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YANG–MILLS FIELDS AND RANDOM HOLONOMY ALONG BROWNIAN BRIDGES

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We characterize Yang–Mills connections in vector bundles in terms of covariant derivatives of stochastic parallel transport along variations of Brownian bridges on the base manifold. In particular, we prove that a connection in a vector bundle E is Yang–Mills if and only if the covariant derivative of parallel transport along Brownian bridges (in the direction of their drift) is a local martingale, when transported back to the starting point. We present a Taylor expansion up to order 3 for stochastic parallel transport in E along small rescaled Brownian bridges and prove that the connection in E is Yang–Mills if and only if all drift terms in the expansion (up to order 3) vanish or, equivalently, if and only if the average rotation of parallel transport along small bridges and loops is of order 4.

1. Introduction.

This article is concerned with the characterization of Yang–Mills connections in a vector bundle E over a compact Riemannian manifold M in terms of stochastic parallel transport along Brownian bridges. Recall that Yang–Mills connections in a vector bundle E with a metric over M are the critical points of the following functional (the so-called Yang–Mills action):

$$\mathrm{YM}(\nabla) := \int_M \|R^\nabla\|^2 \, d\mathrm{vol}, \tag{1.1}$$

where $R^\nabla \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\,(E))$ is the curvature 2-form to a metric connection ∇ in E. The associated Euler–Lagrange equations characterize Yang–Mills connections by the property that

$$(d^\nabla)^* R^\nabla \equiv 0, \tag{1.2}$$

where d^∇ denotes the exterior differential and $(d^\nabla)^*$ its adjoint; see, for example, [10] and [16].

The equations $(d^\nabla)^* R^\nabla = 0$ are called the Yang–Mills equations. Since one always has $d^\nabla R^\nabla = 0$ by Bianchi's identity for R^∇, an equivalent condition to $(d^\nabla)^* R^\nabla = 0$ is $\Delta(R^\nabla) = 0$, where $\Delta = d^\nabla (d^\nabla)^* + (d^\nabla)^* d^\nabla$. The last condition states that the curvature R^∇ is harmonic.

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Stafford [22] proves that the average rotation (or holonomy) of parallel transport in E along a Brownian motion stopped at the first exit time from a ball of radius $r > 0$ and conditioned to hit a fixed point on the boundary of the ball is $O(r^3)$ in general and $O(r^4)$ if and only if the connection in E is Yang–Mills. Bauer [7] gives a new proof for Stafford's result based on Itô's formula for semimartingales in manifolds and estimates for the Green function of the Laplacian. Bauer [6], establishes a characterization of Yang–Mills connections in terms of parallel transport along perturbed Brownian motion: the covariant derivative of parallel transport with respect to variations induced by the flow of a gradient-type vector field on the base manifold, parallel transported back to the starting point, is a martingale if and only if the connection is Yang–Mills. In [3] the present authors give a similar characterization:

THEOREM 1.1 ([3], Proposition 4.10). *Let X be a Brownian motion in M starting at $x_0 \in M$ and let $/\!/_{0,t} : T_{x_0}M \to T_{X_t}M$ be parallel transport in TM along X. For $u \in T_{x_0}M$ and a varying about* 0 *on the real line, let*

$$X_t(a, u) = \exp_{X_t}(a /\!/_{0,t} u)$$

and let $W_t(a, u)$ denote parallel transport in E along $t \mapsto X_t(a, u)$. Consider the random variables

$$\nabla W_t(u) := \nabla_a\big|_{a=0} W_t(a, u) \in \operatorname{Hom}(E_{x_0}, E_{X_t})$$

and $W^{-1}\nabla W \in T^*_{x_0}M \otimes \operatorname{End}(E_{x_0})$. The following conditions are equivalent:

(a) ∇ *is a Yang–Mills connec tion, that is*, $(d^\nabla)^* R^\nabla = 0$;
(b) $W^{-1}\nabla W$ *is a local martingale for such* W.

Moreover, the quadratic variation S of $W^{-1}\nabla W$ is given by

$$S_t = 2\int_0^t \|R^\nabla\|^2 (X_s)\, ds.$$

A slight modification in [4] of this construction yields a martingale representation of the heat equation for Yang–Mills connections. A monotonicity formula for the quadratic variation of the martingale is derived in [4], as well as nonexplosion criteria for the heat equation involving the quadratic variation of the martingale.

In this paper we study the Yang–Mills property in connection with variations of the stochastic parallel transport along Brownian bridges. In particular, we consider perturbations of Brownian bridges induced by a drift vector field along the bridge or induced by a variation of the lifetime of the bridge.

For the remainder of the paper a connection in a vector bundle E over a manifold M is said to be a Yang–Mills connection, and its curvature R^∇ a Yang–Mills field, if ∇ (resp. R^∇) satisfies (1.2). Since

$$(d^\nabla)^* R^\nabla(u) = -\operatorname{tr}\nabla_{\boldsymbol{.}} R^\nabla(\cdot, u), \tag{1.3}$$

no explicit reference to a bundle metric on E is required. In particular, our connections need not be compatible to any metric on E.

The paper is organized as follows. In Section 2 we establish a characterization of Yang–Mills connections in terms of covariant derivatives ∇W of parallel transport W in E along Brownian bridges: we prove that the connection is Yang–Mills if and only if for perturbations in the direction of the drift of the Brownian bridge $W^{-1}\nabla W$ is a local martingale.

In Section 3 we establish characterizations of Yang–Mills connections similar to [7] and [22], but with parallel transport in E along Brownian bridges and loops in M (see Section 3 for the precise definition), and as already pointed out we do not need our connection in E to be metric preserving. These characterizations are consequences of Theorem 3.1, which has been established in [3]. Theorem 3.1 gives the asymptotic expansion in a at $a=0$ of the parallel transport $W(a)=W(a,(u_1,u_2))$ in E along a Brownian bridge $X(a)$ starting from $\exp_{x_0}(au_1)$ and ending at $\exp_{x_0}(au_2)$ at time 1, with quadratic variation a^2mt, where $x_0\in M$, $u_1,u_2\in T_{x_0}M$, $m=\dim M$. It is proved that the family $a\mapsto X(a)$ can be chosen such that $X(0)\equiv x_0$, $\partial_a|_{a=0}X(a)$ is a Brownian bridge in $T_{x_0}M$ starting at u_1 and ending at u_2, and $\nabla_a|_{a=0}\partial_a X(a)\equiv 0$. Under these assumptions we calculate

$$W(0),\quad \nabla_a\big|_{a=0}W(a),\quad \nabla_a\big|_{a=0}\nabla_a W(a),\quad \nabla_a\big|_{a=0}\nabla_a\nabla_a W(a),\tag{1.4}$$

where ∇_a denotes covariant derivative with respect to a (see Notation 1.2). Theorem 3.1 says, in particular, that when $u_2=0$ the first three processes in (1.4) are martingales and the last one is a semimartingale with drift

$$-\int_0 (d^\nabla)^* R^\nabla\big(\partial_a\big|_{a=0}X(a)\big)\,dt,$$

where R^∇ denotes again the curvature of the connection. Theorem 3.1 also gives an asymptotic expansion of the average holonomy of parallel transport along a Brownian bridge: if $u_2=0$,

$$\mathbb{E}[W_1(a)\,\tau_{1,a}]=\mathrm{id}_{E_{x_0}}-\frac{a^3}{12}(d^\nabla)^*R^\nabla(u_1)+O(a^4),\tag{1.5}$$

where $\tau_{1,a}$ is the parallel transport in E along $a\mapsto\exp_{x_0}(au_1)$. In fact, in (1.5) we have $O(a^4)=a^4\varepsilon(a)$ with $\varepsilon(a)$ converging as $a\searrow 0$.

The proof is based on several technical ingredients. To obtain the covariant derivatives of W with respect to a, we use Theorem 1.3, which gives a general commutation formula of Itô covariant derivatives with respect to t and covariant derivatives with respect to a. In addition, we exploit Taylor expansions of the heat kernel in a small neighborhood of x_0. The parallel transport W_t is easily shown to be in L^1 for time $t<1$. This result is extended to time 1 with a time-reversal argument, in order to get formula (1.5).

Two corollaries are easily derived from Theorem 3.1. The first one (Corollary 3.3) shows that $(d^\nabla)^* R^\nabla$ vanishes at x_0 if and only if the transports W constructed in Theorem 3.1 (with $u_2 = 0$) have the property that all covariant derivatives with respect to a at $a = 0$ up to order 3 are martingales. The second one (Corollary 3.4) states that $(d^\nabla)^* R^\nabla$ vanishes at x_0 if and only if the expected rotation

$$\mathbb{E}\big[W_1(a)\,\tau_{1,a} - \mathrm{id}_{E_{x_0}}\big]$$

of parallel transport $W(a) = W(a, (u_1, 0))$ in E is $O(a^4)$ for sufficiently many u_1. For instance, it is sufficient to take m independent vectors $u_1 \in T_{x_0}M$, where $m = \dim M$.

Finally, in Theorem 3.6 we give a result similar to Theorem 3.1, but with the parallel transport in E replaced by deformed parallel transport. The asymptotic expansion analogous to (1.5) then contains a term which is quadratic in a.

Throughout the paper we use the following notation. Let M be a smooth compact manifold equipped with a connection ∇. We denote by R the curvature tensor with respect to ∇, that is, $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\,(TM))$. The same symbol ∇ is used to denote connections in vector bundles over M, as well as for all connections derived naturally from the given ones.

If X is a continuous semimartingale in M and α a C^2 section of T^*M, we write

$$\int \langle \alpha(X), \delta X\rangle$$

for the Stratonovich integral of α along X. We denote by $/\!/_{0,t} : T_{X_0}M \to T_{X_t}M$ the parallel transport in TM along X. The antidevelopment of X is the $T_{X_0}M$-valued process

$$\mathscr{A}(X)_t = \int_0^t \langle /\!/_{0,s}^{-1}, \delta X_s\rangle.$$

The semimartingale X is said to be a ∇-martingale if $\mathscr{A}(X)$ is a local martingale in $T_{X_0}M$. The Itô integral of α along X is defined as

$$\int_0^t \big\langle \alpha(X_s), d^\nabla_{\text{Itô}} X_s\big\rangle := \int_0^t \langle \alpha(X_s) \circ /\!/_{0,s}, d\mathscr{A}(X)_s\rangle,$$

so formally $d^\nabla_{\text{Itô}} X_t = /\!/_{0,t}\, d\mathscr{A}(X)_t$.

Let $\pi : E \to M$ be a vector bundle over M equipped with a covariant derivative ∇. We denote by $R^\nabla \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}\,(E))$ the curvature tensor with respect to ∇. The tangent bundle TE splits naturally into $HE \oplus VE$, where VE is the vertical bundle and $T_e\pi\,|\,H_eE$ defines an isomorphism onto $T_{\pi(e)}M$ for every $e \in E$. Let

$$h_e = (T_e\pi\,|\,H_eE)^{-1} : T_{\pi(e)}M \to H_eE$$

be the "horizontal lift" and $v_e : E_{\pi(e)} \to V_e E$ the vertical lift. Thus every section $s \in \Gamma(E)$ has a canonical vertical lift $s^v \in \Gamma(TE)$ defined by $s^v_e = v_e(s_{\pi(e)})$, and every vector field $u \in \Gamma(TM)$ a horizontal lift $u^h \in \Gamma(TE)$ defined by $u^h_e = h_e(u_{\pi(e)})$.

It is well known (see, e.g., [12]) that there exists a unique connection ∇^h on E, that is, a covariant derivative on TE,

$$\nabla^h : \Gamma(TE)\Gamma(T^*E \otimes TE),$$

satisfying the following properties: for all sections $r, s \in \Gamma(E)$, $u, w \in \Gamma(TM)$,

$$\nabla^h_{r^v} s^v = 0, \qquad \nabla^h_{r^v} w^h = 0, \qquad \nabla^h_{u^h} s^v = (\nabla_u s)^v, \qquad \nabla^h_{u^h} w^h = (\nabla_u w)^h. \tag{1.6}$$

The connection ∇^h will be called the horizontal lift of ∇ to E.

Let J be an E-valued semimartingale. The parallel transport $/\!/^h_{0,t} V$ of a vector $V = u^h + s^v$ along J_t with respect to ∇^h is given by

$$/\!/^h_{0,t} V = h_{J_t}(/\!/_{0,t} u) + v_{J_t}(/\!/^E_{0,t} s), \tag{1.7}$$

where $/\!/_{0,t}$ is parallel translation in TM along $\pi(J_t)$ w.r.t. the connection on M and $/\!/^E_{0,t}$ is parallel translation in E along $\pi(J_t)$ w.r.t. the connection ∇ in E.

Let J be a continuous E-valued semimartingale and $X = \pi \circ J$. As shown in [2], the antidevelopment of J with respect to ∇^h is given by the formula

$$\mathscr{A}^h(J) = h_{J_0}(\mathscr{A}(X)) + v_{J_0}(/\!/^{E\ -1}_{0,\cdot} J - J_0). \tag{1.8}$$

Consequently, the semimartingale J is a ∇^h-martingale if and only if:

(i) $X = \pi \circ J$ is a ∇-martingale in M, and
(ii) $(/\!/^E_{0,t})^{-1} J_t$ is a local martingale in E_{X_0}.

An object of particular interest is the Itô covariant differential of J:

$$DJ = /\!/^E_{0,\cdot} d(/\!/^{E\ -1}_{0,\cdot} J) = v_J^{-1}\big((d^{\nabla^h}_{\text{Itô}} J)^{\text{vert}}\big).$$

Equivalently, DJ is determined by the formula

$$d^{\nabla^h}_{\text{Itô}} J = h_J(d^{\nabla}_{\text{Itô}} X) + v_J(DJ). \tag{1.9}$$

In local coordinates on an open set U, we may decompose the connection in E as $\nabla = d + A$, where A is an $\operatorname{End}(E)$-valued 1-form over U; similarly, $\nabla = d + \Gamma$ for the connection on the base manifold M, where Γ is an $\operatorname{End}(TM)$-valued 1-form. This leads to the following general formulas for $(DJ)^\alpha$ (see [3]):

$$\begin{aligned}(DJ)^\alpha = {} & dJ^\alpha + A^\alpha(d^{\nabla}_{\text{Itô}} X, J) + A^\alpha(dX, dJ) \\ & + \tfrac{1}{2}\big(dA^\alpha(dX, dX, J) + A^\alpha\big(dX, A(dX, J)\big) \\ & - A^\alpha\big(\Gamma(dX, dX), J\big)\big)\end{aligned} \tag{1.10}$$

or, equivalently,

$$(DJ)^\alpha = dJ^\alpha + A^\alpha(d^{\nabla}_{\text{Itô}} X, J) + A^\alpha(dX, DJ) + \tfrac{1}{2}\nabla A^\alpha(dX, dX, J). \tag{1.11}$$

NOTATION 1.2. If $a \mapsto w(a) \in E$ is a C^1 path, we denote by $\nabla_a w$ its covariant derivative,

$$\nabla_a w = \psi(\partial_a w) = \frac{Dw}{da},$$

where $\psi(W) = v_w^{-1}(W^{\mathrm{vert}})$ if $W \in T_w E$. Slightly abusing the notation, we just write $\nabla_{a_0} w$ for $\nabla_a|_{a=a_0} w$.

The following theorem, which has been proved in [3], describes how covariant derivatives with respect to a and t commute. We write $\nabla R^\nabla(v_1, v_2, v_3)$ for $\nabla_{v_1} R^\nabla(v_2, v_3)$, where $v_1, v_2, v_3 \in T_x M$.

THEOREM 1.3. *Let I be an open interval in $\mathbb{R}$, and for each $a \in I$ let $J(a)$ be a semimartingale with values in the vector bundle E. Assume that $a \mapsto J(a)$ is C^1 in the topology of semimartingales. Let $X(a) = \pi(J(a))$. Then*

$$\begin{aligned} D\nabla_a J &= \nabla_a DJ + R^\nabla(d_{\mathrm{Itô}}^\nabla X, \partial_a X)J + R^\nabla(dX, \partial_a X)DJ \\ &\quad - \tfrac{1}{2}\nabla R^\nabla(dX, \partial_a X, dX)J - \tfrac{1}{2} R^\nabla(D\partial_a X, dX)J. \end{aligned}$$

Finally, for a vector field V on M let $\iota_V R^\nabla := R^\nabla(V, \cdot) = -R^\nabla(\cdot, V)$. Note that by definition $\iota_V R^\nabla$ is an $\mathrm{End}(E)$-valued 1-form on M.

2. Covariant derivative of the parallel transport along a Brownian bridge in the direction of the drift. Let M be a compact m-dimensional Riemannian manifold, ∇ be the Levi–Civita connection on M and $\pi: E \to M$ be a vector bundle endowed with a covariant derivative ∇.

Let $T > 0$ and X be a Brownian bridge on M such that $X_0 = x$, $X_T = y$ and

$$d_{\mathrm{Itô}}^\nabla X_t = \Sigma(X_t)\, dB_t + V_t(X_t)\, dt, \tag{2.1}$$

where B is an $\mathbb{R}^r$-valued Brownian motion, $\Sigma \in \Gamma(\mathbb{R}^r \otimes TM)$ satisfies $\Sigma(x)\Sigma(x)^* = \mathrm{id}_{T_x M}$ for every $x \in M$ and

$$V_t(z) = \mathrm{grad}_x \log p(T - t, \cdot, y)(z), \tag{2.2}$$

with $p(t, z, y)$ being the density at y of an M-valued Brownian motion started at z. Let

$$X_t(a) = \exp_{X_t}\big(a\,(T - t)V_t(X_t)\big) \tag{2.3}$$

and let $W_t(a)$ be the parallel transport in E along $t \mapsto X_t(a)$. Denote by $\nabla_0 W$ the covariant derivative of W with respect to a at $a = 0$: if $a \mapsto v(a)$ is a C^1 path in E with projection $a \mapsto X_0(a)$, then

$$(\nabla_0 W_s)v(0) := \nabla_0(W_s v) - W_s(0)(\nabla_0 v). \tag{2.4}$$

Recall that $\nabla_0 = \nabla_a|_{a=0}$. Finally, we write $W^{-1}\nabla_0 W$ for the $\mathrm{End}(E_{X_0})$-valued process $W(0)^{-1}\nabla_a|_{a=0} W(a)$.

PROPOSITION 2.1. *Let $\pi : E \to M$ be a vector bundle with a connection ∇ over a compact Riemannian manifold M. Let X be the Brownian bridge on M with lifetime T constructed as solution to* (2.1) *and let $X(a)$ be the perturbation of X defined by* (2.3). *Further denote*, *$W(a)$ the parallel transport in E along $X(a)$. Then the drift of the* $\operatorname{End}(E_{X_0})$*-valued semimartingale $W^{-1}\nabla_0 W$ is*

$$-\frac{T-t}{2}\, W^{-1}(d^\nabla)^* R^\nabla(V_t(X_t))W\,dt,$$

and its quadratic variation is

$$\int_0 (T-s)^2 \left\| \iota_{V_s(X_s)} R^\nabla \right\|^2 ds. \tag{2.5}$$

The covariant derivative ∇ is Yang–Mills if and only if, for any W as above, $W^{-1}\nabla_0 W$ is a local martingale.

Note that $W^{-1}\nabla_0 W$ is then already a true martingale, as a consequence of the compactness of the underlying manifold. Indeed, it has bounded quadratic variation, as can be seen from (2.5) along with estimate (3.4) below.

PROOF OF PROPOSITION 2.1. From Theorem 1.3 we conclude

$$\begin{aligned} D\nabla_0 W = {} & R(d^\nabla_{\text{Itô}} X, \partial_0 X)W \\ & - \tfrac{1}{2}(d^\nabla)^* R^\nabla(\partial_0 X)W\,dt - \tfrac{1}{2} R^\nabla(D\partial_0 X, dX)W. \end{aligned} \tag{2.6}$$

Since $\partial_0 X_t = (T-t)\, V_t(X_t)$, we have

$$D\partial_0 X_t \wedge dX_t = (T-t)\, DV_t(X_t) \wedge dX_t = -(T-t)\left(dV_t^\flat(X_t)\right)^\sharp dt,$$

where $\sharp$ denotes the canonical isometry $\Lambda^k T^*M \to \Lambda^k TM$ induced by the Riemannian metric, and $\flat$ its inverse. [The derivative with respect to t in $V_t(x)$ disappears because we consider quadratic variations.] Now by (2.2), for every $t \in [0, 1[$ the vector field $x \mapsto V_t(x)$ is of gradient type, which implies $dV_t^\flat = 0$. As a consequence, the last term in (2.6) vanishes, and we get

$$D\nabla_0 W_t = R^\nabla\left(d^\nabla_{\text{Itô}} X_t, (T-t)\, V_t(X_t)\right)W_t - \frac{T-t}{2}(d^\nabla)^* R^\nabla(V_t(X_t))W\,dt.$$

But since R^∇ is antisymmetric, we may replace $d^\nabla_{\text{Itô}} X$ by $\Sigma(X)\,dB$ to obtain

$$D\nabla_0 W_t = R^\nabla\left(\Sigma(X_t)\,dB_t, (T-t)\, V_t(X_t)\right)W_t - \frac{T-t}{2}(d^\nabla)^* R^\nabla(V_t(X_t))W\,dt,$$

which is equivalent to

$$\begin{aligned} d(W^{-1}\nabla_0 W) = {} & W^{-1} R^\nabla\left(\Sigma(X)\,dB, (T-t)\, V(X)\right)W \\ & - \frac{T-t}{2}\, W^{-1}(d^\nabla)^* R^\nabla(V(X))W\,dt. \end{aligned} \tag{2.7}$$

This gives the formulas for the drift and the quadratic variation of $W^{-1}\nabla_0 W$. In addition, we observe from (2.7) that if ∇ is Yang–Mills then $W^{-1}\nabla_0 W$ is a local martingale.

We are left to verify that if all such $W^{-1}\nabla_0 W$ are local martingales then ∇ is Yang–Mills. Let $y \in M$ and $u \in T_y M$. We need to prove that $(d^\nabla)^* R^\nabla(u) = 0$. For $\varepsilon > 0$ and $t \in [0, \varepsilon^2[$ let

$$\hat{V}_t(\varepsilon, y) = \operatorname{grad}\log p(\varepsilon^2 - t, \cdot, y)\big(\exp_y(-\varepsilon u)\big).$$

Let $Y_t(\varepsilon)$ be a Brownian bridge satisfying $Y_0(\varepsilon) = \exp_y(-\varepsilon u)$ and $Y_{\varepsilon^2}(\varepsilon) = y$. Its drift at time 0 is $\hat{V}_0(\varepsilon, x)$; hence, combining (2.7) and the fact that $W^{-1}\nabla_0 W$ is a local martingale, we obtain $(d^\nabla)^* R^\nabla(\varepsilon \hat{V}_0(\varepsilon, x)) = 0$. But there exists a neighborhood $\mathscr{V}$ of y such that, for $s \geq 0$, $x \in \mathscr{V}$,

$$p(s, x, y) = s^{-m/2} \exp\left(-\frac{d^2(x, y)}{2s}\right) \psi(s, x, y), \tag{2.8}$$

where $(s, x) \mapsto \psi(s, x, y)$ is smooth in $[0, \infty[\times \mathscr{V}$, and $\psi(0, y, y) > 0$ (see [5] and [15]). Hence, for $\varepsilon > 0$ sufficiently small,

$$\hat{V}_0(\varepsilon, y) = \operatorname{grad}\left(-\frac{d^2(\cdot, y)}{2\varepsilon^2}\right)\big(\exp_y(-\varepsilon u)\big) + \operatorname{grad}\log\psi(\varepsilon^2, \cdot, y)\big(\exp_y(-\varepsilon u)\big),$$

which implies that $\varepsilon \hat{V}_0(\varepsilon, y)$ converges to u as ε tends to 0. As a consequence, $(d^\nabla)^* R^\nabla(u) = 0$, which completes the proof. $\square$

COROLLARY 2.2. *The covariant derivative ∇ is Yang–Mills if and only if $W^{-1}\nabla_0 W$ is a local martingale for the parallel transport W along any Brownian loop. Then, since M is compact, all $W^{-1}\nabla_0 W$ are already true martingales.*

PROOF. We only have to prove that ∇ is Yang–Mills provided that, for any parallel transport W along a Brownian loop, the process $W^{-1}\nabla_0 W$ is a local martingale. Let $y \in M$ and $u \in T_y M$. We want to show that $(d^\nabla)^* R^\nabla(u) = 0$. To this end consider a loop X with lifetime 1 based at y. Note that X has a smooth positive density $q(s, x)$ with respect to the volume measure on M for $0 < s < 1$. Now let $\varepsilon \in]0, 1[$. We know from (2.7) by taking the derivative of the drift at time $1 - \varepsilon^2$ that

$$(d^\nabla)^* R^\nabla\big(\varepsilon \operatorname{grad}\log p(\varepsilon^2, \cdot, y)(X_{1-\varepsilon^2})\big) = 0. \tag{2.9}$$

Recall that the density q at time $1 - \varepsilon^2$ and at the point $\exp_y(-\varepsilon u)$ is positive. The smoothness of $(s, x) \mapsto q(s, x)$ and $(s, x) \mapsto \operatorname{grad}\log p(1 - s, \cdot, y)(x)$ in a neighborhood of $(1 - \varepsilon^2, \exp_y(-\varepsilon u))$, along with the fact that $(d^\nabla)^* R^\nabla$ is a smooth $\operatorname{End}(E)$-valued 1-form, implies

$$(d^\nabla)^* R^\nabla\big(\varepsilon \operatorname{grad}\log p(\varepsilon^2, \cdot, y)\big(\exp_y(-\varepsilon u)\big)\big) = 0. \tag{2.10}$$

Indeed, otherwise there would be a small time interval I centered about $1-\varepsilon^2$ and a small neighborhood $\mathscr{V}'$ of $\exp_y(-\varepsilon u)$ such that with positive probability, $X_s \in \mathscr{V}'$ for all $s \in I$ and the drift of $W^{-1}\nabla_0 W$ is close to a nonzero value for all $s \in I$, which is impossible. As in the proof of Proposition 2.1, we conclude from (2.10) that $(d^\nabla)^* R^\nabla(u) = 0$. $\square$

REMARK 2.3. In the whole of Section 2 compactness of the manifold is actually not essential. In fact, the characterizations of Theorem 2.1 and Corollary 2.2 equally hold for geodesically complete noncompact manifolds, if stated with local martingales.

3. Asymptotics of the parallel transport along a rescaled Brownian bridge. Let M be a compact m-dimensional Riemannian manifold with its Levi–Civita connection ∇ and let $\pi : E \to M$ be a vector bundle over M endowed with a connection ∇. Let $x_0 \in M$, $u = (u_1, u_2) \in (T_{x_0}M)^2$ and $X(a) = X(a, u)$ be a rescaled Brownian bridge from $\exp_{x_0}(au_1)$ to $\exp_{x_0}(au_2)$ with lifetime 1 defined as follows: $X(a) = X(a, u)$ satisfies $X_0(a) = \exp_{x_0}(au_1)$ and, for $t \in [0, 1[$,

$$d^\nabla_{\text{Itô}} X_t(a) = a\,\Sigma(X_t(a))\,dB_t + b_t(a)\,dt, \tag{3.1}$$

where B is an $\mathbb{R}^r$-valued Brownian motion, $\Sigma \in \Gamma(\mathbb{R}^r \otimes TM)$ is such that, for all $x \in M$, $\Sigma(x)\Sigma(x)^* = \mathrm{id}_{T_xM}$ and $\nabla\Sigma(x_0) = 0$. (Notice that such a choice for Σ is always possible, locally with an orthonormal frame whose covariant derivative vanishes at x_0, and globally with the help of a partition of unity.) The drift b_t in (3.1) is given by

$$b_t(a) = V_t\big(a, X_t(a), u_2\big), \tag{3.2}$$

where

$$V_t(a, x, u_2) = a^2 \operatorname{grad}_x \log p\big(a^2(1-t), x, \exp_{x_0} au_2\big) \tag{3.3}$$

and $p(t, x, \cdot)$ is again the density at time t of a Brownian motion on M started at x.

Note that there exists a constant C depending only on M such that, for all $(s, x, y) \in]0, 1[\times M \times M$,

$$\|\operatorname{grad}_x \log p(s, x, y)\| \le C\left\{\frac{d(x, y)}{s} + \frac{1}{\sqrt{s}}\right\}; \tag{3.4}$$

see [14] and [21].

The process $X(a)$ is called a rescaled Brownian bridge, since $X(0) \equiv x_0$, and for $a > 0$ the rescaled process $t \mapsto X_{t/a^2}(a)$ describes a Brownian motion starting at $\exp_{x_0}(au_1)$ and conditioned to hit the point $\exp_{x_0}(au_2)$ at time a^2. In particular, for $u = (0, 0)$, the process $X(a)$ defines Brownian loops based at x_0 with lifetime a^2.

In the rest of this article we keep the notation ∂_0 for $\partial_a|_{a=0}$ and ∇_0 for $\nabla_a|_{a=0}$. The differentiation of C^1 families of semimartingales is understood in the topology of semimartingales.

There exists a small neighborhood $\mathscr{V}$ of x_0 such that, for all $x, y \in \mathscr{V}$,

$$p(s, x, y) = s^{-m/2} \exp\biggl(-\frac{d^2(x, y)}{2s}\biggr) \psi(s, x, y),$$

where ψ is smooth in $[0, \infty[\times \mathscr{V} \times \mathscr{V}$, and $\psi(0, x, y) > 0$ ([5], formula (27), and [9]). Thus, for $x \in \mathscr{V}$,

$$\begin{aligned} V_t(a, x, u_2) = &-\frac{1}{2(1-t)} \operatorname{grad}_x d^2(\exp a u_2, x) \\ &+ a^2 \operatorname{grad}_x \log \psi\bigl(a^2(1-t), x, \exp a u_2\bigr). \end{aligned} \tag{3.5}$$

Consider an exponential chart centered at x_0 and let $f(x) = d^2(x_0, x)$. We may choose the chart such that $f(x) = \sum_{i=1}^m (x^i)^2$. Denote by (g_{ij}) the metric, by (g^{ij}) its inverse, by Γ_{ij}^k the Christoffel symbols, and let $D_j = \partial/\partial x^j$. Observe that

$$d^2\bigl(x, \exp_{x_0}(a u_2)\bigr) = \sum_{i=1}^m (x^i - a u_2^i)^2 + r(x, a u_2), \tag{3.6}$$

where for all $w, w' \in T_{x_0}M$, with some constant $c > 0$,

$$r(\exp_{x_0} w, w') \le c\,(\|w \wedge w'\|^2) \le c\,(\|w\|\,\|w'\|\,\|w - w'\|^2), \tag{3.7}$$

$\|\cdot\|$ being the Euclidean norm in $T_{x_0}M$. This gives

$$\begin{aligned} &V_t(a, X, u_2) \\ &\quad = -\frac{1}{1-t}\Biggl(\sum_{i,j=1}^m (X^i - a u_2^i) g^{ij}(X) D_j(X) + \frac{1}{2} \operatorname{grad} r(\cdot, a u_2)(X)\Biggr) \\ &\qquad + a^2 \operatorname{grad} \log \psi\bigl(a^2(1-t), \cdot, \exp a u_2\bigr)(X) \end{aligned} \tag{3.8}$$

and shows, in particular, that

$$b_t(0) = 0. \tag{3.9}$$

To calculate $\nabla_0 b_t$, we differentiate in (3.8) the first term on the right:

$$\begin{aligned} &\nabla_a\Biggl(\sum_{i,j=1}^m (X^i - a u_2^i) g^{ij} D_j\Biggr) \\ &\quad = \sum_{k=1}^m \partial_a X^k \Biggl(\sum_{j=1}^m g^{kj} D_j + \sum_{i,j=1}^m (X^i - a u_2^i)\, D_k g^{ij} D_j \\ &\qquad\qquad + \sum_{i,j,\ell=1}^m (X^i - a u_2^i)\, g^{ij} \Gamma_{kj}^\ell D_\ell\Biggr) - \sum_{i,j=1}^m u_2^i g^{ij} D_j. \end{aligned}$$

Since $g_{ij}(x_0) = \delta_{ij}$ and since we may neglect the other terms in (3.8), we conclude

$$\nabla_0 b_t = -\frac{1}{1-t}(\partial_0 X - u_2). \tag{3.10}$$

Differentiating again and taking into account that $dg_{ij}(x_0) = 0$, $\Gamma^k_{ij}(x_0) = 0$, gives

$$\nabla_0 \nabla_a \left(-\frac{1}{1-t} \sum_{i,j=1}^{m} (X^i - au_2^i)\, g^{ij} D_j(X) \right) = -\frac{1}{1-t} \nabla_0 \partial_a X.$$

Since $\nabla_0 \nabla_a \operatorname{grad} r(\cdot, au_2)(X) = 0$, we get

$$\nabla_0 \nabla_a b = -\frac{1}{1-t} \nabla_0 \partial_a X + 2 \operatorname{grad} \log \psi(0, \cdot, x_0)(x_0). \tag{3.11}$$

But for x close to x_0 we have $\psi(0, x, x_0) = (\det(g_{ij})(x))^{-1/4}$ (see, e.g., [8], page 208, or [20], (3.11)), and hence, combined with $dg_{ij}(x_0) = 0$,

$$\operatorname{grad}_x \log \psi(0, \cdot, x_0)(x_0) = 0.$$

Thus (3.11) leads to

$$\nabla_0 \nabla_a b = -\frac{1}{1-t} \nabla_0 \partial_a X. \tag{3.12}$$

Now differentiating (3.1) with respect to a by means of Theorem 2.2 in [3] and taking the covariant derivative according to [3], (4.7), and (1.9) above, we get (using that ∇ is torsion-free)

$$D\partial_a X = a \nabla_{\partial_a X} \Sigma(X)\, dB + \Sigma(X)\, dB + \nabla_a b\, dt - \tfrac{1}{2} R(\partial_a X, dX)\, dX. \tag{3.13}$$

At $a = 0$, since $X_0(a) = \exp_{x_0}(au_1)$, we have $\partial_0 X_0 = u_1$ and

$$D\partial_0 X = \Sigma(x_0)\, dB - \frac{\partial_0 X - u_2}{1-t}\, dt; \tag{3.14}$$

hence $\partial_0 X$ is a Brownian bridge in the Euclidean space $T_{x_0}M$, starting from u_1 and ending at u_2 at time 1. Note that in (3.14) the covariant differential $D\partial_0 X$ equals $d\partial_0 X$ since $X(0) \equiv x_0$.

As the next step, since $X(0) \equiv x_0$, differentiating (3.13) at $a = 0$ with the help of Theorem 1.3 gives

$$D\nabla_0 \partial_a X = 2\nabla_{\partial_0 X} \Sigma(x_0)\, dB + \nabla_0 \nabla_a b\, dt. \tag{3.15}$$

But $\nabla \Sigma(x_0) = 0$; hence, along with (3.12), we conclude from (3.15) that

$$D\nabla_0 \partial_a X = -\frac{1}{1-t} \nabla_0 \partial_a X\, dt. \tag{3.16}$$

On the other hand, $a \mapsto X_0(a) = \exp_{x_0}(au_1)$ is a geodesic curve, and therefore $\nabla_0 \partial_a X_0 = 0$. This observation, together with (3.16) and Gronwall's lemma, yields

$$\nabla_0 \partial_a X_t \equiv 0, \tag{3.17}$$

and, consequently, by means of (3.12),

$$\nabla_0 \nabla_a b_t \equiv 0. \tag{3.18}$$

THEOREM 3.1. *Let M be a compact Riemannian manifold and let E be a vector bundle over M endowed with a covariant derivative ∇. For $u = (u_1, u_2) \in (T_{x_0}M)^2$ let $X(a, u)$ be the rescaled Brownian bridge from $\exp_{x_0}(au_1)$ to $\exp_{x_0}(au_2)$, as defined above by* (3.1), *and let $W(a) = W(a, u)$ be parallel transport in E along $X(a, u)$. Then, in the topology of semimartingales indexed by $t \in [0, 1[$,*

$$W_t(0) = \mathrm{id}_{E_{x_0}}, \tag{3.19}$$

$$\nabla_0 W_t \equiv 0, \tag{3.20}$$

$$\nabla_0 \nabla_a W_t = \int_0^t R^\nabla(d\partial_0 X_s, \partial_0 X_s), \tag{3.21}$$

$$\nabla_0 \nabla_a \nabla_a W_t = 2\int_0^t \nabla R^\nabla(\partial_0 X_s, d\partial_0 X_s, \partial_0 X_s) - (d^\nabla)^* R^\nabla\biggl(\int_0^t \partial_0 X_s\, ds\biggr). \tag{3.22}$$

When $u_2 = 0$, we have the asymptotic expansion at $a = 0$:

$$\mathbb{E}[W_1(a)\,\tau_{1,a}] = \mathrm{id}_{E_{x_0}} - \frac{a^3}{12}(d^\nabla)^* R^\nabla(u_1) + O(a^4), \tag{3.23}$$

where $\tau_{1,a}$ is the parallel transport in E along $a \mapsto \exp_{x_0}(au_1)$, and $O(a^4)$ is uniform in $x_0 \in M$, u_1 varying in a compact subset of $T_{x_0}M$.

REMARK 3.2. (i) Using the antisymmetry of R^∇, the Itô integral in formula (3.21) can be replaced by a Stratonovich integral to give

$$\nabla_0 \nabla_a W_t = \int_0^t R^\nabla(\delta\partial_0 X_s, \partial_0 X_s). \tag{3.24}$$

Equation (3.24) is formula (39) in [5] where the author considers the case $u_1 = 0$. Similarly, formula (3.22) may be written as

$$\nabla_0 \nabla_a \nabla_a W_t = 2\int_0^t \nabla R^\nabla(\partial_0 X_s, \delta\partial_0 X_s, \partial_0 X_s). \tag{3.25}$$

(ii) Equation (3.14) shows that, when $u_2 = 0$, formula (3.21) can be rewritten as

$$\nabla_0 \nabla_a W_t = \int_0^t R^\nabla\bigl(\Sigma(x_0)\, dB_s, \partial_0 X_s\bigr). \tag{3.26}$$

In this case $\nabla_0 \nabla_a W$ turns out to be a martingale. Similarly, again under the assumption $u_2 = 0$, formula (3.22) may be written as

$$\begin{aligned} &\nabla_0 \nabla_a \nabla_a W_t \\ &\quad = 2\int_0^t \nabla R^\nabla\bigl(\partial_0 X_s, \Sigma(x_0)\, dB_s, \partial_0 X_s\bigr) - (d^\nabla)^* R^\nabla\biggl(\int_0^t \partial_0 X_s\, ds\biggr). \end{aligned} \tag{3.27}$$

(iii) The asymptotic expansion for W^{-1} is given by

$$W^{-1}(0) = \mathrm{id}_{E_{x_0}}, \qquad \nabla_0 W^{-1} \equiv 0, \tag{3.28}$$

$$\nabla_0\nabla_a W^{-1} = -\nabla_0\nabla_a W, \qquad \nabla_0\nabla_a\nabla_a W^{-1} = -\nabla_0\nabla_a\nabla_a W, \tag{3.29}$$

which is an easy consequence of $WW^{-1} = \mathrm{id}_{E_{x_0}}$ and formulas (3.19)–(3.22).

PROOF OF THEOREM 3.1. We first calculate the derivatives of W with respect to a in the topology of semimartingales. For brevity, we write Σ for $\Sigma(X(a))$ in the remainder of this article.

First of all note that $W_0(a) = \mathrm{id}_{E_{X_0(a)}}$ and hence all covariant derivatives of W_0 with respect to a vanish. Since (3.19) is obvious, we proceed with proving (3.20). We observe that Theorem 1.3, along with $DW = 0$ and $\nabla_a DW = 0$, gives

$$\begin{aligned} D\nabla_a W = {}& R^\nabla(d^\nabla_{\mathrm{It\hat{o}}}X, \partial_a X)W - \tfrac{1}{2}a^2(d^\nabla)^* R^\nabla(\partial_a X)W\,dt \\ & - \tfrac{1}{2}R^\nabla(D\partial_a X, dX)W. \end{aligned} \tag{3.30}$$

Evaluating (3.30) at $a = 0$ shows $D\nabla_0 W \equiv 0$, which together with $\nabla_0 W_0 = 0$ implies (3.20).

According to (3.1) and (3.13), the last term of (3.30) can be written as

$$-\frac{a^2}{2} R^\nabla(\nabla_a \Sigma\, dB, \Sigma\, dB)W. \tag{3.31}$$

Because $\nabla\Sigma(x_0) = 0$ and thus $\nabla_0\Sigma = 0$, this expression is $O(a^3)$ and may hence be neglected in the calculations of $\nabla_0\nabla_a W$ and $\nabla_0\nabla_a\nabla_a W$. Thus we have

$$D\nabla_a W = R^\nabla(d^\nabla_{\mathrm{It\hat{o}}}X, \partial_a X)W - \tfrac{1}{2}a^2(d^\nabla)^* R^\nabla(\partial_a X)W\,dt + O(a^3), \tag{3.32}$$

where for a continuous semimartingale Y the notation $Y = O(a^k)$, or $dY = O(a^k)$, means that Y/a^k converges in the topology of semimartingales to some continuous semimartingale as $a \searrow 0$.

Differentiating (3.32) with the help of Theorem 1.3 and making use of $dX = O(a)$, $\nabla_a W = O(a)$ and $\nabla_a\partial_a X = O(a)$ which follows from (3.17), we get

$$\begin{aligned} D\nabla_a\nabla_a W &= \nabla_a D\nabla_a W + O(a^2) \\ &= \nabla R^\nabla(\partial_a X, d^\nabla_{\mathrm{It\hat{o}}}X, \partial_a X)W + R^\nabla\big(\nabla_a(d^\nabla_{\mathrm{It\hat{o}}}X), \partial_a X\big)W \\ &\quad - a(d^\nabla)^* R^\nabla(\partial_a X)W\,dt + O(a^2). \end{aligned} \tag{3.33}$$

Differentiating (3.1), on the other hand, yields

$$\nabla_a(d^\nabla_{\mathrm{It\hat{o}}}X) = \Sigma\, dB + a\,\nabla_a\Sigma\, dB + \nabla_a b\, dt; \tag{3.34}$$

in particular,

$$\nabla_0(d^\nabla_{\mathrm{It\hat{o}}}X) = \Sigma(x_0)\, dB + \nabla_0 b\, dt = d\partial_0 X. \tag{3.35}$$

Substituting $a=0$ in (3.33) and using (3.35), we end up with

$$D\nabla_0\nabla_a W = R^\nabla(d\partial_0 X, \partial_0 X),$$

which together with $\nabla_0\nabla_a W_0 = 0$ implies (3.21). Note that differentiating (3.34) at $a=0$ yields

$$\nabla_0\nabla_a(d^\nabla_{\text{Itô}} X) = 0. \tag{3.36}$$

We next differentiate (3.33) by means of Theorem 1.3, this time at $a=0$. Using $d^\nabla_{\text{Itô}} X(0) = 0$, $\nabla_0\partial_a X = 0$, $\nabla_0 W = 0$ and (3.35) and (3.36), we obtain

$$\begin{aligned} D\nabla_0\nabla_a\nabla_a W &= 2\nabla R^\nabla\big(\partial_0 X, \nabla_0(d^\nabla_{\text{Itô}} X), \partial_0 X\big) - (d^\nabla)^* R^\nabla(\partial_0 X)\,dt \\ &= 2\nabla R^\nabla(\partial_0 X, d\partial_0 X, \partial_0 X) - (d^\nabla)^* R^\nabla(\partial_0 X)\,dt. \end{aligned} \tag{3.37}$$

To obtain (3.22), we are left to integrate (3.37) with the initial condition $\nabla_0\nabla_a\nabla_a W_0 = 0$.

To establish the asymptotic expansion (3.23), a careful analysis of the equations at time $t=1$ is required. We divide the proof into two steps. First of all note that, by the Serre–Swan theorem, E is a subbundle of a trivial bundle $M\times\mathbb{R}^n$, and hence $W(a)$ may be considered as taking its values in $\mathbb{R}^n$.

Step 1. Prove that, for any fixed $0<\varepsilon<1$, the map

$$a \mapsto W^{1-\varepsilon}(a) = \big(W_{t\wedge(1-\varepsilon)}(a)\big)_{0\le t\le 1}$$

has a polynomial expansion of every order at $a=0$ in L^p for every $p\ge 1$, where the L^p-norm is given by

$$\|W\|_p = \mathbb{E}\Big[\sup_{0\le t\le 1}\|W_t\|^p_{\mathbb{R}^n}\Big]^{1/p}.$$

Step 2. Evaluate the limit as $t\nearrow 1$ and establish the asymptotic expansion (3.23) by means of a time-reversal argument.

To Step 1: We start by regularizing the equation for W outside a small neighborhood $\mathscr{V}'$ of x_0. To this end, we assume the set $\mathscr{V}$ defined before (3.5) to be a small regular geodesic ball with center x_0 and radius $2\alpha>0$, and take as $\mathscr{V}'$ the geodesic ball with center x_0 and radius α. Let $\tau(a) = \inf\{t\ge 0,\ X_t(a)\notin\mathscr{V}'\}$. We first prove the existence of a constant $C>0$ such that

$$\mathbb{P}\{\tau(a) < 1-\varepsilon\} \le e^{-C/a^2}. \tag{3.38}$$

In the exponential chart introduced after (3.5), and on $\{t<\tau(a)\}$, writing again

$f(x) = d^2(x, x_0)$, the process $f(X) = f(X(a))$ satisfies

$$
\begin{aligned}
df(X) = 2a \sum_{i=1}^{m} X^i \Sigma^i \, dB & \\
& - \frac{2}{1-t} \sum_{i,j=1}^{m} X^i X^j g^{ij} \, dt \\
& + a \frac{2}{1-t} \sum_{i,j=1}^{m} u_2^i X^j g^{ij} \, dt - \frac{1}{1-t} \sum_{i=1}^{m} X^i \operatorname{grad}_x^i r(X, au_2) \, dt \\
& + a^2 \sum_{i=1}^{m} \Biggl(2X^i \operatorname{grad}_x^i \log \psi\bigl(a^2(1-t), X, x_0\bigr) \\
& \qquad\qquad - X^i \Gamma^i(\Sigma, \Sigma) + \sum_{j=1}^{r} (\Sigma_j^i)^2 \Biggr) \, dt.
\end{aligned}
$$

The first term of the drift is nonpositive, the second term is $O(a)$, the third term is $O(a)$ by (3.7) and the sum of the other terms is $O(a^2)$. Consequently, the drift is bounded above by $C_1 a$ for some $C_1 > 0$, and for every $a > 0$ satisfying $a^2 \|u_1\|^2 + C_1 a (1 - \varepsilon) \le \alpha^2/2$, we obtain

$$
\mathbb{P}\{\tau(a) < 1 - \varepsilon\} \le \mathbb{P}\left\{ \int_0^{\tau(a) \wedge (1-\varepsilon)} 2a \sum_{i=1}^{m} X^i \Sigma^i \, dB \ge \alpha^2/2 \right\}.
$$

But since $\sum_{i=1}^m x^i \Sigma^i(x)$ is bounded on $\mathscr{V}'$, by Bernstein's inequality (see, e.g., Exercise 3.16, Chapter 4 in [19]) the right-hand side is bounded by e^{-C/a^2} for some $C > 0$, which gives the claimed estimate (3.38). By compactness of M, for α sufficiently small, for example, less than the injectivity radius of M, the constant C can be chosen independent of x_0.

Next we want to prove that

$$
\sup_{a \in]0,1]} \|W^{1-\varepsilon}(a)\|_p < \infty \qquad \text{for every } p \ge 1, \tag{3.39}
$$

with a uniform bound in x_0. Writing $\nabla = d + A$ in $\mathbb{R}^n$, (1.11) gives

$$
dW = -A(d_{\text{Itô}}^{\nabla} X, W) - \tfrac{1}{2} (\nabla A)(dX, dX, W).
$$

We substitute (3.1) for $d_{\text{Itô}}^{\nabla} X$, along with (3.2) and (3.3) for the drift of $d_{\text{Itô}}^{\nabla} X$, and recall that $V_t(a, x)$ is uniformly bounded in $(t, a, x) \in [0, 1 - \varepsilon] \times]0, 1] \times M$ by estimate (3.4). Then it is easy to see that W is a solution of an equation of the type

$$
dW = \sigma(t, a, X, W) \, dB + c(t, a, X, W) \, dt,
$$

where the coefficients σ and c are linear in W and bounded as linear maps uniformly in $(t, a, X) \in [0, 1-\varepsilon] \times]0, 1] \times M$. This obviously achieves the desired estimate (3.39).

Let $\phi : M \to \mathbb{R}$ be a smooth nonnegative function, compactly supported in $\mathscr{V}$, such that $\phi = 1$ on $\mathscr{V}'$. Let $(X_t'(a), W_t'(a))_{0 \le t \le 1}$ be an $M \times \mathbb{R}^n$-valued process with the same starting point as $(X_t(a), W_t(a))$ such that $X_t'(a)$ solves

$$d^{\nabla}_{\mathrm{It\hat{o}}} X_t' = a\,\phi(X_t')\,\Sigma(X_t')\,dB_t + \phi(X_t')\,V_t(a, X_t', u_2)\,dt$$

and $W_t'(a)$ solves

$$dW' = -A(d^{\nabla}_{\mathrm{It\hat{o}}} X', W') - \tfrac{1}{2}(\nabla A)(dX', dX', W').$$

Observe that all coefficients can be smoothly extended by 0 at $a = 0$. Consequently, by means of Proposition 1.3 in [18] [with the correspondence $(a, x, w) = (x_1, x_2, x_3)$], we see that $a \mapsto ((W_t')^{1-\varepsilon}(a))_{0 \le t \le 1}$ is smooth in L^p for any $p \ge 1$, where $a \in [0, 1]$ (by Whitney's embedding theorem the fact that x is an element of a compact manifold instead of some $\mathbb{R}^\ell$ does not change the situation). Clearly, the L^p covariant derivatives in a at $a = 0$ of $(W')^{1-\varepsilon}$ [resp. $(W'^{-1})^{1-\varepsilon}$] are the semimartingales defined by the right-hand sides of (3.19)–(3.22) [resp. of (3.28) and (3.29), stopped at time $1-\varepsilon$].

Since the processes $W(a)$ and $W'(a)$ coincide on $\{t < \tau(a)\}$, the term

$$\|W^{1-\varepsilon}(a) - (W')^{1-\varepsilon}(a)\|_p$$

is bounded by

$$\big(\|W^{1-\varepsilon}(a)\|_{2p} + \|(W')^{1-\varepsilon}(a)\|_{2p}\big)\big(\mathbb{P}\{\tau(a) < 1-\varepsilon\}\big)^{1/2p},$$

which according to (3.38) and (3.39), along with the corresponding equations for W', is asymptotically less than $C_2 e^{-C_3/a^2}$, where C_2 and C_3 do not depend on x_0. Consequently, $W^{1-\varepsilon}(a)$ and $(W')^{1-\varepsilon}(a)$ share the same polynomial expansion in L^p at $a = 0$. By the same argument, $(W^{1-\varepsilon}(a))^{-1}$ and $((W')^{1-\varepsilon}(a))^{-1}$ have identical polynomial expansions at $a = 0$ as well.

Now let $\tau_{i,a}$ be the parallel transport in E along $a \mapsto \exp_{x_0}(au_i)$ and denote by τ_a^t parallel transport in E along $a \mapsto X_t(a)$. Putting together the results so far, we obtain the following two formulas: for $t \in [0, 1[$ there holds

$$\begin{aligned}(\tau_a^t)^{-1} W_t(a)\,\tau_{1,a} = \mathrm{id}_{E_{x_0}} &+ \frac{a^2}{2}\int_0^t R^{\nabla}(\delta\partial_0 X_s, \partial_0 X_s)\\ &+ \frac{a^3}{3}\int_0^t \nabla R^{\nabla}(\partial_0 X_s, \delta\partial_0 X_s, \partial_0 X_s)\\ &+ a^4\,Y_t(a)\end{aligned} \tag{3.40}$$

and

$$
\begin{aligned}
\tau_{1,a}^{-1} W_t^{-1}(a)\, \tau_a^t = \mathrm{id}_{E_{x_0}} &- \frac{a^2}{2}\int_0^t R^\nabla(\delta\partial_0 X_s, \partial_0 X_s) \\
&- \frac{a^3}{3}\int_0^t \nabla R^\nabla(\partial_0 X_s, \delta\partial_0 X_s, \partial_0 X_s) \\
&+ a^4\, Y_t'(a),
\end{aligned}
\tag{3.41}
$$

where, for every $\varepsilon > 0$ and $p \geq 1$, $Y^{1-\varepsilon}(a)$ and $(Y')^{1-\varepsilon}(a)$ are bounded in L^p, uniformly in a, x_0 and $u = (u_1, u_2)$ varying in a compact subset of $(T_{x_0}M)^2$.

To Step 2: We want to establish (3.23). Note that it is not sufficient to take expectation on both sides of (3.40) since the equation is valid only for time $t < 1$. We proceed with a time reversal. The process $\tilde{X}_t(a) = X_{1-t}(a)$ is a rescaled Brownian bridge starting from $\exp_{x_0}(au_2)$ and ending at $\exp_{x_0}(au_1)$ at time 1; consequently, $\tilde{X}_t(a)$ solves

$$
d^\nabla_{\mathrm{It\hat{o}}}\tilde{X}_t(a) = a\Sigma(\tilde{X}_t(a))\, d\tilde{B}_t(a) + V_t\big(a, \tilde{X}_t(a), u_1\big)\, dt,
\tag{3.42}
$$

where $\tilde{B}(a)$ is an $\mathbb{R}^r$-valued Brownian motion. Observe that $\tilde{B}$ depends on a. We fix $a_0 > 0$ and consider the family of Brownian bridges $\tilde{X}(a, a_0)$ satisfying $\tilde{X}_0(a, a_0) = \exp_{x_0}(au_2)$ and

$$
d^\nabla_{\mathrm{It\hat{o}}}\tilde{X}_t(a, a_0) = a\Sigma\big(\tilde{X}_t(a, a_0)\big)\, d\tilde{B}_t(a_0) + V_t\big(a, \tilde{X}_t(a, a_0), u_1\big)\, dt.
\tag{3.43}
$$

Note that the driving Brownian motion is $\tilde{B}_t(a_0)$ and that $\tilde{X}_1(a, a_0) = \exp_{x_0}(au_1)$. We denote by $\tilde{W}(a, a_0)$ the parallel transport in E along $\tilde{X}(a, a_0)$. The laws of $\tilde{X}(a, a_0)$ and $\tilde{W}(a, a_0)$ are then independent of a_0.

The map $a \mapsto (\tilde{X}(a, a_0), \tilde{W}(a, a_0))$ has the same kind of asymptotic development as $a \mapsto (X(a), W(a))$. In particular, we find that $\tilde{X}(0, a_0) \equiv x_0$, $\partial_0\tilde{X}(\cdot, a_0)$ is a Brownian bridge in $T_{x_0}M$ starting at u_2 and ending at u_1 and $\nabla_0\partial_a\tilde{X}(\cdot, a_0) \equiv 0$. We also conclude that $(\tilde{W}(a, a_0))^{1-\varepsilon}$ and $(\tilde{W}^{-1}(a))^{1-\varepsilon}$ have a polynomial expansion in a at $a = 0$ in any L^p. More precisely, let τ_a^{t,a_0} be parallel transport along $a \mapsto \tilde{X}_t(a, a_0)$. Then for any $t \in [0, 1[$ we have

$$
\begin{aligned}
&\tau_{2,a}^{-1}\tilde{W}_t^{-1}(a, a_0)\tau_a^{t,a_0} \\
&\quad = \mathrm{id}_{E_{x_0}} - \frac{a^2}{2}\int_0^t R^\nabla\big(\delta\partial_0\tilde{X}_s(\cdot, a_0), \partial_0\tilde{X}_s(\cdot, a_0)\big) \\
&\qquad - \frac{a^3}{3}\int_0^t \nabla R^\nabla\big(\partial_0\tilde{X}_s(\cdot, a_0), \delta\partial_0\tilde{X}_s(\cdot, a_0), \partial_0\tilde{X}_s(\cdot, a_0)\big) \\
&\qquad + a^4\, Y_t'(a, a_0),
\end{aligned}
\tag{3.44}
$$

where, for any $0 < \varepsilon < 1$ and $p \geq 1$, the random variables $(Y'(a, a_0))^{1-\varepsilon}$ are bounded in L^p, uniformly in a, a_0, x_0 and u varying in a compact subset of $(T_{x_0}M)^2$ (recall that the laws do not depend on a_0).

By the transfer principle, since W can be defined through a Stratonovich equation, we have

$$\tilde{W}_t^{-1}(a,a) = W_1(a) W_{1-t}^{-1}(a),$$

as well as

$$W_1(a) = \tilde{W}_{1-t}^{-1}(a,a)\, W_t(a).$$

This implies

$$W_1(a) = \tilde{W}_{1-t}^{-1}(a,a)\, \tau_a^{1-t,a}\, (\tau_a^{1-t,a})^{-1}\, \tau_a^t\, (\tau_a^t)^{-1}\, W_t(a). \tag{3.45}$$

We fix $t \in]0,1[$. Then by (3.17) we have $\nabla_0 \partial_a X_t = 0$, and similarly

$$\nabla_0 \partial_a \tilde{X}_{1-t}(\cdot, a) = 0.$$

Hence, in terms of the Taylor expansion of the parallel transports, we get

$$(\tau_a^{1-t,a})^{-1} \tau_a^t = \mathrm{id}_{E_{x_0}} + O(a^4) \tag{3.46}$$

in L^p for every $p \geq 1$. Thus we are left to exploit (3.45), (3.46), (3.40) and (3.44) to get

$$\begin{aligned}
\tau_{2,a}^{-1}\, W_1(a)\, \tau_{1,a} = \mathrm{id}_{E_{x_0}} &+ \frac{a^2}{2} \int_0^t R^{\nabla}(\delta \partial_0 X_s, \partial_0 X_s) \\
&+ \frac{a^3}{3} \int_0^t \nabla R^{\nabla}(\partial_0 X_s, \delta \partial_0 X_s, \partial_0 X_s) \\
&- \frac{a^2}{2} \int_0^{1-t} R^{\nabla}\big(\delta \partial_0 \tilde{X}_s(\cdot, a), \partial_0 \tilde{X}_s(\cdot, a)\big) \\
&- \frac{a^3}{3} \int_0^{1-t} \nabla R^{\nabla}\big(\partial_0 \tilde{X}_s(\cdot, a), \delta \partial_0 \tilde{X}_s(\cdot, a), \partial_0 \tilde{X}_s(\cdot, a)\big) \\
&+ a^4\, Y_t''(a),
\end{aligned} \tag{3.47}$$

where $Y_t''(a)$ is uniformly bounded in L^p.

Now $s \mapsto \partial_0 \tilde{X}_s(\cdot, a)$ is a Brownian bridge in $T_{x_0}M$ starting at u_2 and ending at u_1 at time 1, so it has the same law as $s \mapsto \partial_0 X_{1-s}$. In particular, by the reversibility of Stratonovich integrals,

$$\mathbb{E}\bigg[\int_t^1 R^{\nabla}(\delta \partial_0 X_s, \partial_0 X_s)\bigg] = -\mathbb{E}\bigg[\int_0^{1-t} R^{\nabla}\big(\delta \partial_0 \tilde{X}_s(\cdot, a), \partial_0 \tilde{X}_s(\cdot, a)\big)\bigg]$$

and

$$\begin{aligned}
&\mathbb{E}\bigg[\int_t^1 \nabla R^{\nabla}(\partial_0 X_s, \delta \partial_0 X_s, \partial_0 X_s)\bigg] \\
&\qquad = -\mathbb{E}\bigg[\int_0^{1-t} \nabla R^{\nabla}\big(\partial_0 \tilde{X}_s(\cdot, a), \delta \partial_0 \tilde{X}_s(\cdot, a), \partial_0 \tilde{X}_s(\cdot, a)\big)\bigg].
\end{aligned}$$

Consequently, taking expectations on both sides of (3.47) yields

$$\begin{aligned}
&\mathbb{E}[\tau_{2,a}^{-1} W_1(a)\,\tau_{1,a}] \\
&\quad = \mathrm{id}_{E_{x_0}} + \frac{a^2}{2}\mathbb{E}\biggl[\int_0^1 R^\nabla(\delta\partial_0 X_s, \partial_0 X_s)\biggr] \\
&\qquad + \frac{a^3}{6}\mathbb{E}\biggl[\int_0^1 \nabla R^\nabla(\partial_0 X_s, \delta\partial_0 X_s, \partial_0 X_s)\biggr] + O(a^4).
\end{aligned} \tag{3.48}$$

Now letting $u_2 = 0$ and using formulas (3.26) and (3.27) of Remark 3.2 yields

$$\mathbb{E}[W_1(a)\,\tau_{1,a}] = \mathrm{id}_{E_{x_0}} - \frac{a^3}{6}(d^\nabla)^* R^\nabla\biggl(\int_0^1 \mathbb{E}[\partial_0 X_s]\,ds\biggr) + O(a^4). \tag{3.49}$$

But $\mathbb{E}[\partial_0 X_s] = (1-s)u_1$, and hence $\int_0^1 \mathbb{E}[\partial_0 X_s]\,ds = u_1/2$. Replacing this in (3.49) establishes (3.23). $\square$

As an easy consequence of formulas (3.22) and (3.23), we obtain the two following corollaries.

COROLLARY 3.3. *The notation is the same as in Theorem* 3.1. *The three following conditions are equivalent*:

(i) $(d^\nabla)^* R^\nabla$ *vanishes at* x_0;
(ii) *there exists* $u_1 \in T_{x_0}M$ *such that* $\nabla_0\nabla_a\nabla_a W(a,(u_1,0))$ *is a martingale*;
(iii) *for every* $u_1 \in T_{x_0}M$, $\nabla_0\nabla_a\nabla_a W(a,(u_1,0))$ *is a martingale*.

COROLLARY 3.4. *The notation is the same as in Theorem* 3.1. *The two following conditions are equivalent*:

(i) $(d^\nabla)^* R^\nabla$ *vanishes at* x_0;
(ii) *for every* $u_1 \in T_{x_0}M$,

$$\mathbb{E}\bigl[W_1\bigl(a,(u_1,0)\bigr)\,\tau_{1,a} - \mathrm{id}_{E_{x_0}}\bigr] = O(a^4). \tag{3.50}$$

In Corollary 3.4(ii) it is sufficient to ask (3.50) for vectors u_1 constituting a basis for $T_{x_0}M$.

REMARK 3.5. In [7] and [22] the authors obtained a condition similar to (3.50), but in their result the time is not fixed; it is the first exit time of a ball of radius $a\|u_1\|$. Here we derive the full terms of the asymptotic expansion in a, and the covariant derivative ∇ is not required to be compatible with any metric in E.

We finish this section by giving a result similar to Theorem 3.1 but for deformed parallel transport. Let $\mathscr{R}$ be a smooth section of $T^*M \otimes T^*M \otimes \operatorname{End}(E)$ over M. The deformed parallel transport $\Theta_{0,t}(a,u)$ is the $\operatorname{Hom}(E_{X_0(a,u)}, E_{X_t(a,u)})$-valued semimartingale solution to $\Theta_{0,0}(a,u) = \mathrm{id}_{E_{X_0(a,u)}}$ and

$$
\begin{aligned}
D\Theta_{0,t}(a,u) &= -\frac{1}{2}\mathscr{R}\big(dX(a,u), dX(a,u)\big)\Theta_{0,t}(a,u) \\
&\equiv -\frac{a^2}{2}\operatorname{tr}\mathscr{R}\big(X_t(a,u)\big)\Theta_{0,t}(a,u)\,dt
\end{aligned}
\tag{3.51}
$$

(see, e.g., [3], Section 5). Our main example is $E = TM$ and $\mathscr{R}(u,v)w = R(w,u)v$, which gives $\operatorname{tr}\mathscr{R} = \operatorname{Ric}^\sharp$. In this situation $\Theta_{0,t}$ is the so-called damped parallel transport (or Dohrn–Guerra transport or geodesic transport); see, for example, [11] and [17].

THEOREM 3.6. *Let M be a compact Riemannian manifold and let E be a vector bundle over M endowed with a covariant derivative ∇. For $u = (u_1, u_2) \in (T_{x_0}M)^2$ let $X(a,u)$ be the rescaled Brownian bridge from $\exp_{x_0}(au_1)$ to $\exp_{x_0}(au_2)$, as defined by* (3.1), *and let $\Theta_{0,t}(a) = \Theta_{0,t}(a,u)$ be the deformed parallel transport in E along $X(a,u)$. Then, in the topology of semimartingales indexed by $t \in [0,1[$,*

$$\Theta_{0,t}(0) = \mathrm{id}_{E_{x_0}}, \tag{3.52}$$

$$\nabla_0\Theta_{0,t} \equiv 0, \tag{3.53}$$

$$\nabla_0\nabla_a\Theta_{0,t} = \int_0^t R^\nabla(d\partial_0 X_s, \partial_0 X_s) - t\operatorname{tr}\mathscr{R}(x_0), \tag{3.54}$$

$$
\begin{aligned}
\nabla_0\nabla_a\nabla_a\Theta_{0,t} = {}& 2\int_0^t \nabla R^\nabla(\partial_0 X_s, d\partial_0 X_s, \partial_0 X_s) \\
&- \big((d^\nabla)^* R^\nabla + 3\nabla\operatorname{tr}\mathscr{R}\big)\bigg(\int_0^t \partial_0 X_s\,ds\bigg).
\end{aligned}
\tag{3.55}
$$

In case $u_2 = 0$ we have the asymptotic expansion at $a = 0$:

$$
\begin{aligned}
\mathbb{E}[\Theta_{0,1}(a)\,\tau_{1,a}] = {}& \mathrm{id}_{E_{x_0}} - \frac{a^2}{2}\operatorname{tr}\mathscr{R}(x_0) \\
&- \frac{a^3}{12}\big((d^\nabla)^* R^\nabla + 3\nabla\operatorname{tr}\mathscr{R}\big)(u_1) + O(a^4),
\end{aligned}
\tag{3.56}
$$

where $\tau_{1,a}$ is the parallel transport in E along $a \mapsto \exp_{x_0}(au_1)$, and $O(a^4)$ is uniform in $x_0 \in M$, u_1 varying in a compact subset of $T_{x_0}M$.

The proof is similar to the proof of Theorem 3.1 and hence omitted. Note that the additional terms in (3.54) and (3.55) are, respectively, the second covariant derivative and the third covariant derivative in a at $a = 0$ of the right-hand side of (3.51).

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