THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING

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ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, and let e be a nilpotent element of $\mathfrak{g}$. Denote by $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$ and by $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of $\mathfrak{g}^e$. We say that e is *good* if the nullvariety of some ℓ homogenous elements of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension ℓ. If e is good then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. The main result of this paper stipulates that if for some homogenous generators of $S(\mathfrak{g})^{\mathfrak{g}}$, the initial homogenous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent, with (e, h, f) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, then e is good. As applications, we pursue the investigations of [PPY07] and we produce (new) examples of nilpotent elements that satisfy the above polynomiality condition, in simple Lie algebras of both classical and exceptional types. We also give a counter-example in type $\mathbf{D}_7$.

CONTENTS

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, let $\langle .\,,.\rangle$ be the Killing form of $\mathfrak{g}$ and let G be the adjoint group of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, we denote by $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$. For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$ and by G^x the stabilizer of x in G. Then $\mathrm{Lie}(G^x) = \mathrm{Lie}(G_0^x) = \mathfrak{g}^x$ where G_0^x is the identity component of G^x. Moreover, $S(\mathfrak{g}^x)$ is a $\mathfrak{g}^x$-module and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)^{G_0^x}$. An interesting question, first raised by A. Premet, is the following:

Question 1. *Is* $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *a polynomial algebra in* ℓ *variables?*

In order to answer this question, thanks to the Jordan decomposition, we can assume that x is nilpotent. Besides, if $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit $G.x$ of x. If $x = 0$, it is well-known since Chevalley that $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in ℓ variables. At the

Date: September 23, 2015.
1991 *Mathematics Subject Classification.* 17B35,17B20,13A50,14L24.
Key words and phrases. symmetric invariant, centralizer, polynomial algebra, Slodowy grading.

opposite extreme, if x is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension ℓ, [DV69], and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)$ is polynomial in ℓ variables too.

For the introduction, let us say most simply that $x \in \mathfrak{g}$ *satisfies the polynomiality condition* if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra in ℓ variables.

A positive answer to Question 1 was suggested in [PPY07, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. O. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [Y07], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not satisfy the polynomiality condition. The present paper contains another counter-example in type $\mathbf{D}_7$ (cf. Example 7.8). In particular, we cannot expect a positive answer to [PPY07, Conjecture 0.1] for the simple Lie algebras of classical type. Question 1 still remains interesting and has a positive answer for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

1.2. **Review of known results.** We briefly review in this paragraph what has been achieved so far about Question 1. Recall that the *index* of a finite-dimensional Lie algebra $\mathfrak{q}$, denoted by $\mathrm{ind}\,\mathfrak{q}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{q}$ for the coadjoint representation, (cf. [Di74]):

$$\mathrm{ind}\,\mathfrak{q} := \min\{\dim \mathfrak{q}^{\xi} \ ; \ \xi \in \mathfrak{q}^*\} \ \text{ where } \ \mathfrak{q}^{\xi} := \{x \in \mathfrak{q} \ ; \ \xi([x, \mathfrak{q}]) = 0\}.$$

By [R63], if $\mathfrak{q}$ is algebraic, i.e., $\mathfrak{q}$ is the Lie algebra of some algebraic linear group Q, then the index of $\mathfrak{q}$ is the transcendence degree of the field of Q-invariant rational functions on $\mathfrak{q}^*$. The following result will be important for our purpose.

Theorem 1.1 ([CMo10, Theorem 1.2])**.** *The index of* $\mathfrak{g}^x$ *is equal to* ℓ *for any* $x \in \mathfrak{g}$.

Theorem 1.1 was first conjectured by Elashvili in the 90's motivated by a result of Bolsinov, [Bol91, Theorem 2.1]. It was proven by O. Yakimova when $\mathfrak{g}$ is a simple Lie algebra of classical type, [Y06], and checked by a program by W. de Graaf when $\mathfrak{g}$ is a simple Lie algebra of exceptional type, [DeG08]. Before that, the result was established for some particular classes of nilpotent elements by D. Panyushev, [Pa03].

Theorem 1.1 is deeply related to Question 1. First of all, it implies that if $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, it is so in ℓ variables. Further, according to Theorem 1.1, the main results of [PPY07] that we summarize below apply (see Theorem 1.2).

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into a $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with* e. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $(\mathfrak{g}^e)^*$ with $\mathfrak{g}^f$, through the Killing form $\langle .\,, . \rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$, denote by ${}^e p$ the initial homogenous component of its restriction to $\mathcal{S}_e$. According to [PPY07, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Let $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ be the set of nonregular linear forms $x \in (\mathfrak{g}^e)^*$, i.e.,

$$(\mathfrak{g}^e)^*_{\mathrm{sing}} := \{x \in (\mathfrak{g}^e)^* \mid \dim (\mathfrak{g}^e)^x > \mathrm{ind}\,\mathfrak{g}^e = \ell\}.$$

If $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension at least 2 in $(\mathfrak{g}^e)^*$, we say that $\mathfrak{g}^e$ is *nonsingular*.

Theorem 1.2 ([PPY07, Theorem 0.3])**.** *Suppose that the following two conditions are satisfied:*

(1) *for some homogenous generators* $q_1, \ldots, q_\ell$ *of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*, the polynomial functions* ${}^e q_1, \ldots, {}^e q_\ell$ *are algebraically independent,*
(2) $\mathfrak{g}^e$ *is nonsingular.*

Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra with generators* ${}^e q_1, \ldots, {}^e q_\ell$.

As a consequence of Theorem 1.2, if $\mathfrak{g}$ is simple of type $\mathbf{A}$ or $\mathbf{C}$, then all nilpotent elements of $\mathfrak{g}$ satisfy the polynomiality condition, cf. [PPY07, Theorems 4.2 and 4.4]. The result for the type $\mathbf{A}$ was independently obtained by Brown and Brundan, [BB09]. In [PPY07], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types **B** and **D**, and a few ones in the simple exceptional Lie algebras.

At last, note that the analogue question to Question 1 for the positive characteristic was dealt with by L. Topley for the simple Lie algebras of types **A** and **C**, [T12].

1.3. **Main results.** The main goal of this paper is to continue the investigations of [PPY07]. Let us describe our main results. The following definition is central in our work (cf. Definition 3.2):

Definition 1.3. An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some homogenous sequence $(p_1, \dots, p_\ell)$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \dots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

For example, regular nilpotent elements are good. Indeed, for e in the regular nilpotent orbit of $\mathfrak{g}$ and $(q_1, \dots q_\ell)$ a homogenous generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, it is well-known that ${}^e q_i = \mathrm{d}_e q_i$ for $i = 1, \dots, \ell$ and that $(\mathrm{d}_e q_1, \dots, \mathrm{d}_e q_\ell)$ forms a basis of $\mathfrak{g}^e$, [Ko63]. Hence e is good.

Also, by [PPY07, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type **A** are good. Moreover, according to [Y09, Corollary 8.2], *even*[1] nilpotent elements without odd (respectively even) Jordan blocks of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type **C** (respectively **B** or **D**). We generalize these results (cf. Theorem 5.1, Corollary 5.8 and Remark 5.9).

The good elements satisfy the polynomiality condition (cf. Theorem 3.3):

Theorem 1.4. *Let x be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Furthermore, x is good if and only if so is its nilpotent component in the Jordan decomposition (cf. Proposition 3.5). As a consequence, we can restrict the study to the case of nilpotent elements.

The main result of the paper is the following (cf. Theorem 3.6) whose proof is outlined in Subsection 1.4:

Theorem 1.5. *Suppose that for some homogenous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, $({}^e q_1, \dots, {}^e q_\ell)$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

In other words, Theorem 1.5 says that Condition (1) of Theorem 1.2 is sufficient to ensure the polynomiality of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. However, if only Condition (1) of Theorem 1.2 is satisfied, the (polynomial) algebra $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying Condition (1) and for which $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \dots, {}^e q_\ell$, for any choice of homogenous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ (cf. Remark 5.25).

Theorem 1.5 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras (cf. Section 5), and for some nilpotent orbits in the exceptional Lie algebras (cf. Section 6). For example, according to [PY13, Proposition 6.3] and Theorem 1.5, the elements of the subregular nilpotent orbit of $\mathfrak{g}$ are good.

To state our results for the simple Lie algebras of types **B** and **D**, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ. For an endomorphism x of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of x. Then for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell + 1$, set $q_\ell := Q_{2\ell}$, and

[1] i.e., this means that the Dynkin grading of $\mathfrak{g}$ associated with the nilpotent element has no odd term.

if $\dim \mathbb{V} = 2\ell$, let q_ℓ be the Pfaffian that is a homogenous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$. Denote by $\delta_1, \ldots, \delta_\ell$ the degrees of ${}^e q_1, \ldots, {}^e q_\ell$ respectively. By [PPY07, Theorem 2.1], if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0,$$

then the polynomials ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent. In that event, by Theorem 1.5, e is good and we will say that e is *very good* (cf. Corollary 5.8 and Definition 5.10). The notion of very good element is specific to this setting where there are natural generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.

The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of $\dim \mathbb{V}$ (cf. Lemma 5.11). Theorem 1.5 also allows to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$; for them, there are a few more work to do (cf. Subsection 5.3).

In this way, we obtain a large number of good nilpotent elements, including all even nilpotent elements in type **B**, or in type **D** with odd rank (cf. Corollary 5.8). For the type **D** with even rank, we obtain the statement for some particular cases (cf. Theorem 5.23). On the other hand, there are examples of elements that satisfy the polynomiality condition but that are not good; see Examples 7.5 and 7.6. To deal with them, we use different techniques, more similar to those used in [PPY07]. These alternative methods are presented in Section 7.

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^7)$ are good, and that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 13$, satisfy the polynomiality condition (cf. Table 5). In particular, by [PPY07, §3.9], this provides examples of good nilpotent elements for which $\mathfrak{g}^e$ is singular. For such nilpotent elements, note that [PPY07, Theorem 0.3] (cf. Theorem 1.2) cannot be applied.

Our results do not cover all nilpotent orbits in type **B** and **D**. As a matter of fact, we obtain a counter-example in type $\mathbf{D}_7$ to Premet's conjecture (cf. Example 7.8).

Proposition 1.6. *The nilpotent elements of* $\mathfrak{so}(\mathbb{k}^{14})$ *associated with the partition* $(3, 3, 2, 2, 2, 2)$ *of* 14 *do not satisfy the polynomiality condition.*

1.4. **Outline of the proof of Theorem 1.5.** Let $q_1, \ldots, q_\ell$ be homogenous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \ldots, d_\ell$ respectively. The sequence $(q_1, \ldots, q_\ell)$ is ordered so that $d_1 \leqslant \cdots \leqslant d_\ell$. Assume that the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent.

According to Theorem 1.4, it suffices to show that e is good, and more accurately that the nullvariety of ${}^e q_1, \ldots, {}^e q_\ell$ in $\mathfrak{g}^f$ has codimension ℓ, since ${}^e q_1, \ldots, {}^e q_\ell$ are invariant homogenous polynomials. To this end, it suffices to prove that $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of the $\mathbb{k}$-algebra generated by ${}^e q_1, \ldots, {}^e q_\ell$ (see Proposition 2.5,(ii)). We are led to find a subspace V_0 of S such that the linear map

$$V_0 \otimes_{\mathbb{k}} \mathbb{k}[{}^e q_1, \ldots, {}^e q_\ell] \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism. We explain below the construction of the subspace V_0.

Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \ldots, r$, $[h, x_i] = n_i x_i$ for some nonnegative integer n_i. For $\mathbf{j} = (j_1, \ldots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \qquad |\mathbf{j}|_e := j_1 n_1 + \cdots + j_r n_r + 2|\mathbf{j}|, \qquad x^{\mathbf{j}} = x_1^{j_1} \cdots x_r^{j_r}.$$

The algebra $\mathrm{S}(\mathfrak{g}^e)$ has two gradings: the standard one and the *Slodowy grading*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogenous with respect to these two gradings. It has standard degree $|\mathbf{j}|$ and, by definition, it has Slodowy degree $|\mathbf{j}|_e$. For m nonnegative integer, denote by $\mathrm{S}(\mathfrak{g}^e)^{[m]}$ the subspace of $\mathrm{S}(\mathfrak{g}^e)$ of Slodowy degree m.

Let us simply denote by S the algebra $\mathrm{S}(\mathfrak{g}^e)$ and let t be an indeterminate. For any subspace V of S, set:

$$V[t] := \mathbb{k}[t] \otimes_{\mathbb{k}} V, \qquad V[t, t^{-1}] := \mathbb{k}[t, t^{-1}] \otimes_{\mathbb{k}} V, \qquad V[[t]] := \mathbb{k}[[t]] \otimes_{\mathbb{k}} V, \qquad V((t)) := \mathbb{k}((t)) \otimes_{\mathbb{k}} V,$$

with $\Bbbk((t))$ the fraction field of $\Bbbk[[t]]$. For V a subspace of $S[[t]]$, denote by $V(0)$ the image of V by the quotient morphism

$$S[t] \longrightarrow S, \qquad a(t) \longmapsto a(0).$$

The Slodowy grading of S induces a grading of the algebra $S((t))$ with t having degree 0. Let τ be the morphism of algebras

$$S \longrightarrow S[t], \qquad x_i \mapsto tx_i, \quad i = 1, \dots, r.$$

The morphism τ is a morphism of graded algebras. Denote by $\delta_1, \dots, \delta_\ell$ the standard degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively, and set for $i = 1, \dots, \ell$:

$$Q_i := t^{-\delta_i} \tau(\kappa(q_i)) \quad \text{ with } \quad \kappa(q_i)(x) := q_i(e + x), \quad \forall\, x \in \mathfrak{g}^f.$$

Let A be the subalgebra of $S[t]$ generated by $Q_1, \dots, Q_\ell$. Then $A(0)$ is the subalgebra of S generated by ${}^e q_1, \dots, {}^e q_\ell$. For $(j_1, \dots, j_\ell)$ in $\mathbb{N}^\ell$, $\kappa(q_1^{j_1}) \cdots \kappa(q_\ell^{j_\ell})$ and ${}^e q_1^{j_1} \cdots {}^e q_\ell^{j_\ell}$ are Slodowy homogenous of Slodowy degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ (cf. [Pr02, PPY07] or Proposition 4.1,(i)). Hence, A and $A(0)$ are graded subalgebras of $S[t]$ and S respectively. Denote by $A(0)_+$ the augmentation ideal of $A(0)$, and let V_0 be a graded complement to $SA(0)_+$ in S. The main properties of our data A and $A(0)$ are the following ones:

(1) A is a graded polynomial algebra,
(2) the canonical morphism $A \to A(0)$ is a homogenous isomorphism,
(3) the algebra $S[t, t^{-1}]$ is a free extension of A,
(4) the ideal $S[t, t^{-1}]A_+$ of $S[t, t^{-1}]$ is radical.

With these properties we first obtain that $S[[t]]$ is a free extension of A (cf. Corollary 4.17) and that $S[[t]]$ is a free extension of the subalgebra $\tilde{A}$ of $S[[t]]$ generated by $\Bbbk[[t]]$ and A (cf. Theorem 4.21,(i)). From these results, we deduce that the linear map

$$V_0 \otimes_\Bbbk A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism, as expected; see Theorem 4.21,(iii). The key points of the proof are Lemma 4.2, Lemma 4.5, Proposition 4.9 and Corollary 4.17.

1.5. **A related problem.** Let us now mention a recent result of T. Arakawa and A. Premet which resembles our results, [AP].

Let $V^{\rm cri}(\mathfrak{g}^e)$ be the universal affine Vertex algebra associated with $\mathfrak{g}^e$ at critical level, and let $Z(V^{\rm cri}(\mathfrak{g}^e))$ be the center of $V^{\rm cri}(\mathfrak{g}^e)$. Assume that Conditions (1) et (2) of Theorem 1.2 are satisfied. Then $S(\hat{\mathfrak{g}}^e_-)^{\hat{\mathfrak{g}}^e_+}$ is a polynomial algebra, with $\hat{\mathfrak{g}}^e_- := \mathfrak{g}^e[t^{-1}]t^{-1}$. Moreover, $Z(V^{\rm cri}(\mathfrak{g}^e))$ is a polynomial algebra, and explicit generators can be described.

The particular case where $e = 0$ is an old result of B. Feigin and E. Frenkel, [FF92]. Arakawa and Premet have used *affine W-algebras* to prove the general case.

It would be interesting to extend the results of Arakawa and Premet to the setting of Theorem 1.5, that is to the cases where only the Conditon (1) of Theorem 1.2 is satisfied, at least to the cases where we have explicit generators of $S(\mathfrak{g})^{\mathfrak{g}^e}$, not necessarily of the form ${}^e q_1, \dots, {}^e q_\ell$ for some generators $q_1, \dots, q_\ell$ of $S(\mathfrak{g})^{\mathfrak{g}}$; cf. e.g. Remark 5.25.

1.6. The remainder of the paper will be organized as follows.

Section 2 is about general facts on commutative algebra, useful for the Sections 3 and 4. In Section 3, the notions of good elements and good orbits are introduced, and some properties of good elements are described. Theorem 3.3 asserts that the good elements satisfy the polynomiality condition. The main result (Theorem 3.6) is also stated in this section. Section 4 is devoted to the proof of Theorem 3.6. In Section 5, we give applications of Theorem 3.6 to the simple classical Lie algebras. In Section 6, we give applications to the exceptional Lie algebras of types $\mathbf{E}_6$, $\mathbf{F}_4$ and $\mathbf{G}_2$. This allows us to exhibit a great number of good nilpotent orbits. Other examples, counter-examples, remarks and a conjecture are discussed in Section 7. In this last section, other techniques are developed.

Acknowledgments. We thank Alexander Premet for his important comments on the previous version of this paper. We also thank the referee for careful reading and thoughtful suggestions.

This work was partially supported by the ANR-project 10-BLAN-0110.

2. General facts on commutative algebra

We state in this section preliminary results on commutative algebra. Theorem 2.20 will be particularly important in Sections 3 for the proof of Theorem 3.3. As for Proposition 2.5, it will be used in the proof of Theorem 3.6.

2.1. As a rule, for A a graded algebra over $\mathbb{N}$, we denote by A_+ the ideal of A generated by its homogenous elements of positive degree. For M a graded A-module, we set $M_+ := A_+M$.

Let S be a finitely generated regular graded $\Bbbk$-algebra over $\mathbb{N}$. If E is a finite dimensional vector space over $\Bbbk$, we denote by $\mathrm{S}(E)$ the polynomial algebra generated by E. It is a finitely generated regular $\Bbbk$-algebra, graded over $\mathbb{N}$ by the standard grading. Let A be a graded subalgebra of S, different from S and such that $A = \Bbbk + A_+$. Let X_A and X_S be the affine varieties $\mathrm{Specm}(A)$ and $\mathrm{Specm}(S)$ respectively, and let $\pi_{A,S}$ be the morphism from X_S to X_A whose comorphism is the canonical injection from A into S. Let $\mathcal{N}_0$ be the nullvariety of A_+ in X_S and set

$$N := \dim S - \dim A.$$

The following lemma is well-known. It is an easy consequence of a Chevalley's theorem [H77, Ch. II, Exercise 3.22] for Assertions (i) and (ii), and of [Ma86, Ch. 5, Theorem 13.5] for Assertion (iii).

Lemma 2.1. (i) *The irreducible components of the fibers of $\pi_{A,S}$ have dimension at least N.*

(ii) *If $\mathcal{N}_0$ has dimension N, then the fibers of $\pi_{A,S}$ are equidimensional of dimension N.*

(iii) *Suppose that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space E. If $\mathcal{N}_0$ has dimension N, then for some $x_1, \dots, x_N$ in E, the nullvariety of $x_1, \dots, x_N$ in $\mathcal{N}_0$ is equal to $\{0\}$.*

Let $\overline{A}$ be the algebraic closure of A in S.

Lemma 2.2. *Let M be a graded A-module and let V be a graded subspace of M such that $M = V \oplus M_+$. Denote by τ the canonical map $A \otimes_{\Bbbk} V \longrightarrow M$. Then τ is surjective. Moreover, τ is bijective if and only if M is a flat A-module.*

Proof. Let M' be the image of τ. Since $M = V \oplus M_+ = V + A_+M \subset M' + A_+M$, we get by induction on k,

$$M \subset M' + A_+^k M.$$

Since M is graded and since A_+ is generated by elements of positive degree, $M = M'$.

If τ is bijective, then all basis of V is a basis of the A-module M. In particular, it is a flat A-module. Conversely, let us suppose that M is a flat A-module. For v in M, denote by $\overline{v}$ the element of V such that $v - \overline{v}$ is in A_+M.

Claim 2.3. Let $(v_1, \dots, v_n)$ be a homogenous sequence in M such that $\overline{v_1}, \dots, \overline{v_n}$ are linearly independent over $\Bbbk$. Then $v_1, \dots, v_n$ are linearly independent over A.

Proof of Claim 2.3. Since the sequence $(v_1, \dots, v_n)$ is homogenous, it suffices to prove that for a homogenous sequence $(a_1, \dots, a_n)$ in A,

$$a_1 v_1 + \cdots + a_n v_n = 0 \Longrightarrow a_1 = \cdots = a_n = 0.$$

Prove the statement by induction on n. First of all, by flatness, for some homogenous sequence $(y_1, \dots, y_k)$ in M and for some homogenous sequence $(b_{i,j}, i = 1, \dots, n, j = 1, \dots, k)$,

$$v_i = \sum_{j=1}^{k} b_{i,j} y_j \quad \text{and} \quad \sum_{l=1}^{n} a_l b_{l,m} = 0$$

for $i = 1, \dots, n$ and $m = 1, \dots, k$. For $n = 1$, since $\overline{v_1} \neq 0$, for some j, $b_{1,j}$ is in $\Bbbk^*$ since $A = \Bbbk + A_+$. So $a_1 = 0$. Suppose the statement true for $n-1$. Since $\overline{v_n} \neq 0$, for some j, $b_{n,j}$ is in $\Bbbk^*$, whence

$$a_n = -\sum_{i=1}^{n-1} \frac{b_{i,j}}{b_{n,j}} a_i \quad \text{and} \quad \sum_{i=1}^{n-1} a_i (v_i - \frac{b_{i,j}}{b_{n,j}} v_n) = 0.$$

Since $\overline{v_1}, \dots, \overline{v_n}$ are linearly independent over $\Bbbk$, so are the elements

$$\left(\overline{v_i - (b_{i,j}/b_{n,j}) v_n}, \quad i = 1, \dots, n-1\right).$$

By induction hypothesis, $a_1 = \cdots = a_{n-1} = 0$, whence $a_n = 0$. □

According to Claim 2.3, any homogenous basis of V consists of linearly independent elements over A. Hence any homogenous basis of V is a basis of the A-module M since $M = AV$. □

Corollary 2.4. *Suppose that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space E, and suppose that $\dim \mathcal{N}_0 = N$. Then $\overline{A}$ is the integral closure of A in $\mathrm{S}(E)$. In particular, $\overline{A}$ is finitely generated.*

Proof. Since A is finitely generated, so is its integral closure in $\mathrm{S}(E)$ by [Ma86, §33, Lemma 1]. According to the hypothesis on $\mathcal{N}_0$ and Lemma 2.1,(iii), for some $x_1, \dots, x_N$ in E, the nullvariety of $x_1, \dots, x_N$ in $\mathcal{N}_0$ is equal to $\{0\}$. In particular, $x_1, \dots, x_N$ are algebraically independent over A since E has dimension $N + \dim A$. Let J be the ideal of $\mathrm{S}(E)$ generated by A_+ and $x_1, \dots, x_N$. Then the radical of J is the augmentation ideal of $\mathrm{S}(E)$ so that J has finite codimension in $\mathrm{S}(E)$. For V a homogenous complement to J in $\mathrm{S}(E)$, $\mathrm{S}(E)$ is the $A[x_1, \dots, x_N]$-submodule generated by V by Lemma 2.2. Hence $\mathrm{S}(E)$ is a finite extension of $A[x_1, \dots, x_N]$.

Let p be in $\overline{A}$. Since $A[x_1, \dots, x_N]$ is finitely generated, $A[x_1, \dots, x_N][p]$ is a finite extension of $A[x_1, \dots, x_N]$. Let

$$p^m + a_{m-1} p^{m-1} + \cdots + a_0 = 0$$

an integral dependence equation of p over $A[x_1, \dots, x_N]$. For $i = 0, \dots, m$, a_i is a polynomial in $x_1, \dots, x_N$ with coefficients in A since $x_1, \dots, x_N$ are algebraically independent over A. Denote by $a_i(0)$ its constant coefficient. Since p is in $\overline{A}$, $x_1, \dots, x_N$ are algebraically independent over $A[p]$, whence

$$p^m + a_{m-1}(0) p^{m-1} + \cdots + a_0(0) = 0.$$

As a result, $\overline{A}$ is the integral closure of A in $\mathrm{S}(E)$. □

Most of the following proposition is contained in [Ben93, Corollary 6.2.3]. Since Proposition 2.5 is more extensive, we give a proof.

Proposition 2.5. *Let us consider the following conditions on A:*

1) *A is a polynomial algebra,*
2) *A is a regular algebra,*
3) *A is a polynomial algebra generated by $\dim A$ homogenous elements,*

4) *the A-module S is faithfully flat,*
5) *the A-module S is flat,*
6) *the A-module S is free.*

(i) *The conditions* (1)*,* (2)*,* (3) *are equivalent.*

(ii) *The conditions* (4)*,* (5)*,* (6) *are equivalent. Moreover, Condition* (4) *implies Condition* (2) *and, in that event, $\mathcal{N}_0$ is equidimensional of dimension N.*

(iii) *If $\mathcal{N}_0$ is equidimensional of dimension N, then the conditions* (1)*,* (2)*,* (3)*,* (4)*,* (5)*,* (6) *are all equivalent.*

Proof. Let n be the dimension of A.

(i) The implications (3) ⇒ (1), (1) ⇒ (2) are straightforward. Let us suppose that A is a regular algebra. Since A is graded and finitely generated, there exists a homogenous sequence $(x_1, \dots, x_n)$ in A_+ representing a basis of A_+/A_+^2. Let A' be the subalgebra of A generated by $x_1, \dots, x_n$. Then

$$A_+ \subset A' + A_+^2.$$

So by induction on m,

$$A_+ \subset A' + A_+^m$$

for all positive integer m. Then $A = A'$ since A is graded and A_+ is generated by elements of positive degree. Moreover, $x_1, \dots, x_n$ are algebraically independent over $\Bbbk$ since A has dimension n. Hence A is a polynomial algebra generated by n homogenous elements.

(ii) The implications (4) ⇒ (5), (6) ⇒ (5) are straightforward and (5) ⇒ (6) is a consequence of Lemma 2.2.

(5) ⇒ (4): Recall that $x_0 = A_+$. Let us suppose that S is a flat A-module. Then $\pi_{A,S}$ is an open morphism whose image contains x_0. Moreover, $\pi(X_S)$ is stable under the action of $\mathrm{G_m}$. So $\pi_{A,S}$ is surjective. Hence, by [Ma86, Ch. 3, Theorem 7.2], S is a faithfully flat extension of A.

(4) ⇒ (2): Since S is regular and since S is a faithfully flat extension of A, all finitely generated A-module has finite projective dimension. So by [Ma86, Ch. 7, §19, Lemma 2], the global dimension of A is finite. Hence by [Ma86, Ch. 7, Theorem 19.2], A is regular.

If Condition (4) holds, by [Ma86, Ch. 5, Theorem 15.1], the fibers of $\pi_{A,S}$ are equidimensional of dimension N. So $\mathcal{N}_0$ is equidimensional of dimension N.

(iii) Suppose that $\mathcal{N}_0$ is equidimensional of dimension N. By (i) and (ii), it suffices to prove that (2) ⇒ (5). By Lemma 2.1,(ii), the fibers of $\pi_{A,S}$ are equidimensional of dimension N. Hence by [Ma86, Ch. 8, Theorem 23.1], S is a flat extension of A since S and A are regular. □

2.2. We present in this paragraph some results about algebraic extensions, that are independent of Subsection 2.1. These results are used only in the proof of Proposition 2.15. Our main reference is [Ma86]. For A an algebra and $\mathfrak{p}$ a prime ideal of A, $A_{\mathfrak{p}}$ denotes the localization of A at $\mathfrak{p}$.

Let t be an indeterminate, and let L be a field containing $\Bbbk$. Let B, L_1, B_1 satisfying the following conditions:

(I) L_1 is an algebraic extension of $L(t)$ of finite degree,
(II) L is algebraically closed in L_1,
(III) B is a finitely generated subalgebra of L, L is the fraction field of B and B is integrally closed in L,
(IV) B_1 is the integral closure of $B[t]$ in L_1,
(V) tB_1 is a prime ideal of B_1.

For C a subalgebra of L, containing B, we set:

$$R(C) := C \otimes_B B_1,$$

and we denote by μ_C the canonical morphism $R(C) \to CB_1$. Since C and B_1 are integral algebras, the morphisms $c \mapsto c\otimes 1$ and $b \mapsto 1\otimes b$ from C and B_1 to $R(C)$ respectively are embeddings. So, C and B_1 are identified to subalgebras of $R(C)$ by these embeddings. We now investigate some properties of the algebras $R(C)$.

Lemma 2.6. *Let μ_L be the canonical morphism $R(L) \to LB_1$.*
(i) *The algebra $R(L)$ is reduced and μ_L is an isomorphism.*
(ii) *The ideal tLB_1 of LB_1 is maximal. Furthermore B_1 is a finite extension of $B[t]$.*
(iii) *The algebra LB_1 is the direct sum of L and tLB_1.*
(iv) *The ring LB_1 is integrally closed in L_1.*

Proof. (i) Let a be in the kernel of μ_L. Since L is the fraction field of B, for some b in B, $ba = 1\otimes\mu_L(ba)$ so that $ba = 0$ and $a = 0$. As a result, μ_L is an isomorphism and $R(L)$ is reduced since LB_1 is integral.

(ii) Since t is not algebraic over L and since LB_1 is integral over $L[t]$ by Condition (IV), tLB_1 is strictly contained in LB_1. Let a and b be in LB_1 such that ab is in tLB_1. By Condition (III), for some c in $B \setminus \{0\}$, ca and cb are in B_1. So, by Condition (V), ca or cb is in tB_1. Hence a or b is in tLB_1. As a result, tLB_1 is a prime ideal and the quotient Q of LB_1 by tLB_1 is an integral domain. Denote by ι the quotient morphism. Since L is a field, the restriction of ι to L is an embedding of L into Q. According to Conditions (I) and (IV) and [Ma86, §33, Lemma 1], B_1 is a finite extension of $B[t]$. Then Q is a finite extension of L and tLB_1 is a maximal ideal of LB_1.

(iii) Since L is algebraically closed in L_1, Q and L_1 are linearly disjoint over L. So, $Q\otimes_L L_1$ is isomorphic to the extension of L_1 generated by Q. Denoting this extension by QL_1, QB_1 is a subalgbera of QL_1 and we have the exact sequences

$$0 \longrightarrow tLB_1 \longrightarrow LB_1 \longrightarrow Q \longrightarrow 0$$

$$0 \longrightarrow tQB_1 \longrightarrow QB_1 \longrightarrow Q\otimes_L Q \longrightarrow 0$$

$$0 \longrightarrow tQB_1 \longrightarrow tQB_1 + LB_1 \longrightarrow Q\otimes 1 \longrightarrow 0$$

As a result,

$$Q \subset LB_1 + tQB_1.$$

By (ii), QB_1 is a finite $L[t]$-module. So, by Nakayama's Lemma, for some a in $L[t]$, $(1+ta)QB_1$ is contained in LB_1. As a result, Q is contained in L_1, whence $Q = L$ since L is algebraically closed in L_1. The assertion follows since Q is the quotient of LB_1 by tLB_1.

(iv) Let a be in the integral closure of LB_1 in L_1 and let

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0$$

an integral dependence equation of a over LB_1. For some b in $L\setminus\{0\}$, ba_i is in B_1 for $i = 0, \ldots, m-1$. Then, by Condition (IV), ba is in B_1 since it satisfies an integral dependence equation over B_1. As a result, LB_1 is integrally closed in L_1. □

Let L_2 be the Galois extension of $L(t)$ generated by L_1, and let Γ be the Galois group of the extension L_2 of $L(t)$. Denote by B_2 the integral closure of $B[t]$ in L_2. For C subalgebra of L, containing B, set

$$R_2(C) := C\otimes_B B_2,$$

and denote by $\mu_{C,2}$ the canonical morphism $R_2(C) \to CB_2$. The action of Γ in B_2 induces an action of Γ in $R_2(C)$ given by $g.(c \otimes b) = c \otimes g(b)$.

Lemma 2.7. *Let x be a primitive element of L_1, and let Γ_x be the stabilizer of x in Γ.*

(i) *The subfield L_1 of L_2 is the set of fixed points under the action of Γ_x in L_2, and B_1 is the set of fixed points under the action of Γ_x in B_2.*

(ii) *For C subalgebra of L containing B, the canonical morphism $R(C) \to R_2(C)$ is an embedding and its image is the set of fixed points under the action of Γ_x in $R_2(C)$.*

(iii) *For C subalgebra of L, containing B, $C[t]$ is embedded in $R(C)$ and $R_2(C)$. Moreover, $C[t]$ is the set of fixed points under the action of Γ in $R_2(C)$.*

Proof. (i) Let L'_1 be the set of fixed points under the action of Γ_x in L_2,

$$L'_1 = \{y \in L_2 \mid \Gamma_x.y = y\}.$$

Then L_1 is contained in L'_1, and L_2 is an extension of degree $|\Gamma_x|$ of L'_1. Since x is a primitive element of L_1, the degree of this extension is equal to $|\Gamma.x|$ so that L_2 is an extension of degree $|\Gamma_x|$. Hence $L'_1 = L_1$.

Since B_2 is the integral closure of $B[t]$ in L_2, B_2 is invariant under Γ. Moreover, the intersection of B_2 and L_1 is equal to B_1 by Condition (IV). Hence B_1 is the set of fixed points under the action of Γ_x in B_2.

(ii) For a in B_2 and b in $R_2(C)$, set:

$$a^{\#} := \frac{1}{|\Gamma_x|} \sum_{g \in \Gamma_x} g(a), \quad \overline{b} := \frac{1}{|\Gamma_x|} \sum_{g \in \Gamma_x} g.b.$$

Then $a \mapsto a^{\#}$ is a projection of B_2 onto B_1. Moreover, it is a morphism of B_1-module. Denote by ι the canonical morphism $R(C) \to R_2(C)$, and by φ the morphism

$$R_2(C) \longrightarrow R(C), \quad c \otimes a \longmapsto c \otimes a^{\#}.$$

For b in $R_2(C)$,

$$\varphi(b) = \varphi(\overline{b}) \quad \text{and} \quad \iota \circ \varphi(b) = \overline{b}$$

Then φ is a surjective morphism and the image of $\iota \circ \varphi$ is the set of fixed points under the action of Γ_x in $R_2(C)$. Moreover ι is injective, whence the assertion.

(iii) From the equalities

$$R(C) = (C \otimes_B B[t]) \otimes_{B[t]} B_1 \quad \text{and} \quad C[t] = C \otimes_B B[t]$$

we deduce that $R(C) = C[t] \otimes_{B[t]} B_1$. In the same way, $R_2(C) = C[t] \otimes_{B[t]} B_2$. Then, since $C[t]$ is an integral algebra, the morphism $c \mapsto c \otimes 1$ is an embedding of $C[t]$ in $R(C)$ and $R_2(C)$. Moreover, $C[t]$ is invariant under the action of Γ in $R_2(C)$.

Let a be in $R_2(C)$ invariant under Γ. Then a has an expansion

$$a = \sum_{i=1}^{k} c_i \otimes b_i$$

with $c_1, \ldots, c_k$ in $C[t]$ and $b_1, \ldots, b_k$ in B_2. Since a is invariant under Γ,

$$a = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g.a = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \sum_{i=1}^{k} c_i \otimes g.b_i.$$

For $i = 1, \ldots, k$, set:

$$b'_i := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g.b_i$$

The elements $b'_1, \dots, b'_k$ are in $B[t]$, and

$$a = (\sum_{i=1}^{k} c_i b'_i) \otimes 1 \in C[t],$$

whence the assertion. □

From now on, we fix a finitely generated subalgebra C of L containing B. Denote by $\mathfrak{n}$ the nilradical of $R(C)$.

Lemma 2.8. *Let $\mathfrak{k}$ be the kernel of $\mu_{C,2}$ and let $\mathfrak{n}_2$ be the nilradical of $R_2(C)$.*

(i) *The algebras $R(C)$ and $R_2(C)$ are finitely generated. Furthermore, they are finite extensions of $C[t]$.*

(ii) *For a in $\mathfrak{k}$, $ba = 0$ for some b in $B \setminus \{0\}$.*

(iii) *The ideal $\mathfrak{k}$ is the minimal prime ideal of $R_2(C)$ such that $\mathfrak{k} \cap B = \{0\}$. Moreover, $\mathfrak{k} \cap B[t] = \{0\}$.*

(iv) *The ideal $\mathfrak{n}$ is the kernel of μ_C. Moreover, $\mathfrak{n}_2 = \mathfrak{k}$ and $\mathfrak{n}$ is a prime ideal.*

(v) *The local algebra $R(C)_{\mathfrak{n}}$ is isomorphic to L_1.*

Proof. (i) According to Lemma 2.7,(iii), $R(C)$ is an extension of $C[t]$ and $R(C) = C[t] \otimes_{B[t]} B_1$. Then, by Lemma 2.6,(ii), $R(C)$ is a finite extension of $C[t]$. In particular, $R(C)$ is a finitely generated algebra since so is C. In the same way, $R_2(C)$ is a finite extension of $C[t]$ and it is finitely generated.

(ii) Let a be in $\mathfrak{k}$. Then a has an expansion

$$a = \sum_{i=1}^{k} c_i \otimes b_i$$

with $c_1, \dots, c_k$ in C and $b_1, \dots, b_k$ in B_2. Since C and B have the same fraction field, for some b in $B \setminus \{0\}$, bc_i is in B, whence

$$ba = 1 \otimes (\sum_{i=1}^{k} bc_i b_i).$$

So $ba = 0$ since $\mathfrak{k}$ is the kernel of $\mu_{C,2}$.

(iii) By (i) there are finitely many minimal prime ideals of $R_2(C)$. Denote them by $\mathfrak{p}_1, \dots, \mathfrak{p}_k$. Since $C[t]$ is an integral algebra, $\mathfrak{n}_2 \cap C[t] = \{0\}$ so that $\mathfrak{p}_i \cap C = \{0\}$ for some i. Let i be such that $\mathfrak{p}_i \cap B = \{0\}$ and let a be in $\mathfrak{k}$. By (ii), for some b in $B \setminus \{0\}$, ba is in $\mathfrak{p}_i$. Hence $\mathfrak{k}$ is contained in $\mathfrak{p}_i$. Since CB_2 is an integral algebra, $\mathfrak{k}$ is a prime ideal. Then $\mathfrak{p}_i = \mathfrak{k}$ since $\mathfrak{p}_i$ is a minimal prime ideal, whence the assertion since for some j, $\mathfrak{p}_j \cap C[t] = \{0\}$.

(iv) By (iii), there is only one minimal prime ideal of $R_2(C)$ whose intersection with B is equal to $\{0\}$. So, it is invariant under Γ. Hence $\mathfrak{k}$ is invariant under Γ. As a result, for a in $\mathfrak{k}$,

$$0 = \prod_{g \in \Gamma} (a - g.a) = a^m + a_{m-1} a^{m-1} + \cdots + a_0$$

with $m = |\Gamma|$ and $a_0, \dots, a_{m-1}$ in $\mathfrak{k}$. Moreover, by Lemma 2.7,(iii), $a_0, \dots, a_{m-1}$ are in $C[t]$. So, by (iii), they are all equal to zero so that a is a nilpotent element. Hence $\mathfrak{k}$ is contained in $\mathfrak{n}_2$. Then $\mathfrak{n}_2 = \mathfrak{k}$ by (iii).

By Lemma 2.7,(ii), $R(C)$ identifies with a subalgebra of $R_2(C)$ so that $\mathfrak{n} = \mathfrak{n}_2 \cap R(C)$, and μ_C is the restriction of $\mu_{C,2}$ to $R(C)$. Hence $\mathfrak{n}$ is the kernel of μ_C and $\mathfrak{n}$ is a prime ideal of $R(C)$.

(v) By (iii), $\mathfrak{n} \cap C = \{0\}$. So, by (ii), $\mathfrak{n} R(C)_{\mathfrak{n}} = \{0\}$. As a result, $R(C)_{\mathfrak{n}}$ is a field since $\mathfrak{n} R(C)_{\mathfrak{n}}$ is a maximal ideal of $R(C)_{\mathfrak{n}}$. Moreover, by (iii), it is isomorphic to a subfield of L_1, containing B_1. So, $R(C)_{\mathfrak{n}}$ is isomorphic to L_1. □

For c in $L[t]$, denote by $c(0)$ the constant term of c as a polynomial in t with coefficients in L.

Lemma 2.9. *Assume that C is integrally closed in L. Denote by $\overline{CB_1}$ the integral closure of CB_1 in L_1.*

(i) *Let $i \in \{1, 2\}$. For all positive integer j, the intersection of $C[t]$ and $t^j LB_i$ equals $t^j C[t]$.*

(ii) *The intersection of tLB_1 and $\overline{CB_1}$ equals $t\overline{CB_1}$.*

(iii) *The algebra $\overline{CB_1}$ is contained in $C + t\overline{CB_1}$.*

(iv) *The algebra B_1 is the direct sum of B and tB_1.*

Proof. First of all, CB_1 and $\overline{CB_1}$ are finite extensions of $C[t]$ by Lemma 2.7,(i), and [Ma86, §33, Lemma 1]. So $\overline{CB_1}$ is the integral closure of $C[t]$ in L_1 by Condition (IV). Denote by $\overline{CB_2}$ the integral closure of $C[t]$ in L_2. Since C is integrally closed in L, $C[t]$ is integally closed in $L[t]$. Hence $C[t]$ is the set of fixed points under the action of Γ in $\overline{CB_2}$. Let a be in $\overline{CB_2}$. Then

$$0 = \prod_{g \in \Gamma}(a - g(a)) = a^m + a_{m-1}a^{m-1} + \cdots + a_0$$

with $a_0, \ldots, a_{m-1}$ in $C[t]$.

(i) Since $t^j LB_1$ is contained in $t^j LB_2$ and contains $t^j C[t]$, it suffices to prove the assertion for $i = 2$. Let us prove it by induction on j. Let c be in $C[t]$. Then $c - c(0)$ is in tLB_2. By Lemma 2.6,(ii), $L \cap tLB_2 = \{0\}$ since L is a field, whence $C \cap tLB_2 = \{0\}$ since C is contained in L. As a result, if c is in tLB_2, $c(0) = 0$ and c is in $tC[t]$, whence the assertion for $j = 1$. Suppose the assertion true for $j - 1$. Let c be in $C[t] \cap t^j LB_2$. By induction hypothesis, $c = t^{j-1}c'$ with c' in $C[t]$. Then c' is in $C[t] \cap tLB_2$, whence c is in $t^j C[t]$ by the assertion for $j = 1$.

(ii) Suppose that a is in tLB_1. Since tLB_2 is invariant under Γ, for $i = 0, \ldots, m-1$, a_i is in $t^{m-i}LB_2$. Set for $i = 0, \ldots, m-1$,

$$a'_i := \frac{a_i}{t^{m-i}}.$$

Then by (i), $a'_0, \ldots, a'_{m-1}$ are in $C[t]$. Moreover,

$$(\frac{a}{t})^m + a'_{m-1}(\frac{a}{t^{m-1}})^{m-1} + \cdots + a'_0 = 0,$$

so that a/t is in $\overline{CB_1}$, whence the assertion.

(iii) Suppose that a is in $\overline{CB_1}$. By Lemma 2.6,(iii), L is the quotient of LB_1 by tLB_1. So, denoting by $\overline{a}$ the image of a by the quotient morphism,

$$\overline{a}^m + a_{m-1}(0)\overline{a}^{m-1} + \cdots + a_0(0) = 0.$$

Then $\overline{a}$ is in C since C is integrally closed. Hence a is in $C + tLB_1$. As a result, by (ii), $\overline{CB_1}$ is contained in $C + t\overline{CB_1}$.

(iv) By Condition (III), B is integrally closed in L. So the assertion results from (iii) and Condition (IV) for $C = B$. □

Corollary 2.10. *The ideal $R(C)t$ of $R(C)$ is prime and t is not a zero divisor in $R(C)$.*

Proof. According to Lemma 2.9,(iv), $R(C) = C + R(C)t$. Furthermore, this sum is direct since $C \cap tCB_1 = \{0\}$ by Lemma 2.6,(ii) and since the restriction of μ_C to C is injective. Then $R(C)t$ is a prime ideal of $R(C)$ since C is an integral algebra.

Since $R(C)t$ is a prime ideal, $\mathfrak{n}$ is contained in $R(C)t$. According to Lemma 2.8,(iv), $\mathfrak{n}$ is the kernel of μ_C. Let a be in $\mathfrak{n}$. Then $a = a't$ for some a' in $R(C)$. Since $0 = \mu_C(a't) = \mu_C(a')t$, a' is in $\mathfrak{n}$. As a result, by induction on m, for all positive integer m, $a = a_m t^m$ for some a_m in $\mathfrak{n}$.

For k positive integer, denote by J_k the subset of elements a of $R(C)$ such that $at^k = 0$. Then $(J_1, J_2, \ldots)$ is an increasing sequence of ideals of $R(C)$. For a in J_k, $0 = \mu_C(at^k) = \mu_C(a)t^k$. Hence J_k is contained in

$\mathfrak{n}$. According to Lemma 2.8,(i), the $\Bbbk$-algebra $R(C)$ is finitely generated. So for some positive integer k_0, $J_k = J_{k_0}$ for all k bigger than k_0. Let a be in J_1. Then $a = a_{k_0}t^{k_0}$ for some a_{k_0} in $\mathfrak{k}$. Since $a_{k_0}t^{k_0+1} = 0$, a_{k_0} is in J_{k_0} so that $a = 0$. Hence t is not a zero divisor in $R(C)$. □

Proposition 2.11. *Suppose that C is integrally closed and Cohen-Macaulay. Let $\mathfrak{p}$ be a prime ideal of CB_1, containing t and let $\tilde{\mathfrak{p}}$ be its inverse image by μ_C.*

(i) *The local algebra $(CB_1)_{\mathfrak{p}}$ is normal.*

(ii) *The local algebra $R(C)_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay and reduced. In particular, the canonical morphism $R(C)_{\tilde{\mathfrak{p}}} \to (CB_1)_{\mathfrak{p}}$ is an isomorphism.*

(iii) *The local algebra $(CB_1)_{\mathfrak{p}}$ is Cohen-Macaulay.*

Proof. (i) Let $\overline{CB_1}$ be the integral closure of CB_1 in L_1. Setting $S := CB_1 \setminus \mathfrak{p}$, $(CB_1)_{\mathfrak{p}}$ is the localization of CB_1 with respect to S. Denote by $(\overline{CB_1})_{\mathfrak{p}}$ the localization of $\overline{CB_1}$ with respect to S. Then $(\overline{CB_1})_{\mathfrak{p}}$ is a finite $(CB_1)_{\mathfrak{p}}$-module since $\overline{CB_1}$ is a finite extension of CB_1. According to Lemma 2.8,(iii),

$$\overline{CB_1} \subset CB_1 + t\overline{CB_1}.$$

Then since t is in $\mathfrak{p}$,

$$(\overline{CB_1})_{\mathfrak{p}}/(CB_1)_{\mathfrak{p}} = \mathfrak{p}(\overline{CB_1})_{\mathfrak{p}}/(CB_1)_{\mathfrak{p}}.$$

So, by Nakayama's Lemma, $(\overline{CB_1})_{\mathfrak{p}} = (CB_1)_{\mathfrak{p}}$, whence the assertion.

(ii) According to Corollary 2.10, $R(C)t$ is a prime ideal containing $\mathfrak{n}$. Denote by $\overline{\mathfrak{p}}$ the intersection of $\mathfrak{p}$ and C. Since $\tilde{\mathfrak{p}}$ is the inverse image of $\mathfrak{p}$ by μ_C, $C_{\overline{\mathfrak{p}}}$ is the quotient of $R(C)_{\tilde{\mathfrak{p}}}$ by $R(C)_{\tilde{\mathfrak{p}}}t$. Since C is Cohen-Macaulay, so is $C_{\overline{\mathfrak{p}}}$. As a result, $R(C)_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay since t is not a zero divisor in $R(C)$ by Corollary 2.10 and since $R(C)_{\tilde{\mathfrak{p}}}t$ is a prime ideal of height 1.

Denote by $\mu_{C,\tilde{\mathfrak{p}}}$ the canonical extension of μ_C to $R(C)_{\tilde{\mathfrak{p}}}$. Then $(CB_1)_{\mathfrak{p}}$ is the image of $\mu_{C,\tilde{\mathfrak{p}}}$. According to Lemma 2.8,(iv), the nilradical $\mathfrak{n}R(C)_{\tilde{\mathfrak{p}}}$ of $R(C)_{\tilde{\mathfrak{p}}}$ is the minimal prime ideal of $R(C)_{\tilde{\mathfrak{p}}}$ and it is the kernel of $\mu_{C,\tilde{\mathfrak{p}}}$. By Lemma 2.8,(v), the localization of $R(C)_{\tilde{\mathfrak{p}}}$ at $\mathfrak{n}R(C)_{\tilde{\mathfrak{p}}}$ is a field. In particular, it is regular. Then, by [Bou98, §1, Proposition 15], $R(C)_{\tilde{\mathfrak{p}}}$ is a reduced algebra since it is Cohen-Macaulay. As a result, $\mu_{C,\tilde{\mathfrak{p}}}$ is an isomorphism onto $(CB_1)_{\mathfrak{p}}$.

(iii) results from (ii). □

2.3. Return to the situation of Subsection 2.1, and keep its notations. From now on, and until the end of the section, we assume that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space E. As a rule, if B is a subalgebra of $\mathrm{S}(E)$, we denote by $K(B)$ its fraction field, and we set for simplicity

$$K := K(\mathrm{S}(E)).$$

Furthermore we assume until the end of the section that the following conditions hold:

(a) $\dim \mathcal{N}_0 = N$,

(b) A is a polynomial algebra,

(c) $K(\overline{A})$ is algebraically closed in K.

We aim to prove Theorem 2.20 (see Subsection 2.4). Let $(v_1, \dots, v_N)$ be a sequence of elements of E such that its nullvariety in $\mathcal{N}_0$ equals $\{0\}$. Such a sequence does exist by Lemma 2.1,(iii). Set

$$C := \overline{A}[v_1, \dots, v_N].$$

By Proposition 2.5,(ii), C is a polynomial algebra if and only if so is $\overline{A}$ since C is a faithfully flat extension of $\overline{A}$. Therefore, in order to prove Theorem 2.20, it suffices to prove that $\mathrm{S}(E)$ is a free extension of C, again by Proposition 2.5,(ii). This is now our goal.

Condition (c) is actually not useful for the following lemma:

Lemma 2.12. *The algebra C is integrally closed and* $\mathrm{S}(E)$ *is the integral closure of C in K.*

Proof. Since $\overline{A}$ has dimension $\dim E - N$ and since the nullvariety of $v_1, \dots, v_N$ in $\mathcal{N}_0$ is $\{0\}$, $v_1, \dots, v_N$ are algebraically independent over A and $\overline{A}$. By Serre's normality criterion [Bou98, §1, n°10, Théorème 4], any polynomial algebra over a normal ring is normal. So C is integrally closed since so is $\overline{A}$ by definition. Moreover, C is a homogenous finitely generated subalgebra of $\mathrm{S}(E)$ since so is $\overline{A}$ by Corollary 2.4. Since C has dimension $\dim E$, $\mathrm{S}(E)$ is algebraic over C. Then, by Corollary 2.4, $\mathrm{S}(E)$ is the integral closure of C in K. Indeed, $\mathrm{S}(E)$ is integrally closed as a polynomial algebra and $\{0\}$ is the nullvariety of C_+ in E^*. □

Set $Z_0 := \mathrm{Specm}(\overline{A})$ and $Z := Z_0 \times \Bbbk^N$. Then Z is equal to $\mathrm{Specm}(C)$. Let X_0 be a desingularization of Z_0 and let π_0 be the morphism of desingularization. Such a desingularization does exist by [Hir64]. Set $X := X_0 \times \Bbbk^N$ and denote by π the morphism

$$X \longrightarrow Z, \qquad (x, v) \longmapsto (\pi_0(x), v).$$

Then (X, π) is a desingularization of Z.

Fix x_0 in $\pi_0^{-1}(C_+)$. For $i = 0, \dots, N$, set $X_i := X_0 \times \Bbbk^i$ and let $x_i := (x_0, 0_{\Bbbk^i})$. Define K_i, C'_i, C_i by the induction relations:

(1) $C'_0 := C_0 := \overline{A}$ and K_0 is the fraction field of $\overline{A}$,
(2) $C'_i := C'_{i-1}[v_i]$,
(3) K_i is the algebraic closure of $K_{i-1}(v_i)$ in K and C_i is the integral closure of $C_{i-1}[v_i]$ in K_i.

Lemma 2.13. *Let $i = 1, \dots, N$.*

(i) *The field K_i is a finite extension of $K_{i-1}(v_i)$ and K_{i-1} is algebraically closed in K_i.*
(ii) *The algebra C_i is finitely generated and integrally closed in K. Moreover, K_i is the fraction field of C_i.*
(iii) *The algebra C_i is contained in* $\mathrm{S}(E)$ *and $C_N =$* $\mathrm{S}(E)$*. Moreover, $K_N = K$.*
(iv) *The algebra C_i is a finite extension of C'_i.*
(v) *The algebra C_i is the intersection of* $\mathrm{S}(E)$ *and K_i. Moreover, v_iC_i is a prime ideal of C_i.*

Proof. (i) By Condition (c), K_0 is algebraically closed in K. So K_0 is algebraically closed in K_1. By definition, for $i > 1$, K_{i-1} is algebraically closed in K. So it is in K_i. Since the nullvariety of $v_1, \dots, v_N$ in $\mathcal{N}_0$ equals $\{0\}$, $v_1, \dots, v_N$ are algebraically independent over K_0. Hence $K_{i-1}(v_i, \dots, v_N)$ is a field of rational fractions over K_{i-1}. Moreover, K is an algebraic extension of $K_{i-1}(v_i, \dots, v_N)$ by Lemma 2.12. Since $\mathrm{S}(E)$ is a finitely generated $\Bbbk$-algebra, K is a finite extension of $K_{i-1}(v_i, \dots, v_N)$. By definition, K_i is the algebraic closure of $K_{i-1}(v_i)$ in K. Hence K_i is a finite extension of $K_{i-1}(v_i)$.

(ii) Prove the assertion by induction on i. By definition, it is true for $i = 0$ and C_i is the integral closure of $C_{i-1}[v_i]$ in K_i for $i = 1, \dots, N$, whence the assertion by (i) and [Ma86, §33, Lemma 1].

(iii) Since $\mathrm{S}(E)$ is integrally closed in K, C_i is contained in $\mathrm{S}(E)$ by induction on i. By definition, the field K_N is algebraically closed in K and it contains C. So $K_N = K$ by Lemma 2.12. Since C_N is integrally closed in K_N and it contains C, $C_N = \mathrm{S}(E)$ by Lemma 2.12.

(iv) Prove the assertion by induction on i. By definition, it is true for $i = 0$. Suppose that it is true for $i - 1$. Then C_i is a finite extension of $C'_{i-1}[v_i] = C'_i$.

(v) Prove by induction on i that C_{N-i} is the intersection of $\mathrm{S}(E)$ and K_{N-i} for $i = 0, \dots, N$. By (iii), it is true for $i = 0$. Suppose that it is true for $i - 1$. By induction hypothesis, it suffices to prove that C_{N-i} is the intersection of C_{N-i+1} and K_{N-i}. Let a be in this intersection. Then a satisfies an integral dependence equation over $C_{N-i}[v_{N-i+1}]$:

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0.$$

Denoting by $a_j(0)$ the constant term of a_j as a polynomial in v_{N-i+1} with coefficients in C_{N-i},

$$a^m + a_{m-1}(0)a^{m-1} + \cdots + a_0(0) = 0$$

since a is in K_{N-i} and v_{N-i+1} is algebraically independent over K_{N-i}. Hence a is in C_{N-i} since C_{N-i} is integrally closed in K_{N-i} by (ii).

Let a and b be in C_i such that ab is in v_iC_i. Since v_i is in E, $v_i\mathrm{S}(E)$ is a prime ideal of $\mathrm{S}(E)$. So a or b is in $v_i\mathrm{S}(E)$ since C_i is contained in $\mathrm{S}(E)$. Hence a/v_i or b/v_i are in the intersection of $\mathrm{S}(E)$ and K_i. So a or b is in v_iC_i. □

Remark 2.14. According to Lemma 2.13,(i),(ii),(iv), for $i = 1, \ldots, N$, K_{i-1}, v_i, C_{i-1}, K_i, C_i satisfy Conditions (I), (II), (III), (V) satisfed by L, t, B, L_1, B_1 in Subsection 2.2. Moreover, Condition (IV) is satisfied by construction (cf. Lemma 2.13,(v)).

Proposition 2.15. *Let* $i = 1, \ldots, N$.

(i) *The semi-local algebra* $\mathcal{O}_{X_i,x_i}C_i$ *is normal and Cohen-Macaulay.*

(ii) *The canonical morphism* $\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i \rightarrow \mathcal{O}_{X_i,x_i}C_i$ *is an isomorphism.*

Proof. (i) The local ring $\mathcal{O}_{X_i,x_i}$ is an extension of C'_i and C_i is a finite extension of C'_i by Lemma 2.13,(iv). So $\mathcal{O}_{X_i,x_i}C_i$ is a semi-local ring as a finite extension of the local ring $\mathcal{O}_{X_i,x_i}$. Prove the assertion by induction on i. For $i = 0$, $\mathcal{O}_{X_0,x_0}C_0 = \mathcal{O}_{X_0,x_0}$ and $\mathcal{O}_{X_0,x_0}$ is a regular local algebra. Suppose that it is true for $i-1$ and set $\mathfrak{A}_{i-1} := \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$. Then $\mathfrak{A}_{i-1}$ is a subalgebra of K_{i-1} since $\mathcal{O}_{X_{i-1},x_{i-1}}$ is contained in the fraction field of C'_{i-1}. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_{X_i,x_i}C_i$. The local ring $\mathcal{O}_{X_i,x_i}$ is the localization of $\mathcal{O}_{X_{i-1},x_{i-1}}[v_i]$ at $\mathfrak{m} \cap \mathcal{O}_{X_{i-1},x_{i-1}}[v_i]$. Hence v_i is in $\mathfrak{m}$, and $\mathfrak{m} \cap \mathfrak{A}_{i-1}C_i$ is a prime ideal of $\mathfrak{A}_{i-1}C_i$ such that the localization of $\mathfrak{A}_{i-1}C_i$ at this prime ideal is the localization of $\mathcal{O}_{X_i,x_i}C_i$ at $\mathfrak{m}$. By the induction hypothesis, $\mathfrak{A}_{i-1}$ is normal and Cohen-Macaulay. According to Remark 2.14 and Proposition 2.11,(i) and (iii), the localization of $\mathfrak{A}_{i-1}C_i$ at $\mathfrak{m} \cap \mathfrak{A}_{i-1}C_i$ is normal and Cohen-Macaulay, whence the assertion.

(ii) Prove the assertion by induction on i. For $i = 0$, C_0 is contained in $\mathcal{O}_{X_0,x_0}$. Suppose that it is true for $i-1$. For $j \in \{i-1, i\}$, denote by ν_j the canonical morphism

$$\mathcal{O}_{X_j,x_j} \otimes_{C'_j} C_j \longrightarrow \mathcal{O}_{X_j,x_j}C_j.$$

Recall that $\mathfrak{A}_{i-1} := \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$. By induction hypothesis, the morphism $\nu_{i-1}\otimes \mathrm{id}_{C_i}$,

$$(\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_{i-1}) \otimes_{C_{i-1}} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$$

is an isomorphism. Since C'_{i-1} is contained in $\mathcal{O}_{X_{i-1},x_{i-1}}$,

$$\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C'_{i-1}[v_i] = \mathcal{O}_{X_{i-1},x_{i-1}}[v_i].$$

Furthermore,

$$(\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_{i-1}) \otimes_{C_{i-1}} C_i = \mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_i = (\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C'_{i-1}[v_i]) \otimes_{C'_{i-1}[v_i]} C_i,$$

whence an isomorphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i.$$

Let $\mathfrak{m}$ be as in (i). Set

$$\mathfrak{p} := \mathfrak{m} \cap \mathfrak{A}_{i-1}C_i, \qquad \tilde{\mathfrak{m}} := \nu_i^{-1}(\mathfrak{m}),$$

and denote by $\tilde{\mathfrak{p}}$ the inverse image of $\mathfrak{p}$ by the canonical morphism

$$\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i \longrightarrow \mathfrak{A}_{i-1}C_i.$$

According to Proposition 2.11,(ii), the canonical morphism

$$(\mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1} \otimes_{C_{i-1}} C_i)_{\tilde{\mathfrak{p}}} \longrightarrow (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}}$$

is an isomorphism since $\mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$ is a finitely generated subalgebra of K_{i-1}, containing C_{i-1}, which is Cohen-Macaulay and integrally closed. Let $\mathfrak{p}^{\#}$ be the inverse image of $\tilde{\mathfrak{p}}$ by the isomorphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1} \otimes_{C_{i-1}} C_i.$$

Then the canonical morphism

$$(\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i)_{\mathfrak{p}^{\#}} \longrightarrow (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}}$$

is an isomorphism. From the equalities

$$(\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i)_{\mathfrak{p}^{\#}} = (\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i)_{\tilde{\mathfrak{m}}}, \qquad (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}} = (\mathcal{O}_{X_i,x_i}C_i)_{\mathfrak{m}}$$

we deduce that the support of the kernel of ν_i in $\mathrm{Spec}(\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i)$ does not contain $\tilde{\mathfrak{m}}$. As a result, denoting by $\mathcal{S}_i$ this support, $\mathcal{S}_i$ does not contain the inverse images by ν_i of the maximal ideals of $\mathcal{O}_{X_i,x_i}C_i$.

According to Lemma 2.8,(iv), the kernel of the canonical morphism

$$\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i \longrightarrow \mathcal{O}_{X_{i-1},x_{i-1}}C_i$$

is the nilradical of $\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$. Hence, the kernel of the canonical morphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i \rightarrow \mathcal{O}_{X_{i-1},x_{i-1}}[v_i]C_i$$

is the nilradical of $\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i$ since the canonical map

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$$

is an isomorphism by induction hypothesis. As a result, all element of $\mathcal{S}_i$ is the inverse image of a prime ideal in $\mathcal{O}_{X_i,x_i}C_i$. Hence $\mathcal{S}_i$ is empty, and ν_i is an isomorphism. □

The following Corollary results from Proposition 2.15 and Lemma 2.13,(iii) since $\pi^{-1}(C_+) = \pi_0^{-1}(C_+) \times \{0\}$.

Corollary 2.16. *Let x be in $\pi^{-1}(C_+)$.*

(i) *The semi-local algebra $\mathcal{O}_{X,x}\mathrm{S}(E)$ is normal and Cohen-Macaulay.*

(ii) *The canonical morphism $\mathcal{O}_{X,x} \otimes_C \mathrm{S}(E) \rightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$ is an isomorphism.*

Let d be the degree of the extension K of $K(C)$. Let x be in $\pi^{-1}(C_+)$, and denote by Q_x the quotient of $\mathcal{O}_{X,x}\mathrm{S}(E)$ by $\mathfrak{m}_x\mathrm{S}(E)$, with $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$.

Lemma 2.17. *Let V be a homogenous complement to $\mathrm{S}(E)C_+$ in $\mathrm{S}(E)$.*

(i) *The $\Bbbk$-space V has finite dimension, $\mathrm{S}(E) = CV$ and $K = K(C)V$.*

(ii) *The $\Bbbk$-space Q_x has dimension d. Furthermore, for all subspace V' of dimension d of V such that Q_x is the image of V' by the quotient map, the canonical map*

$$\mathcal{O}_{X,x} \otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

is bijective.

Proof. (i) According to Lemma 2.12, $\mathrm{S}(E)$ is a finite extension of C. Hence, the $\Bbbk$-space V is finite dimensional. On the other hand, we have $\mathrm{S}(E) = V + \mathrm{S}(E)C_+$. Hence, by induction on m, $\mathrm{S}(E) = CV + \mathrm{S}(E)C_+^m$ for any m, whence $\mathrm{S}(E) = CV$ since C_+ is generated by elements of positive degree. As a result, $K = K(C)V$ since the $\Bbbk$-space V is finite dimensional.

(ii) Let d' be the dimension of Q_x. By (i), since C_+ is contained in $\mathfrak{m}_x$,

$$\mathcal{O}_{X,x}\mathrm{S}(E) = V + \mathfrak{m}_x\mathrm{S}(E).$$

As a result, for some subspace V' of dimension d' of V, Q_x is the image of V' by the quotient map. Then,

$$\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V' + \mathfrak{m}_x\mathrm{S}(E),$$

and by Nakayama's Lemma, $\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V'$. Let $(v_1,\dots,v_{d'})$ be a basis of V'. Suppose that the elements $v_1,\dots,v_{d'}$ are not linearly independent over $\mathcal{O}_{X,x}$. A contradiction is expected. Let l be the smallest integer such that

$$a_1v_1 + \cdots + a_{d'}v_{d'} = 0$$

for some sequence $(a_1,\dots,a_{d'})$ in $\mathfrak{m}_x^l$, not contained in $\mathfrak{m}_x^{l+1}$. According to Corollary 2.16,(i) and [Ma86, Ch. 8, Theorem 23.1], $\mathcal{O}_{X,x}\mathrm{S}(E)$ is a flat extension of $\mathcal{O}_{X,x}$ since $\mathcal{O}_{X,x}\mathrm{S}(E)$ is a finite extension of $\mathcal{O}_{X,x}$. So, for some $w_1,\dots,w_m$ in $\mathrm{S}(E)$ and for some sequences $(b_{i,1},\dots,b_{i,m},\ i=1,\dots,d')$ in $\mathcal{O}_{X,x}$,

$$v_i = \sum_{j=1}^{m} b_{i,j}w_j \quad \text{and} \quad \sum_{j=1}^{d'} a_jb_{j,k} = 0$$

for all $i=1,\dots,d'$ and for $k=1,\dots,m$. Since $\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V'$,

$$w_j = \sum_{k=1}^{d'} c_{j,k}v_k$$

for some sequence $(c_{j,k},\ j=1,\dots,m, i=1,\dots,d')$ in $\mathcal{O}_{X,x}$. Setting

$$u_{i,k} = \sum_{j=1}^{m} b_{i,j}c_{j,k}$$

for $i,k=1,\dots,d'$, we have

$$v_i = \sum_{k\in I} u_{i,k}v_k \quad \text{and} \quad \sum_{j\in I} a_ju_{j,i} = 0$$

for all $i=1,\dots,d'$. Since $v_1,\dots,v_{d'}$ are linearly independent modulo $\mathfrak{m}_x\mathrm{S}(E)$,

$$u_{i,k} - \delta_{i,k} \in \mathfrak{m}_x$$

for all (i,k), with $\delta_{i,k}$ the Kronecker symbol. As a result, a_i is in $\mathfrak{m}_x^{l+1}$ for all i, whence a contradiction. Then the canonical map

$$\mathcal{O}_{X,x}\otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

is bijective. Since $K = K(C)\mathrm{S}(E)$ and since $K(C)$ is the fraction field of $\mathcal{O}_{X,x}$, $v_1,\dots,v_{d'}$ is a basis of K over $K(C)$. Hence, $d' = d$ and the assertion follows. □

Recall that K_0 is the fraction field of $\overline{A}$. Let $v_{N+1},\dots,v_{N+r}$ be elements of E such that $v_1,\dots,v_{N+r}$ is a basis of E. Denoting by $t_1,\dots,t_r$ some indeterminates, let ϑ be the morphism of C-algebras

$$C[t_1,\dots,t_r] \longrightarrow \mathrm{S}(E), \qquad t_i \longmapsto v_{N+i},$$

and let $\tilde{\vartheta}$ be the morphism of $K_0[v_1,\dots,v_N]$-algebras

$$K_0[v_1,\dots,v_N,t_1,\dots,t_r] \longrightarrow K_0\otimes_{\overline{A}}\mathrm{S}(E), \qquad t_i \longmapsto v_{N+i}.$$

For $\mathbf{i} = (i_1, \dots, i_N)$ in $\mathbb{N}^N$ and for $\mathbf{j} = (j_1, \dots, j_r)$ in $\mathbb{N}^r$, set:

$$v^{\mathbf{i}} := v_1^{i_1} \cdots v_N^{i_N}, \qquad t^{\mathbf{j}} := t_1^{j_1} \cdots t_r^{j_r}.$$

For a in $\overline{A}$, denote by $\overline{a}$ the polynomial in $\Bbbk[v_1, \dots, v_N, t_1, \dots, t_r]$ such that $\vartheta(\overline{a}) = a$.

Lemma 2.18. *Let I be the ideal of $C[t_1, \dots, t_r]$ generated by the elements $a - \overline{a}$ with a in $\overline{A}$.*

(i) *For all homogenous generating family $(a_1, \dots, a_m)$ of $\overline{A}_+$, I is the ideal generated by the sequence $(a_i - \overline{a_i}\,,\, i = 1, \dots, m)$.*

(ii) *The ideal I is the kernel of ϑ.*

Proof. (i) Let I' be the ideal of $C[t_1, \dots, t_r]$ generated by the sequence $(a_i - \overline{a_i}, i = 1, \dots, m)$. Since the map $a \mapsto \overline{a}$ is linear, it suffices to prove that $a - \overline{a}$ is in I' for all homogenous element a of $\overline{A}_+$. Prove it by induction on the degree of a. For some homogenous sequence $(b_1, \dots, b_m)$ in $\overline{A}$,

$$a = b_1 a_1 + \cdots + b_m a_m$$

so that

$$a - \overline{a} = \sum_{i=1}^{m} b_i(a_i - \overline{a_i}) + \sum_{i=1}^{m} \overline{a_i}(b_i - \overline{b_i}).$$

If a has minimal degree, $b_1, \dots, b_m$ are in $\Bbbk$ and $b_i = \overline{b_i}$ for $i = 1, \dots, m$. Otherwise, for $i = 1, \dots, m$, if b_i is not in $\Bbbk$, b_i has degree smaller than a, whence the assertion by induction hypothesis.

(ii) By definition, I is contained in the kernel of ϑ. Let a be in $C[t_1, \dots, t_r]$. Then a has an expansion

$$a = \sum_{(\mathbf{i},\mathbf{j}) \in \mathbb{N}^N \times \mathbb{N}^r} a_{\mathbf{i},\mathbf{j}} v^{\mathbf{i}} t^{\mathbf{j}}$$

with the $a_{\mathbf{i},\mathbf{j}}$'s in $\overline{A}$, whence

$$a = \sum_{(\mathbf{i},\mathbf{j}) \in \mathbb{N}^N \times \mathbb{N}^r} (a_{\mathbf{i},\mathbf{j}} - \overline{a_{\mathbf{i},\mathbf{j}}}) v^{\mathbf{i}} t^{\mathbf{j}} + \sum_{(\mathbf{i},\mathbf{j}) \in \mathbb{N}^N \times \mathbb{N}^r} \overline{a_{\mathbf{i},\mathbf{j}}} v^{\mathbf{i}} t^{\mathbf{j}}.$$

If $\vartheta(a) = 0$, then

$$\sum_{(\mathbf{i},\mathbf{j}) \in \mathbb{N}^N \times \mathbb{N}^r} \overline{a_{\mathbf{i},\mathbf{j}}} v^{\mathbf{i}} t^{\mathbf{j}} = 0$$

since the restriction of ϑ to $\Bbbk[v_1, \dots, v_N, t_1, \dots, t_r]$ is injective, whence the assertion. □

For x in $\pi^{-1}(C_+)$, denote by ϑ_x the morphism

$$\mathcal{O}_{X,x}[t_1, \dots, t_r] \longrightarrow K, \qquad a t^{\mathbf{j}} \longmapsto a v_{N+1}^{j_1} \cdots v_{N+r}^{j_r}.$$

Proposition 2.19. *Let x be in $\pi^{-1}(C_+)$.*

(i) *The kernel of ϑ_x is the ideal of $\mathcal{O}_{X,x}[t_1, \dots, t_r]$ generated by I. Furthermore, the image of ϑ_x is the subalgebra $\mathcal{O}_{X,x}\mathrm{S}(E)$ of K.*

(ii) *The intersection of $\mathfrak{m}_x\mathrm{S}(E)$ and $\mathrm{S}(E)$ is equal to $C_+\mathrm{S}(E)$.*

Proof. (i) From the short exact sequence

$$0 \longrightarrow I \longrightarrow C[t_1, \dots, t_r] \longrightarrow \mathrm{S}(E) \longrightarrow 0$$

we deduce the exact sequence

$$\mathcal{O}_{X,x} \otimes_C I \longrightarrow \mathcal{O}_{X,x} \otimes_C C[t_1, \dots, t_r] \longrightarrow \mathcal{O}_{X,x} \otimes_C \mathrm{S}(E) \longrightarrow 0.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
\mathcal{O}_{X,x} \otimes_C I & \xrightarrow{\mathrm{d}} & \mathcal{O}_{X,x} \otimes_C C[t_1, \ldots, t_r] & \xrightarrow{\mathrm{d}} & \mathcal{O}_{X,x} \otimes_C \mathrm{S}(E) & \longrightarrow & 0 \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\mathcal{O}_{X,x} I & \xrightarrow{\mathrm{d}} & \mathcal{O}_{X,x}[t_1, \ldots, t_r] & \xrightarrow{\mathrm{d}} & \mathcal{O}_{X,x}\mathrm{S}(E) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

with exact columns by Corollary 2.16,(ii). For a in $\mathcal{O}_{X,x}[t_1, \ldots, t]$ such that $\mathrm{d}a = 0$,

$$a = \delta b, \quad b = \mathrm{d}c \quad \text{with} \quad b \in \mathcal{O}_{X,x} \otimes_C C[t_1, \ldots, t_r], \quad c \in \mathcal{O}_{X,x} \otimes_C I,$$

so that $a = \mathrm{d}\circ\delta c$. Hence $\mathcal{O}_{X,x} I$ is the kernel of ϑ_x.

(ii) Let $a_1, \ldots, a_m$ be a homogenous generating family of $\overline{A}_+$. For $i = 1, \ldots, m$,

$$\overline{a_i} = \sum_{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^N \times \mathbb{N}^r} a_{i,\mathbf{j},\mathbf{k}} v^{\mathbf{j}} t^{\mathbf{k}},$$

with the $a_{i,\mathbf{j},\mathbf{k}}$'s in $\Bbbk$. Set:

$$a'_i := \sum_{\mathbf{k} \in \mathbb{N}^N} a_{i,0,\mathbf{k}} t^{\mathbf{k}}.$$

For $i = 1, \ldots, m$,

$$a'_i \in \overline{a_i} - a_i + C_+[t_1, \ldots, t_r]$$

since a_i is in $\overline{A}_+$ so that $\vartheta(a'_i)$ is in $C_+\mathrm{S}(E)$.

Since C_+ is contained in $\mathfrak{m}_x$, $C_+\mathrm{S}(E)$ is contained in $\mathfrak{m}_x\mathrm{S}(E) \cap \mathrm{S}(E)$. Let a be in $\mathfrak{m}_x[t_1, \ldots, t_r]$ such that $\vartheta_x(a)$ is in $\mathrm{S}(E)$. According to (i),

$$a \in C[t_1, \ldots, t_r] + \mathcal{O}_{X,x} I.$$

So, by Lemma 2.17,(i),

$$a = b + b_1(a_1 - \overline{a_1}) + \cdots + b_m(a_m - \overline{a_m}),$$

with b in $C[t_1, \ldots, t_r]$ and $b_1, \ldots, b_m$ in $\mathcal{O}_{X,x}$. Then,

$$b = b_0 + b_+, \quad \text{with} \quad b_0 \in \Bbbk[t_1, \ldots, t_r] \quad \text{and} \quad b_+ \in C_+[t_1, \ldots, t_r]$$

$$b_i = b_{i,0} + b_{i,+}, \quad \text{with} \quad b_{i,0} \in \Bbbk \quad \text{and} \quad b_{i,+} \in \mathfrak{m}_x$$

for $i = 1, \ldots, m$. Since a is in $\mathfrak{m}_x[t_1, \ldots, t_r]$ and $a_1, \ldots, a_m$ are in C_+,

$$b_0 - b_{1,0}\overline{a_1} - \cdots - b_{m,0}\overline{a_m} \in \mathfrak{m}_x[t_1, \ldots, t_r]$$

Moreover, for $i = 1, \ldots, m$,

$$\overline{a_i} - a'_i \in C_+[t_1, \ldots, t_r].$$

Hence

$$b_0 - b_{1,0}a'_1 - \cdots - b_{m,0}a'_m = 0 \quad \text{since} \quad \mathfrak{m}_x[t_1, \ldots, t_r] \cap \Bbbk[t_1, \ldots, t_r] = 0.$$

As a result, $\vartheta_x(a)$ is in $C_+\mathrm{S}(E)$ since $\vartheta_x(a) = \vartheta_x(b_0) + \vartheta_x(b_+)$. □

2.4. We are now in a position to prove the main result of the section. Recall the main notations: E is a finite dimensional vector space over $\Bbbk$, A is a homogenous subalgebra of $\mathrm{S}(E)$, different from $\mathrm{S}(E)$ and such that $A = \Bbbk + A_+$, $\mathcal{N}_0$ is the nullvariety of A_+ in E^*, K is the fraction field of $\mathrm{S}(E)$ and $K(\overline{A})$ that one of $\overline{A}$, the algebraic closure of A in $\mathrm{S}(E)$.

Theorem 2.20. *Suppose that the following conditions are satisfied:*

(a) *$\mathcal{N}_0$ has dimension N,*
(b) *A is a polynomial algebra,*
(c) *$K(\overline{A})$ is algebraically closed in K.*

Then $\overline{A}$ is a polynomial algebra. Moreover, $\mathrm{S}(E)$ is a free extension of $\overline{A}$.

Proof. Use the notations of Subsection 2.3. In particular, set

$$C = \overline{A}[v_1, \dots, v_N],$$

with $(v_1, \dots, v_N)$ a sequence of elements of E such that its nullvariety in $\mathcal{N}_0$ is equal to $\{0\}$ (cf. Lemma 2.1,(iii)), and let $K(C)$ be the fraction field of C. As already explained, according to Proposition 2.5,(ii), it suffices to prove that $\mathrm{S}(E)$ is a free extension of C. Let V be as in Lemma 2.18, a homogenous complement to $\mathrm{S}(E)C_+$ in $\mathrm{S}(E)$. Recall that X is a desingularization of $Z = \mathrm{Specm}(C)$ and that π is the morphism of desingularization. Let x be in $\pi^{-1}(C_+)$. According to Proposition 2.19,(ii), for some subspace V' of V, V' is a complement to $\mathfrak{m}_x\mathrm{S}(E)$ in $\mathcal{O}_{X,x}\mathrm{S}(E)$. Then, by Lemma 2.17,(ii), V' has dimension the degree of the extension K of $K(C)$ and the canonical map

$$\mathcal{O}_{X,x} \otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

is bijective. Moreover,

$$V' \oplus \mathrm{S}(E)C_+ = \mathrm{S}(E) \quad \text{and} \quad V' = V.$$

Indeed, for $a \in \mathrm{S}(E)$, write $a = b + c$ with $b \in V'$ and $c \in \mathfrak{m}_x\mathrm{S}(E)$. Since V' is contained in $\mathrm{S}(E)$, c is in $\mathrm{S}(E)$, whence c in $\mathrm{S}(E)C_+$ by Proposition 2.19,(ii). In addition, $\mathrm{S}(E) = CV$ as it has been observed in the proof of Lemma 2.17,(i). As a result, the canonical map

$$C \otimes_{\Bbbk} V \longrightarrow \mathrm{S}(E)$$

is bijective. This concludes the proof of the theorem. □

3. Good elements and good orbits

Recall that $\Bbbk$ is an algebraically closed field of characteristic zero. As in the introduction, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ of rank ℓ, $\langle .\, , . \rangle$ denotes the Killing form of $\mathfrak{g}$, and G denotes the adjoint group of $\mathfrak{g}$.

3.1. The notions of good element and good orbit in $\mathfrak{g}$ are introduced in this paragraph.

For x in $\mathfrak{g}$, denote by $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$, by G^x its stabilizer in G, by G_0^x the identity component of G^x and by K_x the fraction field of the symmetric algebra $\mathrm{S}(\mathfrak{g}^x)$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and $K_x^{\mathfrak{g}^x}$ denote the sets of G_0^x-invariant elements of $\mathrm{S}(\mathfrak{g}^x)$ and K_x respectively.

Lemma 3.1. *Let x be in $\mathfrak{g}$. Then $K_x^{\mathfrak{g}^x}$ is the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and $K_x^{\mathfrak{g}^x}$ is algebraically closed in K_x of transcendental degree ℓ over $\Bbbk$.*

Proof. Let a be in K_x, algebraic over $K_x^{\mathfrak{g}^x}$. For all g in G_0^x, $g.a$ satisfies the same equation of algebraic dependence over $K_x^{\mathfrak{g}^x}$ as a. Since a polynomial in one indeterminate has a finite number of roots, the G_0^x-orbit of a is finite. But this orbit is then reduced to $\{a\}$, G_0^x being connected. Hence a is in $K_x^{\mathfrak{g}^x}$. This shows that $K_x^{\mathfrak{g}^x}$ is algebraically closed in K_x. According to [CMo10, Theorem 1.2] (see also Theorem 1.1), the index of $\mathfrak{g}^x$ is equal to ℓ. So, by [R63], the transcendental degree of $K_x^{\mathfrak{g}^x}$ over $\Bbbk$ is equal to ℓ. It remains to prove that $K_x^{\mathfrak{g}^x}$ is the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.

Since $\mathfrak{g}^x$ is the centralizer of x_{n} in the reductive Lie algebra $\mathfrak{g}^{x_{\mathrm{s}}}$, we can suppose x nilpotent. Any rational invariant is a quotient of two semi-invariant polynomials, because of the prime factor decomposition. Each semi-invariant has a central character λ, a character of the center of a Levi subalgebra in $\mathfrak{g}^x$. By [JS10, Lemma 4.6,(i)], there is also a semi-invariant with the character $-\lambda$. Multiplying both numerator and denominator by this invariant, we get the same invariant as a quotient of invariants, whence the lemma. □

Definition 3.2. An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some homogenous elements $p_1, \ldots, p_\ell$ of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$. A G-orbit in $\mathfrak{g}$ is called *good* if it is the orbit of a good element.

Since the nullvariety of $\mathrm{S}(\mathfrak{g})_+^{\mathfrak{g}}$ in $\mathfrak{g}$ is the nilpotent cone of $\mathfrak{g}$, 0 is a good element of $\mathfrak{g}$. For (g, x) in $G \times \mathfrak{g}$ and for a in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $g(a)$ is in $\mathrm{S}(\mathfrak{g}^{g(x)})^{\mathfrak{g}^{g(x)}}$. So, an orbit is good if and only if all its elements are good.

Theorem 3.3. *Let x be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Proof. Let $p_1, \ldots, p_\ell$ be homogenous elements of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ such that the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ. Denote by A the subalgebra of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ generated by $p_1, \ldots, p_\ell$. Then A is a homogenous subalgebra of $\mathrm{S}(\mathfrak{g}^x)$ and the nullvariety of A_+ in $(\mathfrak{g}^x)^*$ has codimension ℓ. So, by Lemma 2.1,(ii), A has dimension ℓ. Hence, $p_1, \ldots, p_\ell$ are algebraically independent and A is a polynomial algebra. Denote by $\overline{A}$ the algebraic closure of A in $\mathrm{S}(\mathfrak{g}^x)$. By Lemma 3.1, $\overline{A}$ is contained in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is algebraically closed in K_x. As a matter of fact, $\overline{A} = \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ since the fraction fields of A and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ have the same transcendental degree. Hence, by Theorem 2.20, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. □

Remark 3.4. The algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ may be polynomial even though x is not good. Indeed, let us consider a nilpotent element e of $\mathfrak{g} = \mathfrak{so}(\Bbbk^{10})$ in the nilpotent orbit associated with the partition $(3, 3, 2, 2)$. Then the algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, generated by elements of degrees $1, 1, 2, 2, 5$. But the nullcone has an irreducible component of codimension at most 4. So, e is not good. We refer the reader to Example 7.5 for more details.

For $x \in \mathfrak{g}$, denote by x_{s} and x_{n} the semisimple and the nilpotent components of x respectively.

Proposition 3.5. *Let x be in $\mathfrak{g}$. Then x is good if and only if x_{n} is a good element of the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$.*

Proof. Let $\mathfrak{z}$ be the center of $\mathfrak{g}^{x_{\mathrm{s}}}$ and let $\mathfrak{a}$ be the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$. Then

$$\mathfrak{g}^x = \mathfrak{z} \oplus \mathfrak{a}^{x_{\mathrm{n}}}, \qquad \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{z}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}.$$

By the first equality, $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ identifies with the orthogonal complement to $\mathfrak{z}$ in $(\mathfrak{g}^x)^*$. Set $d := \dim \mathfrak{z}$. Suppose that x_{n} is a good element of $\mathfrak{a}$ and let $p_1, \ldots, p_{\ell-d}$ be homogenous elements of $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$ whose nullvariety in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Denoting by $v_1, \ldots, v_d$ a basis of $\mathfrak{z}$, the nullvariety of $v_1, \ldots, v_d, p_1, \ldots, p_{\ell-d}$ in $(\mathfrak{g}^x)^*$ is the nullvariety of $p_1, \ldots, p_{\ell-d}$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$. Hence, x is a good element of $\mathfrak{g}$.

Conversely, let us suppose that x is a good element of $\mathfrak{g}$. By Theorem 3.3, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra generated by homogenous polynomials $p_1, \ldots, p_\ell$. Since $\mathfrak{z}$ is contained in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $p_1, \ldots, p_\ell$ can be chosen so that $p_1, \ldots, p_d$ are in $\mathfrak{z}$ and $p_{d+1}, \ldots, p_\ell$ are in $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$. Then the nullvariety of $p_{d+1}, \ldots, p_\ell$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Hence, x_{n} is a good element of $\mathfrak{a}$. □

3.2. In view of Theorem 3.3, we wish to find a sufficient condition for that an element $x \in \mathfrak{g}$ is good. According to Proposition 3.5, it is enough to consider the case where x is nilpotent.

Let e be a nilpotent element of $\mathfrak{g}$, embedded into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Identify the dual of $\mathfrak{g}$ with $\mathfrak{g}$, and the dual of $\mathfrak{g}^e$ with $\mathfrak{g}^f$ through the Killing form $\langle . , . \rangle$ of $\mathfrak{g}$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}]$, denote by $\kappa(p)$ the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e+x)$ and denote by ${}^e p$ its initial homogenous component. According to [PPY07, Proposition 0.1], for p in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

The proof of the following theorem will be achieved in Subsection 4.4.

Theorem 3.6. *Suppose that for some homogenous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, $({}^e q_1, \dots, {}^e q_\ell)$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

The overall idea of the proof is the following.

According to Theorem 3.3, it suffices to prove that e is good, and more accurately that the nullvariety of ${}^e q_1, \dots, {}^e q_\ell$ in $\mathfrak{g}^f$ has codimension ℓ since ${}^e q_1, \dots, {}^e q_\ell$ are invariant homogenous polynomials. As explained in the introduction, we will use the Slodowy grading on $\mathrm{S}(\mathfrak{g}^e)[[t]]$ and $\mathrm{S}(\mathfrak{g}^e)((t))$, induced from that on $\mathrm{S}(\mathfrak{g}^e)$, to deal with this problem. This is the main purpose of Section 4.

4. Slodowy grading and proof of Theorem 1.5

This section is devoted to the proof of Theorem 3.6 (or Theorem 1.5). The proof is achieved in Subsection 4.5. As in the previous section, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Let us simply denote by S the algebra $\mathrm{S}(\mathfrak{g}^e)$.

Let $q_1, \dots, q_\ell$ be homogenous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \dots, d_\ell$ respectively. The sequence $(q_1, \dots, q_\ell)$ is ordered so that $d_1 \leqslant \cdots \leqslant d_\ell$. We assume in the whole section that the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. The aim is to show that e is good (cf. Definition 3.2).

4.1. Let $x_1, \dots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \dots, r$, $[h, x_i] = n_i x_i$ for some nonnegative integer n_i. For $\mathbf{j} = (j_1, \dots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \qquad |\mathbf{j}|_e := j_1 n_1 + \cdots + j_r n_r + 2|\mathbf{j}|, \qquad x^{\mathbf{j}} = x_1^{j_1} \cdots x_r^{j_r}.$$

The algebra S has two gradings: the standard one and the *Slodowy grading*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogenous with respect to these two gradings. It has standard degree $|\mathbf{j}|$ and Slodowy degree $|\mathbf{j}|_e$. In this section, we only consider the Slodowy grading. So, by grading we will always mean Slodowy grading. For m nonnegative integer, denote by $S^{[m]}$ the subspace of S of degree m.

Let t be an indeterminate. For all subspace V of S, set:

$$V[t] := \Bbbk[t] \otimes_{\Bbbk} V, \qquad V[t, t^{-1}] := \Bbbk[t, t^{-1}] \otimes_{\Bbbk} V, \qquad V[[t]] := \Bbbk[[t]] \otimes_{\Bbbk} V, \qquad V((t)) := \Bbbk((t)) \otimes_{\Bbbk} V,$$

with $\Bbbk((t))$ the fraction field of $\Bbbk[[t]]$. For V a subspace of $S[[t]]$, denote by $V(0)$ the image of V by the quotient morphism

$$S[t] \longrightarrow S, \qquad a(t) \longmapsto a(0).$$

The grading of S induces a grading of the algebra $S((t))$ with t having degree 0. For V a homogenous subspace of $S((t))$ and for m a nonnegative integer, let $V^{[m]}$ be its component of degree m. In particular, for V a homogenous subspace of S, $V((t))$ is a homogenous subspace of $S((t))$ and

$$V((t))^{[m]} = V^{[m]}((t)).$$

Let τ be the morphism of algebras,

$$\tau\colon S \longrightarrow S[t], \quad x_i \mapsto tx_i \quad \text{for} \quad i = 1,\dots,r.$$

The morphism τ is a morphism of homogenous algebras. Denote by $\delta_1,\dots,\delta_\ell$ the standard degrees of ${}^eq_1,\dots,{}^eq_\ell$ respectively, and set for $i = 1,\dots,\ell$

$$Q_i := t^{-\delta_i}\tau(\kappa(q_i)).$$

Let A be the subalgebra of $S[t]$ generated by $Q_1,\dots,Q_\ell$. Then observe that $A(0)$ is the subalgebra of S generated by ${}^eq_1,\dots,{}^eq_\ell$. For $\mathbf{j} = (j_1,\dots,j_\ell)$ in $\mathbb{N}^\ell$, set

$$q^{\mathbf{j}} := q_1^{j_1}\cdots q_\ell^{j_\ell}, \qquad \kappa(q)^{\mathbf{j}} := \kappa(q_1^{j_1})\cdots\kappa(q_\ell^{j_\ell}), \qquad {}^eq^{\mathbf{j}} := {}^eq_1^{j_1}\cdots {}^eq_\ell^{j_\ell}, \qquad Q^{\mathbf{j}} := Q_1^{j_1}\cdots Q_\ell^{j_\ell}.$$

Proposition 4.1. (i) *For $\mathbf{j}$ in $\mathbb{N}^\ell$, $\kappa(q)^{\mathbf{j}}$ and ${}^eq^{\mathbf{j}}$ are homogenous of degree $2d_1j_1 + \cdots + 2d_\ell j_\ell$.*

(ii) *The map $Q \mapsto Q(0)$ is an isomorphism of homogenous algebras from A onto $A(0)$.*

Proof. (i) follows from [Pr02, Section 5] or [PPY07, Section 2].

(ii) The set $(Q^{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^\ell)$ is a basis of the $\Bbbk$-space A and the image of $Q^{\mathbf{j}}$ by the map $Q \mapsto Q(0)$ is equal to ${}^eq^{\mathbf{j}}$. Moreover, by (i), $Q^{\mathbf{j}}$ and ${}^eq^{\mathbf{j}}$ are homogenous of degree $2d_1j_1 + \cdots + 2d_\ell j_\ell$ so that $Q \mapsto Q(0)$ is a morphism of graded algebras. By definition, its image is $A(0)$. Since ${}^eq_1,\dots,{}^eq_\ell$ are algebraically independent, it is injective. □

By Proposition 4.1,(ii), A and $A(0)$ are isomorphic homogenous subalgberas of $S[t]$ and S respectively. In particular, A is a polynomial algebra since $A(0)$ is polynomial by our hypothesis.

Denote by A_+ and $A(0)_+$ the ideals of A and $A(0)$ generated by the homogenous elements of positive degree respectively, and denote by $\tilde{A}$ the subalgebra of $S[[t]]$ generated by $\Bbbk[[t]]$ and A, i.e.,

$$\tilde{A} := \Bbbk[[t]]A.$$

Lemma 4.2. (i) *The algebra $\tilde{A}$ is isomorphic to $\Bbbk[[t]] \otimes_\Bbbk A$. In particular, it is regular.*

(ii) *The element t of $\tilde{A}$ is prime.*

(iii) *Each prime element of A is a prime element of $\tilde{A}$.*

Proof. (i) Let $a_m, m \in \mathbb{N}$, be in A such that

$$\sum_{m\in\mathbb{N}} t^m a_m = 0.$$

If $a_m \neq 0$ for some m, then $a_p(0) = 0$ if p is the smallest one such that $a_p \neq 0$. By Proposition 4.1,(ii), it is not possible. Hence, the canonical map

$$\Bbbk[[t]] \otimes_\Bbbk A \longrightarrow \tilde{A}$$

is an isomorphism. As observed just above, A is a polynomial algebra. Then $\tilde{A}$ is a regular algebra by [Ma86, Ch. 7, Theorem 19.5].

(ii) By (i), A is the quotient of $\tilde{A}$ by $t\tilde{A}$ so that t is a prime element of $\tilde{A}$.

(iii) By (i), for a in A, the quotient $\tilde{A}/\tilde{A}a$ is isomorphic to $\Bbbk[[t]] \otimes_\Bbbk A/Aa$. Hence a is a prime element of $\tilde{A}$ if it is a prime element of A. □

As it has been explained in Subsection 3.2, in order to prove Theorem 3.6, we aim to prove that S is a free extension that $A(0)$. As a first step, we describe in Subsections 4.2, 4.3 and 4.4 some properties of the algebra A. We show in Subsection 4.3 that $S((t))$ is a free extension of A (cf. Proposition 4.9,(iii)), and we show in Subsection 4.4 that $S[[t]]$ is a free extension of A (cf. Corollary 4.17). In Subsection 4.5, we consider the algebra $\tilde{A}$ and prove that $S[[t]]$ is a free extension of $\tilde{A}$ (cf. Theorem 4.21,(i)). The expected result will follow from this (cf. Theorem 4.21,(iii)).

4.2. Let θ_e be the map

$$G \times (e + \mathfrak{g}^f) \longrightarrow \mathfrak{g}, \qquad (g, x) \mapsto g(x),$$

and let $\mathfrak{J}_e$ be the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by the elements $\kappa(q_1), \ldots, \kappa(q_\ell)$. The following lemma is known by [Pr02, Theorem 5.4] and the proof of [PPY07, Theorem 2.1].

Lemma 4.3. (i) *The map θ_e is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G.e$.*

(ii) *The nullvariety of $\mathfrak{J}_e$ in $\mathfrak{g}^f$ is equidimensional of dimension $r - \ell$.*

(iii) *The ideal $\mathfrak{J}_e$ of $\mathrm{S}(\mathfrak{g}^e)$ is radical.*

Denote by $\mathcal{V}$ the nullvariety of A_+ in $\mathfrak{g}^f \times \Bbbk$, and by $\mathcal{V}_0$ the nullvariety of $A(0)_+$ in $\mathfrak{g}^f$. Then denote by $\mathcal{V}_*$ the union of the irreducible components of $\mathcal{V}$ which are not contained in $\mathfrak{g}^f \times \{0\}$. Note that $\mathcal{V}_0 \times \{0\}$ is the nullvariety of t in $\mathcal{V}$, and that

$$\mathcal{V} = \mathcal{V}_* \cup \mathcal{V}_0 \times \{0\}.$$

Corollary 4.4. (i) *The variety $\mathcal{V}_*$ is equidimensional of dimension $r + 1 - \ell$. Moreover, for X an irreducible component of $\mathcal{V}_*$ and for z in $\Bbbk$, the nullvariety of $t - z$ in X has dimension $r - \ell$.*

(ii) *The algebra $S[t, t^{-1}]$ is a free extension of A.*

(iii) *The ideal $S[t, t^{-1}]A_+$ of $S[t, t^{-1}]$ is radical.*

Proof. (i) Let $\mathcal{V}'_*$ be the intersection of $\mathcal{V}_*$ and $\mathfrak{g}^f \times \Bbbk^*$ and let X be an irreducible component of $\mathcal{V}'_*$. Then $\mathcal{V}'_*$ is the nullvariety of $Q_1, \ldots, Q_\ell$ in $\mathfrak{g}^f \times \Bbbk^*$ since A_+ is the ideal of A generated by $Q_1, \ldots, Q_\ell$. In particular, X has dimension at least $r + 1 - \ell$. For z in $\Bbbk^*$, denote by X_z the subvariety of $\mathfrak{g}^f$ such that $X_z \times \{z\} = X \cap \mathfrak{g}^f \times \{z\}$. By definition, for $i = 1, \ldots, \ell$, $Q_i = t^{-\delta_i} \tau \circ \kappa(q_i)$. Hence $\mathcal{V}'_*$ is the nullvariety of $\tau \circ \kappa(q_1), \ldots, \tau \circ \kappa(q_\ell)$ in $\mathfrak{g}^f \times \Bbbk^*$ and X_z is the image of X_1 by the homothety $v \mapsto z^{-1} v$. By Lemma 4.3,(ii), X_1 has dimension $r - \ell$. Hence X_z has dimension $r - \ell$ and X has dimension at most $r + 1 - \ell$. As a result, X has dimension $r + 1 - \ell$ and X_z is strictly contained in X, whence the assertion since X is not contained in $\mathfrak{g}^f \times \{0\}$ by definition.

(ii) The algebra $S[t, t^{-1}]$ is graded and A is a homogenous polynomial subalgebra of $S[t, t^{-1}]$. According to (i), the fiber at A_+ of the extension $S[t, t^{-1}]$ of A is equidimensional of dimension $r + 1 - \ell$. Hence, by Proposition 2.5, $S[t, t^{-1}]$ is a free extension of A.

(iii) Let $\mathcal{I}_e$ be the ideal of $S[t, t^{-1}]$ generated by $\tau \circ \kappa(q_1), \ldots, \tau \circ \kappa(q_\ell)$. Since $t^{\delta_i} Q_i = \tau \circ \kappa(q_i)$ for $i = 1, \ldots, \ell$, we get $\mathcal{I}_e = S[t, t^{-1}]A_+$. Denote by $\overline{\tau}$ the endomorphism of the algebra $S[t, t^{-1}]$ defined by

$$\overline{\tau}(t) = t, \quad \overline{\tau}(x_1) = t x_1, \ldots, \overline{\tau}(x_r) = t x_r.$$

Then $\overline{\tau}$ is an automorphism and $\mathcal{I}_e = \overline{\tau}(S[t, t^{-1}]\mathfrak{J}_e)$. So, it suffices to prove that the ideal $S[t, t^{-1}]\mathfrak{J}_e$ is radical.

Let $\mathfrak{J}'_e$ be the radical of $S[t, t^{-1}]\mathfrak{J}_e$. For a in $S[t, t^{-1}]$, a has a unique expansion

$$a = \sum_{m \in \mathbb{Z}} t^m a_m$$

with $(a_m, m \in \mathbb{Z})$ a sequence of finite support in S. Denote by $\nu(a)$ the cardinality of this finite support. Moreover, a is in $S[t, t^{-1}]\mathfrak{J}_e$ if and only if a_m is in $\mathfrak{J}_e$ for all m. Suppose that $S[t, t^{-1}]\mathfrak{J}_e$ is strictly contained in $\mathfrak{J}'_e$. A contradiction is expected. Let a be in $\mathfrak{J}'_e \setminus S[t, t^{-1}]\mathfrak{J}_e$ such that $\nu(a)$ is minimal. Denote by m_0 the smallest integer such that $a_{m_0} \neq 0$. For some positive integer, a^k and $(t^{-m_0} a)^k$ are in $S[t, t^{-1}]\mathfrak{J}_e$ and we have

$$(t^{-m_0} a)^k = a_{m_0}^k + \sum_{m > 0} t^m b_m$$

with the b_m's in $\mathfrak{J}_e$. Then $a_{m_0}^k$ is in $\mathfrak{J}_e$ and by Lemma 4.3,(iii), a_{m_0} is in $\mathfrak{J}_e$. As a result $a' := a - t^{m_0} a_{m_0}$ is an element of $\mathfrak{J}'_e$ such that $\nu(a') < \nu(a)$. By the minimality of $\nu(a)$, a' is in $S[t, t^{-1}]\mathfrak{J}_e$ and so is a, whence the contradiction. ∎

Let $\mathcal{I}_*$ be the ideal of definition of $\mathcal{V}_*$ in $S[t]$. Then $\mathcal{I}_*$ is an ideal of $S[t]$ containing the radical of $S[t]A_+$. It will be shown that $\mathcal{V}_* = \mathcal{V}$ and that $S[t]A_+$ is radical (cf. Theorem 4.21). Thus, $\mathcal{I}_*$ will be at the end equal to $S[t]A_+$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals containing $S[t]A_+$ and let $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ be the primary decomposition of $S[t]A_+$ such that $\mathfrak{p}_i$ is the radical of $\mathfrak{q}_i$ for $i = 1, \dots, m$.

Lemma 4.5. (i) *For a in $S[t]$, a is in $\mathcal{I}_*$ if and only if $t^m a$ is in $S[t]A_+$ for some positive integer m. Moreover, for some nonnegative integer l, $t^l\mathcal{I}_*$ is contained in $S[t]A_+$.*

(ii) *The ideal $\mathcal{I}_*$ is the intersection of the prime ideals $\mathfrak{p}_i$ which do not contain t. Furthermore, for such i, $\mathfrak{p}_i = \mathfrak{q}_i$, i.e. $\mathfrak{q}_i$ is radical.*

Proof. (i) Let a be in $S[t]$. If $t^l a$ is in $S[t]A_+$ for some positive integer l, then a is equal to 0 on $\mathcal{V}_*$ so that a is in $\mathcal{I}_*$. Conversely, if a is in $\mathcal{I}_*$, then ta is in the radical of $S[t]A_+$ since $\mathcal{V}$ is contained in the union of $\mathcal{V}_*$ and $\mathfrak{g}^f \times \{0\}$. According to Corollary 4.4,(iii), for some positive integer m, $t^m(ta)$ is in $S[t]A_+$. Since $\mathcal{I}_*$ is finitely generated as an ideal of $S[t]$, we deduce that for some nonnegative integer l, $t^l\mathcal{I}_*$ is contained in $S[t]A_+$, whence the assertion.

(ii) Let $i \in \{1, \dots, m\}$. Then $\mathfrak{p}_i$ does not contain t if and only if the nullvariety of $\mathfrak{p}_i$ in $\mathfrak{g}^f \times \Bbbk$ is an irreducible component of $\mathcal{V}_*$, whence the first part of the statement.

By (i), for some nonnegative integer l, $t^l\mathcal{I}_*$ is contained in $S[t]A_*$. Let l be the minimal nonnegative integer satisfying this condition. If $l = 0$, $\mathcal{I}_* = S[t]A_+$, whence the assertion. Suppose l positive. Denote by $\mathcal{I}'_*$ the ideal of $S[t]$ generated by t^l and $S[t]A_+$. It suffices to prove that $S[t]A_+$ is the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$. As a matter of fact, if so, the primary decomposition of $S[t]A_+$ is the union of the primary decompositions of $\mathcal{I}_*$ and $\mathcal{I}'_*$ since the minimal prime ideals containing $\mathcal{I}_*$ do not contain t.

Let a be in the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$. Then

$$a = t^l b + \sum_{i=1}^{\ell} a_i Q_i$$

with $b, a_1, \dots, a_l$ in $S[t]$. Since $S[t]A_+$ is contained in $\mathcal{I}_*$, $t^l b$ is in $\mathcal{I}_*$ and b is in $\mathcal{I}_*$ by (i). Hence $t^l b$ and a are in $S[t]A_+$. As a result, $S[t]A_+$ is the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$ since $S[t]A_+$ is contained in this intersection. □

4.3. Let V_0 be a homogenous complement to $SA(0)_+$ in S. We will show that the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism (cf. Theorem 4.21).

Lemma 4.6. *We have $S[[t]] = V_0[[t]] + S[[t]]A_+$ and $S((t)) = V_0((t)) + S((t))A_+$.*

Proof. The equality $S((t)) = V_0((t)) + S((t))A_+$ will follow from the equality $S[[t]] = V_0[[t]] + S[[t]]A_+$. Since $S[[t]]$, $V_0[[t]]$ and $S[[t]]A_+$ are homogenous, it suffices to show that for d a positive integer,

$$S[[t]]^{[d]} \subset V_0[[t]]^{[d]} + (S[[t]]A_+)^{[d]},$$

the inclusion $V_0[[t]] + S[[t]]A_+ \subset S[[t]]$ being obvious.

Let d be a positive integer and let a be in $S[[t]]^{[d]}$. Let $(\varphi_1, \dots, \varphi_m)$ be a basis of the $\Bbbk[[t]]$-module $(S[[t]]A_+)^{[d]}$. Such a basis does exist since $\Bbbk[[t]]$ is a principal ring and $S[[t]]^{[d]}$ is a finite free $\Bbbk[[t]]$-module. Then $\varphi_1(0), \dots, \varphi_m(0)$ generate $(SA(0)_+)^{[d]}$. Since $S^{[d]} = V_0^{[d]} \oplus (SA(0)_+)^{[d]}$,

$$a - a_0 - \sum_{j=1}^{m} a_{0,j}\varphi_j = t\psi_0,$$

with a_0 in $V_0^{[d]}$, $a_{0,1}, \dots, a_{0,m}$ in $\Bbbk$ and $\psi_0 \in S[[t]]^{[d]}$. Suppose that there are sequences $(a_0, \dots, a_n)$ and $(a_{i,1}, \dots, a_{i,m})$, for $i = 0, \dots, n$, in $V_0^{[d]}$ and $\Bbbk$ respectively such that

$$a - \sum_{i=0}^{n} a_i t^i - \sum_{i=0}^{n} \sum_{j=1}^{m} t^i a_{i,j} \varphi_j = t^{n+1} \psi_n$$

for some ψ_n in $S[[t]]^{[d]}$. Then for some a_{n+1} in $V_0^{[d]}$ and $a_{n+1,1}, \dots, a_{n+1,m}$ in $\Bbbk$,

$$\psi_n - a_{n+1} - \sum_{j=1}^{m} a_{n+1,j} \varphi_j \in tS[[t]]$$

so that

$$a - \sum_{i=0}^{n+1} a_i t^i - \sum_{i=0}^{n+1} \sum_{j=1}^{m} a_{i,j} \varphi_j t^i \in t^{n+2} S[[t]].$$

As a result,

$$a \in V_0[[t]]^{[d]} + (S[[t]]A_+)^{[d]}$$

since $S[[t]]^{[d]}$ is a finite $\Bbbk[[t]]$-module. □

Recall that $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are the minimal prime ideals of $S[t]$ containing $S[t]A_+$. Since A_+ is a homogenous subspace of $S[t]$, $S[t]A_+$ is a homogenous ideal of $S[t]$, and so are $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. According to Lemma 4.5,(ii), $\mathfrak{I}_*$ is the intersection of the $\mathfrak{p}_i$'s which do not contain t. Hence, $\mathfrak{I}_*$ is homogenous. Thereby, $\mathfrak{I}_* \cap V_0[t]$ has a homogenous complement in $V_0[t]$. Set

$$W := \mathfrak{I}_* \cap V_0[t].$$

Then $W(0)$ is a homogenous subspace of V_0. Denote by V_0' a homogenous complement to $W(0)$ in V_0. Then set

$$V_0'' := W(0)$$

so that $V_0 = V_0' \oplus V_0''$.

Lemma 4.7. *Let $(v_i, i \in J)$ be a homogenous basis of V_0'.*

(i) *The elements $v_i, i \in J$, are linearly independent over $\Bbbk[t]$.*

(ii) *The sum of W and of $V_0'[t]$ is direct.*

Proof. We prove (i) and (ii) all together.

Let $(c_i, i \in J)$ be a sequence in $\Bbbk[t]$, with finite support J_c, such that

$$\sum_{i \in J} c_i v_i = w$$

for some w in W. Suppose that J_c is not empty. A contradiction is expected. Since V_0' is a complement to V_0'', $c_i(0) = 0$ for all i in J. Then, for i in J_c, $c_i = t^{m_i} c_i'$ with $m_i > 0$ and $c_i'(0) \neq 0$. Denote by m the smallest of the integers m_i, for $i \in J_c$. Then $w = t^m w'$ for some w' in $V_0[t]$, and

$$\sum_{i \in J_c} t^{m_i - m} c_i' v_i = w'.$$

According to Lemma 4.5,(i), w' is in $\mathfrak{I}_*$. So, $c_i'(0) = 0$ for i such that $m_i = m$, whence the contradiction. □

As a rule, for M a $\Bbbk[t]$-submodule of $S[t]$, we denote by $\widehat{M}$ the $\Bbbk[[t]]$-module generated by M, i.e.,

$$\hat{M} = \Bbbk[[t]]M.$$

Lemma 4.8. *Let M be a $\Bbbk[t]$-submodule of $S[t]$.*

(i) *Let a be in the intersection of $S[t]$ and $\widehat{M}$. For some q in $\Bbbk[t]$ such that $q(0) \neq 0$, qa is in M.*

(ii) *For N a $\Bbbk[t]$-submodule of $S[t]$, the intersection of $\widehat{M}$ and $\widehat{N}$ is the $\Bbbk[[t]]$-module generated by $M \cap N$.*

Proof. (i) Denote by $\overline{a}$ the image of a in $S[t]/M$ by the quotient map and by J its annihilator in $\Bbbk[t]$. Then we have a commutative diagram with exact lines and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\mathrm{d}} & S[t] & \xrightarrow{\mathrm{d}} & S[t]/M & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & J & \xrightarrow{\mathrm{d}} & \Bbbk[t] & \xrightarrow{\mathrm{d}} & \Bbbk[t]\overline{a} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, tensoring this diagram by $\Bbbk[[t]]$ gives the following diagram with exact lines and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{M} & \xrightarrow{\mathrm{d}} & S[[t]] & \xrightarrow{\mathrm{d}} & \Bbbk[[t]] \otimes_{\Bbbk[t]} S[t]/M & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & \Bbbk[[t]]J & \xrightarrow{\mathrm{d}} & \Bbbk[[t]] & \xrightarrow{\mathrm{d}} & \Bbbk[[t]]\overline{a} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

For b in $\Bbbk[[t]]$, $(\delta\circ\mathrm{d})b = (\mathrm{d}\circ\delta)b = 0$ since a is in $\widehat{M}$, whence $\mathrm{d}b = 0$. As a result, $\Bbbk[[t]]J = \Bbbk[[t]]$. So qa is in M for some q in $\Bbbk[t]$ such that $q(0) \neq 0$.

(ii) Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, the canonical morphism

$$\Bbbk[[t]] \otimes_{\Bbbk[t]} M \longrightarrow \widehat{M}.$$

is an isomorphism and from the short exact sequence

$$0 \longrightarrow M \cap N \longrightarrow M \oplus N \longrightarrow M + N \longrightarrow 0$$

we deduce the short exact sequence

$$0 \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} M \cap N \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} (M \oplus N) \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} (M + N) \longrightarrow 0,$$

whence the short exact sequence

$$0 \longrightarrow \widehat{M \cap N} \longrightarrow \widehat{M} \oplus \widehat{N} \longrightarrow \widehat{M + N} \longrightarrow 0,$$

and whence the assertion. □

Proposition 4.9. (i) *The space $V_0[[t]]$ is the direct sum of $V_0'[[t]]$ and $\widehat{W}$.*

(ii) *The space $S[[t]]$ is the direct sum of $V_0'[[t]]$ and of $W + S[[t]]A_+$.*

(iii) *The linear map*

$$V_0'((t)) \otimes_{\Bbbk} A \longrightarrow S((t)), \qquad v{\otimes}a \longmapsto v{\otimes}a$$

is a homogenous isomorphism onto $S((t))$.

(iv) *For all nonnegative integer d,*

$$\dim S^{[d]} = \sum_{i=0}^{d} \dim {V'_0}^{[d-i]} \times \dim A^{[i]}.$$

Proof. (i) According to Lemma 4.8,(ii), the intersection of $V'_0[[t]]$ and $\widehat{W}$ is the $\Bbbk[[t]]$-submodule generated by the intersection of $V'_0[t]$ and W. So, by Lemma 4.7,(iii), the sum of $V'_0[[t]]$ and $\widehat{W}$ is direct.

Let $(v_i, i \in J)$ be a homogenous basis of V'_0. Let d be a positive integer and let v be in $V_0^{[d]}$. Denote by J_d the set of indices i such that v_i has degree d. Since V_0 is the direct sum of V'_0 and V''_0, for some w in $W^{[d]}$ and for some $c_i, i \in J_d$, in $\Bbbk$,

$$v - \sum_{i \in J} c_i v_i = w(0).$$

Since $w - w(0)$ is in $tV_0[t]^{[d]}$,

$$v - \sum_{i \in J_d} c_i v_i - w \in tV_0[t]^{[d]}.$$

As a result,

$$V_0^{[d]}[[t]] \subset {V'_0}^{[d]}[[t]] + \widehat{W}^{[d]} + tV_0^{[d]}[[t]].$$

Then by induction on m,

$$V_0^{[d]}[[t]] \subset {V'_0}^{[d]}[[t]] + \widehat{W}^{[d]} + t^m V_0^{[d]}[[t]].$$

So, since $V_0^{[d]}[[t]]$ is a finitely generated $\Bbbk[[t]]$-module,

$$V_0^{[d]}[[t]] = {V'_0}^{[d]}[[t]] + \widehat{W}^{[d]},$$

whence the assertion.

(ii) According to Lemma 4.5,(i), for some nonnegative integer l, $t^l \mathfrak{I}_*$ is contained in $S[t]A_+$. Hence $\widehat{W} + S[[t]]A_+$ is equal to $W + S[[t]]A_+$. So, by (i) and Lemma 4.6,

$$S[[t]] = V'_0[[t]] + W + S[[t]]A_+.$$

According to Lemma 4.7,(ii), the intersection of $V'_0[t]$ and $S[t]A_+$ is equal to $\{0\}$ since $S[t]A_+$ is contained in $\mathfrak{I}_*$. As a result, by Lemma 4.8,(ii), the intersection of $V'_0[[t]]$ and $S[[t]]A_+$ is equal to $\{0\}$. If a is in the intersection of $V'_0[[t]]$ and $W + S[[t]]A_+$, $t^l a$ is in the intersection of $V'_0[[t]]$ and $S[[t]]A_+$. So the sum of $V'_0[[t]]$ and $W + S[[t]]A_+$ is direct.

(iii) According to Lemma 4.5,(i), W is contained in $S((t))A_+$. So, by (ii),

$$S((t)) = V'_0((t)) \oplus S((t))A_+.$$

Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, and since

$$S((t)) = \Bbbk[[t]] \otimes_{\Bbbk[t]} S[t, t^{-1}],$$

we deduce that $S((t))$ is a flat extension of A by Corollary 4.4,(ii). So, by Lemma 2.2, all basis of $V'_0[[t]]$ over $\Bbbk$ consists of linearly independent elements over A. The assertion follows.

(iv) First of all, the canonical map

$$\Bbbk((t)) \otimes_{\Bbbk} A \longrightarrow \Bbbk((t))A$$

is an isomorphism by Lemma 4.2,(i). As a result, we have the canonical isomorphism

$$V'_0((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow V'_0((t)) \otimes_{\Bbbk((t))} (\Bbbk((t)) \otimes_{\Bbbk} A),$$

and for all nonnegative integer i,

$$\dim A^{[i]} = \dim_{\Bbbk((t))} (\Bbbk((t))A)^{[i]}.$$

From the above isomorphism, it results that the canonical morphism

$$V'_0((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow V'_0((t)) \otimes_{\Bbbk} A$$

is an isomorphism of graded spaces since $V'_0((t)) \otimes_{\Bbbk((t))} \Bbbk((t)) = V'_0((t))$. As a result, by (iii), the canonical morphism

$$V'_0((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow S((t))$$

is a homogenous isomorphism. So, for all nonnegative integer d,

$$\dim_{\Bbbk((t))} S((t))^{[d]} = \sum_{i=0}^{d} \dim_{\Bbbk((t))} V'_0((t))^{[d-i]} \times \dim_{\Bbbk((t))} (\Bbbk((t))A)^{[i]},$$

whence the assertion since $\dim S^{[i]} = \dim_{\Bbbk((t))} S((t))^{[i]}$ and $\dim V'^{[i]}_0 = \dim_{\Bbbk((t))} V'_0((t))^{[i]}$ for all i. □

4.4. Let $(w_k, k \in K)$ be a homogenous sequence in W such that $(w_k(0), k \in K)$ is a basis of $V''_0 = W(0)$. For k in K, denote by m_k the smallest integer such that $t^{m_k} w_k$ is in $S[t]A_+$. According to Lemma 4.5,(i), m_k is finite for all k. Moreover, m_k is positive since $W(0) \cap SA(0)_+ = \{0\}$. Set

$$\Theta := \{(k, i) \mid k \in K,\ i \in \{0, \dots, m_k - 1\}\},$$

and set for all (k, i) in Θ,

$$w_{k,i} := t^i w_k.$$

Let E_Θ be the $\Bbbk$-subspace of $V_0[t]$ generated by the elements $w_{k,i}$, $(k, i) \in \Theta$.

Set

$$\widehat{\mathcal{I}_*} := \Bbbk[[t]]\mathcal{I}_*.$$

It is an ideal of $S[[t]]$.

Lemma 4.10. (i) *For some q in $\Bbbk[t]$ such that $q(0) \neq 0$, $q\mathcal{I}_*$ is contained in $W + S[t]A_+$.*

(ii) *The space W is contained in $E_\Theta + S[t]A_+$. Moreover, $\widehat{\mathcal{I}_*}$ is the sum of E_Θ and $S[[t]]A_+$.*

(iii) *The sequence $(w_{k,i}, (k, i) \in \Theta)$ is a homogenous basis of E_Θ.*

(iv) *For all nonnegative integer i, $E^{[i]}_\Theta$ has finite dimension.*

(v) *For i a nonnegative integer, there exists a nonnegative integer l_i such that $t^{l_i} E^{[i]}_\Theta$ is contained in $V'_0[[t]]A_+$.*

Proof. (i) Let a be in $\mathcal{I}_*$. According to Lemma 4.6 and Lemma 4.8,(i), for some q in $\Bbbk[t]$ such that $q(0) \neq 0$, $qa \in \mathcal{I}_*$ and $qa = a_1 + a_2$ with a_1 in $V_0[t]$ and a_2 in $S[t]A_+$. Then a_1 is in $\mathcal{I}_*$ since so are a_2 and qa. So $a_1 \in \mathcal{I}_* \cap V_0[t] = W$. The assertion follows because $\mathcal{I}_*$ is finitely generated.

(ii) Let us prove the first assertion. It suffices to prove

$$W \subset E_\Theta + S[t]A_+ + t^m S[t]$$

for all m. Indeed, W, E_Θ, $S[t]A_+$ are contained in $\mathcal{I}_*$. So, if $w = e + a + t^m b$, with $w \in W$, $e \in E_\Theta$ and $b \in S[t]$, then b is in $\mathcal{I}_*$ and so, for m big enough, it is in $S[t]A_+$ by Lemma 4.5,(i).

Prove now the inclusion by induction on m. The inclusion is tautological for $m = 0$, and it is true $m = 1$ because $E_\Theta(0) = V''_0$. Suppose that it is true for $m > 0$. Let w be in W. By induction hypothesis,

$$w = a + b + t^m c, \quad \text{with} \quad a \in E_\Theta,\ b \in S[t]A_+,\ c \in S[t].$$

Since E_Θ and $S[t]A_+$ are contained in $\mathcal{I}_*$, c is in $\mathcal{I}_*$ by Lemma 4.5,(i). According to (i), for some q in $\Bbbk[t]$ such that $q(0) \neq 0$, $qc = a' + b'$ with a' in W and b' in $S[t]A_+$. Since the inclusion is true for $m = 1$,

$$t^m(a' + b') \in t^m E_\Theta + S[t]A_+ + t^{m+1} S[[t]],$$

and by definition, $t^m E_\Theta$ is contained in $E_\Theta + S[t]A_+$. Moreover, $q(0)c$ is in $qc + tS[t]$. Then

$$t^m c \in E_\Theta + S[t]A_+ + t^{m+1}S[t] \quad \text{and} \quad w \in E_\Theta + S[t]A_+ + t^{m+1}S[t],$$

whence the statement.

Turn to the second assertion. By (i), $\widehat{\mathcal{I}_*}$ is the sum of $\widehat{W}$ and $S[[t]]A_+$. An element of $\widehat{W}$ is the sum of terms $t^m w_m$, with $m \in \mathbb{N}$ and $w_n \in W$. For m big enough, $t^m w_m \in S[t]A_+$ by Lemma 4.5,(i). So $\widehat{\mathcal{I}_*}$ is the sum of W and $S[[t]]A_+$, whence the assertion by the previous inclusion.

(iii) By definition, the elements $w_{k,i}, (k,i) \in \Theta$, are homogenous. So it suffices to prove that they are linearly independent over $\Bbbk$. Let $(c_{k,i}, (k,i) \in \Theta)$ be a sequence in $\Bbbk$, with finite support, such that

$$\sum_{k\in K} \sum_{i=0}^{m_k-1} c_{k,i} w_{k,i} = 0.$$

Let us prove that $c_{k,i} = 0$ for all (k,i). Suppose $c_{k,i} \neq 0$ for some (k,i). A contradiction is expected. Let K' be the set of k such that $c_{k,i} \neq 0$ for some i. Denote by i_0 the smallest integer such that $c_{k,i_0} \neq 0$ for some k in K' and set:

$$K'_0 := \{k \in K' \mid c_{k,i_0} \neq 0\}.$$

Then

$$\sum_{k\in K'_0} c_{k,i_0} w_k(0) = 0,$$

whence the contradiction since the elements $(w_k(0), k \in K)$ are linearly independent.

(iv) Let K_i be the set of k such that w_k is in $S[t]^{[i]}$. For such k, $w_k(0)$ is in $S^{[i]}$. Hence K_i is finite since $S^{[i]}$ has finite dimension and since the elements $(w_k(0), k \in K)$ are linearly independent. For k in K, $\Bbbk[t]w_k \cap E_\Theta$ has dimension m_k by (iii). Hence $E_\Theta^{[i]}$ has finite dimension.

(v) Let k be in K_i. Set

$$\Theta^{[i]} := \Theta \cap (K_i \times \mathbb{N}).$$

By Proposition 4.9,(iii), $t^{l+m_k} w_k$ is in $V'_0[[t]]A_+$ since $t^{m_k} w_k$ is in $S[t]A_+$ by definition, whence the assertion since $E_\Theta^{[i]}$ is generated by the finite sequence $(w_{k,j}, (k,j) \in \Theta^{[i]})$. □

Definition 4.11. We say that a subset T of Θ is *complete* if

$$(k,i) \in T \Longrightarrow (k,j) \in T, \ \forall j \in \{0,\dots,i\}.$$

For T subset of Θ, denote by K_T the image of T by the projection $(k,i) \mapsto k$, and by E_T the subspace of E_Θ generated by the elements $w_{k,i}$, $(k,i) \in T$. In particular, $K_\Theta = K$.

Lemma 4.12. *For some complete subset T of Θ such that $K_T = K$, the subspace E_T is a complement to $S[t]A_+$ in $E_\Theta + S[t]A_+$. In particular, the sum of E_T and $S[t]A_+$ is direct.*

Proof. Since $V''_0 \cap SA(0)_+ = \{0\}$, the sum of $E_{K\times\{0\}}$ and $S[t]A_+$ is direct. Let $\mathcal{T}$ be the set of complete subsets T of Θ satisfying the following conditions:

(1) for all k in K, $(k,0)$ is in T,
(2) the sum of E_T and $S[t]A_+$ is direct.

Since the sum of $E_{K\times\{0\}}$ and $S[t]A_+$ is direct, $\mathcal{T}$ is not empty. If $(T_j, j \in \mathfrak{J})$ is an increasing sequence of elements of $\mathcal{T}$, with respect to the inclusion, its union is in $\mathcal{T}$. Then, by Zorn's Lemma, $\mathcal{T}$ has a maximal element. Denote it by T_*. It remains to prove that $w_{k,i}$ is in $E_{T_*} + S[t]A_+$ for all (k,i) in Θ.

Let k be in K. Denote by i the biggest integer such that (k, i) is in T_*. Prove by induction on i' that for $m_k > i' > i$, $w_{k,i'}$ is in $E_{T_*} + S[t]A_+$. By maximality of T_* and i, $w_{k,i+1}$ is in $E_{T_*} + S[t]A_+$. Suppose that $w_{k,i'}$ is in $E_{T_*} + S[t]A_+$. Then, for some a in $S[t]A_+$ and $c_{m,j}, (m, j) \in T_*$ in $\Bbbk$,

$$w_{k,i'} = \sum_{(m,j)\in T_*} c_{m,j} w_{m,j} + a,$$

whence

$$w_{k,i'+1} = \sum_{(m,j)\in T_*} c_{m,j} t^{j+1} w_m + ta.$$

By maximality of T_*, $t^{j+1}w_m$ is in $E_{T_*} + S[t]A_+$ for all (m, j) such that $t^j w_m$ is in T_*. Hence $w_{k,i'+1}$ is in $E_{T_*} + S[t]A_+$. The lemma follows. □

Fix a complete subset T_* of Θ such that

$$K_{T_*} = K \qquad \text{and} \qquad E_\Theta + S[t]A_+ = E_{T_*} \oplus S[t]A_+,$$

and set

$$E := E_{T_*}.$$

Such a set T_* does exist by Lemma 4.12.

Corollary 4.13. (i) *The space $S[[t]]$ is the direct sum of $V_0'[[t]]$, E and $S[[t]]A_+$.*

(ii) *The space $S[[t]]$ is the sum of EA and $V_0'[[t]]A$.*

Proof. (i) According to Proposition 4.9,(ii), $S[[t]]$ is the direct sum of $V_0'[[t]]$ and $W + S[[t]]A_+$. By Lemma 4.10,(ii) (and its proof), $W + S[[t]]A_+$ is equal to $E_\Theta + S[[t]]A_+$. Since $E_\Theta + S[t]A_+$ is the direct sum of E and $S[t]A_+$, we deduce that $W + S[[t]]A_+$ is the direct sum of E and $S[[t]]A_+$. Hence, $S[[t]]$ is the direct sum of $V_0'[[t]]$, E and $S[[t]]A_+$.

(ii) By (i) and by induction on m,

$$S[[t]] \subset V_0'[[t]]A + EA + S[[t]]A_+^m.$$

Hence $S[[t]]$ is the sum of $V_0'[[t]]A$ and EA since $S[[t]]$ is graded and A_+ is generated by elements of positive degree. □

Definition 4.14. For k in K, denote by ν_k the degree of w_k. For T and T' subsets of Θ, we say that T *is smaller than* T', and we denote $T \prec T'$, if the following conditions are satisfied:

(1) T is contained in T'
(2) if for k in K_T and k' in $K_{T'}$, we have $\nu_{k'} < \nu_k$, then k' is in K_T.

Let μ be the linear map

$$E \otimes_\Bbbk A \oplus V_0'[[t]] \otimes_\Bbbk A \longrightarrow S[[t]], \qquad w \otimes a + v \otimes b \longmapsto wa + vb.$$

For T a subset of T_*, denote by μ_T the restriction of μ to the subspace

$$E_T \otimes_\Bbbk A \oplus V_0'[[t]] \otimes_\Bbbk A.$$

Lemma 4.15. *Let $\mathcal{T}_*$ be the set of subsets T of T_* such that μ_T is injective.*

(i) *The set $\mathcal{T}_*$ is not empty.*

(ii) *The set $\mathcal{T}_*$ has a maximal element with respect to the order $\prec$.*

(iii) *The set T_* is in $\mathcal{T}_*$.*

Proof. (i) For k in K, set $T_k := \{(k, 0)\}$. Suppose that T_k is not in $\mathcal{T}_*$. A contradiction is expected. Then for some a in $A \setminus \{0\}$, $w_k a$ is in $V'_0[[t]]A_+$, whence

$$w_k a = \sum_{i \in J} v_i b_i$$

with $(b_i, i \in J)$ in $\Bbbk[[t]]A_+$ with finite support. By Lemma 4.10,(v), for some positive integer, $t^l w_k$ is in $V'_0[[t]]A_+$. Then

$$t^l w_k = \sum_{i \in J} v_i c_i$$

with $(c_i, i \in J)$ in $\Bbbk[[t]]A_+$ with finite support. Hence

$$\sum_{i \in J} v_i t^l b_i = \sum_{i \in J} v_i c_i a.$$

According to Proposition 4.9,(iii), $t^l b_i = c_i a$ for all i. Since $a \neq 0$, $a(0) \neq 0$ by Proposition 4.1,(ii). Then, by Lemma 4.2,(ii), $c_i = t^l c'_i$ for some c'_i in $\tilde{A} = \Bbbk[[t]]A$. As a result,

$$w_k = \sum_{i \in J} v_i c'_i,$$

whence the contradiction by Corollary 4.13,(i).

(ii) Let $(T_l, l \in L)$ be a net in $\mathcal{T}_*$ with respect to $\prec$. Let T be the union of the sets T_l, $l \in L$. Since E_T is the space generated by the subspaces $E_{T_l}, l \in L$, the map μ_T is injective. Let l_0 be in L and k in K_T such that $\nu_k < \nu_{k'}$ for some k' in $K_{T_{l_0}}$. Since K_T is the union of the sets K_{T_l}, $l \in L$, we deduce that k is in K_{T_l} for some l in L. By properties of the nets, for some l' in L, $T_l \prec T_{l'}$ and $T_{l_0} \prec T_{l'}$ so that k is in $K_{T_{l'}}$. Hence, k is in $K_{T_{l_0}}$, whence $T_{l_0} \prec T$. As a result, $\prec$ is an inductive order in $\mathcal{T}_*$, and by Zorn's Theorem, it has a maximal element.

(iii) Let T be a maximal element of $\mathcal{T}_*$ with respect to $\prec$. Suppose T strictly contained in T_*. A contradiction is expected. Let k be in K such that (k, i) is not in T and (k, i) is in T_* for some i. We can suppose that ν_k is minimal under this condition. Let i_* be the smallest integer such that (k, i_*) is not in T and (k, i_*) is in T_*. Then $T \prec T \cup \{(k, i_*)\}$. So, by the maximality of T, for some a in $A \setminus \{0\}$,

$$w_{k,i_*} a \in E_T A + V'_0[[t]]A.$$

Since E_T, $V'_0[[t]]$, A, w_{k,i_*} are homogenous, we can suppose that a is homogenous. Then a has positive degree. Otherwise, $w_{k,i_*} \in E_T A + V'_0[[t]]A \subset E_T + V'_0[[t]] + S[[t]]A_+$, and we deduce from Corollary 4.13,(i), that $w_{k,i_*} \in E_T$ since $w_{k,i_*} \in E_{T_*}$. This is impossible by the choice of (k, i_*). Thus, by Corollary 4.13,(ii),

$$w_{k,i_*} a \in E_T A_+ + V'_0[[t]]A_+.$$

Hence

$$w_{k,i_*} a = \sum_{(n,j) \in T} w_{n,j} a_{n,j} + \sum_{i \in J} v_i b_i$$

with $(a_{n,j}, (n, j) \in T)$ in A_+ and $(b_i, i \in J)$ in $\tilde{A}_+$ with finite support.

By Corollary 4.13,(ii),

$$t^{m_k} w_k = \sum_{(l,s) \in T_*} w_{l,s} a_{l,s,k} + \sum_{i \in J} v_i b_{i,k}$$

with $(a_{l,s,k}, (l, s) \in T_*)$ in A_+ and $(b_{i,k}, i \in J)$ in $\tilde{A}_+$ with finite support. Moreover these two sequences are homogenous, so that $a_{l,s,k} = 0$ if $\nu_l \geqslant \nu_k$. By minimality of ν_k, (l, s) is in T if $a_{l,s,k} \neq 0$. For (n, j) in T such that $m_k - i_* + j \geqslant m_n$,

$$t^{m_k - i_*} w_{n,j} = \sum_{(l,s)\in T_*} w_{l,s} a_{l,s,n,j} + \sum_{i\in J} v_i b_{i,n,j}$$

with $(a_{l,s,n,j}, (l,s) \in T_*)$ in A_+ and $(b_{i,n,j}, i \in J)$ in $\tilde{A}_+$ with finite support. Moreover these two sequences are homogenous, so that $a_{l,s,n,j} = 0$ if $\nu_l \geqslant \nu_n$. So, by minimality of ν_k, (l,s) is in T if $a_{l,s,n,j} \neq 0$ and $\nu_n \leqslant \nu_k$. As a result,

$$\begin{aligned}\sum_{(l,s)\in T} w_{l,s} a_{l,s,k} a + \sum_{i\in J} v_i b_{i,k} a &= \sum_{(n,j)\in T} w_{n,j} t^{m_k - i_*} a_{n,j} + \sum_{i\in J} v_i t^{m_k - i_*} b_i \\ &= \sum_{\substack{(n,j)\in T \\ m_k - i_* + j < m_n}} w_{n, m_k - i_* + j} a_{n,j} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} w_{l,s} a_{l,s,n,j} a_{n,j} \\ &\quad + \sum_{i\in J} v_i t^{m_k - i^*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} \sum_{i\in J} v_i b_{i,n,j} a_{n,j}\end{aligned}$$

whence

$$\begin{aligned}\sum_{(l,s)\in T} w_{l,s} a_{l,s,k} a + \sum_{i\in J} v_i b_{i,k} a &= \sum_{\substack{(n,j)\in T \\ m_k - i_* + j < m_n}} w_{n, m_k - i_* + j} a_{n,j} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} \sum_{(l,s)\in T} w_{l,s} a_{l,s,n,j} a_{n,j} \\ &\quad + \sum_{i\in J} v_i (t^{m_k - i_*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} b_{i,n,j} a_{n,j}).\end{aligned}$$

Since μ_T is injective, for all i in J,

$$t^{m_k - i_*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} b_{i,n,j} a_{n,j} - b_{i,k} a = 0, \tag{1}$$

and for all (l,s) in T,

$$a_{l, s + i_* - m_k} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} a_{n,j} a_{l,s,n,j} - a_{l,s,k} a = 0. \tag{2}$$

with $a_{l,s} = 0$ if $s < 0$.

Claim 4.16. For all (l,s) in T, a divides $a_{l,s}$ in A.

Proof of Claim 4.16. Prove the claim by induction on ν_l. Let l be in K_T such that

$$\nu_{l'} > \nu_l \quad \text{and} \quad (l', s') \in T \Longrightarrow a_{l',s'} = 0.$$

Then by Equality (2), $a_{l, s+i_* - m_k} = a_{l,s,k} a$, whence the satement for l. Suppose that a divides $a_{l',s'}$ in A for all (l', s') in T such that $\nu_{l'} > \nu_l$. By Equality (2) and the induction hypothesis, a divides $a_{l, s+i_*-m_k}$ in A since $a_{l,s,n,j} = 0$ for $\nu_n \leqslant \nu_l$, whence the claim. □

By Claim 4.16 and Equality (1), for all i in J, a divides $t^{m_k - i_*} b_i$ in $\Bbbk[[t]]A$. Since a has positive degree, all prime divisor of a in A has positive degree and does not divide t since t has degree 0. Then, by Lemma 4.2,(iii), a divides b_i in $\Bbbk[[t]]A$. As a result,

$$w_{k,i_*} \in E_T A + V'_0[[t]]A$$

whence

$$w_{k,i_*} \in V'_0[[t]] + E_T + S[[t]]A_+.$$

Since w_{k,i_*} is in E, w_{k,i_*} is in E_T by Corollary 4.13,(i). We get a contradiction because (k, i_*) is not in T. □

Corollary 4.17. *The canonical map*

$$E \otimes_{\Bbbk} A \oplus V_0'[[t]] \otimes_{\Bbbk} A \longrightarrow S[[t]]$$

is an isomorphism. In particular, $S[[t]]$ is a free extension of A.

Proof. By Lemma 4.15, T_* is the biggest element of $\mathcal{T}_*$. Hence μ is injective. Then, by Corollary 4.13,(ii), μ is bijective. As a matter of fact, μ is an isomorphism of A-modules, whence the corollary. □

4.5. Recall that $\tilde{A}$ is the subalgebra of $S[[t]]$ generated by $\Bbbk[[t]]$ and A. Our next aim is to show that $S[[t]]$ is a free extension of $\tilde{A}$ (cf. Theorem 4.21). Theorem 3.6 will then follows.

For I an ideal of $\tilde{A}$, denote by σ_I and ν_I the canonical morphisms

$$S[[t]] \otimes_A I \xrightarrow{\sigma_I} S[[t]] \otimes_A \tilde{A} \qquad\qquad S[[t]] \otimes_{\tilde{A}} I \xrightarrow{\nu_I} S[[t]]I\ .$$

Consider on $S[[t]]\otimes_A I$ and $S[[t]]\otimes_{\tilde{A}} I$ the linear topologies such that $\{t^n(S[[t]]\otimes_A I)\}_{n\in\mathbb{N}}$ and $\{t^n(S[[t]]\otimes_{\tilde{A}} I)\}_{n\in\mathbb{N}}$ are systems of neighborhood of 0 in these $S[[t]]$-modules. Denote by φ_I the canonical morphism

$$S[[t]] \otimes_A I \xrightarrow{\varphi_I} S[[t]] \otimes_{\tilde{A}} I$$

and by $\mathcal{K}_I$ its kernel. Then φ_I is continuous with respect to the above topologies.

Lemma 4.18. *Let I be an ideal of $\tilde{A}$.*

(i) *The morphism σ_I is injective.*

(ii) *The module $\mathcal{K}_I$ is the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the elements $r\otimes a - 1\otimes ra$ with r in $\Bbbk[[t]]$ and a in I.*

Proof. (i) According to Corollary 4.17, $S[[t]]$ is a flat extension of A. The assertion follows since I is contained in $\tilde{A}$.

(ii) Let $\mathcal{K}_I'$ be the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the elements $r\otimes a - 1\otimes ra$ with r in $\Bbbk[[t]]$ and a in I. Then $\mathcal{K}_I'$ is contained in $\mathcal{K}_I$. Let us prove the opposite inclusion.

Let (x, y) be in $S[[t]] \times I$ and let a be in $\tilde{A}$. According to (i), a has an expansion

$$a = \sum_{i=1} r_i a_i$$

with $r_1, \dots, r_m$ in $\Bbbk[[t]]$ and $a_1, \dots, a_m$ in A. Then, in $S[[t]] \otimes_A I$,

$$x\otimes ay - ax\otimes y = \sum_{i=1}^{m} x\otimes r_i a_i y - r_i x\otimes a_i y = \sum_{i=1}^{m} x(1\otimes r_i a_i y - r_i\otimes a_i y) \in \mathcal{K}_I'.$$

As a result, $\mathcal{K}_I = \mathcal{K}_I'$ since $\mathcal{K}_I$ is the S[[t]]-submodule of $S[[t]] \otimes_A I$ generated by the $xa\otimes y - x\otimes ay$'s. □

Corollary 4.19. *Let I be an ideal of $\tilde{A}$. The module $\mathcal{K}_I$ is the closure of the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the set $\{t\otimes a - 1\otimes ta\}_{a\in I}$.*

Proof. Let $\mathcal{L}_I$ be the $S[[t]]$-submodule generated by the set $\{t\otimes a - 1\otimes ta\}_{a\in I}$. Prove by induction on n that $t^n\otimes a - 1\otimes t^n a$ is in $\mathcal{L}_I$ for all a in I. The statement is straightforward for $n = 0, 1$. Suppose $n \geqslant 2$ and the statement true for $n - 1$. For a in I,

$$t^n a - 1\otimes t^n a = t^{n-1}(t\otimes a - 1\otimes ta) + t^{n-1}\otimes ta - 1\otimes t^{n-1} ta.$$

By induction hypothesis, $t^{n-1}\otimes ta - 1\otimes t^{n-1} ta$ is in $\mathcal{L}_I$, whence $t^n\otimes a - 1\otimes t^n a$ is in $\mathcal{L}_I$. As a result, for r in $\Bbbk[t]$, $r\otimes a - 1\otimes ra$ is in $\mathcal{L}_I$. So, for r in $\Bbbk[[t]]$, $r\otimes a - 1\otimes ra$ is in the closure of $\mathcal{L}_I$ in $S[[t]]\otimes_A I$. Since φ_I is continuous, $\mathcal{K}_I$ is a closed submodule of $S[[t]] \otimes_A I$, whence the corollary by Lemma 4.18,(iii). □

Proposition 4.20. *Let I be an ideal of $\tilde{A}$.*

(i) *The canonical morphism*

$$V'_0\tilde{A} \otimes_{\tilde{A}} I \longrightarrow S[[t]] \otimes_{\tilde{A}} I$$

is an embedding.

(ii) *For the structure of $S[[t]]$-module on $S[[t]] \otimes_{\tilde{A}} I$, t is not a divisor of* 0 *in $S[[t]] \otimes_{\tilde{A}} I$.*

Proof. (i) We have the commutative diagram

$$\begin{array}{ccc} V'_0\tilde{A} \otimes_{\tilde{A}} I & \xrightarrow{\mathrm{d}} & S[[t]] \otimes_{\tilde{A}} I \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ V'_0 I & \xrightarrow{\mathrm{d}} & S[[t]]I \end{array}$$

with canonical arrows d and δ. According to Proposition 4.9,(iii), the left down arrow δ is an isomorphism. Let a be in $V'_0\tilde{A} \otimes_{\tilde{A}} I$ such that $\mathrm{d}a = 0$. Then $\mathrm{d}\circ\delta a = 0$, whence $\delta a = 0$ since the bottom horizontal arrow d is an embedding so that $a = 0$.

(ii) Let a be in $S[[t]] \otimes_A I$ such that $t\varphi_I(a) = 0$. According to Corollary 4.19, for l in $\mathbb{N}$ such that $l \geqslant 2$,

$$ta - \sum_{i=1}^{m} b_i(t\otimes a_i - 1\otimes ta_i) \in t^l S[[t]] \otimes_A I$$

for some $b_1, \ldots, b_m$ in $S[[t]]$ and for some $a_1, \ldots, a_m$ in I. For $i = 1, \ldots, m$,

$$b_i = b_{i,0} + tb'_i$$

with $b_{i,0}$ in S and b'_i in $S[[t]]$, whence

$$t(a - \sum_{i=1}^{m} b'_i(t\otimes a_i - 1\otimes ta_i)) - \sum_{i=1}^{m} b_{i,0}(t\otimes a_i - 1 - \otimes ta_i) \in t^l S[[t]] \otimes_A I.$$

Set:

$$a' := a - \sum_{i=1}^{m} b'_i(t\otimes a_i - 1\otimes ta_i) \quad \text{and} \quad a'' = \sum_{i=1}^{m} b_{i,0}(t\otimes a_i - 1\otimes ta_i).$$

Then $\varphi_I(a) = \varphi_I(a')$ and $\sigma_I(a'')$ is in $tS[[t]] \otimes_{\Bbbk} \Bbbk[[t]]$. Moreover, for $i = 1, \ldots, m$, a_i has a unique expansion

$$a_i = \sum_{n\in\mathbb{N}} t^n a_{i,n}$$

with $a_{i,n}, n \in \mathbb{N}$, in A. Then

$$\begin{aligned} \sigma_I(a'') &= \sum_{i=1}^{m} b_{i,0}(\sum_{n\in\mathbb{N}} ta_{i,n}\otimes t^n - a_{i,n}\otimes t^{n+1}) \\ &= t\sum_{i=1}^{m} a_{i,0}b_{i,0}\otimes 1 + \sum_{n\in\mathbb{N}^*}\sum_{i=1}^{m} b_{i,0}(ta_{i,n} - a_{i,n-1})\otimes t^n. \end{aligned}$$

Since the right hand side is divisible by t in $S[[t]] \otimes_{\Bbbk} \Bbbk[[t]]$, for all positive integer n,

$$\sum_{i=1}^{m} b_{i,0}a_{i,n-1} = 0$$

since $b_{i,0}$ and $a_{i,n-1}$ are in S for all i. Hence $\sigma_I(a'') = 0$ and $a'' = 0$ by Lemma 4.18,(i). Thus,

$$a' \in t^{l-1} S[[t]] \otimes_A I.$$

As a result, $\varphi_I(a)$ is in $t^l S[[t]] \otimes_{\tilde{A}} I$ for all positive integer l. Since the $S[[t]]$-module $S[[t]] \otimes_{\tilde{A}} I$ is finitely generated, by a Krull's theorem [Ma86, Ch. 3, Theorem 8.9], for some b in $S[[t]]$, $(1+tb)\varphi_I(a) = 0$, whence $\varphi_I(a) = 0$ since $t\varphi_I(a) = 0$. □

Remind that $\mathcal{V}_0$ is the nullvariety of $A(0)_+$ in $\mathfrak{g}^f$, and that $\tilde{A} = \Bbbk[[t]]A$.

Theorem 4.21. (i) *The algebra $S[[t]]$ is a free extension of $\tilde{A}$.*

(ii) *The varieties $\mathcal{V}$ and $\mathcal{V}_*$ are equal. Moreover, $\mathcal{V}_0$ is equidimensional of dimension $r-\ell$.*

(iii) *The $A(0)$-module S is free and $V_0 = V_0'$. In particular, the canonical morphism*

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \quad v \otimes a \longmapsto va$$

is an isomorphism.

Proof. (i) First of all, prove that $S[[t]]$ is a flat extension of $\tilde{A}$. Then the freeness of the extension will result from the equality $V_0 = V_0'$, Lemma 4.6 and Proposition 4.9,(iii).

By the criterion of flatness [Ma86, Ch. 3, Theorem 7.7], it is equivalent to say that for all ideal I of $\tilde{A}$, the canonical morphism ν_I,

$$S[[t]] \otimes_{\tilde{A}} I \longrightarrow S[[t]]I$$

is injective. Let a be in the kernel of ν_I. Consider the commutative diagram

$$\begin{array}{ccc} V_0'\tilde{A} \otimes_{\tilde{A}} I & \xrightarrow{\mathrm{d}} & S[[t]] \otimes_{\tilde{A}} I \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ V_0' I & \xrightarrow{\mathrm{d}} & S[[t]]I \end{array}$$

of the proof of Proposition 4.20,(i). According to Lemma 4.10,(v), for l sufficiently big, $t^l a = \mathrm{d}b$ for some b in $V_0'\tilde{A} \otimes_{\tilde{A}} I$. Then $\delta b = 0$ since $\nu_I(t^l a) = 0$. By Proposition 4.9,(iii), δ is an isomorphism. Hence $b = 0$ and $t^l a = 0$. Then, by Proposition 4.20,(ii), $a = 0$, whence the the flatness.

(ii) Denote by $\Bbbk[t]_0$ the localization of $\Bbbk[t]$ at $t\Bbbk[t]$. Then $\Bbbk[[t]]$ is a faithfully flat extension of $\Bbbk[t]_0$. Hence, $S[[t]]$ is a faithfully flat extension of

$$S[t]_0 := \Bbbk[t]_0 \otimes_{\Bbbk[t]} S.$$

Set

$$\tilde{A}_0 := \Bbbk[t]_0 \otimes_{\Bbbk} A.$$

Then

$$\tilde{A} = \Bbbk[[t]] \otimes_{\Bbbk[t]_0} \tilde{A}_0$$

so that $\tilde{A}$ is faithfully flat extension of $\tilde{A}_0$. For M a $\tilde{A}_0$-module, we have

$$\Bbbk[[t]] \otimes_{\Bbbk[t]_0} (S[t]_0 \otimes_{\tilde{A}_0} M) = (\Bbbk[[t]] \otimes_{\Bbbk[t]_0} S[t]_0) \otimes_{\tilde{A}_0} M = S[[t]] \otimes_{\tilde{A}} (\tilde{A} \otimes_{\tilde{A}_0} M).$$

Hence, $S[t]_0$ is a flat extension of $\tilde{A}_0$ since so is the extension $S[[t]]$ of $\tilde{A}$.

The variety $\mathcal{V}$ is the union of $\mathcal{V}_*$ and $\mathcal{V}_0 \times \{0\}$. Moreover $\mathcal{V}_0 \times \{0\}$ is the nullvariety in $\mathfrak{g}^f \times \Bbbk$ of the ideal of $\Bbbk[t]A$ generated by t and A_+. Then, by [Ma86, Ch. 5, Theorem 15.1], $\mathcal{V}_0$ is equidimensional of dimension $r-\ell$ since $S[t]_0$ is a flat extension of $\tilde{A}_0$ by (i) and since $\tilde{A}_0$ has dimension $\ell+1$. Since $\mathcal{V}$ is the nullvariety of ℓ functions, all irreducible component of $\mathcal{V}$ has dimension at least $r+1-\ell$ by [Ma86, Ch. 5, Theorem 13.5]. Hence any irreducible component of $\mathcal{V}_0 \times \{0\}$ is not an irreducible component of $\mathcal{V}$. As a result, $\mathcal{V}_0 \times \{0\}$ is contained in $\mathcal{V}_*$ and so $\mathcal{V} = \mathcal{V}_*$.

(iii) Since $A(0)$ is a polynomial algebra, S is a free extension of $A(0)$ by (ii) and Proposition 2.5. Moreover, by Lemma 2.2, the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a homogenous isomorphism with respect to the grading of $V_0 \otimes_{\Bbbk} A(0)$ induced by those of V_0 and $A(0)$. As a result, for all nonnegative integer i,

$$\dim S^{[i]} = \sum_{j=0}^{i} \dim V_0^{[i-j]} \times \dim A(0)^{[j]},$$

whence $\dim V_0^{[i]} = \dim V_0'^{[i]}$ for all i by Proposition 4.9,(iv) since $\dim A^{[i]} = \dim A(0)^{[i]}$ for all i by Proposition 4.1,(ii). Then $V_0 = V_0'$. □

As explained in Subsection 3.2, by Theorem 3.3 and Proposition 2.5,(ii), Theorem 3.6 results from Theorem 4.21,(ii).

Remark 4.22. According to the part (ii) of Theorem 4.21, $\mathcal{J}_*$ is the radical of $S[t]A_+$. Hence $S[t]A_+$ is radical by Lemma 4.5,(ii), and then $\mathcal{J}_* = S[t]A_+$.

5. Consequences of Theorem 1.5 for the simple classical Lie algebras

This section concerns some applications of Theorem 1.5 to the simple classical Lie algebras.

5.1. The first consequence of Theorem 3.6 is the following.

Theorem 5.1. *Assume that $\mathfrak{g}$ is simple of type* **A** *or* **C**. *Then all the elements of $\mathfrak{g}$ are good.*

Proof. This follows from [PPY07, Theorems 4.2 and 4.4], Theorem 3.6 and Proposition 3.5. Further, in type **A**, the result is given by [PPY07, Theorem 5.4]. □

5.2. In this subsection and the next one, $\mathfrak{g}$ is assumed to be simple of type **B** or **D**. More precisely, we assume that $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell+1$ or 2ℓ. Then $\mathfrak{g}$ is embedded into $\tilde{\mathfrak{g}} := \mathfrak{gl}(\mathbb{V}) = \mathrm{End}(\mathbb{V})$. For x an endomorphism of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of x. Then, for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $(q_1, \dots, q_\ell)$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell-1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell+1$, set $q_\ell = Q_{2\ell}$ and if $\dim \mathbb{V} = 2\ell$, let q_ℓ be the Pfaffian that is a homogenous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$.

Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Following the notations of Subsection 3.2, for $i \in \{1, \dots, \ell\}$, denote by ${}^e q_i$ the initial homogenous component of the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto q_i(e+x)$, and by δ_i the degree of ${}^e q_i$. According to [PPY07, Theorem 2.1], ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

Our first aim in this subsection is to describe the sum $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell)$ in term of the partition of $\dim \mathbb{V}$ associated with e.

Remark 5.2. The sequence of the degrees $(\delta_1, \dots, \delta_\ell)$ is described by [PPY07, Remark 4.2].

For $\lambda = (\lambda_1, \dots, \lambda_k)$ a sequence of positive integers, with $\lambda_1 \geqslant \cdots \geqslant \lambda_k$, set:

$$|\lambda| := k, \qquad r(\lambda) := \lambda_1 + \cdots + \lambda_k.$$

Assume that the partition λ of $r(\lambda)$ is associated with a nilpotent orbit of $\mathfrak{so}(\Bbbk^{r(\lambda)})$. Then the even integers of λ have an even multiplicity, [CMc93, §5.1]. Thus k and $r(\lambda)$ have the same parity. Moreover, there is an involution $i \mapsto i'$ of $\{1, \ldots, k\}$ such that $i = i'$ if λ_i is odd, and $i' \in \{i-1, i+1\}$ if λ_i is even. Set:

$$S(\lambda) := \sum_{i=i',\, i \text{ odd}} i - \sum_{i=i',\, i \text{ even}} i$$

and denote by n_λ the number of even integers in the sequence λ.

From now on, assume that λ is the partition of $\dim \mathbb{V}$ associated with the nilpotent orbit $G.e$.

Lemma 5.3. (i) *If* $\dim \mathbb{V}$ *is odd, i.e., k is odd, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda).$$

(ii) *If* $\dim \mathbb{V}$ *is even, i.e., k is even, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda).$$

Proof. (i) If $\dim \mathbb{V}$ is odd, then by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda),$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell + 1$.

(ii) If $\dim \mathbb{V}$ is even, then $\delta_\ell = k/2$ by [PPY07, Remark 4.2] and by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) + k = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda)$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell$. □

The sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ satisfies one of the following five conditions:

1) λ_k and λ_{k-1} are odd,
2) λ_k and λ_{k-1} are even,
3) $k > 3$, λ_k and λ_1 are odd and λ_i is even for any $i \in \{2, \ldots, k-1\}$,
4) $k > 4$, λ_k is odd and there is $k' \in \{2, \ldots, k-2\}$ such that $\lambda_{k'}$ is odd and λ_i is even for any $i \in \{k'+1, \ldots, k-1\}$,
5) $k = 1$ or λ_k is odd and λ_i is even for any $i < k$.

For example, $(4,4,3,1)$ satisfies Condition (1); $(6,6,5,4,4)$ satisfies Condition (2); $(7,6,6,4,4,4,4,3)$ satisfies Condition (3); $(8,8,7,5,4,4,2,2,3)$ satisfies Condition (4) with $k' = 4$; (9) and $(6,6,4,4,3)$ satisfy Condition (5). If $k = 2$, then one of the conditions (1) or (2) is satisfied.

Definition 5.4. Define a sequence λ^* of positive integers, with $|\lambda^*| \leqslant |\lambda|$, as follows:

- if $k = 2$ or if Condition (3) or (5) is satisfied, then set $\lambda^* := \lambda$,
- if Condition (1) or (2) is satisfied, then set:

$$\lambda^* := (\lambda_1, \ldots, \lambda_{k-2}),$$

- if $k > 3$ and if Condition (4) is satisfied, then set

$$\lambda^* := (\lambda_1, \ldots, \lambda_{k'-1}).$$

In any case, the partition of $r(\lambda^*)$ corresponding to λ^* is associated with a nilpotent orbit of $\mathfrak{so}(\mathbb{k}^{r(\lambda^*)})$. Recall that n_λ is the number of even integers in the sequence λ.

Definition 5.5. Denote by d_λ the integer defined by:

- if $k = 2$, then $d_\lambda := n_\lambda$,
- if $k > 2$ and if Condition (1) or (4) is satisfied, then $d_\lambda := d_{\lambda^*}$,
- if $k > 2$ and if Condition (2) is satisfied, then $d_\lambda := d_{\lambda^*} + 2$,
- if $k > 2$ and if Condition (3) is satisfied, then $d_\lambda := 0$,
- if Condition (5) is satisfied, then $d_\lambda := 0$.

Lemma 5.6. (i) *Assume that k is odd. If Condition* (1), (2) *or* (5) *is satisfied, then*

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*).$$

Otherwise,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*) + k - |\lambda^*| - 2.$$

(ii) *If k is even, then*

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + S(\lambda^*) + d_\lambda - d_{\lambda^*}.$$

Proof. (i) If Condition (3) or (5) is satisfied, there is nothing to prove. If Condition (1) is satisfied,

$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) + 1.$$

Then

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + 1$$

whence the assertion. If Condition (2) is satisfied,

$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*)$$

whence the assertion. If Condition (4) is satisfied,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + k - |\lambda^*| - 1.$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + k - |\lambda^*| - 1$$

whence the assertion.

(ii) If $k = 2$ or if $k > 2$ and Condition (3) or (5) is satisfied, there is nothing to prove. Let us suppose that $k > 3$. If Condition (1) is satisfied,

$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) - 1.$$

Then

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 1 + S(\lambda^*) - 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. If Condition (2) is satisfied,

$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 2 + S(\lambda^*)$$

whence the assertion since $d_\lambda - d_{\lambda^*} = 2$. If Condition (4) is satisfied,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + |\lambda^*| + 1 - k.$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + k - |\lambda^*| - 1 + S(\lambda^*) + |\lambda^*| - k + 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. □

Lemma 5.7. (i) *If λ_1 is odd and if λ_i is even for $i \geqslant 2$, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0$.

(ii) *If k is odd, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_\lambda - d_\lambda$.

(iii) *If k is even, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = d_\lambda$.

Proof. (i) By the hypothesis, $n_\lambda = k - 1$ and $S(\lambda) = 1$, whence the assertion by Lemma 5.3,(i).

(ii) Let us prove the assertion by induction on k. For $k = 3$, if λ_1 and λ_2 are even, $n_\lambda = 2$, $d_\lambda = 0$ and $S(\lambda) = 3$, whence the equality by Lemma 5.3,(i). Assume that $k > 3$ and suppose that the equality holds for the integers smaller than k. If Condition (1) or (2) is satisfied, then by Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*}.$$

But if Condition (1) or (2) is satisfied, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. If Condition (5) is satisfied, then

$$n_\lambda = k - 1, \qquad S(\lambda) = k, \qquad d_\lambda = 0,$$

whence the equality by Lemma 5.3,(i). Let us suppose that Condition (4) is satisfied. By Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*} + k - |\lambda^*| - 2 = n_\lambda - d_\lambda$$

whence the assertion since Condition (3) is never satisfied when k is odd.

(iii) The statement is clear for $k = 2$ by Lemma 5.3,(ii). Indeed, if Condition (1) is satisfied, then $d_\lambda = n_\lambda = 0$ and $S(\lambda) = -1$ and if Condition (2) is satisfied, then $d_\lambda = n_\lambda = 2$ and $S(\lambda) = 0$. If Condition (3) is satisfied, $n_\lambda = k - 2$ and $S(\lambda) = 1 - k$, whence the statement by Lemma 5.3,(ii). When Condition (4) is satisfied, by induction on $|\lambda|$, the statement results from Lemma 5.3,(ii) and Lemma 5.6,(ii), whence the assertion since Condition (5) is never satisfied when k is even. □

Corollary 5.8. (i) *If λ_1 is odd and if λ_i is even for all $i \geqslant 2$, then e is good.*

(ii) *If k is odd and if $n_\lambda = d_\lambda$, then e is good. In particular, if $\mathfrak{g}$ is of type* **B**, *then the even nilpotent elements of $\mathfrak{g}$ are good.*

(iii) *If k is even and if $d_\lambda = 0$, then e is good. In particular, if $\mathfrak{g}$ is of type* **D** *and of odd rank, then the even nilpotent elements of $\mathfrak{g}$ are good.*

Proof. As it has been already noticed, by [PPY07, Theorem 2.1], the polynomials ${}^eq_1, \ldots, {}^eq_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

So, by Theorem 3.6 and Lemma 5.7, if either λ_1 is odd and λ_i is even for all $i \geqslant 2$, or if k is odd and $n_\lambda = d_\lambda$, or if k is even and $d_\lambda = 0$, then e is good.

Suppose that e is even. Then the integers $\lambda_1, \dots, \lambda_k$ have the same parity, cf. e.g. [Ca85, §1.3.1]. Moreover, $n_\lambda = d_\lambda = 0$ whenever $\lambda_1, \dots, \lambda_k$ are all odd (cf. Definition 5.5). This in particular occurs if either $\mathfrak{g}$ is of type **B**, or if $\mathfrak{g}$ is of type **D** with odd rank. □

Remark 5.9. The fact that the even nilpotent elements of $\mathfrak{g}$ without (only) even Jordan blocks are good if $\mathfrak{g}$ is of type **B** or **D** was already observed by O. Yakimova in [Y09, Corollary 8.2] in a different formulation. Corollary 5.8 is more general.

Definition 5.10. A sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ is said to be *very good* if $n_\lambda = d_\lambda$ whenever k is odd and if $d_\lambda = 0$ whenever k is even. A nilpotent element of $\mathfrak{g}$ is said to be *very good* if it is associated with a very good partition of $\dim \mathbb{V}$.

According to Corollary 5.8, if e is very good then e is good. The following lemma characterizes the very good sequences.

Lemma 5.11. (i) *If k is odd then λ is very good if and only if λ_1 is odd and if $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences satisfying Conditions* (1) *or* (2) *with $k = 2$.*

(ii) *If k is even then λ is very good if and only if λ is a concatenation of sequences satisfying Condition* (3) *or Condition* (1) *with $k = 2$.*

For example, the partitions $(5, 3, 3, 2, 2)$ and $(7, 5, 5, 4, 4, 3, 1, 1)$ of 15 and 30 respectively are very good.

Proof. (i) Assume that λ_1 is odd and that $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences satisfying Conditions (1) or (2) with $k = 2$. So, if $k > 1$, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. Then, a quick induction proves that $n_\lambda - d_\lambda = n_{(\lambda_1)} - d_{(\lambda_1)} = 0$ since λ_1 is odd. The statement is clear for $k = 1$.

Conversely, assume that $n_\lambda - d_\lambda = 0$. If λ satisfies Conditions (1) or (2), then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$ and $|\lambda^*| < |\lambda|$. So, we can assume that λ does not satisfy Conditions (1) or (2). Since k is odd, λ cannot satisfy Condition (3). If λ satisfies Condition (4), then $n_\lambda - d_\lambda = n_\lambda - d_{\lambda^*} > n_{\lambda^*} - d_{\lambda^*} \geqslant 0$. This is impossible since $n_\lambda - d_\lambda = 0$. If λ satisfies Condition (5), then $n_\lambda - d_\lambda = n_\lambda$. So, $n_\lambda - d_\lambda = 0$ if and only if $k = 1$. Thereby, the direct implication is proven.

(ii) Assume that λ is a concatenation of sequences satisfying Condition (3) or Condition (1) with $k = 2$. In particular, λ does not satisfy Condition (2). Moreover, Condition (5) is not satisfied since k is even. Then $d_\lambda = 0$ by induction on $|\lambda|$, whence e is very good.

Conversely, suppose that $d_\lambda = 0$. If $k = 2$, Condition (1) is satisfied and if $k = 4$, then either Condition (3) is satisfied, or $\lambda_1, \dots, \lambda_4$ are all odd. Suppose $k > 4$. Condition (2) is not satisfied since $d_\lambda = d_{\lambda_*} + 2$ in this case. If Condition (1) is satisfied then $d_{\lambda_*} = 0$ and λ is a concatenation of λ^* and $(\lambda_{k-1}, \lambda_k)$. If Condition (4) is satisfied, then $d_{\lambda_*} = 0$ and λ is a concatenation of λ_* and a sequence satisfying Condition (3), whence the assertion by induction on $|\lambda|$ since Condition (5) is not satisfied when k is even. □

5.3. Assume in this subsection that $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfies the following condition:

($*$) *For some $k' \in \{2, \dots, k\}$, λ_i is even for all $i \leqslant k'$, and $(\lambda_{k'+1}, \dots, \lambda_k)$ is very good.*

In particular, k' is even and by Lemma 5.11, $\lambda_{k'+1}$ is odd and λ is not very good. For example, the sequences $\lambda = (6, 6, 4, 4, 3, 2, 2)$ and $(6, 6, 4, 4, 3, 3, 3, 2, 2, 1)$ satisfy the condition ($*$) with $k' = 4$. Define a sequence $\nu = (\nu_1, \dots, \nu_k)$ of integers of $\{1, \dots, \ell\}$ by

$$\forall\, i \in \{1, \dots, k'\}, \qquad \nu_i := \frac{\lambda_1 + \cdots + \lambda_i}{2}.$$

If $k' = k$, then $\nu_k = (\lambda_1 + \cdots + \lambda_k)/2 = r(\lambda)/2 = \ell$. Define elements $p_1, \ldots, p_{k'}$ of $S(\mathfrak{g}^e)$ as follows:

- if $k' < k$, set for $i \in \{1, \ldots, k'\}$, $p_i := {}^e q_{\nu_i}$,
- if $k' = k$, set for $i \in \{1, \ldots, k'-1\}$, $p_i := {}^e q_{\nu_i}$ and set $p_k := ({}^e q_{\nu_k})^2$. In this case, set also $\tilde{p}_k := {}^e q_{\nu_k}$.

Remind that δ_i is the degree of ${}^e q_i$ for $i = 1, \ldots, \ell$. The following lemma is a straightforward consequence of [PPY07, Remark 4.2]:

Lemma 5.12. (i) *For all $i \in \{1, \ldots, k'\}$,* $\deg\, p_i = i$.

(ii) *Set $\nu_0 := 0$. Then for $i \in \{1, \ldots, k'\}$ and $r \in \{1, \ldots, \nu_{k'} - 1\}$,*

$$\delta_r = i \iff \nu_{i-1} < r \leqslant \nu_i.$$

In particular, for $r \in \{1, \ldots, \nu_{k'} - 2\}$, $\delta_r < \delta_{r+1}$ if and only if r is a value of the sequence ν.

Example 5.13. Consider the partition $\lambda = (8, 8, 4, 4, 4, 4, 2, 2, 1, 1)$ of 38. Then $k = 10$, $k' = 8$ and $\nu = (4, 8, 10, 12, 14, 16, 17, 18)$. We represent in Table 1 the degrees of the polynomials $p_1, \ldots, p_8$ and ${}^e q_1, \ldots, {}^e q_{18}$. Note that $\deg\, {}^e q_{19} = 5$. In the table, the common degree of the polynomials appearing on the ith column is i.

	${}^e q_4 = p_1$	${}^e q_8 = p_2$						
	${}^e q_3$	${}^e q_7$						
	${}^e q_2$	${}^e q_6$	${}^e q_{10} = p_3$	${}^e q_{12} = p_4$	${}^e q_{14} = p_5$	${}^e q_{16} = p_6$		
	${}^e q_1$	${}^e q_5$	${}^e q_9$	${}^e q_{11}$	${}^e q_{13}$	${}^e q_{15}$	${}^e q_{17} = p_7$	${}^e q_{18} = p_8$
degrees	1	2	3	4	5	6	7	8

TABLE 1.

Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{g}$ generated by e, h, f and decompose $\mathbb{V}$ into simple $\mathfrak{s}$-modules $\mathbb{V}_1, \ldots, \mathbb{V}_k$ of dimension $\lambda_1, \ldots, \lambda_k$ respectively. One can order them so that for $i \in \{1, \ldots, k'/2\}$, $\mathbb{V}_{(2(i-1)+1)'} = \mathbb{V}_{2i}$. For $i \in \{1, \ldots, k\}$, denote by e_i the restriction to $\mathbb{V}_i$ of e and set $\varepsilon_i := e_i^{\lambda_i - 1}$. Then e_i is a regular nilpotent element of $\mathfrak{gl}(\mathbb{V}_i)$ and $(\mathrm{ad}\, h)\varepsilon_i = 2(\lambda_i - 1)\varepsilon_i$.

For $i \in \{1, \ldots, k'/2\}$, set

$$\mathbb{V}[i] := \mathbb{V}_{2(i-1)+1} + \mathbb{V}_{2i}$$

and set

$$\mathbb{V}[0] := \mathbb{V}_{k'+1} \oplus \cdots \oplus \mathbb{V}_k.$$

Then for $i \in \{0, 1, \ldots, k'/2\}$, denote by $\mathfrak{g}_i$ the simple Lie algebra $\mathfrak{so}(\mathbb{V}[i])$. For $i \in \{1, \ldots, k'/2\}$, $e_{2(i-1)+1} + e_{2i}$ is an even nilpotent element of $\mathfrak{g}_i$ with Jordan blocks of size $(\lambda_{2(i-1)+1}, \lambda_{2i})$. Let $i \in \{1, \ldots, k'/2\}$ and set:

$$z_i := \varepsilon_{2(i-1)+1} + \varepsilon_{2i}.$$

Then z_i lies in the center of $\mathfrak{g}^e$ and

$$(\mathrm{ad}\, h)z_i = 2(\lambda_{2(i-1)+1} - 1)z_i = 2(\lambda_{2i} - 1)z_i.$$

Moreover, $2(\lambda_{2i} - 1)$ is the highest weight of $\mathrm{ad}\, h$ acting on $\mathfrak{g}_i^e := \mathfrak{g}_i \cap \mathfrak{g}^e$, and the intersection of the $2(\lambda_{2i} - 1)$-eigenspace of $\mathrm{ad}\, h$ with $\mathfrak{g}_i^e$ is spanned by z_i, see for instance [Y09, §1]. Set

$$\overline{\mathfrak{g}} \quad := \quad \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k'/2} \quad = \quad \mathfrak{so}(\mathbb{V}[0]) \oplus \mathfrak{so}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{so}(\mathbb{V}[k'/2])$$

and denote by $\bar{\mathfrak{g}}^e$ (resp. $\bar{\mathfrak{g}}^f$) the centralizer of e (resp. f) in $\bar{\mathfrak{g}}$. For $p \in \mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\bar{\mathfrak{g}}^f \simeq (\bar{\mathfrak{g}}^e)^*$; it is an element of $\mathrm{S}(\bar{\mathfrak{g}}^e)$. Our goal is to describe the elements $\overline{p}_1, \ldots, \overline{p}_{k'}$ (see Proposition 5.18). The motivation comes from Lemma 5.14.

Let $\mathfrak{g}^f_{\mathrm{reg}}$ (resp. $\bar{\mathfrak{g}}^f_{\mathrm{reg}}$) be the set of elements $x \in \mathfrak{g}^f$ (resp. $\bar{\mathfrak{g}}^f$) such that x is a regular linear form on $\mathfrak{g}^e$ (resp. $\bar{\mathfrak{g}}^e$).

Lemma 5.14. (i) *The intersection $\mathfrak{g}^f_{\mathrm{reg}} \cap \bar{\mathfrak{g}}^f$ is a dense open subset of $\bar{\mathfrak{g}}^f_{\mathrm{reg}}$.*

(ii) *The morphism*

$$\theta: \quad G_0^e \times \bar{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \quad (g, x) \longmapsto g.x$$

is a dominant morphism from $G_0^e \times \bar{\mathfrak{g}}^f$ to $\mathfrak{g}^f$.

Proof. (i) Since λ satisfies the condition $(*)$, it satisfies the condition (1) of the proof of [Y06, §4, Lemma 3] and so, $\mathfrak{g}^f_{\mathrm{reg}} \cap \bar{\mathfrak{g}}^f$ is a dense open subset of $\bar{\mathfrak{g}}^f$. Moreover, since $\mathfrak{g}^e$ and $\bar{\mathfrak{g}}^e$ have the same index by [Y06, Theorem 3], $\mathfrak{g}^f_{\mathrm{reg}} \cap \bar{\mathfrak{g}}^f$ is contained in $\bar{\mathfrak{g}}^f_{\mathrm{reg}}$.

(ii) Let $\mathfrak{m}$ be the orthogonal complement to $\bar{\mathfrak{g}}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .\,, . \rangle$. Since the restriction to $\bar{\mathfrak{g}} \times \bar{\mathfrak{g}}$ of $\langle .\,, . \rangle$ is nondegenerate, $\mathfrak{g} = \bar{\mathfrak{g}} \oplus \mathfrak{m}$ and $[\bar{\mathfrak{g}}, \mathfrak{m}] \subset \mathfrak{m}$. Set $\mathfrak{m}^e := \mathfrak{m} \cap \mathfrak{g}^e$. Since the restriction to $\bar{\mathfrak{g}}^f \times \bar{\mathfrak{g}}^e$ of $\langle .\,, . \rangle$ is nondegenerate, we get the decomposition

$$\mathfrak{g}^e = \bar{\mathfrak{g}}^e \oplus \mathfrak{m}^e$$

and $\mathfrak{m}^e$ is the orthogonal complement to $\bar{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Moreover, $[\bar{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$.

By (i), $\mathfrak{g}^f_{\mathrm{reg}} \cap \bar{\mathfrak{g}}^f \neq \varnothing$. Let $x \in \mathfrak{g}^f_{\mathrm{reg}} \cap \bar{\mathfrak{g}}^f$. The tangent map at $(1_{\mathfrak{g}}, x)$ of θ is the linear map

$$\mathfrak{g}^e \times \bar{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (u, y) \longmapsto u.x + y,$$

where $u.$ denotes the coadjoint action of u on $\mathfrak{g}^f \simeq (\mathfrak{g}^e)^*$. The index of $\bar{\mathfrak{g}}^e$ is equal to the index of $\mathfrak{g}^e$ and $[\bar{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$. So, the stabilizer of x in $\bar{\mathfrak{g}}^e$ coincides with the stabilizer of x in $\mathfrak{g}^e$. In particular, $\dim \mathfrak{m}^e.x = \dim \mathfrak{m}^e$. As a result, θ is a submersion at $(1_{\mathfrak{g}}, x)$ since $\dim \mathfrak{g}^f = \dim \mathfrak{m}^e + \dim \bar{\mathfrak{g}}^f$. In conclusion, θ is a dominant morphism from $G_0^e \times \bar{\mathfrak{g}}^f$ to $\mathfrak{g}^f$. □

Let $(\mu_1, \ldots, \mu_m)$ be the strictly decreasing sequence of the values of the sequence $(\lambda_1, \ldots, \lambda_{k'})$ and let $k_1, \ldots, k_m$ be the multiplicity of $\mu_1, \ldots, \mu_m$ respectively in this sequence. By our assumption, the integers $\mu_1, \ldots, \mu_m, k_1, \ldots, k_m$ are all even. Notice that $k_1 + \cdots + k_m = k'$. The set $\{1, \ldots, k'\}$ decomposes into parts $K_1, \ldots, K_m$ of cardinality $k_1, \ldots, k_m$ respectively given by:

$$\forall\, s \in \{1, \ldots, m\}, \qquad K_s := \{k_0 + \cdots + k_{s-1} + 1, \ldots, k_0 + \cdots + k_s\}.$$

Here, the convention is that $k_0 := 0$.

Remark 5.15. For $s \in \{1, \ldots, m\}$ and $i \in K_s$,

$$\nu_i := k_0(\frac{\mu_0}{2}) + \cdots + k_{s-1}(\frac{\mu_{s-1}}{2}) + j(\frac{\mu_s}{2}),$$

where $j = i - (k_0 + \cdots + k_{s-1})$ and $\mu_0 = 0$.

Decompose also the set $\{1, \ldots, k'/2\}$ into parts $I_1, \ldots, I_m$ of cardinality $k_1/2, \ldots, k_m/2$ respectively, with

$$\forall\, s \in \{1, \ldots, m\}, \qquad I_s := \{\frac{k_0 + \cdots + k_{s-1}}{2} + 1, \ldots, \frac{k_0 + \cdots + k_s}{2}\}.$$

For $p \in \mathrm{S}(\mathfrak{g}^e)$ an eigenvector of $\mathrm{ad}\, h$, denote by $\mathrm{wt}(p)$ its $\mathrm{ad}\, h$-weight.

Lemma 5.16. *Let $s \in \{1,\dots,m\}$ and $i \in K_s$.*
(i) *Set $j = i - (k_0 + \cdots + k_{s-1})$. Then,*

$$\mathrm{wt}(\overline{p}_i) = 2(2\nu_i - i) = \sum_{l=1}^{s-1} 2k_l(\mu_l - 1) + 2j(\mu_s - 1).$$

Moreover, if $p \in \{{}^e q_1,\dots,{}^e q_{\ell-1},({}^e q_\ell)^2\}$ is of degree i, then $\mathrm{wt}(p) = \mathrm{wt}(\overline{p}) \leqslant 2(2\nu_i - i)$ and the equality holds if and only if $p = p_i$.
(ii) *The polynomial $\overline{p}_i$ is in $\Bbbk[z_l,\ l \in I_1 \cup \ldots \cup I_s]$.*

Proof. (i) This is a consequence of [PPY07, Lemma 4.3] (or [Y09, Theorem 6.1]), Lemma 5.12 and Remark 5.15.
(ii) Let $\tilde{\mathfrak{g}}^f$ be the centralizer of f in $\tilde{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$, and let ${}^e\overline{Q}_{2\nu_i}$ be the initial homogenous component of the restriction to

$$(\mathfrak{gl}(\mathbb{V}[0]) \oplus \mathfrak{gl}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{gl}(\mathbb{V}[k'/2])) \cap \tilde{\mathfrak{g}}^f$$

of the polynomial function $x \mapsto Q_{2\nu_i}(e + x)$. Since $\overline{p}_i \neq 0$, $\overline{p}_i$ is the restriction to $\tilde{\mathfrak{g}}^f$ of ${}^e\overline{Q}_{2\nu_i}$ and we have

$$\mathrm{wt}({}^e\overline{Q}_{2\nu_i}) = \mathrm{wt}(\overline{p}_i) = 2(2\nu_i - i), \qquad \deg {}^e\overline{Q}_{2\nu_i} = \deg \overline{p}_i = i.$$

Then, by (i) and [PPY07, Lemma 4.3], ${}^e\overline{Q}_{2\nu_i}$ is a sum of monomials whose restriction to $\tilde{\mathfrak{g}}^f$ is zero and of monomials of the form

$$(\varepsilon_{\varsigma^{(1)}1} \ldots \varepsilon_{\varsigma^{(1)}k_1}) \cdots (\varepsilon_{\varsigma^{(s-1)}1} \ldots \varepsilon_{\varsigma^{(s-1)}k_{s-1}})(\varepsilon_{\varsigma^{(s)}j_1} \ldots \varepsilon_{\varsigma^{(s)}j_i})$$

where $j_1 < \cdots < j_i$ are integers of K_s, and $\varsigma^{(1)},\dots,\varsigma^{(s-1)}$, $\varsigma^{(s)}$ are permutations of $K_1,\dots,K_{s-1}$, $\{j_1,\dots,j_i\}$ respectively. Hence, $\overline{p}_i$ is in $\Bbbk[z_l,\ l \in I_1 \cup \ldots \cup I_s]$. More precisely, for $l \in I_1 \cup \ldots \cup I_s$, the element z_l appears in $\overline{p}_i$ with a multiplicity at most 2 since $z_l = \varepsilon_{2(l-1)+1} + \varepsilon_{2l}$. □

Let $s \in \{1,\dots,m\}$ and $i \in K_s$. In view of Lemma 5.16,(ii), we aim to give an explicit formula for $\overline{p}_i$ in term of the elements $z_1,\dots,z_{k'/2}$. Besides, according to Lemma 5.16,(ii), we can assume that $s = m$. As a first step, we state inductive formulae. If $k' > 2$, set

$$\overline{\mathfrak{g}}' := \mathfrak{so}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{so}(\mathbb{V}[k'/2 - 1]),$$

and let $\overline{p}'_1,\dots,\overline{p}'_{k'}$ be the restrictions to $(\overline{\mathfrak{g}}')^f := \overline{\mathfrak{g}}' \cap \mathfrak{g}^f$ of $\overline{p}_1,\dots,\overline{p}_{k'}$ respectively. Note that $\overline{p}'_{k'-1} = \overline{p}'_{k'} = 0$. Set by convention $k_0 := 0$, $p_0 := 1$, $p'_0 := 1$ and $p_{-1} := 0$. It will be also convenient to set

$$k^* := k_0 + \cdots + k_{m-1}.$$

Lemma 5.17. (i) *If $k_m = 2$, then*

$$\overline{p}_{k^*+1} = -2\,\overline{p}'_{k^*}\, z_{k'/2} \quad \text{and} \quad \overline{p}_{k^*+2} = \overline{p}'_{k^*}\,(z_{k'/2})^2.$$

(ii) *If $k_m > 2$, then*

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2}$$

and for $j = 2,\dots,k_m$,

$$\overline{p}_{k^*+j} = \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\,(z_{k'/2})^2.$$

Proof. For $i = 1, \dots, k'/2$, let w_i be the element of $\mathfrak{g}_i^f := \mathfrak{g}_i \cap \mathfrak{g}^f$ such that

$$(\mathrm{ad}\, h)w_i = -2(\lambda_{2i} - 1)w_i \quad \text{ and } \quad \det(e_i + w_i) = 1.$$

Remind that $p_i(y)$, for $y \in \mathfrak{g}^f$, is the initial homogenous component of the coefficient of the term $T^{\dim \mathbb{V} - 2\nu_i}$ in the expression $\det(T - e - y)$. By Lemma 5.16,(ii), in order to describe $\overline{p}_i$, it suffices to compute $\det(T - e - s_1 w_1 - \dots - s_{k'/2} w_{k'/2})$, with $s_1, \dots, s_{k'/2}$ in $\Bbbk$.

1) To start with, consider the case $k' = k_m = 2$. By Lemma 5.16, $p_1 = a z_1$ and $p_2 = b z_1^2$ for some $a, b \in \Bbbk$. One has,

$$\det(T - e - s_1 w_1) = T^{2\mu_1} - 2 s_1 T^{\mu_1} + s_1^2.$$

As a result, $a = -2$ and $b = 1$. This proves (i) in this case.

2) Assume from now that $k' > 2$. Setting $e' := e_1 + \dots + e_{k'/2-1}$, observe that

$$\begin{aligned}
(3) \qquad \det(T - e - s_1 w_1 - \dots - s_{k'/2} w_{k'/2}) \\
&= \det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1}) \det(T - e_{k'/2} - s_{k'/2} w_{k'/2}) \\
&= \det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1}) (T^{2\mu_m} - 2 s_{k'/2} T^{\mu_m} + s_{k'/2}^2)
\end{aligned}$$

where the latter equality results from Step (1).

(i) If $k_m = 2$, then $k^* = k' - 2$ and the constant term in $\det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1})$ is $\overline{p}'_{k^*}$. By Lemma 5.16,(i),

$$\mathrm{wt}(\overline{p}_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}(z_{k'/2})$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1})$ of this weight. Similarly,

$$\mathrm{wt}(\overline{p}_{k^*+2}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}((z_{k'/2})^2)$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1})$ of this weight. As a consequence, the equalities follow.

(ii) Suppose $k_m > 2$. Then by Lemma 5.16,(i),

$$\mathrm{wt}(\overline{p}_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}(z_{k'/2}).$$

Moreover, $\overline{p}'_{k^*+1}$ and $\overline{p}'_{k^*}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1})$ of this weight with degree $k^* + 1$ and k^* respectively. Similarly, by Lemma 5.16,(i), for $j \in \{2, \dots, k_m\}$,

$$\mathrm{wt}(\overline{p}_{k^*+j}) = \mathrm{wt}(\overline{p}'_{k^*+j}) = \mathrm{wt}(\overline{p}'_{k^*+j-1}) + \mathrm{wt}(z_{k'/2}) = \mathrm{wt}(\overline{p}'_{k^*+j-2}) + \mathrm{wt}((z_{k'/2})^2).$$

Moreover, $\overline{p}'_{k^*+j}$, $\overline{p}'_{k^*+j-1}$ and $\overline{p}'_{k^*+j-2}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1 w_1 - \dots - s_{k'/2-1} w_{k'/2-1})$ of this weight with degree $k^* + j$, $k^* + j - 1$ and $k^* + j - 2$ respectively.

In both cases, this forces the inductive formula (ii) through the factorization (3). □

For a subset $I = \{i_1, \dots, i_l\} \subseteq \{1, \dots, k'/2\}$ of cardinality l, denote by $\sigma_{I,1}, \dots, \sigma_{I,l}$ the elementary symmetric functions of $z_{i_1}, \dots, z_{i_l}$:

$$\forall\, j \in \{1, \dots, l\}, \qquad \sigma_{I,j} = \sum_{1 \leqslant a_1 < a_2 < \dots < a_j \leqslant l} z_{i_{a_1}} z_{i_{a_2}} \dots z_{i_{a_j}}.$$

Set also $\sigma_{I,0} := 1$ and $\sigma_{I,j} := 0$ if $j > l$ so that $\sigma_{I,j}$ is well defined for any nonnegative integer j. Set at last $\sigma_{I,j} := 1$ for any j if $I = \varnothing$. If $I = I_s$, with $s \in \{1, \dots, m\}$, denote by $\sigma_j^{(s)}$, for $j \geqslant 0$, the elementary symmetric function $\sigma_{I_s,j}$.

Proposition 5.18. *Let $s \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_s\}$. Then*

$$\overline{p}_{k_0+\cdots+k_{s-1}+j} = (-1)^j \, \overline{p}_{k_0+\cdots+k_{s-1}} \sum_{r=0}^{j} \sigma^{(s)}_{j-r}\sigma^{(s)}_{r} = (-1)^j \, (\sigma^{(1)}_{k_0/2} \dots \sigma^{(s-1)}_{k_{s-1}/2})^2 \sum_{r=0}^{j} \sigma^{(s)}_{j-r}\sigma^{(s)}_{r}.$$

Example 5.19. If $m = 1$, then $k' = k_1$ and

$$p_1 = -\sigma^{(1)}_1\sigma^{(1)}_0 - \sigma^{(1)}_0\sigma^{(1)}_1 = -2\sigma^{(1)}_1 = -2(z_1 + \cdots + z_{k'/2}),$$

$$p_2 = \sigma^{(1)}_2\sigma^{(1)}_0 + (\sigma^{(1)}_1)^2 + \sigma^{(1)}_0\sigma^{(1)}_2 = 2\sigma^{(1)}_2 + (\sigma^{(1)}_1)^2,$$

$$\dots,$$

$$\overline{p}_{k'} = (\sigma^{(1)}_{k'/2})^2 = (z_1 z_2 \dots z_{k'/2})^2.$$

Proof. By Lemma 5.16,(ii), we can assume that $s = m$. Retain the notations of Lemma 5.17. In particular, set again

$$k^* := k_0 + \cdots + k_{m-1}.$$

We prove the statement by induction on $k'/2$. If $k' = 2$, then $m = 1$, $k_m = k' = 2$ and the statement follows from by Lemma 5.17,(i). Assume now that $k' > 2$ and the statement true for the polynomials $\overline{p}'_1, \dots, \overline{p}'_{k'-1}$.

If $k_m = 2$, the statement follows from Lemma 5.17,(i).

Assume $k_m > 2$. For any $r \geqslant 0$, we set $\sigma'_r := \sigma_{I',r}$ where $I' = \{\frac{k^*}{2} + 1, \dots, \frac{k'}{2} - 1\} \subset I_m$. In particular, $\sigma'_0 = 1$ by convention. Observe that for any $r \geqslant 1$,

$$\sigma^{(m)}_r = \sigma'_r + \sigma'_{r-1} z_{k'/2}.$$

Setting $\sigma'_{-1} := 0$, the above equality remains true for $r = 0$. By the induction hypothesis and by Lemma 5.17,(ii), for $j \in \{2, \dots, k_m\}$,

$$\begin{aligned}
\overline{p}_{k^*+j} &= \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\,(z_{k'/2})^2 \\
&= \overline{p}_{k^*}((-1)^j \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r - 2(-1)^{j-1} \sum_{r=0}^{j-1} \sigma'_{j-r-1}\sigma'_r\, z_{k'/2} + (-1)^{j-2} \sum_{r=0}^{j-2} \sigma'_{j-r-2}\sigma'_r\, z^2_{k'/2}). \\
&= (-1)^j\,\overline{p}_{k^*}(\sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r + 2\,(\sum_{r=0}^{j-1} \sigma'_{j-r-1}\sigma'_r)\, z_{k'/2} + (\sum_{r=0}^{j-2} \sigma'_{j-r-2}\sigma'_r)\, z^2_{k'/2})
\end{aligned}$$

since $\overline{p}'_{k^*} = \overline{p}_{k^*}$. On the other hand, we have

$$\begin{aligned}
\sum_{r=0}^{j} \sigma^{(m)}_{j-r}\sigma^{(m)}_r &= \sum_{r=0}^{j} (\sigma'_{j-r} + \sigma'_{j-r-1} z_{k'/2})(\sigma'_r + \sigma'_{r-1} z_{k'/2}) \\
&= \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r + (\sum_{r=0}^{j} \sigma'_{j-r-1}\sigma'_r + \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_{r-1})\, z_{k'/2} + (\sum_{r=0}^{j} \sigma'_{j-r-1}\sigma'_{r-1})\, z^2_{k'/2} \\
&= \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r + 2\,(\sum_{r=0}^{j-1} \sigma'_{j-r-1}\sigma'_r)\, z_{k'/2} + (\sum_{r=0}^{j-2} \sigma'_{j-r-2}\sigma'_r)\, z^2_{k'/2}.
\end{aligned}$$

Thereby, for any $j \in \{2, \dots, k_m\}$, we get

$$\overline{p}_{k^*+j} = (-1)^j\,\overline{p}_{k^*} \sum_{r=0}^{j} \sigma^{(m)}_{j-r}\sigma^{(m)}_r.$$

For $j = 1$, since $\overline{p}'_{k^*} = \overline{p}_{k^*}$, by Lemma 5.17,(ii), and our induction hypothesis,

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma'_1) - 2\,\overline{p}_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma_1^{(m)}).$$

This proves the first equality of the proposition.

For the second one, it suffices to prove by induction on $s \in \{1, \ldots, m\}$ that

$$\overline{p}_{k_0+\cdots+k_{s-1}} = (\sigma^{(1)}_{k_0/2} \ldots \sigma^{(s-1)}_{k_{s-1}/2})^2.$$

For $s = 1$, then $\overline{p}_{k_0+\cdots+k_{s-1}} = \overline{p}_0 = 1$ and $\sigma_{\varnothing,0} = 1$ by convention. Assume $s > 2$ and the statement true for $1, \ldots, s-1$. By the first equality with $j = k_s$, $\overline{p}_{k_0+\cdots+k_s} = (-1)^{k_s}\, \overline{p}_{k_0+\cdots+k_{s-1}}(\sigma^{(s)}_{k_s/2})^2$, whence the statement by induction hypothesis since k_s is even. □

Remark 5.20. Remind that the polynomial $\tilde{p}_k$ was defined before Lemma 5.12. As a by product of the previous formula, whenever $k' = k$, we obtain

$$\overline{\tilde{p}}_k = \sigma^{(1)}_{k_0/2} \ldots \sigma^{(m)}_{k_m/2}.$$

For $s \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_s\}$, set

$$\rho_{k_0+\cdots+k_{s-1}+j} := \frac{\overline{p}_{k_0+\cdots+k_{s-1}+j}}{\overline{p}_{k_0+\cdots+k_{s-1}}}.$$

Proposition 5.18 says that $\rho_{k_0+\cdots+k_{s-1}+j}$ is an element of $\mathrm{Frac}(\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}) \cap \mathrm{S}(\mathfrak{g}^e) = \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Lemma 5.21. *Let $s \in \{1, \ldots, m\}$ and $j \in \{k_s/2 + 1, \ldots, k_s\}$. There is a polynomial $R^{(s)}_j$ of degree j such that*

$$\rho_{k_0+\cdots+k_{s-1}+j} = R^{(s)}_j(\rho_{k_0+\cdots+k_{s-1}+1}, \ldots, \rho_{k_0+\cdots+k_{s-1}+k_s/2}).$$

In particular, for any $j \in \{k_1/2 + 1, \ldots, k_1\}$, we have

$$\overline{p}_j = R^{(1)}_j(\overline{p}_1, \ldots, \overline{p}_{k_1/2}).$$

Proof. 1) Prove by induction on $j \in \{1, \ldots, k_s/2\}$ that for some polynomial $T^{(s)}_j$ of degree j,

$$\sigma^{(s)}_j = T^{(s)}_j(\rho_{k_0+\cdots+k_{s-1}+1}, \ldots, \rho_{k_0+\cdots+k_{s-1}+j}).$$

By Proposition 5.18, $\rho_{k_0+\cdots+k_{s-1}+1} = -(\sigma^{(s)}_1\sigma^{(s)}_0 + \sigma^{(s)}_0\sigma^{(s)}_1) = -2\sigma^{(s)}_1$. Hence, the statement is true for $j = 1$. Suppose $j \in \{2, \ldots, k_s/2\}$ and the statement true for $\sigma^{(s)}_1, \ldots, \sigma^{(s)}_{j-1}$. Since $j \leqslant k_s/2$, $\sigma^{(s)}_j \neq 0$, and by Proposition 5.18,

$$\rho_{k_0+\cdots+k_{s-1}+j} \;=\; (-1)^j(\sigma^{(s)}_j\sigma^{(s)}_0 + \sigma_0\sigma^{(s)}_j) + (-1)^j\sum_{r=1}^{j-1}\sigma^{(s)}_{j-r}\sigma^{(s)}_r \;=\; 2(-1)^j\sigma^{(s)}_j + (-1)^j\sum_{r=1}^{j-1}\sigma^{(s)}_{j-r}\sigma^{(s)}_r.$$

So, the statement for j follows from our induction hypothesis.

2) Let $j \in \{k_s/2 + 1, \ldots, k_s\}$. Proposition 5.18 shows that $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in $\sigma^{(s)}_1, \ldots, \sigma^{(s)}_{k_s/2}$. Hence, by Step 1), $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in

$$\rho_{k_0+\cdots+k_{s-1}+1}, \ldots, \rho_{k_0+\cdots+k_{s-1}+k_s/2}.$$

Furthermore, by Proposition 5.18 and Step (1), this polynomial has degree j. □

Remark 5.22. By Remark 5.20 and the above proof, if $k' = k$ then for some polynomial $\tilde{R}$ of degree $k_m/2$,

$$\frac{\overline{\tilde{p}}_k}{\sigma^{(1)}_{k_0/2} \ldots \sigma^{(m-1)}_{k_{m-1}/2}} = \sigma^{(m)}_{k_m/2} = \tilde{R}(\rho_{k_0+\cdots+k_{m-1}+1}, \ldots, \rho_{k_0+\cdots+k_{m-1}+k_m/2}).$$

Theorem 5.23. (i) *Assume that λ satisfies the condition $(*)$ and that $\lambda_1 = \cdots = \lambda_{k'}$. Then e is good.*

(ii) *Assume that $k = 4$ and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even. Then e is good.*

For example, $(6,6,6,6,5,3)$ satisfies the hypothesis of (i) and $(6,6,4,4)$ satisfies the hypothesis of (ii).

Remark 5.24. If λ satisfies the condition $(*)$ then by Lemma 5.7,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = k'.$$

Indeed, if k is odd, then $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'}$ where $\lambda' = (\lambda_1, \ldots, \lambda_{k'}, \lambda_{k'+1})$ so that $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'} = n_{\lambda'} = k'$ since $\lambda_{k'+1}$ is odd. If k is even, then $d_\lambda = n_{\lambda'} = k'$ where $\lambda' = (\lambda_1, \ldots, \lambda_{k'})$.

Proof. (i) In the previous notations, the hypothesis means that $m = 1$ and $k' = k_m$. According to Lemma 5.21 and Lemma 5.14, for $j \in \{k'/2+1, \ldots, k'-1\}$,

$$p_j = R_j^{(1)}(p_1, \ldots, p_{k'/2}),$$

where $R_j^{(1)}$ is a polynomial of degree j. Moreover, if $k' = k$, then by Remark 5.22 and Lemma 5.14,

$$\tilde{p}_k = \tilde{R}(p_1, \ldots, p_{k/2}),$$

where $\tilde{R}$ is a polynomial of degree $k/2$.

- If $k' < k$, set for any $j \in \{k'/2+1, \ldots, k'\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \ldots, q_{\nu_{k'/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k'/2+1, \ldots, k'\}, \quad \deg {}^e r_j \geqslant j+1.$$

- If $k' = k$, set for $j \in \{k/2+1, \ldots, k'-1\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \ldots, q_{\nu_{k'/2}}) \text{ and } r_k := q_{\nu_k} - \tilde{R}(q_{\nu_1}, \ldots, q_{\nu_{k/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k/2+1, \ldots, k-1\}, \quad \deg {}^e r_j \geqslant j+1 \quad \text{and} \quad \deg {}^e r_k \geqslant k/2+1.$$

In both cases,

$$\{q_j \mid j \in \{1, \ldots, \ell\} \smallsetminus \{\nu_{k'/2+1}, \ldots, \nu_{k'}\}\} \ \cup\ \{r_{k'/2+1}, \ldots, r_{k'}\}$$

is a homogenous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denote by $\hat{\delta}$ the sum of the degrees of the polynomials

$${}^e q_j,\ j \in \{1, \ldots, \ell\} \smallsetminus \{\nu_{k'/2+1}, \ldots, \nu_{k'}\}, \quad {}^e r_{k'/2+1}, \ldots, {}^e r_{k'}.$$

The above discussion shows that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + k'/2$. By Remarks 5.24, we obtain

$$\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0.$$

In conclusion, by [PPY07, Theorem 2.1] and Theorem 3.6, e is good.

(ii) In the previous notations, the hypothesis means that $k' = k = 4$. If $m = 1$ the statement is a consequence of (i). Assume that $m = 2$. Then by Proposition 5.18, $\overline{p}_1 = -2z_1$, $\overline{p}_2 = z_1^2$, $\overline{p}_3 = -2z_1^2 z_2$ and $\overline{p}_4 = (z_1 z_2)^2$. Moreover, $\overline{\tilde{p}}_4 = z_1 z_2$. Hence, by Lemma 5.14, $p_2 = \frac{1}{4}p_1^2$ and $p_3 = p_1 \tilde{p}_4$. Set $r_2 := q_{\nu_2} - \frac{1}{4}q_{\nu_1}^2$ and $r_3 := q_{\nu_3} - q_{\nu_1} q_{\nu_4}$. Then $\deg {}^e r_2 \geqslant 3$ and $\deg {}^e r_3 \geqslant 4$. Moreover,

$$\{q_1, \ldots, q_\ell\} \smallsetminus \{q_{\nu_2}, q_{\nu_3}\} \cup \{r_2, r_3\}$$

is a homogenous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denoting by $\hat{\delta}$ the sum of the degrees of the polynomials

$$\{{}^e q_1, \ldots, {}^e q_\ell\} \smallsetminus \{{}^e q_{\nu_2}, {}^e q_{\nu_3}\} \cup \{{}^e r_2, {}^e r_3\},$$

we obtain that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + 2$. But $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = k' = 4$ by Remark 5.24. So, $\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0$. In conclusion, by [PPY07, Theorem 2.1] and Theorem 3.6, e is good. □

Remark 5.25. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$, with $\dim \mathbb{V} = 12$, and that e belongs to the nilpotent orbit of $\mathfrak{g}$ associated with the partition $(5,5,1,1)$ of 12. Then the degrees of ${}^e q_1$, ${}^e q_2$, ${}^e q_3$, ${}^e q_4$, ${}^e q_5$, ${}^e q_6$ are $1,1,2,2,2,2$ respectively. Since $10 = 1{+}1{+}2{+}2{+}2{+}2 = (\dim \mathfrak{g}^e {+} \ell)/2$, the polynomial functions ${}^e q_1$, ${}^e q_2$, ${}^e q_3$, ${}^e q_4$, ${}^e q_5$, ${}^e q_6$ are algebraically independent, and by Theorem 3.6, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. One can satisfy that ${}^e q_5 = z^2$ for some z in the center $\mathfrak{z}(\mathfrak{g}^e)$ of $\mathfrak{g}^e$. Since $\mathfrak{z}(\mathfrak{g}^e)$ has dimension 3, for any other choice of homogenous generators $q_1, \ldots, q_\ell$ of $S(\mathfrak{g})^{\mathfrak{g}}$, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ cannot be generated by the elements ${}^e q_1$, ${}^e q_2$, ${}^e q_3$, ${}^e q_4$, ${}^e q_5$, ${}^e q_6$ for degree reasons.

This shows that Condition (2) of Theorem 1.2 cannot be removed to ensure that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra in the variables ${}^e q_1$, ${}^e q_2$, ${}^e q_3$, ${}^e q_4$, ${}^e q_5$, ${}^e q_6$. However, one can sometimes, as in this example, provide explicit generators.

6. Examples in simple exceptional Lie algebras

We give in this section examples of good nilpotent elements in simple exceptional Lie algebras of type $\mathbf{E}_6$, $\mathbf{F}_4$ or $\mathbf{G}_2$ which are not covered by [PPY07]. These examples are all obtained through Theorem 3.6.

According to [PPY07, Theorem 0.4] and Theorem 3.6, the elements of the minimal nilpotent orbit of $\mathfrak{g}$, for $\mathfrak{g}$ not of type $\mathbf{E}_8$, are good. In addition, as it is explained in the Introduction, the elements of the regular, or subregular, nilpotent orbit of $\mathfrak{g}$ are good. So we do not consider here these cases.

Example 6.1. Suppose that $\mathfrak{g}$ has type $\mathbf{E}_6$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 27 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{27}(\mathbb{k})$. For x in $\mathfrak{sl}_{27}(\mathbb{k})$ and for $i = 2, \ldots, 27$, let $p_i(x)$ be the coefficient of T^{27-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then $(q_2, q_5, q_6, q_8, q_9, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e,h,f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e,h,f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{27}(\mathbb{k})$. We denote by ${}^e p_i$ the initial homogenous component of the restriction to $e + \tilde{\mathfrak{g}}^f$ of p_i where $\tilde{\mathfrak{g}}^f$ is the centralizer of f in $\mathfrak{sl}_{27}(\mathbb{k})$. As usual, ${}^e q_i$ denotes the initial homogenous component of the restriction to $e + \mathfrak{g}^f$ of q_i. For $i = 2, 5, 6, 8, 9, 12$,

$$\deg {}^e p_i \leqslant \deg {}^e q_i.$$

In some cases, from the knowledge of the maximal eigenvalue of the restriction of $\mathrm{ad}\, h$ to $\mathfrak{g}$ and the $\mathrm{ad}\, h$-weight of ${}^e p_i$, it is possible to deduce that $\deg {}^e p_i < \deg {}^e q_i$. On the other hand,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} \leqslant \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

with equality if and only if ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent. From this, it is possible to deduce in some cases that e is good. These cases are listed in Table 2 where the nine columns are indexed in the following way:

1: the label of the orbit $G.e$ in the Bala-Carter classification,
2: the weighted Dynkin diagram of $G.e$,
3: the dimension of $\mathfrak{g}^e$,
4: the partition of 27 corresponding to the nilpotent element e of $\mathfrak{sl}_{27}(\mathbb{k})$,
5: the degrees of ${}^e p_2$, ${}^e p_5$, ${}^e p_6$, ${}^e p_8$, ${}^e p_9$, ${}^e p_{12}$,
6: their $\mathrm{ad}\, h$-weights,
7: the maximal eigenvalue ν of the restriction of $\mathrm{ad}\, h$ to $\mathfrak{g}$,
8: the sum Σ of the degrees of ${}^e p_2$, ${}^e p_5$, ${}^e p_6$, ${}^e p_8$, ${}^e p_9$, ${}^e p_{12}$,
9: the sum $\Sigma' = \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$.

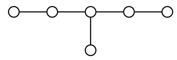
	Label		$\dim \mathfrak{g}^e$	partition	$\deg {}^e p_i$	weights	ν	Σ	Σ'
1.	D_5	2 0 2 0 2 2	10	(11,9,5,1,1)	1,1,1,1,1,1	2,8,10,14,16,22	14	6	8
2.	$E_6(a_3)$	2 0 2 0 2 0	12	$(9,7,5^2,1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	9
3.	$D_5(a_1)$	1 1 0 1 1 2	14	(8,7,6,3,2,1)	1,1,1,1,2,2	2,8,10,14,14,20	10	8	10
4.	A_5	2 1 0 1 2 1	14	$(9,6^2,5,1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	10
5.	A_4+A_1	1 1 0 1 1 1	16	$(7,6,5,4,3,2)$	1,1,1,2,2,2	2,8,10,12,14,20	8	9	11
6.	D_4	0 0 2 0 0 2	18	$(7^3,1^6)$	1,1,1,2,2,2	2,8,10,12,14,20	10	9	12
7.	$D_4(a_1)$	0 0 2 0 0 0	20	$(5^3,3^3,1^3)$	1,1,2,2,2,3	2,8,8,12,14,18	6	11	13
8.	$2A_2+A_1$	1 0 1 0 1 0	24	$(5,4^2,3^3,2^2,1)$	1,1,2,2,2,3	2,8,8,12,14,18	5	11	15

TABLE 2. Data for $\mathbf{E}_6$

In all cases, we observe that $\Sigma < \Sigma'$, i.e.,

$$\deg {}^e p_2 + \deg {}^e p_5 + \deg {}^e p_6 + \deg {}^e p_8 + \deg {}^e p_9 + \deg {}^e p_{12} < \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

So, we need some arguments that we give below.

1. Since $14 < 16$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 9, 12$.
2. Since $10 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 8, 9$.
3. Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
4. Since $10 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 8, 9$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
5. Since $8 < 10$ and $2 \times 8 < 20$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 6, 12$.
6. Since the center of $\mathfrak{g}^e$ has dimension 2 and the weights of h in the center are 2 and 10, $\deg {}^e p_5 < \deg {}^e q_5$. Moreover, since the weights of h in $\mathfrak{g}^e$ are $0, 2, 6, 10$, $\deg {}^e p_9 < \deg {}^e q_9$ and since the multiplicity of the weight 10 equals 1, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
7. Since $6 < 8$ and $2 \times 6 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 9$.
8. Since $5 < 8$, $2 \times 5 < 12$ and $3 \times 5 < 18$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 8, 9, 12$.

In cases **1**, **2**, **5**, **7**, **8**, the discussion shows that

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence, ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent and by Theorem 3.6, e is good. In cases **3**, **4**, **6**, if the above equality does not hold, then for some a in $\Bbbk^*$,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e (q_{12} - a q_6^2) = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence ${}^{e}q_2$, ${}^{e}q_5$, ${}^{e}q_6$, ${}^{e}q_8$, ${}^{e}q_9$, ${}^{e}(q_{12} - aq_6^2)$ are algebraically independent and by Theorem 3.6, e is good.

In conclusion, it remains nine unsolved nilpotent orbits in type $\mathbf{E}_6$.

Example 6.2. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{F}_4$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_4 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 26 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{26}(\mathbb{k})$. For x in $\mathfrak{sl}_{26}(\mathbb{k})$ and for $i = 2, \dots, 26$, let $p_i(x)$ be the coefficient of T^{26-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then (q_2, q_6, q_8, q_{12}) is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{26}(\mathbb{k})$. As in Example 6.1, in some cases, it is possible to deduce that e is good. These cases are listed in Table 3, indexed as in Example 6.1.

	Label	○—○⇒○—○	$\dim \mathfrak{g}^e$	partition	$\deg {}^{e}p_i$	weights	ν	Σ	Σ'
1.	$F_4(a_2)$	0 2 0 2	8	$(9, 7, 5^2)$	1,1,1,2	2,10,14,20	10	5	6
2.	C_3	1 0 1 2	10	$(9, 6^2, 5)$	1,1,1,2	2,10,14,20	10	5	7
3.	B_3	2 2 0 0	10	$(7^3, 1^5)$	1,1,2,2	2,10,12,20	10	6	7
4.	$F_4(a_3)$	0 2 0 0	12	$(5^3, 3^3, 1^2)$	1,2,2,3	2,8,12,18	6	8	8
5.	$C_3(a_1)$	1 0 1 0	14	$(5^2, 4^2, 3, 2^2, 1)$	1,2,2,3	2,8,12,18	6	8	9
6.	$\tilde{A}_2 + A_1$	0 1 0 1	16	$(5, 4^2, 3^3, 2^2)$	1,2,2,3	2,8,12,18	5	8	10

TABLE 3. Data for $\mathbf{F}_4$

For the orbits **1**, **2**, **3**, **5**,**6**, we observe that $\Sigma < \Sigma'$. So, we need some more arguments to conclude as in Example 6.1.

1. Since $10 < 14$, $\deg {}^{e}p_8 < \deg {}^{e}q_8$.

2. Since $10 < 14$, $\deg {}^{e}p_8 < \deg {}^{e}q_8$. Moreover, the multiplicity of the weight 10 equals 1 so that $\deg {}^{e}q_6 > 1$ or $\deg {}^{e}q_{12} > 2$ or ${}^{e}q_{12} \in \mathbb{k}\, {}^{e}q_6^2$.

3. The multiplicity of the weight 10 equals 1. So, either $\deg {}^{e}q_6 > 1$, or $\deg {}^{e}q_{12} > 2$, or ${}^{e}q_{12} \in \mathbb{k}\, {}^{e}q_6^2$.

5. Suppose that ${}^{e}q_2$, ${}^{e}q_6$, ${}^{e}q_8$, ${}^{e}q_{12}$ have degree $1, 2, 2, 3$. We expect a contradiction. Since the center has dimension 2 and since the multiplicity of the weight 6 equals 1, for z of weight 6 in the center, ${}^{e}q_6 \in \mathbb{k}ez$, ${}^{e}q_8 \in \mathbb{k}z^2$, ${}^{e}q_{12} \in \mathbb{k}z^3$. So, for some a and b in $\mathbb{k}^*$,

$$ {}^{e}q_2^2\, {}^{e}q_8 - a\, {}^{e}q_6^2 = 0, \qquad {}^{e}q_{12}^2 - b\, {}^{e}q_8^3 = 0 $$

Hence, $q_2, q_6, q_2^2 q_8 - aq_6^2, q_{12}^2 - bq_8^3$ are algebraically independent element of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that

$$ \deg {}^{e}q_2 + \deg {}^{e}q_6 + \deg {}^{e}(q_2^2 q_8 - aq_6^2) + \deg {}^{e}(q_{12}^2 - bq_8^3) \geqslant 1 + 2 + 5 + 7 > 2 + 3 + 9 $$

whence a contradiction by [PPY07, Theorem 2.1] (see Lemma 7.1).

6. Since $2{\times}5 < 12$ and $3{\times}5 < 18$, $\deg {}^{e}q_8 > \deg {}^{e}p_8$ and $\deg {}^{e}q_{12} > \deg {}^{e}p_{12}$.

In conclusion, it remains six unsolved nilpotent orbits in type $\mathbf{F}_4$.

Example 6.3. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{G}_2$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 7 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_7(\Bbbk)$. For x in $\mathfrak{sl}_7(\Bbbk)$ and for $i = 2, \dots, 7$, let $p_i(x)$ be the coefficient of T^{7-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then q_2, q_6 is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_7(\Bbbk)$. There is only one nonzero nilpotent orbit which is neither regular, subregular or minimal. For e in it, we deduce that e is good from Table 4, indexed as in Example 6.1, since $\Sigma = \Sigma'$.

Label			$\dim \mathfrak{g}^e$	partition	$\deg {}^e p_i$	weights	ν	Σ	Σ'
$\tilde{A}_1$	1	0	6	$(3, 2^2)$	1,3	2,6	3	4	4

Table 4. Data for $\mathbf{G}_2$

In conclusion, all elements are good in type $\mathbf{G}_2$.

7. Other examples, remarks and a conjecture

This section provides examples of nilpotent elements which satisfy the polynomiality condition but that are not good. We also obtain an example of nilpotent element in type $\mathbf{D}_7$ which does not satisfy the polynomiality condition (cf. Example 7.8). Then we conclude with some remarks and a conjecture.

7.1. **Some general results.** In this subsection, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. For p in $\mathrm{S}(\mathfrak{g})$, ${}^e p$ is the initial homogenous component of the restriction of p to the Slodowy slice $e + \mathfrak{g}^f$. Recall that $\Bbbk[e + \mathfrak{g}^f]$ identifies with $\mathrm{S}(\mathfrak{g}^e)$ by the Killing form $\langle .\,, . \rangle$ of $\mathfrak{g}$.

Let $\eta_0 \in \mathfrak{g}^e \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining the Poisson bracket on $\mathrm{S}(\mathfrak{g}^e)$ induced from the Lie bracket. According to the main theorem of [Pr02], $\mathrm{S}(\mathfrak{g}^e)$ is the graded algebra relative to the Kazhdan filtration of the finite W-algebra associated with e so that $\mathrm{S}(\mathfrak{g}^e)$ inherits another Poisson structure. The so-obtained graded algebra structure is the Slodowy graded algebra structure (see Subsection 4.1). Let $\eta \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^2 \mathfrak{g}^f$ be the bivector defining this other Poisson structure. According to [Pr02, Proposition 6.3] (see also [PPY07, §2.4]), η_0 is the initial homogenous component of η. Denote by r the dimension of $\mathfrak{g}^e$ and set:

$$\omega := \eta^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge\nolimits^{r-\ell} \mathfrak{g}^f, \qquad \omega_0 := \eta_0^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge\nolimits^{r-\ell} \mathfrak{g}^f.$$

Then ω_0 is the initial homogenous component of ω.

Let $v_1, \dots, v_r$ be a basis of $\mathfrak{g}^f$. For μ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^i \mathfrak{g}^e$, denote by $j(\mu)$ the image of $v_1 \wedge \dots \wedge v_r$ by the right interior product of μ so that

$$j(\mu) \in \mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^{r-i} \mathfrak{g}^f.$$

Lemma 7.1. *Let $q_1, \dots, q_\ell$ be some homogenous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and let $r_1, \dots, r_\ell$ be algebraically independent homogenous elements of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.*

(i) *For some homogenous element p of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*

$$\mathrm{d}r_1 \wedge \dots \wedge \mathrm{d}r_\ell = p\, \mathrm{d}q_1 \wedge \dots \wedge \mathrm{d}q_\ell.$$

(ii) *The following inequality holds:*

$$\sum_{i=1}^{\ell} \deg {}^e r_i \leqslant \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

(iii) *The polynomials* ${}^er_1, \dots, {}^er_\ell$ *are algebraically independent if and only if*

$$\sum_{i=1}^{\ell} \deg {}^er_i = \deg {}^ep + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

Proof. (i) Since $q_1, \dots, q_\ell$ are generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, for $i \in \{1, \dots, \ell\}$, $r_i = R_i(q_1, \dots, q_\ell)$ where R_i is a polynomial in ℓ indeterminates, whence the assertion with

$$p = \det(\frac{\partial R_i}{\partial q_j},\ 1 \leqslant i, j \leqslant \ell).$$

(ii) Remind that for p in $\mathrm{S}(\mathfrak{g})$, $\kappa(p)$ denotes the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e + x)$. According to [PPY07, Theorem 1.2],

$$j(\mathrm{d}\kappa(q_1) \wedge \cdots \wedge \mathrm{d}\kappa(q_\ell)) = a\omega$$

for some a in $\Bbbk^*$. Hence by (i),

$$j(\mathrm{d}\kappa(r_1) \wedge \cdots \wedge \mathrm{d}\kappa(r_\ell)) = a\kappa(p)\omega.$$

The initial homogenous component of the right-hand side is $a\,{}^ep\omega_0$ and the degree of the initial homogenous component of the left-hand side is at least

$$\deg {}^er_1 + \cdots + \deg {}^er_\ell - \ell.$$

The assertion follows since ω_0 has degree

$$\frac{1}{2}(\dim \mathfrak{g}^e - \ell).$$

(iii) If ${}^er_1, \dots, {}^er_\ell$ are algebraically independent, then the degree of the initial homogenous component of $j(\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell)$ equals

$$\deg {}^er_1 + \cdots + \deg {}^er_\ell - \ell$$

whence

$$\deg {}^er_1 + \cdots + \deg {}^er_\ell = \deg {}^ep + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

by the proof of (ii). Conversely, if the equality holds, then

$$j(\mathrm{d}\,{}^er_1 \wedge \cdots \wedge \mathrm{d}\,{}^er_\ell) = a\,{}^ep\omega_0 \tag{4}$$

by the proof of (ii). In particular, ${}^er_1, \dots, {}^er_\ell$ are algebraically independent. □

Corollary 7.2. *For* $i = 1, \dots, \ell$, *let* $r_i := R_i(q_1, \dots, q_i)$ *be a homogenous element of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ *such that* $\dfrac{\partial R_i}{\partial q_i} \neq 0$. *Then* ${}^er_1, \dots, {}^er_\ell$ *are algebraically independent if and only if*

$$\deg {}^er_1 + \cdots + \deg {}^er_\ell = \sum_{i=1}^{\ell} \deg {}^ep_i + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

with $p_i = \dfrac{\partial R_i}{\partial q_i}$ *for* $i = 1, \dots, \ell$.

Proof. Since $\dfrac{\partial R_i}{\partial q_i} \neq 0$ for all i, $r_1, \dots, r_\ell$ are algebraically independent and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = \prod_{i=1}^{\ell} \frac{\partial R_i}{\partial q_i} \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell$$

whence the corollary by Lemma 7.1,(iii). □

Let $\mathfrak{g}^f_{\mathrm{sing}}$ be the set of nonregular elements of the dual $\mathfrak{g}^f$ of $\mathfrak{g}^e$. Recall that if $\mathfrak{g}^f_{\mathrm{sing}}$ has codimension at least 2 in $\mathfrak{g}^f$, we will say that $\mathfrak{g}^e$ is *nonsingular*.

Corollary 7.3. *Let $r_1, \ldots, r_\ell$ and p be as in Lemma 7.1 and such that ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent.*

(i) *If ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$, then $\mathfrak{g}^e$ is nonsingular.*

(ii) *Assume that there are homogenous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the following conditions:*

1) *${}^e r_1, \ldots, {}^e r_\ell$ are in $\Bbbk[p_1, \ldots, p_\ell]$,*
2) *if d is the degree of a greatest divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)$, then*

$$\deg p_1 + \cdots + \deg p_\ell = d + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

Then $\mathfrak{g}^e$ is nonsingular.

Proof. (i) Suppose that ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$. Then for some ω_1 in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$ whose nullvariety in $\mathfrak{g}^f$ has codimension at least 2,

$$\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell = {}^e p\, \omega_1.$$

Therefore $j(\omega_1) = a\omega_0$ by Equality (4). Since x is in $\mathfrak{g}^f_{\mathrm{sing}}$ if and only if $\omega_0(x) = 0$, we get (i).

(ii) By Condition (1),

$$\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell = q\, \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$$

for some q in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, and for some greatest divisor q' of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$,

$$\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell = q' \omega_1.$$

So, by Equality (4),

$$qq'\, j(\omega_1) = a\, {}^e p\, \omega_0, \tag{5}$$

so that ${}^e p$ divides qq' in $\mathrm{S}(\mathfrak{g}^e)$. By Condition (2) and Equality (5), ω_0 and ω_1 have the same degree. Then qq' is in $\Bbbk^*\, {}^e p$, and for some a' in $\Bbbk^*$,

$$j(\omega_1) = a' \omega_0,$$

whence (ii), again since x is in $\mathfrak{g}^f_{\mathrm{sing}}$ if and only if $\omega_0(x) = 0$. □

The following proposition is a particular case of [JS10, §5.7]. More precisely, part (i) follows from [JS10, Remark 5.7] and part (ii) follows from [JS10, Theorem 5.7].

Proposition 7.4. *Suppose that $\mathfrak{g}^e$ is nonsingular.*

(i) *If there exist algebraically independent homogenous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \ldots, p_\ell$.

(ii) *Suppose that the semiinvariant elements of $\mathrm{S}(\mathfrak{g}^e)$ are invariant. If $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra then it is generated by homogenous polynomials $p_1, \ldots, p_\ell$ such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

7.2. **New examples.** To produce new examples, our general strategy is to apply Proposition 7.4,(i). To that end, we first apply Corollary 7.3 in order to prove that $\mathfrak{g}^e$ is nonsingular. Then, we search for independent homogenous polynomials $p_1, \dots, p_\ell$ in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the conditions of Corollary 7.3,(ii) with $d = 0$.

Example 7.5. Let e be a nilpotent element of $\mathfrak{so}(\mathbb{k}^{10})$ associated with the partition $(3, 3, 2, 2)$. Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra but e is not good as explained below.

In this case, $\ell = 5$ and let $q_1, \dots, q_5$ be as in Subsection 5.2. The degrees of ${}^eq_1, \dots, {}^eq_5$ are $1, 2, 2, 3, 2$ respectively. By a computation performed by Maple, ${}^eq_1, \dots, {}^eq_5$ satisfy the algebraic relation:

$$ {}^eq_4^2 - 4\,{}^eq_3\,{}^eq_5^2 = 0. $$

Set:

$$ r_i := \begin{cases} q_i & \text{if} \quad i = 1, 2, 3, 5 \\ q_4^2 - 4q_3q_5^2 & \text{if} \quad i = 4. \end{cases} $$

The polynomials $r_1, \dots, r_5$ are algebraically independent over $\mathbb{k}$ and

$$ \mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_5 = 2\, q_4\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_5 $$

Moreover, ${}^er_4$ has degree at least 7. Then, by Corollary 7.2, ${}^er_1, \dots, {}^er_5$ are algebraically independent since

$$ \frac{1}{2}(\dim \mathfrak{g}^e + 5) + 3 = 14 = 1 + 2 + 2 + 2 + 7, $$

and by Lemma 7.1,(ii) and (iii), ${}^er_4$ has degree 7.

A precise computation performed by Maple shows that ${}^er_3 = p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, and that ${}^er_4 = p_4\,{}^er_5$ for some polynomial p_4 of degree 5 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^er_i$ for $i = 1, 2, 5$, the polynomials $p_1, \dots, p_5$ are algebraically independent homogenous polynomials of degree $1, 2, 1, 5, 2$ respectively. Furthermore, a computation performed by Maple proves that the greatest divisors of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_5$ in $S(\mathfrak{g}^e)$ have degree 0, and that p_4 is in the ideal of $S(\mathfrak{g}^e)$ generated by p_3 and p_5. So, by Corollary 7.3,(ii), $\mathfrak{g}^e$ is nonsingular, and by Proposition 7.4,(i), $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \dots, p_5$. Moreover, e is not good since the nullvariety of $p_1, \dots, p_5$ in $(\mathfrak{g}^e)^*$ has codimension at most 4.

Example 7.6. In the same way, for the nilpotent element e of $\mathfrak{so}(\mathbb{k}^{11})$ associated with the partition $(3, 3, 2, 2, 1)$, we can prove that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by polynomials of degree $1, 1, 2, 2, 7$, $\mathfrak{g}^e$ is nonsingular but e is not good.

We also obtain that for the nilpotent element e of $\mathfrak{so}(\mathbb{k}^{12})$ (resp. $\mathfrak{so}(\mathbb{k}^{13})$) associated with the partition (5,3,2,2) or (3,3,2,2,1,1) (resp. (5,3,2,2,1), (4,4,2,2,1), or (3,3,2,2,1,1,1)), $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra, $\mathfrak{g}^e$ is nonsingular but e is not good.

We can summarize our conclusions for the small ranks. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$ associated with the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of $\dim \mathbb{V}$. If $\ell \leqslant 6$, our previous results (Corollary 5.8, Lemma 5.11, Theorem 5.23, Examples 7.5 and 7.6) show that either e is good, or e is not good but $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is nevertheless a polynomial algebra and $\mathfrak{g}^e$ is nonsingular. We describe in Table 5 the partitions λ corresponding to good e, and those corresponding to the case where e is not good. The third column of the table gives the degrees of the generators in the latter case.

Remark 7.7. The above discussion shows that there are good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Indeed, by [PPY07, §3.9], for some nilpotent element e' in $\mathbf{B}_3$, the codimension of $(\mathfrak{g}^{e'})^*_{\mathrm{sing}}$ in $(\mathfrak{g}^{e'})^*$ is 1 but, in $\mathbf{B}_3$, all nilpotent elements are good (cf. Table 5).

Type	e is good	$S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, $\mathfrak{g}^e$ is nonsingular but e is not good	degrees of the generators
$\mathbf{B}_n, \mathbf{D}_n, n \leqslant 4$	any λ	$\varnothing$	
$\mathbf{B}_5$	$\lambda \neq (3,3,2,2,1)$	$\lambda = (3,3,2,2,1)$	1, 1, 2, 2, 7
$\mathbf{D}_5$	$\lambda \neq (3,3,2,2)$	$\lambda = (3,3,2,2)$	1,1,2,2,5
$\mathbf{B}_6$	$\lambda \notin \{(5,3,2,2,1),(4,4,2,2,1),(3,3,2,2,1,1,1)\}$	$\lambda \in \{(5,3,2,2,1),(4,4,2,2,1),(3,3,2,2,1,1,1)\}$	{1, 1, 1, 2, 2, 7; 1, 1, 2, 2, 3, 6; 1, 1, 2, 2, 6, 7}
$\mathbf{D}_6$	$\lambda \notin \{(5,3,2,2),(3,3,2,2,1,1)\}$	$\lambda \in \{(5,3,2,2),(3,3,2,2,1,1)\}$	{1,1,1,2,2,5; 1,1,2,2,3,7}

TABLE 5. Conclusions for $\mathfrak{g}$ of type $\mathbf{B}_\ell$ or $\mathbf{D}_\ell$ with $\ell \leqslant 6$

7.3. **A counter-example.** From the rank 7, there are elements that do no satisfy the polynomiality condition. The following example provides a new counter-example to Premet's conjecture.

Example 7.8. Let e be a nilpotent element of $\mathfrak{so}(\Bbbk^{14})$ associated with the partition $(3,3,2,2,2,2)$. Then e does not satisfy the polynomiality condition.

In this case, $\ell = 7$ and let $q_1,\dots,q_7$ be as in Subsection 5.2. The degrees of ${}^eq_1,\dots,{}^eq_7$ are $1,2,2,3,4,5,3$ respectively. By a computation performed by Maple, we can prove that ${}^eq_1,\dots,{}^eq_7$ satisfy the two following algebraic relations:

$$16\,{}^eq_3^2\,{}^eq_5{}^2 + {}^eq_4^4 - 8\,{}^eq_3\,{}^eq_5\,{}^eq_4^2 - 64\,{}^eq_3^3\,{}^eq_7{}^2 = 0, \qquad {}^eq_3\,{}^eq_6^2 - {}^eq_7^2\,{}^eq_4{}^2 = 0$$

Set:

$$r_i := \begin{cases} q_i & \text{if} \quad i = 1,2,3,4,7 \\ 16\,q_3^2q_5{}^2 + q_4^4 - 8\,q_3q_5q_4^2 - 64\,q_3^3q_7{}^2 & \text{if} \quad i = 5 \\ q_3q_6^2 - q_7^2q_4^2 & \text{if} \quad i = 6 \end{cases}$$

The polynomials $r_1,\dots,r_7$ are algebraically independent over $\Bbbk$ and

$$\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_7 = 2q_3q_6\,(32q_3^2q_5 - 8q_3q_4^2)\,\mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}q_7$$

Moreover, ${}^er_5$ and ${}^er_6$ have degree at least 13 and ${}^e(2q_3q_6(32q_3^2q_5 - 8q_3q_4^2))$ has degree 15. Then, by Corollary 7.2, ${}^er_1,\dots,{}^er_7$ are algebraically independent since

$$\frac{1}{2}(\dim\mathfrak{g}^e + 7) + 15 = 37 = 1 + 2 + 2 + 3 + 3 + 26$$

and by Lemma 7.1,(ii) and (iii), ${}^er_5$ and ${}^er_6$ have degree 13.

A precise computation performed by Maple shows that ${}^er_3 = p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, ${}^er_4 = p_3p_4$ for some polynomial p_4 of degree 2 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, ${}^er_5 = p_3^3\,{}^eq_7p_5$ for some polynomial p_5 of degree 7 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, and ${}^er_6 = p_4\,{}^er_7p_6$ for some polynomial p_6 of degree 8 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^er_i$ for $i = 1,2,7$, the polynomials $p_1,\dots,p_7$ are algebraically independent homogenous polynomials of degree $1,2,1,2,7,8,3$ respectively. Let $\mathfrak{l}$ be a reductive factor of $\mathfrak{g}^e$. According to [Ca85, Ch. 13],

$$\mathfrak{l} \simeq \mathfrak{so}_2(\Bbbk)\times\mathfrak{sp}_4(\Bbbk) \simeq \Bbbk\times\mathfrak{sp}_4(\Bbbk).$$

In particular, the center of $\mathfrak{l}$ has dimension 1. Let $\{x_1,\dots,x_{37}\}$ be a basis of $\mathfrak{g}^e$ such that x_{37} lies in the center of $\mathfrak{l}$ and such that $x_1,\dots,x_{36}$ are in $[\mathfrak{l},\mathfrak{l}] + \mathfrak{g}^e_{\mathfrak{u}}$ with $\mathfrak{g}^e_{\mathfrak{u}}$ the nilpotent radical of $\mathfrak{g}^e$. Then p_2 is a polynomial in $\Bbbk[x_1,\dots,x_{37}]$ depending on x_{37}. As a result, by [DDV74, Theorems 3.3 and 4.5], the semiinvariant polynomials of $S(\mathfrak{g}^e)$ are invariant.

Claim 7.9. The algebra $\mathfrak{g}^e$ is nonsingular.

Proof. [Proof of Claim 7.9] The space $\Bbbk^{14}$ is the orthogonal direct sum of two subspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of dimension 6 and 8 respectively and such that e, h, f are in $\overline{\mathfrak{g}} := \mathfrak{so}(\mathbb{V}_1) \oplus \mathfrak{so}(\mathbb{V}_2)$. Then $\overline{\mathfrak{g}}^e = \overline{\mathfrak{g}} \cap \mathfrak{g}^e$ is a subalgebra of dimension 21 containing the center of $\mathfrak{g}^e$. For p in $S(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\overline{\mathfrak{g}}^f$. The partition $(3,3,2,2,2,2,2)$ satisfies Condition (1) of the proof of [Y06, §4, Lemma 3]. So, the proof of Lemma 5.14 remains valid, and the morphism

$$G_0^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (g, x) \longmapsto g(x)$$

is dominant. As a result, for p in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, the differential of $\overline{p}$ is the restriction to $\overline{\mathfrak{g}}^f$ of the differential of p. A computation performed by Maple proves that $\overline{p_3}^{10}$ is a greatest divisor of $d\overline{p_1} \wedge \cdots \wedge d\overline{p_7}$ in $S(\overline{\mathfrak{g}^e})$. If q is a greatest divisor of $dp_1 \wedge \cdots \wedge dp_7$ in $S(\mathfrak{g}^e)$, then q is in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since the semiinvariant polynomials are invariant. So $q = p_3^d$ for some nonnegative integer d. One can suppose that $\{x_1, \ldots, x_{16}\}$ is a basis of the orthogonal complement to $\overline{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Then the Pfaffian of the matrix

$$\Big([x_i, x_j],\ 1 \leqslant i, j \leqslant 16\Big)$$

is in $\Bbbk^* p_3^8$ so that p_3^2 is a greatest divisor of $dp_1 \wedge \cdots \wedge dp_7$ in $S(\mathfrak{g}^e)$. Since

$$\deg p_1 + \cdots + \deg p_7 = 2 + 22 = 2 + \frac{1}{2}(\dim \mathfrak{g}^e + \ell),$$

we conclude that $\mathfrak{g}^e$ is nonsingular by Corollary 7.3,(ii). □

Claim 7.10. Suppose that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. Then for some homogenous polynomials p_5' and p_6' of degrees at least 5 and at most 8 respectively, $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is generated by p_1, p_2, p_3, p_4, p_5', p_6', p_7. Furthermore, the possible values for $(\deg p_5', \deg p_6')$ are $(5, 8)$ or $(6, 7)$.

Proof. [Proof of Claim 7.10] Since the semiinvariants are invariants, by Claim 7.9 and Proposition 7.4,(ii), there are homogenous generators $\varphi_1, \ldots, \varphi_\ell$ of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that

$$\deg \varphi_1 \leqslant \cdots \leqslant \deg \varphi_\ell,$$

and

$$\deg \varphi_1 + \cdots + \deg \varphi_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell) = 22.$$

According to [Mo06c, Theorem 1.1.8] or [Y06b], the center of $\mathfrak{g}^e$ has dimension 2. Hence, φ_1 and φ_2 have degree 1. Thereby, we can suppose that $\varphi_1 = p_1$ and $\varphi_2 = p_3$ since p_1 and p_3 are linearly independent elements of the center of $\mathfrak{g}^e$. Since p_2 and p_4 are homogneous elements of degree 2 such that $p_1, \ldots, p_4$ are algebraically indepent, φ_3 and φ_4 have degree 2 and we can suppose that $\varphi_3 = p_2$ and $\varphi_4 = p_4$. Since p_7 has degree 3, φ_5 has degree at most 3 and at least 2 since the center of $\mathfrak{g}^e$ has dimension 2. Suppose that φ_5 has degree 2. A contradiction is expected. Then

$$\deg \varphi_6 + \deg \varphi_7 = 22 - (1 + 1 + 2 + 2 + 2) = 14.$$

Moreover, since $p_1, \ldots, p_7$ are algebraically independent, φ_7 has degree at most 8 and φ_6 has degree at least 6. Hence p_7 is in the ideal of $\Bbbk[p_1, p_3, \varphi_3, \varphi_4, \varphi_5]$ generated by p_1 and p_3. But a computation shows that the restriction of p_7 to the nullvariety of p_1 and p_3 in $\mathfrak{g}^f$ is different from 0, whence the expected contradiction. As a result, φ_5 has degree 3 and

$$\deg \varphi_6 + \deg \varphi_7 = 13.$$

One can suppose $\varphi_5 = p_7$ and the possible values for $(\deg \varphi_6, \deg \varphi_7)$ are $(5, 8)$ and $(6, 7)$ since φ_7 has degree at most 8. □

Suppose that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. A contradiction is expected. Let p'_5 and p'_6 be as in Claim 7.10 and such that $\deg p'_5 < \deg p'_6$. Then $(\deg p'_5, \deg p'_6)$ equals $(5,8)$ or $(6,7)$. A computation shows that we can choose a basis $\{x_1,\ldots,x_{37}\}$ of $\mathfrak{g}^e$ with $x_{37} = p_3$, with p_1, p_2, p_3, p_4, p_7 in $\Bbbk[x_3,\ldots,x_{37}]$ and with p_5, p_6 of degree 1 in x_1. Moreover, the coefficient of x_1 in p_5 is a prime element of $\Bbbk[x_3,\ldots,x_{37}]$, the coefficient of x_1 in p_6 is a prime element of $\Bbbk[x_2,\ldots,x_{37}]$ having degree 1 in x_2, and the coefficient of x_1x_2 in p_6 equals $a^2p_3^2$ with a a prime homogenous polynomial of degree 2 such that a, p_1, p_2, p_3, p_4 are algebraically independent. In particular, a is not invariant. If p'_5 has degree 5, then

$$p_5 = p'_5 r_0 + r_1$$

with r_0 in $\Bbbk[p_1,p_2,p_3,p_4]$ and r_1 in $\Bbbk[p_1,p_2,p_3,p_4,p_7]$ so that p'_5 has degree 1 in x_1, and the coefficient of x_1 in p_5 is the product of r_0 and the coefficient of x_1 in p'_5. But this is impossible since this coefficient is prime. So, p'_5 has degree 6 and p'_6 has degree 7. We can suppose that $p'_6 = p_5$. Then

$$p_6 = p_5 r_0 + p'_6 r_1 + r_2$$

with r_0 homogenous of degree 1 in $\Bbbk[p_1,p_3]$, r_1 homogenous of degree 2 in $\Bbbk[p_1,p_2,p_3,p_4]$, and r_2 homogenous of degree 8 in $\Bbbk[p_1,p_2,p_3,p_4,p_7]$. According to the above remarks on p_5 and the coefficient of x_1x_2 in p_6, r_1 is in $\Bbbk^* p_3^2$ since r_1 has degree 2.

For p in $S(\mathfrak{g}^e)$, denote by $\overline{p}$ its image in $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$. A computation shows that for some u in $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$,

$$\overline{p_5} = \overline{p_4}^2 u, \qquad \overline{p_6} = -\overline{p_4p_7}u.$$

Furthermore, $\overline{p_4}$ and $\overline{p_7}$ are different prime elements of $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$ and the coefficient u_1 of x_1 in u is the product of two different polynomials of degree 1. The coefficient of x_1 in $\overline{p_6}$ is $u_1\overline{p_4}^2\overline{r_0}$ since

$$\overline{p_6} = \overline{p_5r_0} + \overline{r_2}.$$

On the other hand, the coefficient of x_1 in $\overline{p_6}$ is $-u_1\overline{p_4p_7}$, whence the contradiction since r_0 has degree 1.

7.4. **A conjecture.** All examples of good elements we achieved satisfy the hypothesis of Theorem 3.6.

Conjecture 7.11. *Let $\mathfrak{g}$ be a simple Lie algebra and let e be a nilpotent of $\mathfrak{g}$. If e is good then for some homogenous generating sequence $(q_1,\ldots,q_\ell)$ in $S(\mathfrak{g})^{\mathfrak{g}}$, ${}^eq_1,\ldots,{}^eq_\ell$ are algebraically independent over $\Bbbk$. In other words, the converse implication of Theorem 3.6 holds.*

Notice that it may happen that for some $r_1,\ldots,r_\ell$ in $S(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^er_1,\ldots,{}^er_\ell$ are algebraically independent over $\Bbbk$, and that however e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}(\Bbbk^{12})$ associated with the partition $(5,3,2,2)$, cf. Example 7.6.

In fact, according to [PPY07, Corollary 2.3], for any nilpotent e of $\mathfrak{g}$, there exist $r_1,\ldots,r_\ell$ in $S(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^er_1,\ldots,{}^er_\ell$ are algebraically independent over $\Bbbk$.

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