mathcal{S}_P\backslash\mathcal{S}_{P'}$, $\pi_*(\alpha_M^\vee)=0$.

Conversely, if $U\subset H\subset H'$, we can define P, P', M, M', π^*, N, N' and π_* as above. And we get the following characterization.

Proposition 1.20. *Let (X,x) and (Y,y) be two G-horospherical embeddings associated to the triples $(P,M,\mathbb{F}_X)$ and $(P',M',\mathbb{F}_Y)$ respectively. Then, there exists a G-equivariant morphism $f:X\longrightarrow Y$ with $f(x)=y$ if and only if the two following assertions are satisfied:*

1. *$P\subset P'$ and M' is a sublattice of M;*
2. *for any colored cone $(\mathcal{C},\mathcal{F})\in\mathbb{F}_X$, there exists a colored cone $(\mathcal{C}',\mathcal{F}')\in\mathbb{F}_Y$ such that $\pi_*(\mathcal{C})\subset\mathcal{C}'$ and $\mathcal{F}\cap\{D_\alpha\mid\alpha\in\mathcal{S}_{P'}\}\subset\mathcal{F}'$.*

In the case of G-equivariant birational morphisms (i.e. when $H = H'$, or when $P = P'$ and $M = M'$), we can rewrite the proposition as follows.

Corollary 1.21. *Let G/H be a horospherical homogeneous space and let (X, x), (Y, y) be two G/H-embeddings. Then, there exists a G-equivariant morphism $f : X \longrightarrow Y$ with $f(x) = y$ if and only if for any colored cone $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, there exists a colored cone $(\mathcal{C}', \mathcal{F}') \in \mathbb{F}_Y$ such that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{F} \subset \mathcal{F}'$.*

In other words, a dominant G-equivariant birational morphism $f : X \longrightarrow Y$ exists if and only if $\mathbb{F}_X$ is obtained from $\mathbb{F}_Y$ by subdividing the cones of $\mathbb{F}_Y$ and by deleting colors of $\mathbb{F}_Y$. In particular, we obtain a desingularization of a horospherical variety by subdividing cones and deleting colors in its colored fan.

Below, we give an example of birational morphisms for a particular horospherical homogenous space of rank 2.

Example 1.22. Let $G = \mathrm{SL}_3$. We consider the horospherical homogenous space such that the parabolic subgroup P is a Borel subgroup and M is the lattice $\mathfrak{X}(P)$ of characters of P. In other words, we consider the homogenous space SL_3/U, where U is a maximal unipotent subgroup of G (for example upper triangular matrices with ones on the diagonal). Denote by α and β the two simple roots of G. Then M is generated by the fundamental weights ϖ_α and ϖ_β and N is generated by $\alpha^\vee_M = \alpha^\vee$ and $\beta^\vee_M = \beta^\vee$.

We represent $N_\mathbb{Q}$ with the two colors of G/H as follows.

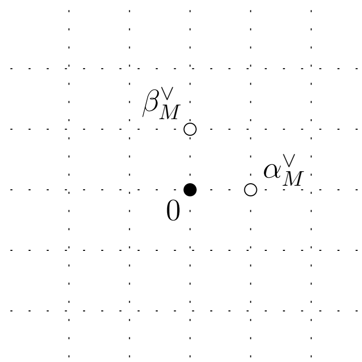
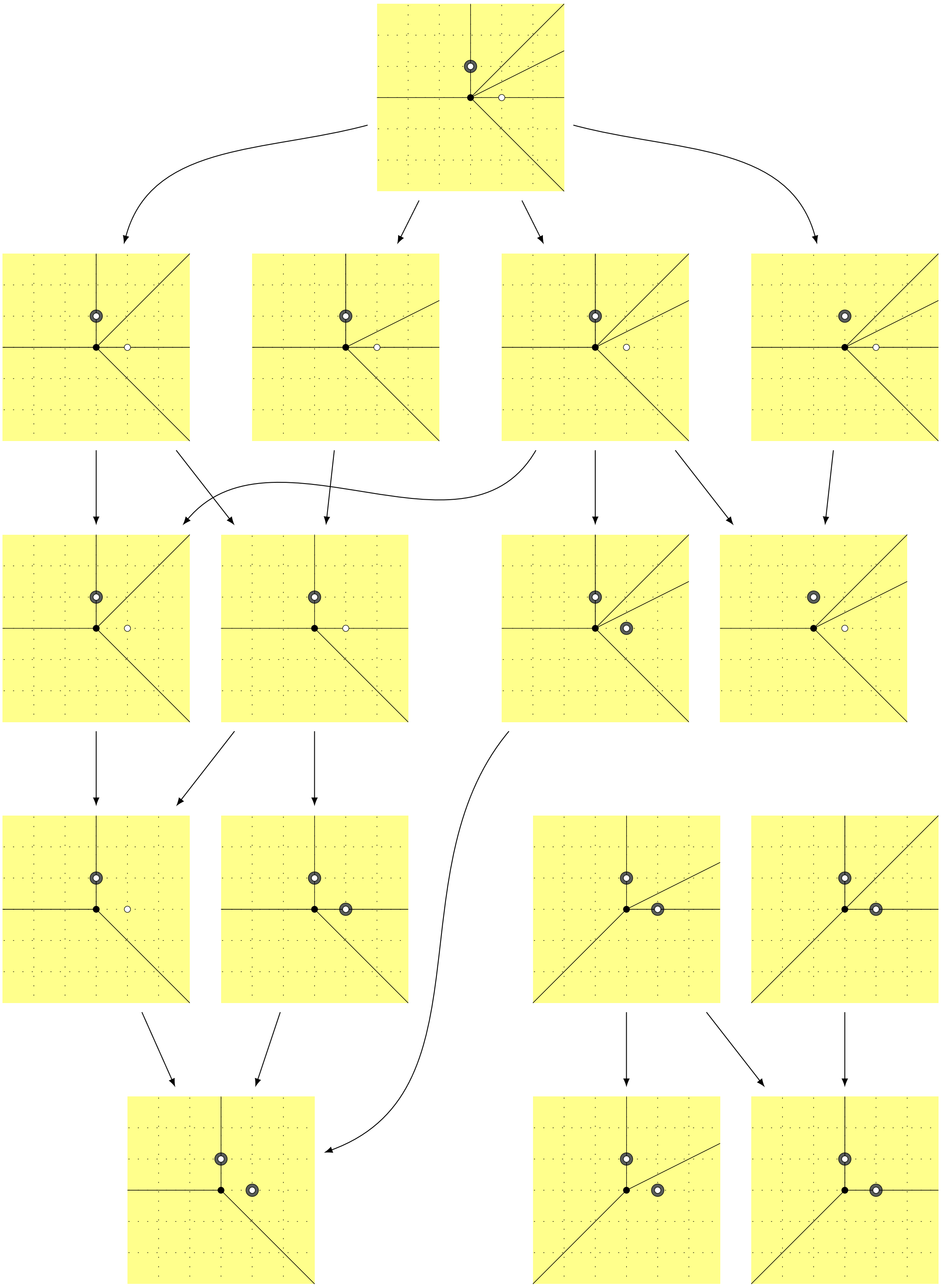


Figure 1: The lattice N for SL_3/U with the images of the two colors of SL_3/U

We will only consider complete colored fans. Note that, here, the images of the colors are all distinct. Then, to represent a colored fan, we only draw the edges of the fan and we represent a color of the fan by bordering in grey the white circle corresponding to the image of the color of G/H. In Figure 2, we draw several SL_3/U-embeddings and all (birational) SL_3-equivariant morphisms between them.

Figure 2: Birational SL_3-equivariant morphisms between some SL_3/U-embeddings



2 Projective horospherical varieties

In this section, we give a classification of projective horospherical varieties in terms of polytopes. In fact, we classify polarized projective horospherical varieties.

Recall that a divisor or a $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier if there exists a positive integer k such that kD is a Cartier divisor.

Definition 2.1. A polarized projective G-horospherical embedding is a triple (X, x, D), where (X, x) is a G-horospherical embedding and D is a B-stable and ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of X. Two polarized projective G-horospherical embeddings are isomorphic if the two G-horospherical embeddings are isomorphic and the divisors are linearly equivalent.

Before to classify polarized projective G-horospherical embeddings, we recall the criteria of Cartier, globally generated and ample divisors of horospherical varieties (due to M. Brion, and also known in the more general case of spherical varieties [Bri89]).

Recall that a G-horospherical embedding (X, x) is associated to a triple $(P, M, \mathbb{F}_X)$ from which we define H, N, σ, ...

2.1 Divisors of horospherical varieties

Let (X, x) be a complete G-horospherical embedding (associated to a triple $(P, M, \mathbb{F}_X)$). Denote by $X_1, \dots, X_m$ the irreducible G-stable divisors of X ($m \geq 0$). We still denote by D_α, with $\alpha \in \mathcal{S}_P$ the irreducible B-stable divisors of X that are not G-stable (i.e. the closures in X of the colors D_α of G/H). For any $i \in \{1, \dots, m\}$, we denote by x_i the image by σ of the valuation associated to X_i. Recall that it is a primitive element of an edge of $\mathbb{F}_X$ that has no color (i.e. contains no $\alpha_M^\vee$ with $\alpha \in \mathcal{F}_X$).

Proposition 2.2. *1. Any Weil divisor of X is linearly equivalent to a B-stable divisor, i.e. of the form $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha \in \mathcal{S}_P} b_\alpha D_\alpha$.*

2. Such a divisor is Cartier if and only if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$ there exists $\chi \in M$ such that for any $x_i \in \mathcal{C}$, $\langle \chi, x_i \rangle = a_i$ and for any $\alpha \in \mathcal{F}$, $\langle \chi, \alpha_M^\vee \rangle = b_\alpha$.

Then any Cartier divisor $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha \in \mathcal{S}_P} b_\alpha D_\alpha$ of a complete horospherical variety X is associated to a unique piecewise linear function h_D, linear on each cone in $\mathbb{F}_X$, such that $\forall i \in \{1, \dots, m\}$, $h_D(x_i) = a_i$ and $\forall \alpha \in \mathcal{F}_X$, $h_D(\alpha_M^\vee) = b_\alpha$. In that case, for any maximal cone $\mathcal{C}$ of $\mathbb{F}_X$ (i.e. for any maximal colored cone $(\mathcal{C}, \mathcal{F})$), we denote by $\chi_{\mathcal{C},D}$ the element of M associated to the linear function defining h_D on $\mathcal{C}$.

Definition 2.3. The piecewise linear function h_D is convex if for any maximal cone $\mathcal{C}$ of $\mathbb{F}_X$, and any $x \in N_\mathbb{Q}$, we have $h_D(x) \geq \langle \chi_{\mathcal{C},D}, x \rangle$.

We say that it is strictly convex if for any maximal cone $\mathcal{C}$ of $\mathbb{F}_X$, and any $x \in N_\mathbb{Q} \backslash \mathcal{C}$, we have $h_D(x) > \langle \chi_{\mathcal{C},D}, x \rangle$.

Proposition 2.4. *A Cartier divisor $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha \in \mathcal{S}_P} b_\alpha D_\alpha$ of a complete horospherical variety X is globally generated (respectively ample) if and only if h_D is convex (respectively strictly convex) and for any $\alpha \in \mathcal{S}_P \backslash \mathcal{F}_X$, $h_D(\alpha_M^\vee) \leq b_\alpha$ (respectively $h_D(\alpha_M^\vee) < b_\alpha$).*

Remark 2.5. 1. Any globally generated Cartier divisor D of a complete horospherical variety X is linearly equivalent to an effective B-stable divisor. Indeed, we can suppose that D is B-stable (Proposition 2.2), and then if we pick any maximal cone $\mathcal{C}$ of $\mathbb{F}_X$, and any B-eigenvector $f_{\chi_{\mathcal{C},D}}$ in $\mathbb{C}(X)$ of weight $\chi_{\mathcal{C},D}$ (unique up to scalar, see Definition 1.12(1)) then $D + \mathrm{div}(f_{\chi_{\mathcal{C},D}}) = D + \sum_{i=1}^m \langle \chi_{\mathcal{C},D}, x_i\rangle X_i + \sum_{\alpha\in\mathcal{S}_P} \langle \chi_{\mathcal{C},D}, \alpha_M^\vee\rangle D_\alpha$ is effective.

2. Let D be an effective B-stable Cartier divisor of a complete horospherical variety X. Then D is globally generated if and only if $Q_D^* := \{u \in N_\mathbb{Q} \mid h_D(u) \leq 1\}$ is convex and contains the points $\frac{\alpha_M^\vee}{b_\alpha}$ with $\alpha \in \mathcal{S}_P$. And D is moreover ample if and only if Q_D^* is "strictly convex with respect to $\mathbb{F}_X$" (i.e. the maximal cones of $\mathbb{F}_X$ are the closures of the cones generated by the faces of Q_D^*, plus the direction cone of Q_D^* if it is of maximal dimension, and the colors of $\mathbb{F}_X$ are the $\alpha \in \mathcal{S}_P$ such that $\frac{\alpha_M^\vee}{b_\alpha}$ is in the boundary of Q_D^*).

3. In particular, for any globally generated effective B-stable Cartier divisor $D = \sum_{i=1}^m a_i X_i + \sum_{\alpha\in\mathcal{S}_P} b_\alpha D_\alpha$ of a complete horospherical variety X, the set Q_D^* is a polyhedron whose dual $\tilde{Q}_D := \{v \in M_\mathbb{Q} \mid \forall u \in Q_D^*, \langle v, u\rangle \geq -1\}$ is also defined by

$$\tilde{Q}_D = \{v \in M_\mathbb{Q} \mid \forall i \in \{1, \dots, m\}, \langle v, x_i\rangle \geq -a_i \text{ and } \forall \alpha \in \mathcal{F}_X \ \langle v, \alpha_M^\vee\rangle \geq -a_\alpha\}.$$

4. If moreover, D is strictly effective (i.e. $\forall i \in \{1, \dots, m\}$, $a_i > 0$ and $\forall \alpha \in \mathcal{F}_X$, $b_\alpha > 0$), then Q_D^* is a polytope. And if D is ample and strictly effective, the colored fan of X is the set of colored cones $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C}$ is the cone generated by a face F of Q_D^* and $\mathcal{F}$ is the set of $\alpha \in \mathcal{S}_P$ such that $\frac{\alpha_M^\vee}{b_\alpha}$ is in F.

Remark 2.5 motivates the classification of polarized horospherical varieties in terms of polytopes in $M_\mathbb{Q}$ (or in $\mathfrak{X}(P)$). But, it also motivates the classification of Fano horospherical varieties in terms of polytopes in $N_\mathbb{Q}$.

Note that it is easy to generalize this section to $\mathbb{Q}$-divisor, $\mathbb{Q}$-Cartier divisor and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Example 2.6. Consider the horospherical SL_3/U-embeddings of Example 1.22. And for any of these horospherical varieties X, denote by $-K_X$ the strictly effective divisor of X defined by the sum of G-stable divisors of X plus $2D_\alpha + 2D_\beta$. We will see in Proposition 3.7 that $-K_X$ is in fact the anticanonical divisor of X.

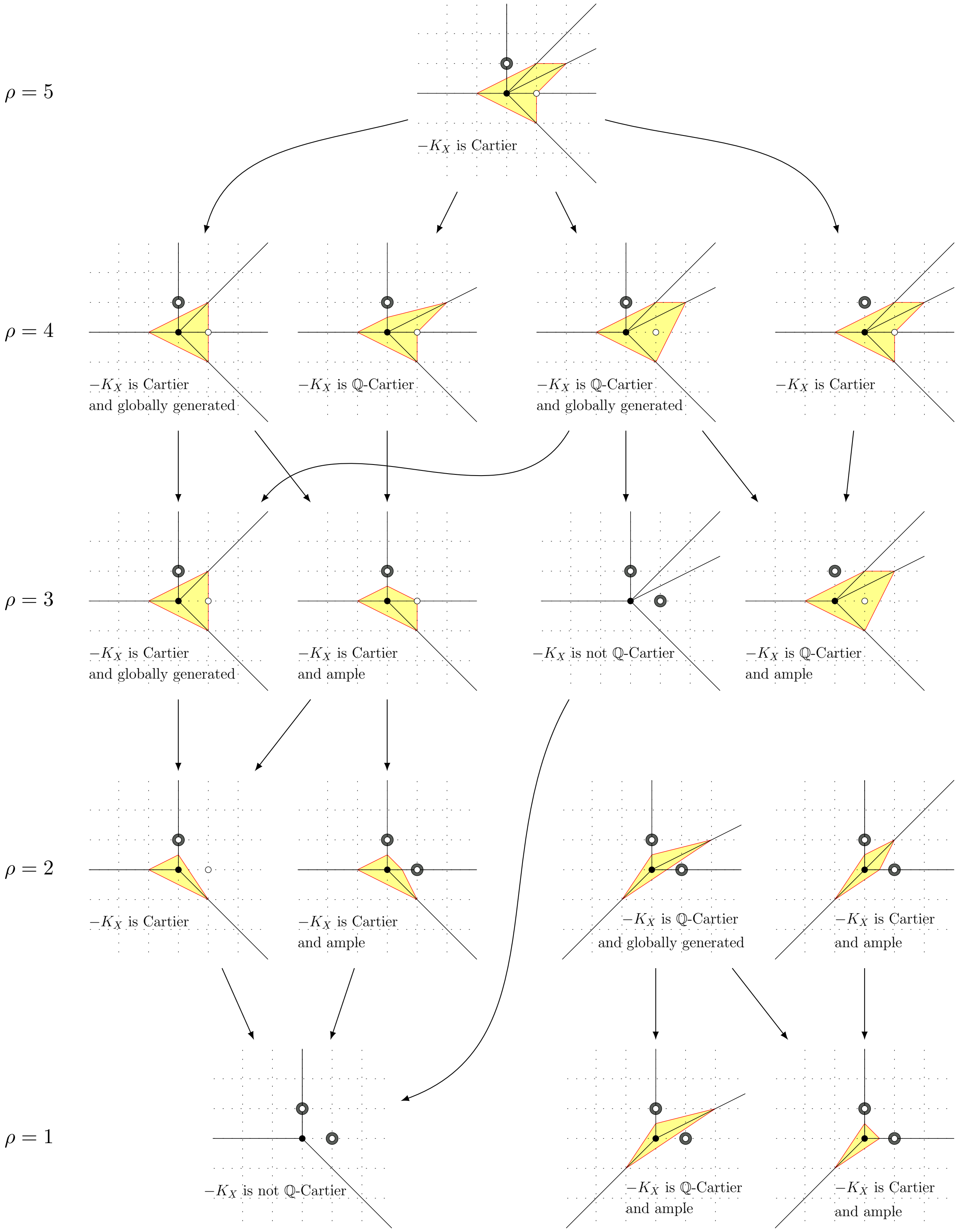
In Figure 3, when $-K_X$ is $\mathbb{Q}$-Cartier, we draw the sets $Q_{-K_X}^*$. Also notice that the rank ρ of the Picard group of X equals the height where we represent the colored fan of X.

2.2 A classification in terms of polytopes

In Corollary 1.17, we gave a classification of G-horospherical embeddings in terms of triples consisting of a parabolic subgroup of G, a sublattice of characters and a colored fan. Now, we give a classification of polarized projective G-horospherical embeddings in terms of triples consisting of a parabolic subgroup of G, a sublattice of characters and a polytope.

Recall that, for any parabolic subgroup P of G containing B, $\mathfrak{X}(P)$ denotes the set of characters of P. We denote by $\mathfrak{X}(P)^+$ the set of dominant weights of P. Also denote by $\mathfrak{X}(P)_\mathbb{Q}$ and $\mathfrak{X}(P)_\mathbb{Q}^+$ the respective vector space and cone. For any simple root α of (G, B, T), we denote by ϖ_α the fundamental weight associated to α. In particular, $\mathfrak{X}(P)_\mathbb{Q}^+ = \{\sum_{\alpha\in\mathcal{S}_P} \lambda_\alpha \varpi_\alpha \mid \forall \alpha \in \mathcal{S}_P, \lambda_\alpha \in$

Figure 3: The anticanonical divisor of some SL_3/U-embeddings



$\mathbb{Q}_{\geq 0}\}$. The cone $\mathfrak{X}(P)^+_{\mathbb{Q}}$ is called the dominant chamber of P, and for any $\alpha \in \mathcal{S}_P$ the set $\{\chi \in \mathfrak{X}(P)^+_{\mathbb{Q}} \mid \langle \chi, \alpha^\vee \rangle = 0\}$ is called a wall of the dominant chamber.

Definition 2.7. A moment triple is a triple (P, M, Q) where P is a parabolic subgroup of G containing B, M is a sublattice of $\mathfrak{X}(P)$, and Q is a polytope in $\mathfrak{X}(P)^+_{\mathbb{Q}}$ that satisfies the two following conditions.

1. Q intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$.
2. There exists $\varpi \in \mathfrak{X}(P)_{\mathbb{Q}}$ such that $\varpi + Q$ is a polytope of maximal dimension in $M_{\mathbb{Q}}$ (i.e. it is contained in $M_{\mathbb{Q}}$ and its interior in $M_{\mathbb{Q}}$ is not empty).

We say that (P, M, Q) and (P', M', Q') are in the same class if $P = P'$, $M = M'$ and if there exists $\chi \in M_{\mathbb{Q}}$ such that $Q = \chi + Q'$.

Proposition 2.8. *Let (X, x, D) be a polarized projective G-horospherical embedding. Denote by $(P, M, \mathbb{F}_X)$ the triple associated to (X, x) (see Corollary 1.17).*

If $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha \in \mathcal{S}_P} d_\alpha D_\alpha$, then denote $\varpi_D = \sum_{\alpha \in \mathcal{S}_P} d_\alpha \varpi_\alpha \in \mathfrak{X}(P)_{\mathbb{Q}}$. And define the pseudo-moment polytope of (X, x, D) by

$$\tilde{Q}_D := \{v \in M_{\mathbb{Q}} \mid \forall i \in \{1, \dots, m\},\ \langle v, x_i \rangle \geq -d_i \ \text{ and } \forall \alpha \in \mathcal{S}_P,\ \langle v, \alpha_M^\vee \rangle \geq -d_\alpha\}.$$

And define the moment polytope of (X, x, D) by $Q_D := \varpi_D + \tilde{Q}_D$.

Then the pseudo-moment polytope of (X, x, D) is a polytope of maximal dimension in $M_{\mathbb{Q}}$. And the moment polytope of (X, x, D) is contained in $\mathfrak{X}(P)^+_{\mathbb{Q}}$ and intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$.

Moreover, the map from the set of isomorphic classes of polarized projective G-horospherical embeddings to the set of classes of moment triples, that maps (X, x, D) to (P, M, Q_D) as above, is a well-defined bijection.

The construction of $\mathbb{F}_X$ from a moment triple (P, M, Q) is the following. Let $\varpi \in \mathfrak{X}(P)_{\mathbb{Q}}$ such that $\varpi + Q$ is a polytope in $M_{\mathbb{Q}}$. For any vertex v of Q, we define $\mathcal{C}_v$ to be the cone in $N_{\mathbb{Q}}$ generated by inward-pointing normal vectors of all facets of $\varpi + Q$ containing the vertex $\varpi + v$ of $\varpi + Q$, and we also define $\mathcal{F}_v$ to be the set of $\alpha \in \mathcal{S}_P$ such that $\alpha_M^\vee \neq 0$ and $\langle v, \alpha^\vee \rangle = 0$. Then $\mathbb{F}_X$ is the set consisting of the colored cones $(\mathcal{C}_v, \mathcal{F}_v)$ and their colored faces.

Also, the divisor D can be computed as follows. For any facet F of Q that is not contained in a wall of the dominant chamber of P, the primitive inward-pointing normal vector x_F (in N) of $\varpi + F$ corresponds to an irreducible G-stable divisor X_F of X, and we denote by d_F the value of $-x_F$ on the facet $\varpi + F$ of $\varpi + Q$. If $\varpi = -\sum_{\alpha \in \mathcal{S}_P} d_\alpha \varpi_\alpha$, then $D = \sum_F d_F X_F + \sum_{\alpha \in \mathcal{S}_P} d_\alpha D_\alpha$ (where F lives in the set of facets of Q that are not contained in a wall of the dominant chamber of P).

Note that for any $\alpha \in \mathcal{F}_X$, d_α is also the value of $-\alpha_M^\vee$ on any vertex $\varpi + v$ of $\varpi + Q$ such that $\langle v, \alpha^\vee \rangle = 0$. The two conditions on Q (it intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ and $\varpi + Q^0$ is a polytope of maximal dimension in $M_{\mathbb{Q}}$) implies, with the construction of D, that D is an ample divisor of X.

Moreover, if we choose another element ϖ' in $\mathfrak{X}(P)_{\mathbb{Q}}$ such that $\varpi' + Q$ is contained in M, we clearly get the same colored fan from $\varpi' + Q$, and the divisor D' obtained from this choice is linearly equivalent to D. Indeed, we have

$$D' = D + \sum_F \langle \varpi - \varpi', x_F \rangle X_F + \sum_{\alpha \in \mathcal{S}_P} \langle \varpi - \varpi', \alpha_M^\vee \rangle D_\alpha.$$

Then, if we pick a positive integer k such that $k(\varpi - \varpi')$ is in M, then $kD' = kD + \mathrm{div}(f_{k(\varpi-\varpi')})$, where $f_{k(\varpi-\varpi')}$ is a B-eigenvector vector in $\mathbb{C}(X)$ of weight $k(\varpi - \varpi')$.

The above construction of the G-variety X from a moment triple (P, M, Q) is not really practical. We give bellow an easy way to describe X geometrically from (P, M, Q) as the closure of a G-orbit in a projective G-space.

Remark 2.9. Suppose that D is a Cartier divisor. Then, all vertices of $\tilde{Q}_D = -\varpi_D + Q_D$ are in M. In particular the vertices of Q_D are in $\mathfrak{X}(P)$.

In that case, the moment polytope of (X, x, D) characterizes the set of global sections of D. Indeed,

$$H^0(X, D) = \bigoplus_{\chi \in M \cap Q_D} V(\chi),$$

where $V(\chi)$ denotes the irreducible G-module of highest weight χ. (See [Bri89, Proposition 3.3] for the description of global sections of Cartier B-stable divisors of any spherical variety.)

Theorem 2.10. *[Pas06, Chapitre 5] Let X be a spherical variety of rank n. Let D be an ample Cartier divisor. Then, if $n \leq 1$, D is very ample, and if $n \geq 2$, $(n-1)D$ is very ample.*

If X is locally factorial, then D is very ample.

Corollary 2.11. *Let (X, x) be a G-horospherical embedding associated to $(P, M, \mathbb{F}_X)$. Denote by n the rank of X (i.e. the rank of M), and let $n' := \max(1, n-1)$. Let D be an ample B-stable Cartier divisor. Then the map*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(\bigoplus_{\chi \in M \cap (n'-1)Q_D} V(\chi)^\vee) \\ y & \longmapsto & (s \mapsto s(y)) \end{array}$$

is a closed immersion. Then (in particular because x is fixed by U, see [Pas14a, Proposition 2.11]), X is the closure of $G \cdot \sum_{\chi \in M \cap (n'-1)Q_D} v_\chi^\vee$ in $\mathbb{P}(\bigoplus_{\chi \in M \cap (n'-1)Q_D} V(\chi)^\vee)$, where $v_\chi^\vee$ is a highest vector of the dual G-module $V(\chi)^\vee$ of $V(\chi)$.

Remark 2.12. [Pas14a, Remark 2.12] With the notation of Corollary 2.11, the G-variety X is isomorphic to the closure of $G \cdot \sum_{\chi \in M \cap (n'-1)Q_D} v_\chi$ in $\mathbb{P}(\bigoplus_{\chi \in M \cap (n'-1)Q_D} V(\chi))$, where v_χ is a highest vector of $V(\chi)$.

Example 2.13. Consider $G = \mathrm{SL}_4$, the maximal parabolic subgroup P of G such that $\mathcal{S}_P = \{\alpha_2\}$ and $M = \mathfrak{X}(P)$.

Let Q be the segment $[0, \varpi_2]$ in $\mathfrak{X}(P)$. Then the horospherical G-variety X associated to the moment triple (P, M, Q) is the projective cone over the grassmannian $\mathrm{Gr}(2, 4)$, embedded in $\mathbb{P}(\mathbb{C} \oplus \bigwedge^2 \mathbb{C}^4)$, that we already consider in Examples 1.8 and 1.18.

Let Q' be the segment $[\varpi_2, 2\varpi_2]$ in $\mathfrak{X}(P)$. Then the horospherical G-variety X' associated to the moment triple (P, M, Q') is the blow-up of the vertex in the projective cone X. It is naturally embedded in $\mathbb{P}(V(\varpi_2) \oplus V(2\varpi_2)) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4 \oplus S^2(\bigwedge^2 \mathbb{C}^4))$. It is the horospherical G-variety corresponding to $(P, M, \mathbb{F}')$ where $\mathbb{F}'$ is defined in Example 1.18.

Example 2.14. We again consider the horospherical SL_3/U-embeddings that we have already considered in Examples 1.22 and 2.6. For any of these varieties, we can find an ample B-stable $\mathbb{Q}$-Cartier divisor D such that their moment polytopes are as represented in Figure 4. When the anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor, we choose D to be the anticanonical divisor (we can observe that when it is not Cartier, the moment polytope has a vertex not in M). Otherwise, we choose D to be Cartier (and ample), such that its moment polytope is as small as possible and as near as possible to the walls of the dominant chamber. (In this example, such a divisor is unique, but that it is not the case in general, even for 2-dimensional toric varieties.)

Recall that M is here the lattice of characters of a Borel subgroup of SL_3 and that the dominant chamber is the cone generated by ϖ_α and ϖ_β. Also recall that the inward-pointing normal vectors of the facets of a moment polytope correspond to edges of the colored fan $\mathbb{F}_X$, and that the moment polytope meets the wall $\langle\cdot,\alpha^\vee\rangle = 0$ (resp. $\langle\cdot,\beta^\vee\rangle = 0$) if and only if D_α (resp. D_β) is a color of the variety X.

We can compute the divisors D (up to linear equivalence) by using the construction described after Proposition 2.8. Denote by X_1, X_2, X_3, X_4, X_5, and X_6 the G-stable irreducible divisors of X (if they exist) that corresponds to the edges of $\mathbb{F}_X$ respectively generated by $(1,0)$, $(2,1)$, $(1,1)$, $(-1,0)$, $(1,-1)$ and $(-1,-1)$. Then, by choosing $\varpi = 0$, the divisors D are respectively:

$$-X_1 - 5X_2 - 3X_3 + 4X_4 + 3X_5$$

$$-X_1 - X_3 + 3X_4 + X_5 \quad -X_1 - 4X_2 + 3X_4 + 2X_5 \quad -5X_2 - 3X_3 + 4X_4 + 2X_5 \quad -X_1 - 5X_2 - 3X_3 + 3X_4 + 3X_5$$

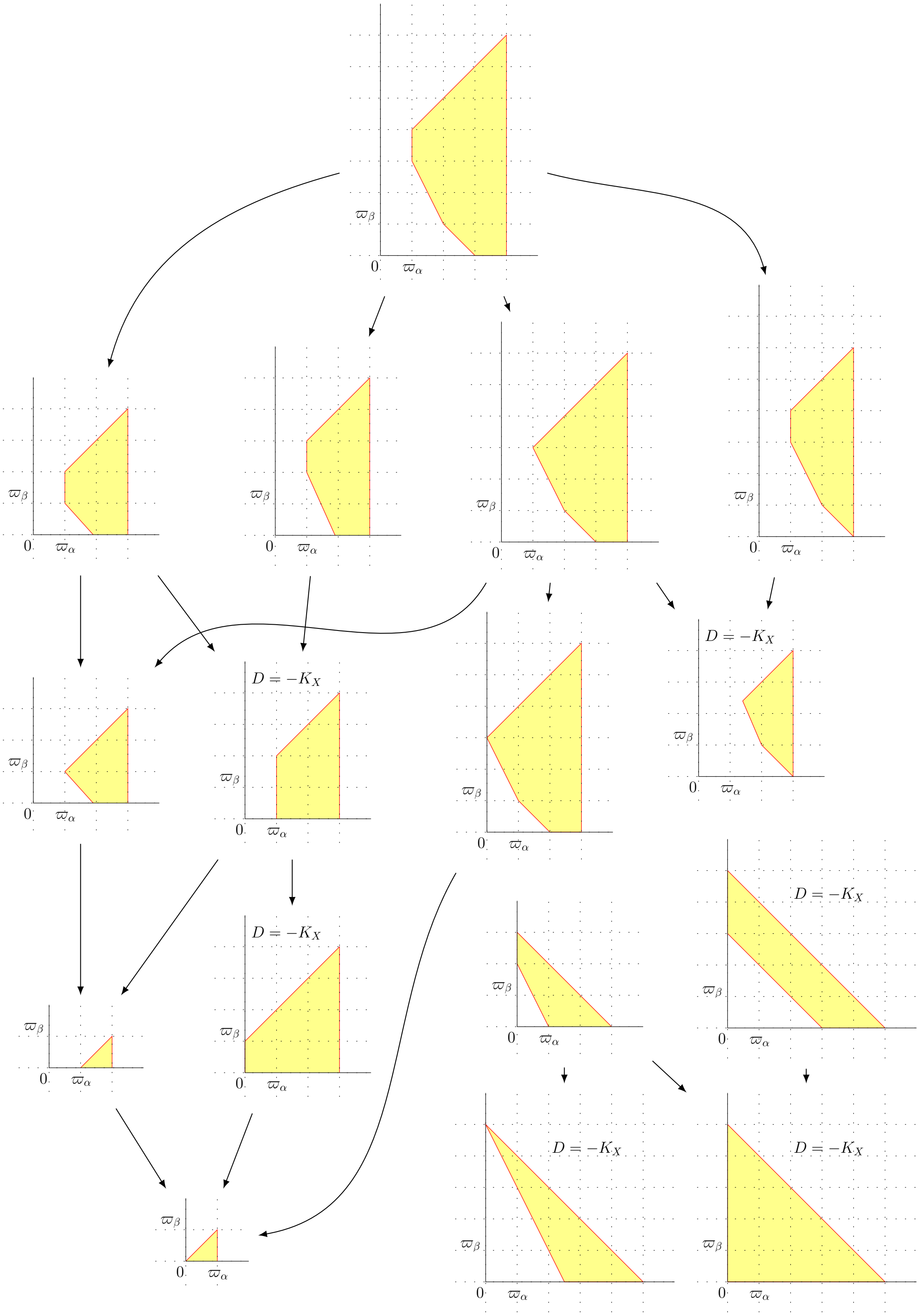
$$-X_3 + 3X_4 \quad -X_1 + 3X_4 + X_5 \sim -K_X \quad -3X_2 - X_3 + 3X_4 + 3X_5 \quad -5X_2 - 3X_3 + 3X_4 + X_5 \sim -K_X$$

$$2X_4 - X_5 \quad 3X_4 + X_5 \sim -K_X \quad -2X_2 + 3X_6 \quad -3X_3 + 5X_6 \sim -K_X$$

$$X_4 \quad -5X_2 + 5X_6 \sim -K_X \quad 5X_6 \sim -K_X.$$

To get effective divisors (linearly equivalent to these divisors), pick respectively ϖ in the polytopes instead of 0. And to get strictly effective divisors, pick ϖ in the interior of the polytopes.

Figure 4: Some moment polytopes of the SL_3/U-embeddings of Example 1.22



3 Singularities of horospherical varieties

In birational geometry, we generally deal with the following singularities:

- $\mathbb{Q}$-factorial;
- terminal, canonical;
- klt (Kawamata log terminal).

But we also consider (for example in the study of Fano varieties) the following singularities:

- smooth;
- locally factorial;
- Gorenstein, $\mathbb{Q}$-Gorenstein.

All these singularities can be characterized combinatorially in terms of properties of colored fans and root systems.

The smooth criterion is the most complicated one and was obtained simultaneously in [Pas06] and [Tim11]. It mixes the combinatorial aspects of colored fans and root systems. Let us begin with the easiest ones: locally factorial and $\mathbb{Q}$-factorial criteria.

Definition 3.1. A variety X is locally factorial if all Weil divisors of X are Cartier.

A variety X is $\mathbb{Q}$-factorial if all Weil divisors of X are $\mathbb{Q}$-Cartier.

Using the criterion of Cartier divisors of horospherical varieties (Proposition 2.2), we get the following result.

Proposition 3.2. *Let (X, x) be a G-horospherical embedding associated to $(P, M, \mathbb{F}_X)$. Then X is locally factorial (respectively $\mathbb{Q}$-factorial) if and only if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, the colors of $\mathcal{F}$ have distinct images in $N_{\mathbb{Q}}$ and there exists a basis $(u_1, \dots, u_k) \cup (\alpha_M^\vee)_{\alpha\in\mathcal{F}}$ of the lattice N (respectively of the vector space $N_{\mathbb{Q}}$) such that $\mathcal{C}$ is generated by the family $(u_1, \dots, u_{k'}) \cup (\alpha_M^\vee)_{\alpha\in\mathcal{F}}$ (where $k' \leq k$ are non-negative integers).*

To state the smooth criterion, we need a little more notations and definitions.

Denote by $\mathcal{S}$ the set of simple roots of (G, B, T) and by $\Gamma_{\mathcal{S}}$ the Dynkin diagram of G (whose vertices are indexed by elements of $\mathcal{S}$). For any subset $\mathcal{I} \subset \mathcal{S}$, we denote by $\Gamma_{\mathcal{I}}$ the subgraph of $\Gamma_{\mathcal{S}}$ with vertices indexed by elements of $\mathcal{I}$ and with all arrows between vertices indexed by elements of $\mathcal{I}$.

Definition 3.3. Let $\mathcal{I}$ and $\mathcal{J}$ be two disjoint subset of $\mathcal{S}$. The couple $(\mathcal{I}, \mathcal{J})$ is said to be regular if any connected component Γ of $\Gamma_{\mathcal{I}\sqcup\mathcal{J}}$ satisfies one of the following conditions.

- Γ is of type A and the only vertex of Γ indexed by an element of $\mathcal{J}$ is an extremity of Γ.
- Γ is of type C and the only vertex of Γ indexed by an element of $\mathcal{J}$ is the first vertex (the extremity indexed by a small root).
- Every vertex of Γ is indexed by an element in $\mathcal{I}$.

Theorem 3.4. *Let (X, x) be a G-horospherical embedding associated to $(P, M, \mathbb{F}_X)$. Suppose that X is locally factorial. Then X is smooth if and only if, for any colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathbb{F}_X$, the couple $(\mathcal{S}\backslash\mathcal{S}_P, \mathcal{F})$ is regular.*

We remark that a locally factorial horospherical variety X with no color (i.e. $\mathcal{F}_X = \emptyset$) is smooth. In particular locally factorial toric varieties are smooth. And, we can always obtain a desingularization of any horospherical variety X by subdividing cones of $\mathbb{F}_X$ and by deleting all colors (see also Section 1.4).

Example 3.5. The projective cone over the grassmannian $\mathrm{Gr}(2, 4)$ (that we have already considered in Examples 1.8, 1.18 and 2.13) is locally factorial but not smooth because the couple $(\{\alpha_1, \alpha_3\}, \{\alpha_2\})$ is not regular.

Definition 3.6. A normal variety X is Gorenstein (respectively $\mathbb{Q}$-Gorenstein) if the anticanonical divisor $-K_X$ is Cartier (respectively $\mathbb{Q}$-Cartier).

To get a criterion of these types of singularities for horospherical varieties, we still use the criterion of Cartier divisors of horospherical varieties (Proposition 2.2) but also the following result.

Proposition 3.7 ([Bri97])**.** *An anticanonical divisor of a G-horospherical embedding (X, x) associated to $(P, M, \mathbb{F}_X)$ is*

$$-K_X = \sum_{i=1}^{m} X_i + \sum_{\alpha\in\mathcal{S}_P} a_\alpha D_\alpha,$$

where $X_1,\ldots,X_m$ are the irreducible G-stable divisors of X and $\forall\alpha \in \mathcal{S}_P$, $a_\alpha = \langle\sum_{\alpha\in\mathcal{R}_P^+} \alpha, \alpha^\vee\rangle \geq 2$, where $\mathcal{R}_P^+$ is the set of positive roots with at least one non-zero coefficient for a simple root of $\mathcal{S}_P$.

Proposition 3.8. *Let (X, x) be a G-horospherical embedding associated to $(P, M, \mathbb{F}_X)$. Then X is Gorenstein (respectively $\mathbb{Q}$-Gorenstein) if and only if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, there exists $m_\mathcal{C} \in M$ (respectively $m_\mathcal{C} \in M_\mathbb{Q}$) such that, for any primitive element x of an edge of $\mathcal{C}$ that is not generated by some $\alpha_M^\vee$ with $\alpha \in \mathcal{F}$, $\langle m_\mathcal{C}, x\rangle = 1$, and for any $\alpha \in \mathcal{F}$, $\langle m_\mathcal{C}, \alpha_M^\vee\rangle = a_\alpha$.*

We now consider singularities appearing in the Minimal Model Program.

Definition 3.9. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Let $f : V \longrightarrow X$ be a desingularization of X (i.e. f is birational and V is smooth). Then $K_V - f^*(K_X) = \sum_{i\in\mathcal{I}} a_i E_i$ where $\{E_i \mid i \in \mathcal{I}\}$ is the set of exceptional divisors of f.

We say that X has

- canonical singularities if, for any $i \in \mathcal{I}$, $a_i \geq 0$;
- terminal singularities if, for any $i \in \mathcal{I}$, $a_i > 0$.

Note that the definition does not depend on the choice of the desingularization. Moreover, if X is horospherical, recall that we can construct a desingularization by deleting the colors of X and by taking subdivision of the cones of $\mathbb{F}_X$. Then, still with the criterion of Cartier divisors on horospherical varieties, we get the following characterizations of canonical and terminal singularities.

Proposition 3.10. *Let (X,x) be a $\mathbb{Q}$-Gorenstein G-horospherical embedding associated to the triple $(P,M,\mathbb{F}_X)$. For any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, denote by $h_{\mathcal{C}}$ the linear function such that for any $\alpha\in\mathcal{F}$, $h_{\mathcal{C}}(\alpha_M^\vee)=a_\alpha$ and, for any primitive element u of an edge of $\mathcal{C}$ that is not generated by some $\alpha_M^\vee$ with $\alpha\in\mathcal{F}$, $h_{\mathcal{C}}(u)=1$.*

- *X has canonical singularities if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, for any $x\in\mathcal{C}\cap N$, $h_{\mathcal{C}}(x)\geq 1$.*
- *X has terminal singularities if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, for any $x\in\mathcal{C}\cap N$, $h_{\mathcal{C}}(x)\leq 1$ implies that x is the primitive element of an edge of $\mathcal{C}$ that is not generated by some $\alpha_M^\vee$ with $\alpha\in\mathcal{F}$ (i.e. $x=x_i$ with our notation).*

Definition 3.11. Let X be a normal variety and let D be an effective $\mathbb{Q}$-divisor such that K_X+D is $\mathbb{Q}$-Cartier. The pair (X,D) is said to be klt (Kawamata log terminal) if for any desingularization $f:V\longrightarrow X$ of X such that $K_V=f^*(K_X+D)+\sum_{i\in\mathcal{I}}a_iE_i$, we have $a_i>-1$ for any $i\in\mathcal{I}$.

We say that X has log terminal singularities if X is $\mathbb{Q}$-Gorenstein and $(X,0)$ is klt.

Remark 3.12.

1. In fact, it is enough to check the above property for one log-resolution to say that a pair (X,D) is klt.
2. The condition "$a_i>-1$ for any $i\in\mathcal{I}$" can be replaced by: $\lfloor D\rfloor=0$ and for any $i\in\mathcal{I}$ such that E_i is exceptional for f, $a_i>-1$.

Still with the criterion of Cartier divisors of horospherical varieties (Proposition 2.2), we get the following result.

Proposition 3.13. *Any $\mathbb{Q}$-Gorenstein horospherical variety has log terminal singularities.*

It is a particular case of the following result.

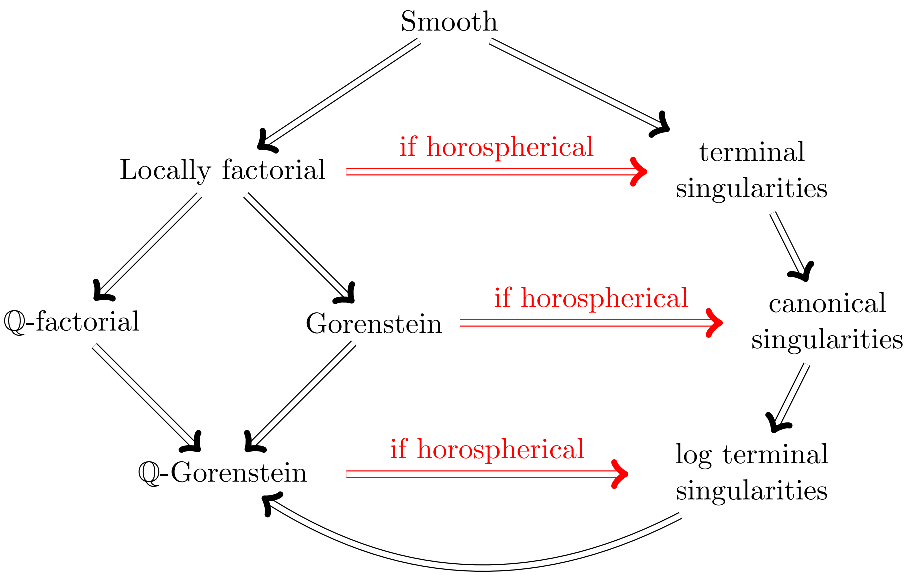
Theorem 3.14. *[Pas15a] Let X be a horospherical variety and let D be an effective $\mathbb{Q}$-divisor such that K_X+D is $\mathbb{Q}$-Cartier. The pair (X,D) is klt if and only if $\lfloor D\rfloor=0$.*

We also easily have the following result;

Proposition 3.15. *Any locally factorial horospherical variety has terminal singularities.*

And any Gorenstein horospherical variety has canonical singularities.

We can summarize this section by the following diagram and example.



Example 3.16. In Example 2.6, we already gave examples of Gorenstein, $\mathbb{Q}$-Gorenstein and not $\mathbb{Q}$-Gorenstein SL_3/U-embeddings.

We can complete this example with Figure 5, by pointing those who are smooth, locally factorial, $\mathbb{Q}$-factorial or not $\mathbb{Q}$-factorial, with terminal or canonical singularities, or only with log terminal singularities. When the variety X is not $\mathbb{Q}$-Gorenstein, we can also precise if there exists, or not, a $\mathbb{Q}$-divisor D such that the pair (X, D) is klt.

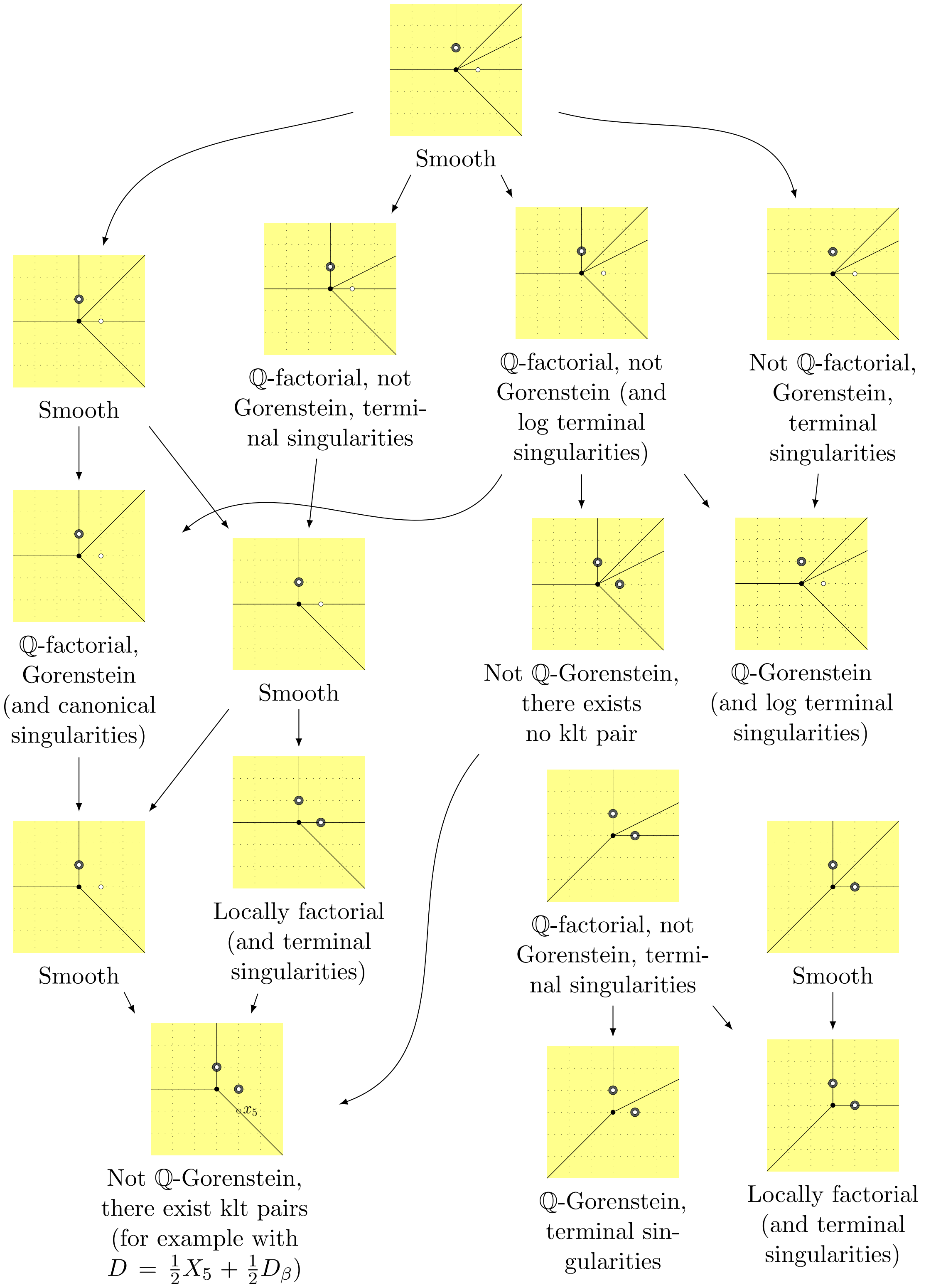
We only write the optimal regularities.

For projective horospherical varieties, we could ask if the criteria of types of singularities can be characterized in terms of properties of polytopes. The answer is yes for $\mathbb{Q}$-factorial singularities, but for other singularities there is no simple characterization on polytopes, essentially because we cannot see directly the anticanonical divisor by looking at a moment polytope (except for Fano varieties polarized by their anticanonical divisor) and we cannot directly distinguish Cartier divisors from not Cartier but $\mathbb{Q}$-Cartier divisors.

Proposition 3.17. *Let (X, x, D) be a polarized G-horospherical embedding. Denote by (P, M, Q) the moment triple of (X, x, D).*

Then X is $\mathbb{Q}$-factorial if and only if Q is simple (i.e. every vertex of Q is exactly in $\dim(Q)$ facets of Q) and for any $\alpha \in \mathcal{S}_P$, the intersection of Q with the hyperplane $\{\chi \in \mathfrak{X}(P)_{\mathbb{Q}} \mid \langle \chi, \alpha^\vee \rangle = 0\}$ is either empty or a facet of Q.

Figure 5: Singularities of the SL_3/U-embeddings of Example 1.22



4 MMP for horospherical varieties

The MMP is based on the Cone Theorem and Contraction Theorem (Theorem 4.2 below) due to Mori in the smooth case, and Y. Kawamata, J. Kollàr, M. Reid, V.V. Shokurov and others. The MMP consists on doing finitely many birational transformations in order to obtain a minimal model or a Mori fibration. It can be conjecturally applied to any projective variety with terminal singularities or more generally to any projective klt pair. In fact, the MMP works under two conditions: the existence of flips and the termination of flips. The first one was not known in general for a long time, but it is now known [BCHM10]. And the second one is still conjectural in general.

The Minimal Model Program (MMP) for $\mathbb{Q}$-factorial spherical varieties is due to M. Brion [Bri93]. In [Pas14a], a new approach to the MMP for projective horospherical varieties is given. The idea is to compute a path of the MMP, from a given projective horospherical variety X to a Mori fibration, by looking at an affine deformation of the moment polytope of an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of X. For D sufficiently general, this description works with $\mathbb{Q}$-factorial singularities assumption. But, for any ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, it also works with $\mathbb{Q}$-Gorenstein singularities assumption (by adapting the definitions of divisorial contractions and flips). The non-$\mathbb{Q}$-factorial MMP is known for the specialists since the 80's, but no proof of it was written, even if O. Fujino recently consider it in [Fuj06], [Fuj07] and [Fuj14]. A proof, which follows the proof of the $\mathbb{Q}$-factorial case produced in [Mat02] (or [KMM87]), can be now found in [Pas14b]. In that latter paper, we also prove that, in the non-$\mathbb{Q}$-factorial for spherical varieties, flips always exist and there is no infinite sequence of flips.

4.1 Introduction to the $\mathbb{Q}$-factorial MMP

Let X be a $\mathbb{Q}$-factorial, normal and projective variety.

We denote by $NE(X)$ the cone generated by effective 1-cycles of X modulo numerical equivalence. And we denote by $NE(X)_{K_X<0}$ (respectively $NE(X)_{K_X\geq 0}$ and $NE(X)_{K_X>0}$) the subcone of $NE(X)$ of classes of 1-cycles that are negative (respectively non-negative and positive) on the canonical divisor K_X. We denote by $\overline{NE}(X)$, $\overline{NE}(X)_{K_X<0}$, $\overline{NE}(X)_{K_X\geq 0}$ and $\overline{NE}(X)_{K_X>0}$ their closures (in the $\mathbb{Q}$-vector space of numerical classes of 1-cycles of X).

Definition 4.1. A morphism $\phi : X \longrightarrow Y$ between normal varieties is a contraction if it is projective and $\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$.

Theorem 4.2 (see for example [KMM87])**.** *Let X be a normal projective variety with terminal singularities (in particular X is $\mathbb{Q}$-Gorenstein).*

Then, there exists a discrete family $(R_j)_{j\in J}$ of extremal rays in $\overline{NE}(X)_{K_X<0}$, such that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X\geq 0} + \sum_{j\in J} R_j,$$

and for any $j \in J$, R_j is generated by the class of an irreducible curve of X.

Let F be a face of $\overline{NE}(X)$ contained in $\overline{NE}(X)_{K_X<0} \cup \{0\}$ (for example one of the extremal rays R_j).

Then, there exists a unique normal projective variety Y and a unique contraction $\phi_F : X \longrightarrow Y$ such that, for any irreducible curve C in X, $\phi_F(C)$ is a point if and only if the class of C in $\overline{NE}(X)$ is in F.

In the $\mathbb{Q}$-factorial MMP, we distinguish three types of contractions of extremal rays of the cone $\overline{NE}_{K_X<0}$.

Definition 4.3. 1. A contraction is of divisorial type (or a divisorial contraction) if its exceptional locus is an irreducible divisor.

2. A contraction is of flipping type if the exceptional locus is at least of codimension 2.

   A flip of a contraction of flipping type $\phi : X \longrightarrow Y$ is a contraction $\phi^+ : X^+ \longrightarrow Y$, from a normal $\mathbb{Q}$-Gorenstein variety X^+, of an extremal ray of $\overline{NE}(X^+)_{K_{X^+}>0}$ and such that the exceptional locus of ϕ^+ is at least of codimension 2.

3. A contraction $\phi : X \longrightarrow Y$ is of fibre type (or a Mori fibration) if $\dim(Y) < \dim(X)$.

Denote by $\mathcal{H}$ the set of normal projective varieties with $\mathbb{Q}$-factorial and terminal singularities. The principle of the $\mathbb{Q}$-factorial MMP is summarized in Figure 6.

If we assume the existence of flips such that X^+ in $\mathcal{H}$, and the finiteness of sequences of flips, it is well-known for a long time that the MMP works and ends in finitely many steps.

Theorem 4.4 (see for example [KMM87])**.** *Let X be a normal projective variety with $\mathbb{Q}$-factorial and terminal singularities. Let $\phi : X \longrightarrow Y$ be a contraction of an extremal ray of flipping type. Then, there exists a flip $\phi^+ : X^+ \longrightarrow Y$ if and only if $\mathcal{A} := \oplus_{m\geq 0}\phi_*\mathcal{O}_X(mK_X)$ is finitely generated as an $\mathcal{O}_Y$-algebra. Moreover, in that case, $X^+ = \mathrm{Proj}(\mathcal{A})$ is unique and has $\mathbb{Q}$-factorial and terminal singularities. And the Picard number of X^+ equals the Picard number of X.*

4.2 The non-$\mathbb{Q}$-factorial MMP

We can remark that in Theorem 4.2, there is no $\mathbb{Q}$-factorial assumption. Then, we can adapt the $\mathbb{Q}$-factorial MMP as follows to get a non-$\mathbb{Q}$-factorial MMP (i.e. for $\mathbb{Q}$-Gorenstein varieties).

Definition 4.5. 1. A contraction is of generalized divisorial type (or a generalized divisorial contraction) if it contracts a Cartier divisor (not necessarily irreducible).

2. A contraction is of generalized flipping type if it does not contract a Cartier divisor (but it could contract a Weil divisor).

   A generalized flip of a contraction of generalized flipping type $\phi : X \longrightarrow Y$ is a contraction $\phi^+ : X^+ \longrightarrow Y$ from a normal $\mathbb{Q}$-Gorenstein variety X^+, such that the exceptional locus of ϕ^+ is at least of codimension 2, and for any curve C^+ of X^+ that is contracted by ϕ^+, we have $K_{X^+} \cdot C^+ > 0$.

We denote by $\mathcal{G}$ the set of normal projective varieties with $\mathbb{Q}$-Gorenstein and terminal singularities. Then, similarly to the $\mathbb{Q}$-factorial MMP, the non-$\mathbb{Q}$-factorial MMP is summarized in Figure 7.

The following result is known since the beginning of the $\mathbb{Q}$-factorial theory. It was recently mentioned in [Fuj06], [Fuj07] and [Fuj14]. And a complete proof can be found in [Pas14b].

Figure 6: $\mathbb{Q}$-factorial MMP

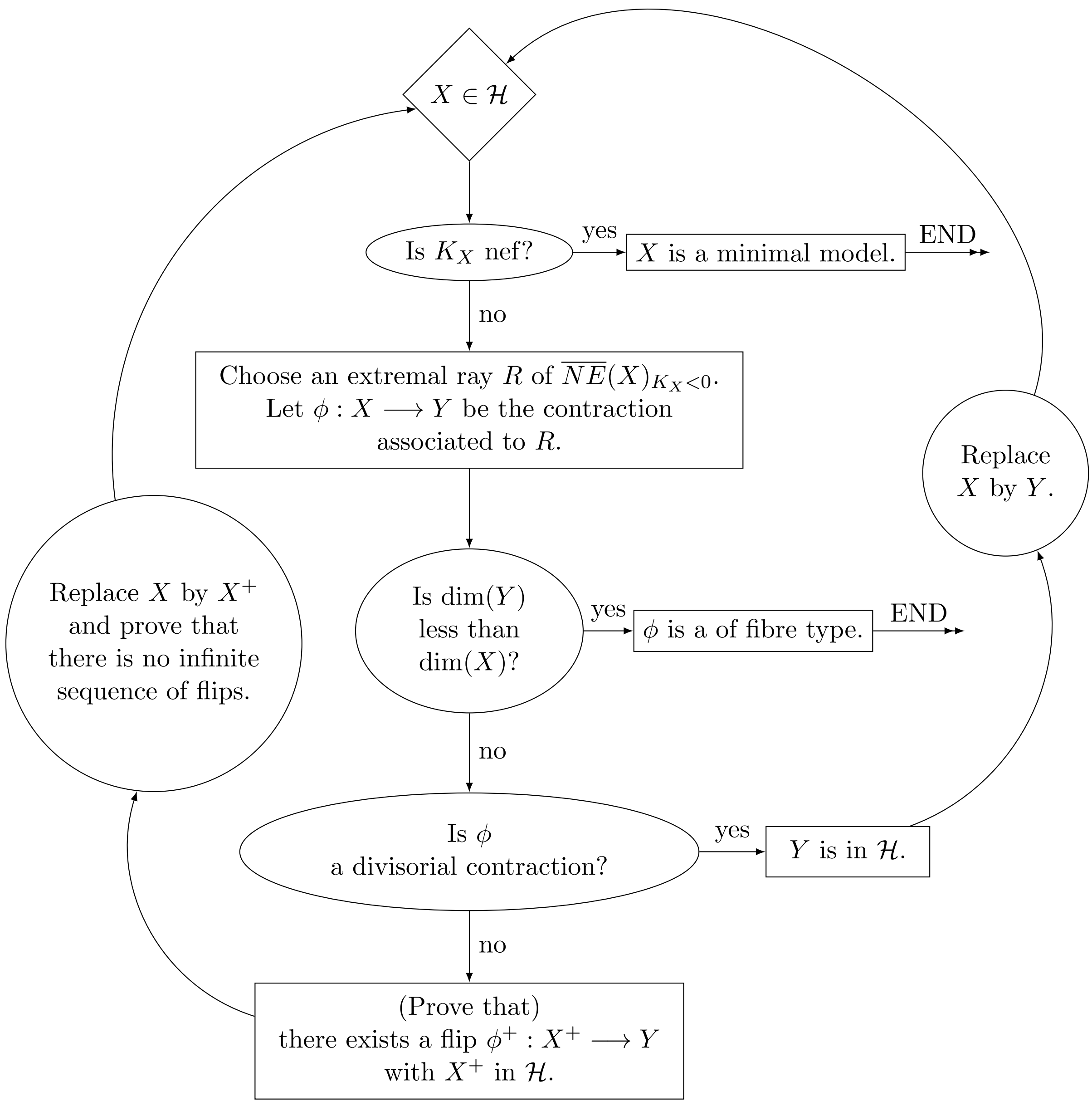
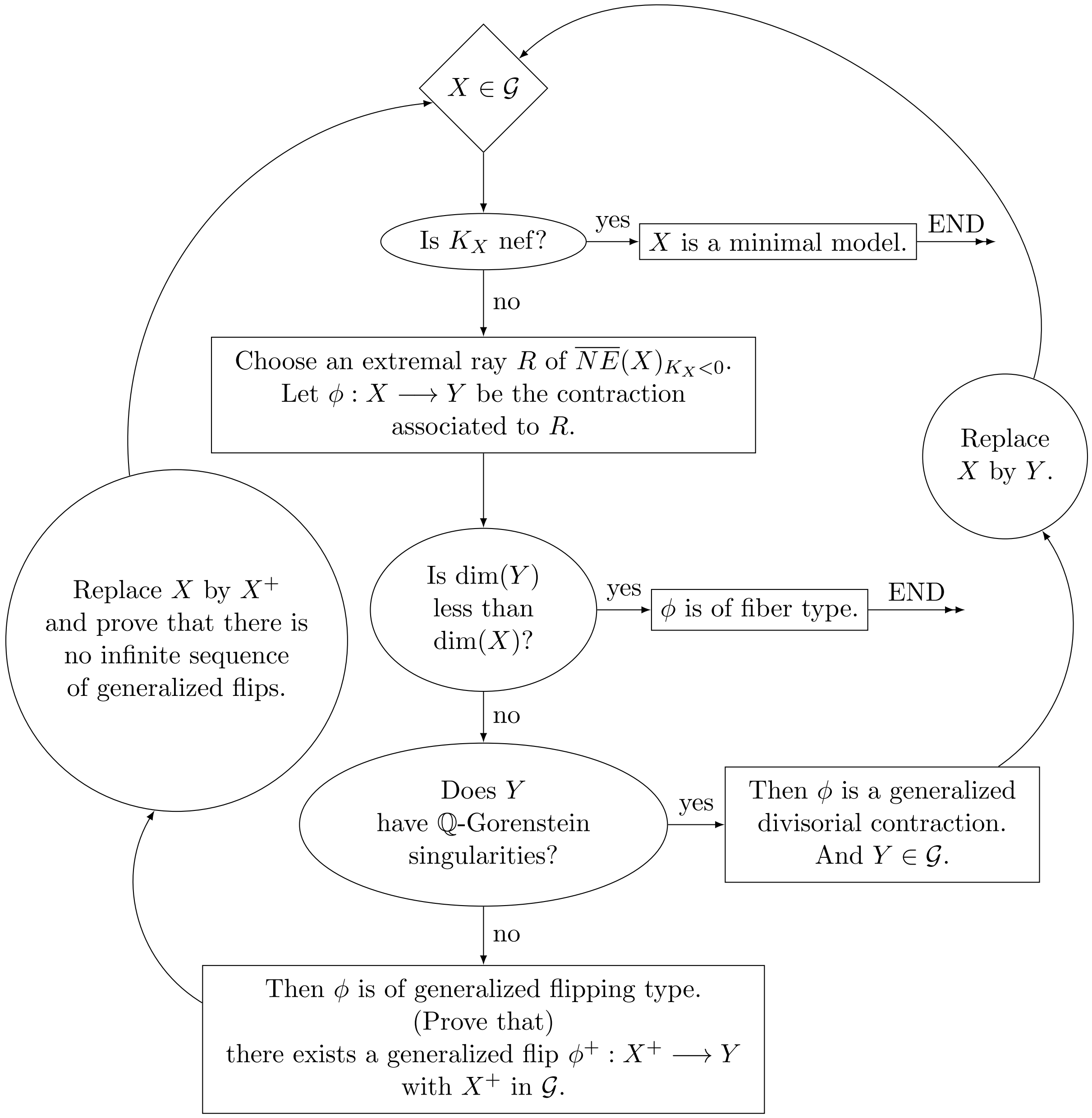


Figure 7: Non-$\mathbb{Q}$-factorial MMP



Theorem 4.6. *Under the assumptions of the existence of generalized flips and the finiteness of sequences of generalized flips, the non-$\mathbb{Q}$-factorial MMP works and ends in finitely many steps.*

Let $\phi : X \longrightarrow Y$ be a contraction of generalized flipping type. Then there exists a generalized flip $\phi^+ : X^+ \longrightarrow Y$ if and only if $\mathcal{A} := \oplus_{m\geq 0}\phi_*\mathcal{O}_X(mK_X)$ is finitely generated as an $\mathcal{O}_Y$-algebra. In that case, $X^+ = \mathrm{Proj}(\mathcal{A})$, it is unique and it has terminal singularities.

Remark that the Picard number of X^+ could be greater than the Picard number of X.

And note that if X is $\mathbb{Q}$-factorial, contractions of generalized divisorial type and generalized flips are respectively contractions of divisorial type and flips. So that, if X is $\mathbb{Q}$-factorial, the non-$\mathbb{Q}$-factorial MMP is the same as the $\mathbb{Q}$-factorial MMP.

Proposition 4.7. *[Pas14b, Proposition 10] For spherical varieties, the assumptions of Theorem 4.6 are satisfied. In particular, in that case, the non-$\mathbb{Q}$-factorial MMP works and ends in finitely many steps.*

Remark 4.8. If X is a G-variety, the contraction ϕ is G-equivariant by Proposition 1.19, then $\mathcal{A}$ is a G-module and G acts naturally on X^+, so that ϕ^+ is G-equivariant and then X^+ is spherical (respectively horospherical and toric) as soon as X is spherical (respectively horospherical and toric).

4.3 How to concretely run the MMP for horospherical varieties via moment polytopes

In that section, we explain how we can run the $\mathbb{Q}$-factorial and non-$\mathbb{Q}$-factorial MMP for horospherical varieties, just by considering a one-parameter family of polytopes. For the proofs and the details, see [Pas14a].

Note that for any spherical variety X, the cone $NE(X)$ is closed and polyhedral [Bri93].

The strategy of running the MMP from a polarized projective G-horospherical embedding (X, x, D) is to consider the family of moment polytopes of $(X, x, D + \epsilon K_X)$ for $\epsilon \geq 0$ small enough (i.e. such that $D + \epsilon K_X$ is still ample) and to extend it to any $\epsilon \geq 0$. As long as a polytope satisfies the two conditions of Definition 2.7, it corresponds to a polarized projective G-horospherical embedding with the same open G-orbit as X.

The varieties we obtain this way are the varieties that can appear in the MMP before getting a Mori fibration.

We now precise the construction of the one-parameter family of polytopes and the result we obtain.

Recall that for any $\mathbb{Q}$-Gorenstein G-horospherical embedding (X, x, D), an anticanonical divisor is $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha\in\mathcal{S}_P} a_\alpha D_\alpha$ (see Proposition 3.7).

Definition 4.9. Let (X, x, D) be a polarized projective $\mathbb{Q}$-Gorenstein G-horospherical embedding. Denote $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha\in\mathcal{S}_P} d_\alpha D_\alpha$ and $\varpi_D = \sum_{\alpha\in\mathcal{S}_P} d_\alpha\varpi_\alpha$. The moment family of (X, x, D) is a family of (may be empty) polytopes Q^ϵ in $\mathfrak{X}(P)^+_{\mathbb{Q}}$, with $\epsilon \geq 0$, defined as follows:

$$Q^\epsilon := \varpi_D - \epsilon\left(\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha\right) + \left\{\begin{array}{r} x \in M_{\mathbb{Q}} \mid \forall i \in \{1, \ldots, m\},\, \langle x, x_i\rangle \geq -d_i + \epsilon \\ \text{and } \forall\alpha \in \mathcal{S}_P,\, \langle x, \alpha^\vee_M\rangle \geq -d_\alpha + \epsilon a_\alpha \end{array}\right\}.$$

As long as Q^ϵ intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$, and $-\varpi_D+\epsilon(\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha)+Q^\epsilon$ is of maximal dimension in $M_{\mathbb{Q}}$, we denote by (X^ϵ, x^ϵ) the projective G-horospherical embedding associated to the moment triple (P, M, Q^ϵ).

Let $\epsilon>0$ such that Q^ϵ is not empty but does not intersect the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ or such that $-\varpi_D+\epsilon(\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha)+Q^\epsilon$ is not of maximal dimension in $M_{\mathbb{Q}}$. Denote by P' the parabolic subgroup of G containing P such that Q^ϵ intersects the interior of $\mathfrak{X}(P')^+_{\mathbb{Q}}$. And denote by M' the maximal sublattice of M such that $-\varpi_D+\epsilon(\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha)+Q^\epsilon$ is of maximal dimension in $M'_{\mathbb{Q}}$. Then we denote by (X^ϵ, x^ϵ) the projective G-horospherical embedding associated to the moment triple (P', M', Q^ϵ). (Note here that, by hypothesis, $P\neq P'$ or $M\neq M'$.)

Remark that the definition depends on the choice of D in its linearly equivalent class and that

$$Q^\epsilon=\left\{\begin{array}{r} y\in\varpi_D-\epsilon\left(\displaystyle\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha\right)+M_{\mathbb{Q}}\mid \\ \forall i\in\{1,\dots,m\},\ \langle y,x_i\rangle\geq -d_i+\langle\varpi_D,x_i\rangle+\epsilon(1-\langle\displaystyle\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha,x_i\rangle) \\ \text{and }\forall\alpha\in\mathcal{S}_P,\ \langle y,\alpha_M^\vee\rangle\geq 0\end{array}\right\}.$$

However the family of projective G-horospherical embeddings (X^ϵ, x^ϵ) does not depend on the choice of D in its linearly equivalent class.

Theorem 4.10. *Let (X,x,D) be a polarized projective $\mathbb{Q}$-Gorenstein G-horospherical embedding. There exist $k\geq 1$ and rational numbers $0=\epsilon_0<\epsilon_1<\cdots<\epsilon_k$ such that Q^{ϵ_k} does not intersect the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ or $-\varpi_D+\epsilon_k(\sum_{\alpha\in\mathcal{S}_P} a_\alpha\varpi_\alpha)+Q^{\epsilon_k}$ is not of maximal dimension in $M_{\mathbb{Q}}$ and for any $i\in\{1,\dots,k\}$:*

1. *for any η_1 and η_2 in $]\epsilon_{i-1},\epsilon_i[$, the projective horospherical G-varieties X^{η_1} and X^{η_2} are isomorphic and $\mathbb{Q}$-Gorenstein;*
2. *let $\eta\in]\epsilon_{i-1},\epsilon_i[$, there exists a G-equivariant morphism ϕ_i from X^η to X^{ϵ_i}, which is the contraction of a face of $NE(X^\eta)_{K_{X^\eta}<0}$;*
3. *if X^{ϵ_i} is isomorphic to X^η with $\eta\in]\epsilon_i,\epsilon_{i+1}[$ $(i<k)$, then ϕ_i contracts a Cartier divisor;*
4. *if X^{ϵ_i} is not isomorphic to X^η with $\eta\in]\epsilon_i,\epsilon_{i+1}[$ $(i<k)$, then X^{ϵ_i} is not $\mathbb{Q}$-Gorenstein, there exists a G-equivariant morphism ϕ_i^+ from X^η to X^{ϵ_i}, which is the contraction of a face of $NE(X^\eta)_{K_{X^\eta}>0}$ and such that the exceptional locus of ϕ_i^+ is at least of codimension 2.*

Moreover, for general D, the maps ϕ_i are contractions of extremal rays. And, if D is general and X is $\mathbb{Q}$-factorial, then, for all $i\in\{1,\dots,k\}$, for all $\eta\in]\epsilon_{i-1},\epsilon_i[$, the horospherical G-varieties X^η are also $\mathbb{Q}$-factorial and the maps ϕ_i^+ are contractions of extremal rays of $NE(X^\eta)_{K_{X^\eta}>0}$.

And for general D, if X is $\mathbb{Q}$-factorial, the general fibre of ϕ_k is a $\mathbb{Q}$-factorial projective horospherical G'-variety of Picard number one, where G' is a Levi of the stabilizer of x^{ϵ_1} in G.

Also note that Q^η is empty for any $\eta>\epsilon_k$, and X is isomorphic to X^η for any $\eta<\epsilon_1$.

Example 4.11. In [Pas14a, Chapter 1], we give an example with a 3-dimensional toric variety. We observe that, if D is not general, it could happen that the contractions are not contractions of extremal rays (but only of faces), and it could also happen that such a contraction goes from a $\mathbb{Q}$-factorial variety to a $\mathbb{Q}$-Gorenstein, but not $\mathbb{Q}$-factorial, variety.

Example 4.12. In Figure 8, we illustrate the MMP via moment polytopes with a horospherical variety of rank 2.

More precisely, we consider again the first SL_3/U-embedding X of Example 1.22 (also considered in Examples 2.6, 2.14 and 3.16). We only keep the SL_3/U-embeddings that could be obtained from the first one by a sequence of contractions of extremal rays negative along the canonical divisors (i.e. the contractions ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_7, ϕ_7 and ϕ_{11} in Figure 8).

We denote by X_1, X_2, X_3, X_4 and X_5 the irreducible G-stable divisor of X respectively corresponding to the primitive elements $x_1 = (1,0)$, $x_2 = (2,1)$, $x_3 = (1,1)$, $x_4 = (-1,0)$ and $x_5 = (1,-1)$. Recall that we have two colors D_α and D_β of SL_3/U, we still denote by D_α and D_β their closures in X, and their images in N are respectively $(1,0)$ and $(0,1)$. Also recall that $-K_X = X_1 + X_2 + X_3 + X_4 + X_5 + 2D_\alpha + 2D_\beta$.

If we choose the ample $\mathbb{Q}$-divisor D to be $X_1 + 2X_2 + 2X_3 + 4X_4 + 2X_5 + 3D_\alpha + 4D_\beta$, then the MMP via moment polytopes gives the three divisorial contractions ϕ_1, ϕ_3 and ϕ_8, and the Mori fibration ϕ_{12}.

If we choose the ample $\mathbb{Q}$-divisor D to be $X_1+2X_2+2X_3+4X_4+2X_5+3D_\alpha+3D_\beta$, then the MMP via moment polytopes gives the two contractions of 2-dimensional faces $\phi_3 \circ \phi_1 = \phi_4 \circ \phi_2$ and $\phi_{11} \circ \phi_8 = \phi_{11}^+ \circ \phi_7$, the flip ϕ_{11}^+ and the Mori fibration ϕ_{10}.

If we add $\frac{1}{10}X_1$ to the latter divisor D, then the MMP via moment polytopes gives the four contractions of extremal rays ϕ_2, ϕ_4, ϕ_8 and ϕ_{11}, the flip ϕ_{11}^+ and the Mori fibration ϕ_{10}.

But if we add instead $-\frac{1}{10}X_1$, the MMP via moment polytopes gives the three contractions of extremal rays ϕ_1, ϕ_3 and ϕ_7, and the Mori fibration ϕ_{10}.

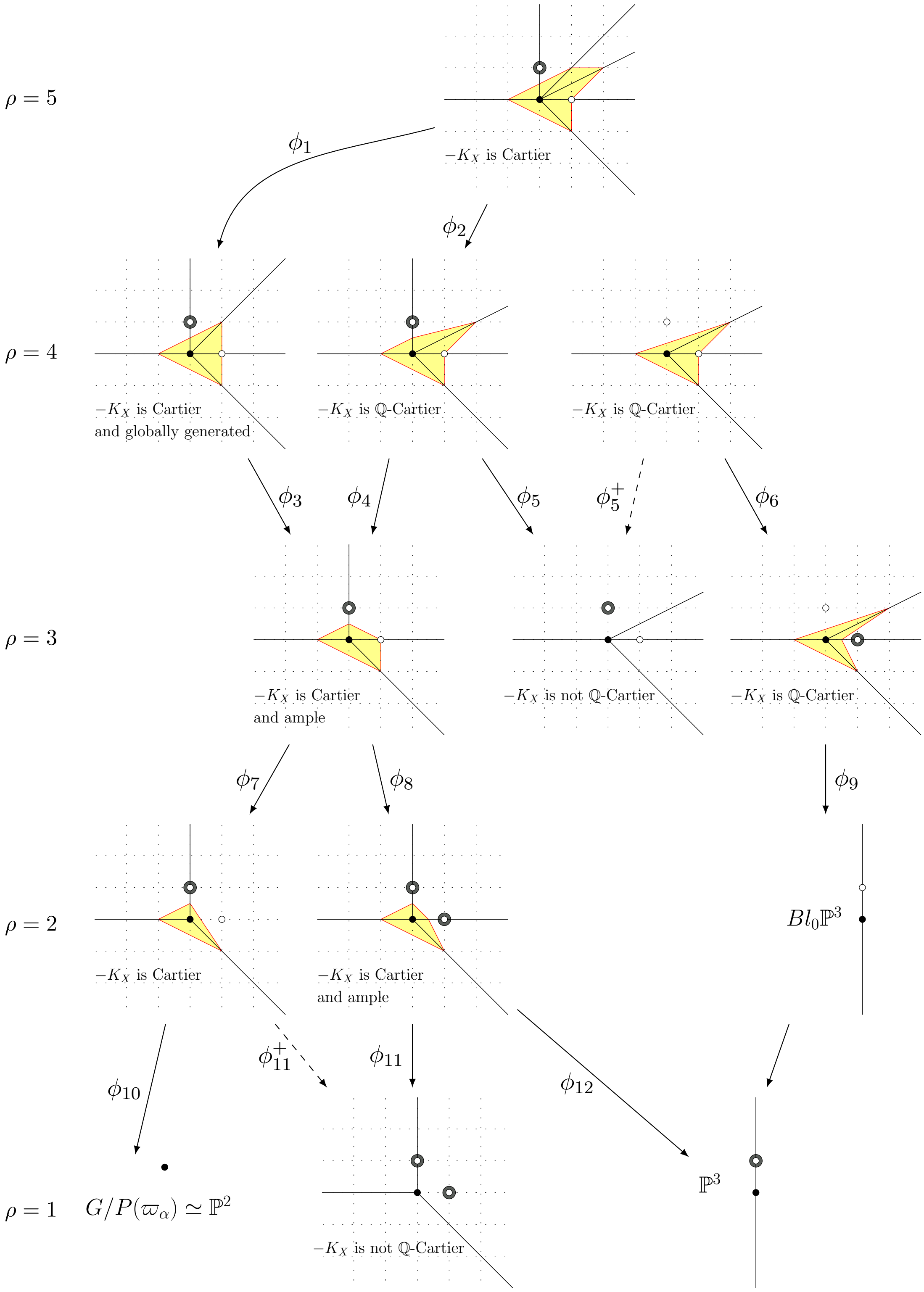
Now, if we choose the ample $\mathbb{Q}$-divisor D to be $30X_1 + 138X_2 + 135X_3 + 42D_\alpha + 134D_\beta$, then the MMP via moment polytopes gives the divisorial contraction ϕ_2, the contraction of flipping type ϕ_5, the flip ϕ_5^+, the divisorial contraction ϕ_6, and the Mori fibration ϕ_9.

Note that $\mathbb{P}^3$ and $Bl_0\mathbb{P}^3$ are both SL_3/H-embeddings where H is the kernel of ϖ_β in $P(\varpi_\beta)$. Moreover the general fibres of ϕ_9 and ϕ_{12} are both isomorphic to the SL_2/U-embedding $\mathbb{P}^2$, and the general fibres of ϕ_{10} are isomorphic to the $(\mathrm{SL}_2/U) \times \mathbb{C}^*$-embedding $\mathbb{P}^3$.

The base of a Mori fibration from a $\mathbb{Q}$-factorial klt pair is also $\mathbb{Q}$-factorial (see [KM98, Corollary 3.18] or [Rei83, Theorem 2.4] for toric varieties). In particular, in Theorem 4.10, for general D, if X is $\mathbb{Q}$-factorial then X^{ϵ_k} is also $\mathbb{Q}$-factorial. But if X is only $\mathbb{Q}$-Gorenstein, X^{ϵ_k} is not necessarily $\mathbb{Q}$-Gorenstein as we can see in the following example.

Example 4.13. Consider $G = \mathrm{SL}_5$. Choose the parabolic subgroup P such that $\mathcal{S}_P = \{\alpha_2, \alpha_3\}$, and M to be the intersection of the kernels of $5\varpi_{\alpha_2}$ and $5\varpi_{\alpha_3}$. Then $N \simeq \mathbb{Z}^2$ with $\alpha_{2M}^\vee = (5,0)$ and $\alpha_{3M}^\vee = (0,5)$. Moreover $a_{\alpha_2} = a_{\alpha_3} = 3$. Then the complete colored fan $\mathbb{F}$ with edges generated

Figure 8: MMP via moment polytopes from the same SL_3/U-embedding, with different polarizations



by $x_1 = (2,1)$, $x_2 = (-1,2)$, $x_3 = (-2,-1)$ and $x_4 = (1,-2)$ (no edges with color), and with set of colors $\{\alpha_2, \alpha_3\}$, is the fan of a $\mathbb{Q}$-Gorenstein horospherical G-variety (Fano, of Picard number 2, and of dimension 10).

Let $D = X_1 + X_2 + 2X_3 + X_4 + 3D_{\alpha_2} + 3D_{\alpha_3}$, it is an ample $\mathbb{Q}$-Cartier divisor of X. And $D+\epsilon K_X$ is ample for any $\epsilon < 1$. The nef divisor $D+K_X = X_3$ define the Mori fribration $X \longrightarrow Y$ where Y is the horospherical G-variety defined by the triple $(P, M', \mathbb{F}')$ where M' is the lattice generated by $10\varpi_{\alpha_2} + 5\varpi_{\alpha_3}$ in $\mathfrak{X}(P)$ and $\mathbb{F}'$ is the unique complete colored fan with set of colors $\{\alpha_2, \alpha_3\}$. Indeed, the polytopes Q^ϵ with $\epsilon \in [0,1]$ are the rectangles

$$Q^\epsilon = (2-2\epsilon)\varpi_{\alpha_2}, (8-6\epsilon)\varpi_{\alpha_2} + (3-2\epsilon)\varpi_{\alpha_3}] \times [(2-2\epsilon)\varpi_{\alpha_2}, (4-4\epsilon)\varpi_{\alpha_3}],$$

which is the segment $[0, 2\varpi_{\alpha_2} + \varpi_{\alpha_3}]$ if $\epsilon = 1$.

Now, identifying M' to $\mathbb{Z}$, we compute that $\alpha_{2M'}^\vee = 10$ and $\alpha_{3M'}^\vee = 5$ then, the two colors of $\mathbb{F}'$ belongs to the same color half-line and there is no $\chi \in M'$ such that $\langle \chi, \alpha_{2M'}^\vee \rangle = a_{\alpha_2} = 3$ and $\langle \chi, \alpha_{3M'}^\vee \rangle = a_{\alpha_3} = 3$. Hence $-K_Y$ is not $\mathbb{Q}$-Cartier. Note here that general fibres are isomorphic to $\mathbb{P}^1$, and we only have one special fibre, which is G-stable and of dimension 9.

4.4 An algorithm and SAGE programs

In author's Web Page, http://www.math.univ-montp2.fr/∼pasquier/MMPSAGE.html, you can find SAGE programs that compute all steps of the MMP from any $\mathbb{Q}$-Gorenstein projective horospherical variety, either polarized by any given ample $\mathbb{Q}$-divisor or randomly polarized.

The program will ask you to define a $\mathbb{Q}$-Gorenstein projective G-horospherical variety X associated to a triple $(P, M, \mathbb{F}_X)$ with the following steps:

- give the type of the semi-simple part of G;
- give the radical part of G;
- define the parabolic subgroup P by given $\mathcal{S}_P$;
- give a basis of the lattice M;
- give all the primitive elements of colored edges of $\mathbb{F}_X$ that have an empty set of colors;
- give the set of maximal colored cones of the colored fan $\mathbb{F}_X$.

Note that the program will ask you if you have already defined the $\mathbb{Q}$-Gorenstein projective G-horospherical variety X, so you could run several times the MMP from the same X with different D, without defining again X.

With MMP.sage, you will also need to give a B-stable ample divisor D of X. With Random-MMP.sage, the B-stable ample $\mathbb{Q}$-divisor will be randomly picked.

Remark that if X is not $\mathbb{Q}$-Gorenstein, the program returns you an error. And if D is not $\mathbb{Q}$-Cartier, not globally generated or not ample, the program also returns you an error. But the program does not check that the colored fan you give is a well-defined and complete colored fan.

Remark 4.14. Let (X,x) be a G-horospherical embedding associated to $(P,M,\mathbb{F}_X)$. Then there exist an integer $r \geq 0$ and a product $\tilde{G}$ of simply connected simple groups, so that, G is the quotient of $\tilde{G}\times(\mathbb{C}^*)^r$ by a central subgroup $\tilde{Z}$ and (X,x) is isomorphic to the $\tilde{G}\times(\mathbb{C}^*)^r$-horospherical embedding associated to $(\tilde{P}\times(\mathbb{C}^*)^r, M, \mathbb{F}_X)$, where $\tilde{P}\times(\mathbb{C}^*)^r$ is the parabolic subgroup of $\tilde{G}\times(\mathbb{C}^*)^r$ whose quotient by $\tilde{Z}$ is P (in particular the lattice of characters of P is the same as the lattice of characters of $\tilde{P}\times(\mathbb{C}^*)^r$).

Hence, we always can suppose that the reductive group G is the product of simply connected simple groups and a torus.

5 Fano horospherical varieties

In the previous section, we saw that Fano varieties take an important place in the MMP, and particularly those with Picard number one. In that section, we study horospherical Fano varieties (expanding the well-known theory of toric Fano varieties).

Definition 5.1. Let X be a normal projective variety. We say that X is Fano if the anticanonical divisor $-K_X$ of X is $\mathbb{Q}$-Cartier and ample.

If moreover $-K_X$ is Cartier, we say that X is Gorenstein Fano.

5.1 A classification

Let (X,x) be a G-horospherical embedding (associated to a triple $(P,M,\mathbb{F}_X)$). We choose $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha\in\mathcal{S}_P} a_\alpha D_\alpha$ as in Proposition 3.7. If $-K_X$ is $\mathbb{Q}$-Cartier and ample, then we construct as in Section 4.3, the pseudo-moment polytope of $(X,x,-K_X)$:

$$\tilde{Q} := \{x\in M_\mathbb{Q} \mid \forall i\in\{1,\dots,m\},\ \langle x,x_i\rangle \geq -1 \text{ and } \forall\alpha\in\mathcal{S}_P,\ \langle x,\alpha_M^\vee\rangle \geq -a_\alpha\}.$$

Its dual $\tilde{Q}^* := \{y\in N_\mathbb{Q} \mid \forall x\in\tilde{Q},\ \langle x,y\rangle\geq -1\}$ is a polytope in $N_\mathbb{Q}$ also defined by the set of points $y\in N_\mathbb{Q}$ such that $h_{-K_X}(y)\leq 1$, where h_{-K_X} is the piecewise linear function associated to the $\mathbb{Q}$-Cartier divisor $-K_X$ (see 2.1).

In particular, $\tilde{Q}^*$ is the convex hull of $(x_i)_{i\in\{1,\dots,m\}}$ and $(\frac{\alpha_M^\vee}{a_\alpha})_{\alpha\in\mathcal{F}_X}$ in $N_\mathbb{Q}$. And it contains all $\frac{\alpha_M^\vee}{a_\alpha}$ with $\alpha\in\mathcal{S}_P$.

Hence, to classify Fano horospherical varieties, it is natural to define the following family of polytopes.

Definition 5.2. Let P be a parabolic subgroup of G containing B and M be a sublattice of $\mathfrak{X}(P)$. Denote by $\mathrm{Prim}(N)$ the set of primitive elements of N.

A polytope in $N_\mathbb{Q}$ is said to be Fano if:

- its vertices are in $\mathrm{Prim}(N)\cup\{\frac{\alpha_M^\vee}{a_\alpha} \mid \alpha\in\mathcal{S}_P\}$;
- it contains 0 in its interior;
- it contains $\{\frac{\alpha_M^\vee}{a_\alpha} \mid \alpha\in\mathcal{S}_P\}$.

It is said to be Gorenstein Fano if moreover its dual has all its vertices in M.

Proposition 5.3. *[Pas06] The map (defined in the beginning of the section) from the set of isomorphic classes of Fano G-horospherical embeddings to the set of Fano polytopes, that sends (X,x) to $\tilde{Q}^*$, is a bijection. It restricts to a bijection from the set of isomorphic classes of Gorenstein Fano G-horospherical embeddings to the set of Gorenstein Fano polytopes.*

If we only consider toric varieties (i.e. $G=P=(\mathbb{C}^*)^n$ and $M=\mathfrak{X}(P)$), then the Gorenstein Fano polytopes are the reflexive polytopes in $\mathbb{Q}^n\supset\mathbb{Z}^n$ defined by V. Batyrev [Bat94].

5.2 Some results on Fano horospherical varieties

In [Pas06] we generalized, to horospherical varieties, two results known for toric varieties ([Deb03] and [Cas06] respectively)

Theorem 5.4 (Upper bound of the degree)**.** *Let X be a locally factorial Fano horospherical variety. Denote by d its dimension, by ρ its Picard number and by n its rank. The degree of X is the intersection number $(-K_X)^d$.*

If $\rho > 1$, then

$$(-K_X)^d \leq d!d^{d\rho+n}.$$

If $\rho = 1$, then

$$(-K_X)^d \leq d!(d+1)^{d+n}.$$

Theorem 5.5 (Upper bound of the Picard number)**.** *Let X be a $\mathbb{Q}$-factorial Gorenstein Fano horospherical variety. Denote by d its dimension, by ρ its Picard number and by n its rank.*

Then

$$\rho \leq n + d \leq 2d.$$

Moreover $\rho = 2d$ if and only if d is even and X is isomorphic to $(S_3)^{\frac{d}{2}}$ where S_3 is the blow-up of 3 general points in $\mathbb{P}^2$.

Note that Theorem 5.4 was recently extended to spherical varieties [GH15].

In [Pas10a] we also generalized, to horospherical varieties, another inequality known for toric varieties [Cas06], and that was conjectured for any smooth Fano variety [BCDD03].

Definition 5.6. Let X be a Gorenstein Fano variety. The pseudo-index ι of X is the positive integer defined by

$$\iota := \min\{-K_X \cdot C \mid C \text{ is a rational curve in } X\}.$$

Theorem 5.7. *Let X be a $\mathbb{Q}$-factorial, Gorenstein Fano horospherical variety. Denote by d its dimension, by ρ its Picard number and by ι its pseudo-index.*

Then

$$(\iota - 1)\rho \leq d.$$

Moreover $(\iota - 1)\rho = d$ if and only if X is isomorphic to $(\mathbb{P}^{\iota-1})^\rho$.

Note that Theorem 5.7 was recently also proved in the case of symmetric varieties [GH14].

5.3 Smooth Fano horospherical varieties with Picard number one

In [Pas09], we classified the smooth projective horospherical varieties with Picard number one, as follows. Note that such varieties are necessarily Fano (because horospherical varieties are rational).

Theorem 5.8. *Let X be a smooth projective horospherical variety with Picard number one. Then X is either homogeneous under its automorphism group, or it is given by one of the G-horospherical embeddings of rank one associated to triples $(P, M, \mathbb{F})$, where $\mathbb{F}$ is the unique complete colored fan with two colors, and G, P and M are as follows (we use the notation of [Bou75]):*

1. $G = \mathrm{Spin}(2m+1)$ *with* $m \geq 3$*,* $\mathcal{S}_P = \{\alpha_{m-1}, \alpha_m\}$*, and* $M = \mathrm{Ker}(\varpi_{m-1} - \varpi_m)$*;*

2. $G = \mathrm{Spin}(3)$*,* $\mathcal{S}_P = \{\alpha_1, \alpha_3\}$ *and* $M = \mathrm{Ker}(\varpi_1 - \varpi_3)$*;*

3. $G = \mathrm{Sp}(2m)$ *with* $m \geq 2$*,* $\mathcal{S}_P = \{\alpha_i, \alpha_{i+1}\}$ *with* $i \in \{1, \dots, m-1\}$*, and* $M = \mathrm{Ker}(\varpi_i - \varpi_{i+1})$*;*

4. $G = F_4$*,* $\mathcal{S}_P = \{\alpha_2, \alpha_3\}$*, and* $M = \mathrm{Ker}(\varpi_2 - \varpi_3)$*;*

5. $G = G_2$*,* $\mathcal{S}_P = \{\alpha_2, \alpha_1\}$*, and* $M = \mathrm{Ker}(\varpi_2 - \varpi_1)$*.*

We completed this classification by the following result.

Theorem 5.9. *Let X be a smooth projective variety with Picard number one. Denote by $\mathbb{G}$ the identity component of the automorphism group of X.*

Then, there exists two smooth spherical (and not horospherical) varieties $\mathbb{X}_1$ and $\mathbb{X}_2$, of rank 1, such that the following assertions are equivalent.

1. *X has two $\mathbb{G}$-orbits, the closed $\mathbb{G}$-orbit Z in X is at least of codimension 2, and the blow-up of Z in X still has two $\mathbb{G}$-orbits.*

2. *X is one of the horospherical varieties listed in Theorem 5.8, or X is one of the two spherical varieties $\mathbb{X}_1$ and $\mathbb{X}_2$.*

Moreover, X is one of the horospherical varieties listed in Theorem 5.8 if and only if $\mathbb{G}$ is not reductive. And, in that case, Z is respectively isomorphic to $G/P(\varpi_m)$, $G/P(\varpi_3)$, $G/P(\varpi_{i+1})$, $G/P(\varpi_3)$ and $G/P(\varpi_1)$; and $\mathbb{G}$ is respectively the semi-direct product of $(G\times\mathbb{C}^*)/\tilde{C}$ with $V(\varpi_m)$, $V(\varpi_3)$, $V(\varpi_1)$, $V(\varpi_4)$ and $V(\varpi_1)$, where $\tilde{C}$ is a subgroup of the center of $G \times \mathbb{C}^*$.

See [Pas09, Definitions 2.11 and 2.12] for explicit definitions of $\mathbb{X}_1$ and $\mathbb{X}_2$.

In a joint work with N. Perrin [PP10], we completed the study of these varieties with the following result.

Theorem 5.10. *Let X be a variety as in Theorem 5.9. If X is the horospherical G_2-variety (Case 5 of Theorem 5.8), then it admits a unique non trivial deformation, which is a deformation to a homogeneous G_2-variety. Otherwise, $H^1(X, T_X) = 0$, where T_X is the tangent bundle of X; in particular X is locally rigid (i.e. admits no local deformation to its complex structure).*

Note that the $\mathrm{Sp}(2m)$-varieties of Theorem 5.8 (Case 3) are the odd symplectic grassmannians (see Example 1.9), and then the results of Theorems 5.9 and 5.10 were already proved in that case in [Mih07].

6 A research program

6.1 The MMP for spherical varieties

The idea is to consider $\mathbb{Q}$-factorial MMP and non-$\mathbb{Q}$-factorial MMP, via moment polytopes for spherical varieties. We will be in front of two main difficulties.

First, contrary to horospherical varieties, if G/H is a fixed spherical homogeneous space, a polytope of $M_{\mathbb{Q}}$ is not necessarily a moment polytope of a G/H-embedding. Then, we need to characterize in a simple way the polytopes of $M_{\mathbb{Q}}$ that are moment polytopes of a G/H-embedding.

Secondly, in order to understand general fibres of Mori fibration, we need to realize spherical varieties as subvarieties of some projective spaces. Note that spherical but not horospherical G-varieties are not a closure in a projective space of a G-orbit of a sum of highest weight vectors.

The second difficulty could be really not easy, so it could be judicious to begin with some subfamilies of spherical varieties as symmetric varieties (for example $(\mathrm{SL}_n \times \mathrm{SL}_n)/\mathrm{Diag}(\mathrm{SL}_n)$-embeddings).

6.2 Upper bound of the degree of Fano spherical varieties

Fano spherical varieties has been classified by G. Gagliardi and J. Hofscheier [GH15], they also found an upper bound of the Picard number of $\mathbb{Q}$-factorial Gorenstein Fano varieties. Can we also find an upper bound of the degree of locally factorial (or smooth) Fano varieties as for horospherical varieties?

The degree of a spherical variety is given by a general formula (as an integral on a polytope) [Bri89, Theorem 4.1]. A possibility is to bound first the volume of the polytope and secondly the function we integrate.

6.3 The conjecture on the pseudo-index for spherical varieties

The conjecture on the pseudo-index was generalized by G. Gagliardi and J. Hofscheier [GH14] and proved for symmetric varieties. Is it still true for any locally factorial (or smooth) spherical Fano variety?

The proof for symmetric varieties is done by computations for each type of symmetric homogeneous spaces. An idea should be to find another proof without a case-by-case computation, which could be easily generalized.

6.4 Projective spherical varieties with Picard number one

In [Pas09], we observe two particular smooth, projective, spherical varieties with Picard number one, and they are the only two smooth, projective, spherical but not horospherical varieties with Picard number one that satisfies some additional condition.

The first natural question is: what occurs if we do not ask this additional condition? In other words, could we classified smooth, projective, spherical varieties with Picard number one? And then, how many orbits have they got? What are their ranks? Can we compute their automorphism groups (at least the identity component of their automorphism groups)?

A second question is: what occurs if we do not ask the varieties to be smooth? Indeed, in the MMP (even the $\mathbb{Q}$-factorial MMP), the general fibres of Mori fibration are not necessarily smooth (but $\mathbb{Q}$-factorial at best). Then, it would be great to know all $\mathbb{Q}$-factorial (and with terminal singularities), projective, horospherical varieties with Picard number one.

Without the smoothness hypothesis, we could have non-homogeneous horospherical varieties with Picard number one, and of rank greater than one. It could be judicious to first consider horospherical varieties with small rank. Note also that, since the automorphism group of a variety fixes the singular locus, it is not necessarily more difficult to compute the automorphism group of a singular variety than a smooth variety.

6.5 G-varieties of complexity one

The first way to generalized spherical varieties, which are of complexity zero, is to consider normal G-varieties of complexity one (i.e. such that the minimal codimension of B-orbits is one, where B is a Borel subgroup of G).

K. Langlois and R. Terpereau already studied normal horospherical G-varieties of complexity one (i.e. normal algebraic G-varieties of complexity one such that the isotropy group of any point contains a maximal unipotent subgroup of G). They gave characterizations of some singularities, in particular they gave a smooth criterion similar to the one I gave for horospherical varieties. And they also gave an anticanonical divisor of any normal horospherical G-varieties of complexity one.

If we can get some Cartier and ampleness criteria for any Weil divisor of normal G-varieties of complexity one, then it will be not difficult to get a classification of (normal) Gorentein Fano horospherical G-varieties of complexity one.

And then, it is natural to ask if we could generalize all the results I obtained on horospherical varieties, to Gorentein Fano horospherical G-varieties of complexity one.

To describe the MMP for these varieties, a first step should be to study their curves and their intersections with Cartier divisors.

6.6 Singularities of spherical varieties

(This work was done between the writing of this text and the defense of the HDR [Pas15b].)

With the criterion of Cartier divisors on spherical varieties, it is quite easy to get combinatorial characterizations of locally factoriality, $\mathbb{Q}$-factoriality, Gorenstein, $\mathbb{Q}$-Gorenstein, canonical, terminal and klt singularities. Most of these characterizations have been already written or mentioned. Moreover, as we can see in this manuscript, these types of singularities play a significant role in the MMP or in the theory of (not necessarily smooth) Fano varieties.
Then it will be helpful to write a survey on singularities of spherical varieties, as I did in Section 3, and with complete proofs.

Note also that it was really more difficult to obtain smooth criteria for spherical varieties. M. Brion gives a first criterion in [Bri91], and recently G. Gagliardi gives another more practical one in [Gag15]. This latter criterion becomes even simpler if we assume some conjecture (which is true for horospherical and symmetric varieties) that implies the conjecture on the pseudo-index for spherical varieties [GH14].

6.7 The cohomology of lines bundles on horospherical varieties

This project is independent of the results summarized in this manuscript.

The description of the cohomology of lines bundles over flag varieties (in zero characteristic) is a particular case of Bott's Theorem [Bot57]. And the cohomology of a line bundle over a smooth toric variety (in any characteristic) equals the cohomology of a complex associated to the fan of the toric variety and the line bundle [Dem70] (see also [Ful93]), which is easier to compute. For example, this result of M. Demazure enables me to get a vanishing result for the cohomology of lines bundles over Bott-Samelson varieties [Pas10b] (in any characteristic).

For spherical varieties, the only known results are some vanishing results due to M. Brion [Bri90] and several works of A. Tchoudjem on the cohomology of line bundles over compactifications of reductive groups [Tch04], over wonderful varieties of minimal rank [Tch07] and over complete symmetric varieties [Tch10].

Then, it is natural to ask if we could describe the cohomology of any line bundle over any smooth complete horospherical variety. We could begin with toroidal horospherical varieties, which are toric fibrations over flag varieties.

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RÉSUMÉ

Une variété horosphérique est une variété algébrique normale dans laquelle un groupe algébrique réductif agit avec une orbite ouverte fibrée en tores sur une variété de drapeaux. En particulier, les variétés de drapeaux et les variétés toriques sont horosphériques. La géométrie de ces variétés peut souvent être caractérisée combinatoirement par des éventails (coloriés), des polytopes rationnels et par des systèmes de racines. Par exemple, on connaît maintenant assez bien les variétés horosphériques de Fano et en particulier celles de nombre de Picard 1. On sait caractériser différents types de singularités des variétés horosphériques. Et on peut décrire complétement toutes les étapes du Programme du Modèle Minimal appliqué à une variété horosphérique $\mathbb{Q}$-Gorenstein, en utilisant les polytopes moments associés à des diviseurs $\mathbb{Q}$-Cartier. Plusieurs questions ouvertes se posent, en généralisant les variétés horosphériques aux variétés sphériques et même aux variétés de complexité 1 ; ou aussi en considérant des résultats connus dans d'autres contextes.

ABSTRACT

A horospherical variety is a normal algebraic variety where a reductive algebraic group acts with an open orbit which is a torus bundle over a flag variety. For example, toric varieties and flag varieties are horospherical. The geometry of these varieties can be often characterized in terms of (colored) fans, rational polytopes and root systems. For example, we now know quite well Fano horospherical varieties, in particular those of Picard number one. We are able to characterize different types of singularities of horospherical varieties. And we can completely describe all steps of the Minimal Model Program from a $\mathbb{Q}$-Gorenstein horospherical variety, by using moment polytopes associated to $\mathbb{Q}$-Cartier divisors. Several open questions arise, by generalizing horospherical varieties to spherical varieties, or even varieties of complexity one ; and also by considering results known in other contexts.

MOTS-CLÉS

Variétés sphériques, Programme du Modèle Minimal, Variétés de Fano, polytopes rationnels.

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