THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING II

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ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, and let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Denote by $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$ and by $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of $\mathfrak{g}^e$. We say that e is good if the nullvariety of some ℓ homogeneous elements of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension ℓ. If e is good then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. In this paper, we prove that the converse of the main result of [CM16] is true. Namely, we prove that e is good if and only if for some homogeneous generating sequence $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the initial homogeneous components of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent over $\Bbbk$.

CONTENTS

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, let $\langle .\,, . \rangle$ be the Killing form of $\mathfrak{g}$ and let G be the adjoint group of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, we denote by $\mathrm{S}(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$. For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$ and by G^x the stabilizer of x in G. Then $\mathrm{Lie}(G^x) = \mathrm{Lie}(G_0^x) = \mathfrak{g}^x$ where G_0^x is the identity component of G^x. Moreover, $\mathrm{S}(\mathfrak{g}^x)$ is a $\mathfrak{g}^x$-module and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)^{G_0^x}$.

In [CM16], we continued the works of [PPY07] and we studied the question on whether the algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in ℓ variables; see [Y07, CM10, JS10, Y16] for other references related to the topic.

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1.2. Let us first summarize the main results of [CM16].

Definition 1.1 ([CM16, Definition 1.3]). An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some homogeneous sequence $(p_1, \ldots, p_\ell)$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.

Thus an element $x \in \mathfrak{g}$ is good if the nullcone of $\mathrm{S}(\mathfrak{g}^x)$, that is, the nullvariety in $(\mathfrak{g}^x)^*$ of the augmentation ideal $\mathrm{S}(\mathfrak{g}^x)_+^{\mathfrak{g}^x}$ of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, is a complete intersection in $(\mathfrak{g}^x)^*$ since the transcendence degree over $\Bbbk$ of the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is ℓ by the main result of [CM10].

For example, regular nilpotent elements are good; see the introduction of [CM16] for more details and other examples.

Theorem 1.2 ([CM16, Theorem 3.3]). *Let x be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

An element x is good if and only if so is its nilpotent component in the Jordan decomposition [CM16, Proposition 3.5]. As a consequence, we can restrict the study to the case of nilpotent elements.

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and $\mathfrak{g}^f$ with $(\mathfrak{g}^e)^*$, through the Killing isomorphism $\mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \langle x\,, .\rangle$. Thus we have the following algebra isomorphisms: $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$ and $\mathrm{S}(\mathfrak{g}^e) \simeq \Bbbk[(\mathfrak{g}^e)^*] \simeq \Bbbk[\mathfrak{g}^f]$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with e*, and let $T_e : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto e + x$ be the translation map. It induces an isomorphism of affine varieties $\mathfrak{g}^f \simeq \mathcal{S}_e$, and the comorphism T_e^* induces an isomorphism between the coordinate algebras $\Bbbk[\mathcal{S}_e]$ and $\Bbbk[\mathfrak{g}^f]$.

Let p be a homogeneous element of $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}]$. Then its restriction to $\mathcal{S}_e$ is an element of $\Bbbk[\mathcal{S}_e] \simeq \Bbbk[\mathfrak{g}^f] \simeq \mathrm{S}(\mathfrak{g}^e)$ through the above isomorphisms. For p in $\mathrm{S}(\mathfrak{g})$, we denote by $\kappa(p)$ its restriction to $\mathcal{S}_e$ so that $\kappa(p) \in \mathrm{S}(\mathfrak{g}^e)$. Denote by ${}^e p$ the initial homogeneous component of $\kappa(p)$. According to [PPY07, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Theorem 1.3 ([CM16, Theorem 1.5]). *Suppose that for some homogeneous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, ${}^e q_1, \ldots, {}^e q_\ell$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

In other words, Theorem 1.3 provides a sufficient condition for that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. By [PPY07], one knows that for homogeneous elements $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent if and only if

$$\sum_{i=1}^{\ell} \deg {}^e q_i = \frac{\dim \mathfrak{g}^e + \ell}{2}. \tag{1}$$

So we have a practical criterion to verify the sufficient condition of Theorem 1.3. However, even if the condition of Theorem 1.3 holds, that is, if (1) holds, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not necessarily generated by the polynomial functions ${}^e q_1, \ldots, {}^e q_\ell$. As a matter of fact, there are nilpotent elements e satisfying this condition and for which $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not generated by some ${}^e q_1, \ldots, {}^e q_\ell$, for any choice of homogeneous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ (cf. [CM16, Remark 2.25]).

Theorem 1.3 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras, and for some nilpotent orbits in the exceptional Lie algebras, see [CM16, Sections 5 and 6]. We also provided in [CM16, Example 7.8] an example of a nilpotent element e for which $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is not polynomial, with $\mathfrak{g}$ of type D_7.

1.3. In this note, we prove that the converse of Theorem 1.3 also holds. Namely, our main result is the following theorem.

Theorem 1.4. *The nilpotent element e of $\mathfrak{g}$ is good if and only if for some homogeneous generating sequence $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$.*

Theorem 1.4 was conjectured in [CM16, Conjecture 7.11]. Notice that it may happen that for some $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\Bbbk$, and that however e is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}(\Bbbk^{12})$ associated with the partition $(5, 3, 2, 2)$, cf. [CM16, Example 7.6]. In fact, according to [PPY07, Corollary 2.3], for any nilpotent element e of $\mathfrak{g}$, there exist $r_1, \ldots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \ldots, {}^e r_\ell$ are algebraically independent over $\Bbbk$. So the assumption that $q_1, \ldots, q_\ell$ generate $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is crucial.

1.4. We introduce in this subsection the main notations of the paper and we outline our strategy to prove Theorem 1.4.

First of all, recall that $\mathfrak{g}^f$ identifies with the dual of $\mathfrak{g}^e$ through the Killing isomorphism so that $\mathrm{S}(\mathfrak{g}^e)$ is the algebra $\Bbbk[\mathfrak{g}^f]$ of polynomial functions on $\mathfrak{g}^f$, and that $\Bbbk[\mathfrak{g}^f]$ identifies with the coordinate algebra of the Slodowy slice $\mathcal{S}_e = e + \mathfrak{g}^f$.

Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \ldots, r$, $[h, x_i] = n_i x_i$ with n_i a nonnegative integer. For $\mathbf{j} = (j_1, \ldots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \quad |\mathbf{j}|_e := j_1(n_1 + 2) + \cdots + j_r(n_r + 2), \quad x^{\mathbf{j}} := x_1^{j_1} \cdots x_r^{j_r}.$$

There are two gradings on $\mathrm{S}(\mathfrak{g}^e)$: the standard one and the Slodowy grading. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ has standard degree $|\mathbf{j}|$ and, by definition, it has Slodowy degree $|\mathbf{j}|_e$. Denoting by $t \mapsto \rho(t)$ the one-parameter subgroup of G generated by $\mathrm{ad}\, h$, the

Slodowy slice $e + \mathfrak{g}^f$ is invariant under the one-parameter subgroup $t \mapsto t^{-2}\rho(t)$ of G. Hence the one-parameter subgroup $t \mapsto t^{-2}\rho(t)$ induces an action on $\Bbbk[\mathcal{S}_e]$. Let $j \in \{1,\ldots,r\}$, y in $\mathfrak{g}^f$ and t in $\Bbbk^*$. Viewing the element x_j of $\mathfrak{g}^e \subset \mathrm{S}(\mathfrak{g}^e)$ as an element $\Bbbk[\mathcal{S}_e]$, we have:

$$x_j(t^{-2}\rho(t)(e+y)) = x_j(e + t^{-2}\rho(t)(y)) = t^{-2}\rho(t^{-1})(x_j)(e+y) = t^{-2-n_j}x_j(e+y),$$

whence for all $\mathbf{j}$ in $\mathbb{N}^r$ and for all y in $\mathfrak{g}^f$,

$$x^{\mathbf{j}}(t^{-2}\rho(t)(e+y)) = t^{-|\mathbf{j}|_e}x^{\mathbf{j}}(e+y).$$

This means that $x^{\mathbf{j}}$, as a regular function on $\mathcal{S}_e$, is homogeneous of degree $|\mathbf{j}|_e$ for the Slodowy grading.

Let t be an indeterminate and let R be the polynomial algebra $\Bbbk[t]$. The polynomial algebra

$$\mathrm{S}(\mathfrak{g}^e)[t] := \Bbbk[t] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g}^e)$$

identifies with the algebra of polynomial functions on $\mathfrak{g}^f \times \Bbbk$. The grading of $\mathrm{S}(\mathfrak{g}^e)$ induces a grading of $\mathrm{S}(\mathfrak{g}^e)[t]$ such that t has degree 0. Denote by ε the evaluation map at $t = 0$ so that ε is a graded morphism from $\mathrm{S}(\mathfrak{g}^e)[t]$ onto $\mathrm{S}(\mathfrak{g}^e)$. Let τ be the embedding of $\mathrm{S}(\mathfrak{g}^e)$ into $\mathrm{S}(\mathfrak{g}^e)[t]$ such that $\tau(x_i) := tx_i$ for $i = 1,\ldots,r$.

Recall that for p in $\mathrm{S}(\mathfrak{g})$, $\kappa(p)$ denotes the restriction to $\mathcal{S}_e$ of p so that $\kappa(p) \in \mathrm{S}(\mathfrak{g}^e)$. Denote by A the intersection of $\mathrm{S}(\mathfrak{g}^e)[t]$ with the sub-$\Bbbk[t,t^{-1}]$-module of

$$\mathrm{S}(\mathfrak{g}^e)[t,t^{-1}] := \Bbbk[t,t^{-1}] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g}^e)$$

generated by $\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$, and let A_+ be its augmentation ideal. Let $\mathcal{V}$ be the nullvariety of A_+ in $\mathfrak{g}^f \times \Bbbk$ and $\mathcal{V}_*$ the union of the irreducible components of $\mathcal{V}$ which are not contained in $\mathfrak{g}^f \times \{0\}$. Let $\mathcal{N}$ be the nullvariety of $\varepsilon(A)_+$ in $\mathfrak{g}^f$, with $\varepsilon(A)_+$ the augmentation ideal of $\varepsilon(A)$. Then $\mathcal{V}$ is the union of $\mathcal{V}_*$ and $\mathcal{N} \times \{0\}$.

The properties of the varieties $\mathcal{V}$ and $\mathcal{V}_*$ allow us to prove the following result.

Theorem 1.5. *Suppose that $\mathcal{N}$ has dimension $r - \ell$. Then for some homogeneous generating sequence $q_1,\ldots,q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1,\ldots,{}^e q_\ell$ are algebraically independent over $\Bbbk$.*

The key point is to show that, under the hypothesis of Theorem 1.5, $\varepsilon(A)$ is the subalgebra of $\mathrm{S}(\mathfrak{g}^e)$ generated by the family ${}^e p$, $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, and hence that $\mathcal{N}$ coincides with the nullvariety in $\mathfrak{g}^f$ of ${}^e q_1,\ldots,{}^e q_\ell$. So, if $\mathcal{N}$ has dimension $r-\ell$, then the elements ${}^e q_1,\ldots,{}^e q_\ell$ must be algebraically independent over $\Bbbk$.

The remainder of the paper is organized as follows. In Section 2, we state useful results on commutative algebra of independent interest. Some of these results are probably well-known. Since we have not found appropriate references, proofs are provided. Moreover, we formulate them as they are used in the paper. We study in Section 3 properties of the varieties $\mathcal{V}$ and $\mathcal{V}_*$. The proof of Theorem 1.5 is achieved in Section 3. Theorem 1.4 is a consequence of Theorem 1.5, and it is proven in Section 4.

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2. Some results on commutative algebra

In this section t is an indeterminate and the base ring R is $\Bbbk$, $\Bbbk[t]$ or $\Bbbk[[t]]$. For M a graded space over $\mathbb{N}$ and for j in $\mathbb{N}$, denote by $M^{[j]}$ the space of degree j and by M_+ the sum of $M^{[j]}$, $j > 0$. Let A be a finitely generated graded R-algebra over $\mathbb{N}$ such that $A^{[0]} = R$ and such that $A^{[j]}$ is a free R-module of finite rank for any $j \in \mathbb{N}$. Moreover, A is an integral domain. Denote by $\dim A$ the Krull dimension of A and set[1]:

$$\ell := \begin{cases} \dim A & \text{if} \quad R = \Bbbk \\ \dim A - 1 & \text{if} \quad R = \Bbbk[t] \text{ or } \Bbbk[t]]. \end{cases}$$

As a rule, for B an integral domain, we denote by $K(B)$ its fraction field.

The one-dimensional multiplicative group of $\Bbbk$ is denoted by $\mathrm{G_m}$.

2.1. Let B be a graded subalgebra of A.

Lemma 2.1. (i) *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be pairwise different graded prime ideals contained in A_+. If they are the minimal prime ideals containing their intersection, then for some homogeneous element p of A_+, the element p is not in the union of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$.*

(ii) *For some homogeneous sequence $p_1, \ldots, p_\ell$ in A_+, A_+ is the radical of the ideal generated by $p_1, \ldots, p_\ell$.*

(iii) *Suppose that A_+ is the radical of AB_+. Then for some homogeneous sequence $p_1, \ldots, p_\ell$ in B_+, A_+ is the radical of the ideal generated by $p_1, \ldots, p_\ell$.*

Proof. (i) Prove by induction on j that for some homogeneous element p_j of A_+, p_j is not in the union of $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$. Since $\mathfrak{p}_1$ is a graded ideal strictly contained in A_+, it is true for $j = 1$. Suppose that it is true for $j - 1$. If p_{j-1} is not in $\mathfrak{p}_j$, there is nothing to prove. Suppose that p_{j-1} is in $\mathfrak{p}_j$. According to the hypothesis, $\mathfrak{p}_j$ is stricly contained in A_+ and it does not contain the intersection of $\mathfrak{p}_1, \ldots, \mathfrak{p}_{j-1}$. So, since $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$ are graded ideals, for some homogeneous sequence r, q in A_+,

$$r \in \bigcap_{k=1}^{j-1} \mathfrak{p}_k \setminus \mathfrak{p}_j, \quad \text{and} \quad q \in A_+ \setminus \mathfrak{p}_j.$$

Denoting by m and n the respective degrees of p_{j-1} and rq, $p_{j-1}^n + (rq)^m$ is homogeneous of degree mn and it is not in $\mathfrak{p}_1, \ldots, \mathfrak{p}_j$ since these ideals are prime.

[1] Since the Lie algebra $\mathfrak{g}$ does not appear in this section, there will be no possible confusion between ℓ and the rank of $\mathfrak{g}$, denoted ℓ, in the introduction too. However, the notation will be justified in the next sections.

(ii) Prove by induction on i that for some homogeneous sequence $p_1, \dots, p_i$ in A_+, the minimal prime ideals of A containing $p_1, \dots, p_i$ have height i. Let p_1 be in $A_+ \setminus \{0\}$. By [Ma86, Ch. 5, Theorem 13.5], all minimal prime ideal containing p_1 has height 1. Suppose that it is true for $i-1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals containing $p_1, \dots, p_{i-1}$. Since A_+ has height $\ell > i-1$, A_+ strictly contains $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. By (i), there exists a homogeneous element p_i in A_+ not in the union of $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then, by [Ma86, Ch. 5, Theorem 13.5], the minimal prime ideals containing $p_1, \dots, p_i$ have height i. For $i = \ell$, the minimal prime ideals containing $p_1, \dots, p_\ell$ have height ℓ. Hence they are equal to A_+ since A_+ is a prime ideal of height ℓ containing $p_1, \dots, p_\ell$, whence the assertion.

(iii) The ideal AB_+ is generated by a homogeneous sequence $a_1, \dots, a_m$ in B_+. Denote by B' the subalgebra of A generated by $a_1, \dots, a_m$. Then B' is a finitely generated graded subalgbera of A such that A_+ is the radical of AB'_+. If $R = \Bbbk$, denote by d its dimension and if $t \in R$, denote by $d+1$ its dimension. By (ii), for some homogeneous sequence $p_1, \dots, p_d$ in B'_+, B'_+ is the radical of the ideal generated by $p_1, \dots, p_d$. Then A_+ is the radical of the ideal of A generated by $p_1, \dots, p_d$. Since A_+ has height ℓ, $\ell \leqslant d$ by [Ma86, Ch. 5, Theorem 3.5]. Since B' is a subalgebra of A, its dimension is at most $\dim A$. Hence $d = \ell$. □

Proposition 2.2. *Suppose that A_+ is the radical of AB_+. Then B is finitely generated and A is a finite extension of B.*

Proof. Since A is a noetherian ring, for some homogeneous sequence $a_1, \dots, a_m$ in B_+, AB_+ is the ideal generated by this sequence. Denote by C the R-subalgebra of A generated by $a_1, \dots, a_m$. Then C is a graded subalgebra of A. Denote by π the morphism

$$\mathrm{Specm}(A) \xrightarrow{\pi} \mathrm{Specm}(C)$$

whose comorphism is the canonical injection $C \hookrightarrow A$. Let $\overline{A}$ and $\overline{C}$ be the respective integral closures of A and C in $K(A)$. Since C is contained in A, $\overline{C}$ is contained in $\overline{A}$. Let α and β be the morphisms

$$\mathrm{Specm}(\overline{A}) \xrightarrow{\alpha} \mathrm{Specm}(A) \quad \text{and} \quad \mathrm{Specm}(\overline{C}) \xrightarrow{\beta} \mathrm{Specm}(C)$$

whose comorphisms are the canonical injections $A \hookrightarrow \overline{A}$ and $C \hookrightarrow \overline{C}$ respectively. Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Specm}(\overline{A}) \arrow[r, "\overline{\pi}"] \arrow[d, "\alpha"'] & \mathrm{Specm}(\overline{C}) \arrow[d, "\beta"] \\ \mathrm{Specm}(A) \arrow[r, "\pi"] & \mathrm{Specm}(C) \end{tikzcd}$$

with $\overline{\pi}$ the morphism whose comorphism is the canonical injection $\overline{C} \to \overline{A}$.

The action of $\mathrm{G_m}$ in A extends to an action of $K(A)$, and $\overline{A}$ is invariant under this action. Denoting by $\overline{R}$ the integral closure of R in $K(A)$, $\overline{R}$ is the set of fixed points under the action of $\mathrm{G_m}$ in $\overline{A}$. Since C is invariant under $\mathrm{G_m}$ so is $\overline{C}$. For $\mathfrak{m}$ a maximal ideal of $\overline{R}$, the ideal $\mathfrak{m}+\overline{C}_+$ is the maximal ideal of $\overline{C}$ containing $\mathfrak{m}$ and invariant under $\mathrm{G_m}$. Then, for $\mathfrak{p}$ a maximal ideal of $\overline{C}$, $\mathfrak{p}\cap\overline{R}+\overline{C}_+$ is in the closure of the orbit of $\mathfrak{p}$ under $\mathrm{G_m}$. Moreover,

$$\{\mathfrak{m}+\overline{A}_+\} = \overline{\pi}^{-1}\{\mathfrak{m}+\overline{C}_+\}$$

for all maximal ideal $\mathfrak{m}$ of $\overline{R}$. Hence $\overline{\pi}$ is quasi finite. Moreover $\overline{\pi}$ is birational. Then, by Zariski's main theorem [Mu88], $\overline{\pi}$ is an open immersion. The image of $\overline{\pi}$ contains fixed points for the $\mathrm{G_m}$-action, and the closure of each $\mathrm{G_m}$-orbit contains fixed points. As a result, $\overline{\pi}$ is surjective since it is $\mathrm{G_m}$-equivariant. Hence $\overline{\pi}$ is an isomorphism and $\overline{A}=\overline{C}$. As a result, $\overline{A}$ is a finite extension of C since β is a finite morphism. As submodules of the finite module $\overline{A}$ over the noetherian ring C, A and B are finite C-modules. Hence A is a finite extension of B. Denoting by $\omega_1,\ldots,\omega_d$ a generating family of the C-module B, B is the subalgebra of A generated by $a_1,\ldots,a_m,\omega_1,\ldots,\omega_d$. □

Denote by $\Bbbk[t]_*$ the localization of $\Bbbk[t]$ at the prime ideal $t\Bbbk[t]$ and set:

$$R_* := \begin{cases} \Bbbk & \text{if} \quad R=\Bbbk \\ \Bbbk[t]_* & \text{if} \quad R=\Bbbk[t] \\ \Bbbk[[t]] & \text{if} \quad R=\Bbbk[[t]] \end{cases} \qquad \widehat{R} := \begin{cases} \Bbbk & \text{if} \quad R=\Bbbk \\ \Bbbk[[t]] & \text{if} \quad R=\Bbbk[t] \\ \Bbbk[[t]] & \text{if} \quad R=\Bbbk[[t]]. \end{cases}$$

For M a R-module, set $\widehat{M} := \widehat{R}\otimes_R M$.

Lemma 2.3. *Suppose $R=\Bbbk[t]$. Let M be a torsion free R-module and let N be a submodule of M. Then for a in $\widehat{N}\cap M$, ra is in N for some r in R such that $r(0)\neq 0$.*

Proof. Since M is torsion free, the canonical map $M\to\widehat{M}$ is an embedding. Moreover, the canonical map $\widehat{N}\to\widehat{M}$ is an embedding since $\widehat{R}$ is flat over R. Let a be in $\widehat{N}\cap M$ and let $\overline{a}$ be its image in M/N by the quotient map. Denote by J_a the annihilator of $\overline{a}$ in R, whence a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\mathrm{d}"] & M \arrow[r, "\mathrm{d}"] & M/N \arrow[r] & 0 \\ 0 \arrow[r] & J_a \arrow[r, "\mathrm{d}"] & R \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & R\overline{a} \arrow[r] \arrow[u, "\delta"] & 0 \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

with exact lines and columns. Since $\widehat{R}$ is a flat extension of R, tensoring this diagram by R gives the following diagram with exact lines and columns:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{N} \arrow[r, "\mathrm{d}"] & \widehat{M} \arrow[r, "\mathrm{d}"] & \widehat{R}\otimes_R M/N \arrow[r] & 0 \\
0 \arrow[r] & \widehat{R}J_a \arrow[r, "\mathrm{d}"] & \widehat{R} \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & \widehat{R}\overline{a} \arrow[r] \arrow[u, "\delta"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

For b in $\widehat{R}$, $(\delta\circ\mathrm{d})b = (\mathrm{d}\circ\delta)b = 0$ since a is in $\widehat{N}$, whence $\mathrm{d}b = 0$. As a result, $\widehat{R}J_a = \widehat{R}$. Hence J_a contains an element r, invertible in $\widehat{R}$, that is $r(0) \neq 0$, whence the lemma. □

Set

$$A_* := R_* \otimes_R A \qquad \text{and} \qquad \widehat{A} := \widehat{R} \otimes_R A.$$

Since $A^{[0]} = R$, the grading on A extends to gradings on A_* and $\widehat{A}$ such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$. When $R = \Bbbk$ or $R = \Bbbk[[t]]$, $A_* = A$ and $\widehat{A} = A$.

For $p_1, \dots, p_\ell$ a homogeneous sequence in A_+ set:

$$\underline{p} := \begin{cases} p_1, \dots, p_\ell & \text{if} \quad R = \Bbbk \\ t, p_1, \dots, p_\ell & \text{if} \quad R = \Bbbk[[t]], \end{cases}$$

and denote by $J_{\underline{p}}$ the ideal of A generated by the sequence $\underline{p}$.

Lemma 2.4. *Suppose that A is Cohen-Macaulay. Let $p_1, \dots, p_\ell$ be a homogeneous sequence in A_+ such that A_+ is the radical of the ideal of A generated by $p_1, \dots, p_\ell$ and let V be a graded complement in A to the $\Bbbk$-subspace $J_{\underline{p}}$.*

(i) *The space V has finite dimension.*
(ii) *The space A_* is equal to $VR_*[p_1, \dots, p_\ell]$.*
(iii) *The algebra A is a flat extension of $R[p_1, \dots, p_\ell]$.*
(iv) *For all homogeneous elements $a_1, \dots, a_n$ in A, linearly independent over $\Bbbk$ modulo $J_{\underline{p}}$, $a_1, \dots, a_n$ are linearly independent over $R[p_1, \dots, p_\ell]$.*
(v) *The linear map*

$$V \otimes_\Bbbk R_*[p_1, \dots, p_\ell] \longrightarrow A_*, \qquad v \otimes a \longmapsto va$$

is an isomorphism.

Proof. According to Lemma 2.1(ii), the sequence p does exist.

(i) Let J_p be the ideal of A generated by $p_1, \dots, p_\ell$. Since A_+ is the radical of J_p, $A^{[d]} = J_p^{[d]}$ for d sufficiently big. When $t \in R$, for all d, then $tA^{[d]}$ has finite codimension in $A^{[d]}$ since $A^{[d]}$ is a finite free R-module. Hence $J_{\underline{p}}$ has finite codimension in A so that V has finite dimension.

(ii) Suppose that t is in R. First of all, we prove by induction on d the inclusion

$$A^{[d]} \subset (VR[p_1,\dots,p_\ell])^{[d]} + tA^{[d]}.$$

Since $A^{[0]}$ is the direct sum of $V^{[0]}$ and $J_{\underline{p}}^{[0]}$, $V^{[0]}$ is contained in $\Bbbk + tR$, whence the inclusion for $d = 0$. Suppose that it is true for all j smaller than d. Since $p_1,\dots,p_\ell$ have positive degrees, by induction hypothesis,

$$J_{\underline{p}}^{[d]} \subset (VR[p_1,\dots,p_\ell])^{[d]} + tA^{[d]},$$

whence the inclusion for d. Then, by induction on m,

$$A^{[d]} \subset (VR[p_1,\dots,p_\ell])^{[d]} + t^m A^{[d]}.$$

As a result, since $A^{[d]}$ is a finite R-module,

$$A^{[d]} \subset (V\widehat{R}[p_1,\dots,p_\ell])^{[d]},$$

whence $\widehat{A} = V\widehat{R}[p_1,\dots,p_\ell]$. This equality remains true when $R = \Bbbk$ by an analogous and simpler argument.

When $R = \Bbbk[t]$, according to Lemma 2.3, for a in A, ra is in $VR[p_1,\dots,p_\ell]$ for some r in R such that $r(0) \neq 0$. As a result, $A_* = VR_*[p_1,\dots,p_\ell]$.

(iii) By Proposition 2.2, A is a finite extension of $R[p_1,\dots,p_\ell]$. In particular, $R[p_1,\dots,p_\ell]$ has dimension $\ell + \dim R$ so that $p_1,\dots,p_\ell$ are algebraically independent over R. Hence $R[p_1,\dots,p_\ell]$ is a regular algebra, whence the assertion by [Ma86, Ch. 8, Theorem 23.1].

(iv) Prove the assertion by induction on n. Since A is an integral domain, the assertion is true for $n = 1$. Suppose the assertion true for $n-1$. Let $(b_1,\dots,b_n)$ be a homogeneous sequence in $R[p_1,\dots,p_\ell]$ such that

$$b_1a_1 + \cdots + b_na_n = 0.$$

Let K and I be the kernel and the image of the linear map

$$R[p_1,\dots,p_\ell]^n \longrightarrow R[p_1,\dots,p_\ell], \qquad (c_1,\dots,c_n) \longmapsto c_1b_1 + \cdots + c_nb_n,$$

whence the short exact sequence of $R[p_1,\dots,p_\ell]$ modules

$$0 \longrightarrow K \longrightarrow R[p_1,\dots,p_\ell]^n \longrightarrow I \longrightarrow 0.$$

The grading of $R[p_1,\dots,p_\ell]$ induces a grading of $R[p_1,\dots,p_\ell]^n$ and K is a graded submodule of $R[p_1,\dots,p_\ell]^n$ since $b_1,\dots,b_n$ is a homogeneous sequence in $R[p_1,\dots,p_\ell]$. Denote by $y_1,\dots,y_m$ a generating homogeneous sequence of the $R[p_1,\dots,p_\ell]$-module K. By (iii), the short sequence of A-modules

$$0 \longrightarrow A \otimes_{R[p_1,\dots,p_\ell]} K \longrightarrow A^n \longrightarrow A \otimes_{R[p_1,\dots,p_\ell]} I \longrightarrow 0$$

is exact. So, for some homogeneous sequence $x_1,\dots,x_m$ in A,

$$a_i = \sum_{j=1}^{m} x_j y_{j,i}$$

for $i = 1, \dots, n$. Since a_n is not in $J_{\underline{p}}$, for some j_*, the element $y_{j_*,i}$ is an invertible element of R_*, whence

$$b_n y_{j_*,n} = -\sum_{i=1}^{n-1} b_i y_{j_*,i} \quad \text{and} \quad \sum_{i=1}^{n-1} b_i(y_{j_*,n}a_i - a_n y_{j_*,i}) = 0.$$

So, by induction hypothesis,

$$b_1 = \cdots = b_{n-1} = 0$$

since the elements

$$y_{j_*,n}a_1 - a_n y_{j_*,1}, \dots, y_{j_*,n}a_{n-1} - a_n y_{j_*,n-1}$$

are linearly independent over $\Bbbk$ modulo $J_{\underline{p}}$. Then $b_n = 0$ since $y_{j_*,n}$ is invertible.

(v) Let $(v_1, \dots, v_n)$ be a homogeneous basis of V. Since the space of relations of linear dependence over $R[p_1, \dots, p_\ell]$ of $v_1, \dots, v_n$ is graded, it is equal to $\{0\}$ by (iv), whence the assertion by (ii). □

Corollary 2.5. (i) *The algebra A_* is Cohen-Macaulay if and only if for some homogeneous sequence $p_1, \dots, p_\ell$ in A_+, the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$.*

(ii) *Suppose that A_* is Cohen-Macaulay. For a homogeneous sequence $q_1, \dots, q_\ell$ in A_+, A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$ if and only if R_*A_+ is the radical of the ideal of A_* generated by $q_1, \dots, q_\ell$.*

Proof. (i) The "only if" part results from Lemma 2.4(v). Suppose that for some homogeneous sequence $p_1, \dots, p_\ell$ in A_+, the algebra A_* is a finite free extension of $R_*[p_1, \dots, p_\ell]$. In particular, $R_*[p_1, \dots, p_\ell]$ is a polynomial algebra over R_* since A_* has dimension $\dim A$. Let $\mathfrak{p}$ be a prime ideal of A_* and let $\mathfrak{q}$ be its intersection with $R_*[p_1, \dots, p_\ell]$. Denote by $A_{\mathfrak{p}}$ and $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ the localizations of A_* and $R_*[p_1, \dots, p_\ell]$ at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Since A_* is a finite extension of $R_*[p_1, \dots, p_\ell]$, these local rings have the same dimension. Denote by d this dimension. By flatness, any regular sequence $a_1, \dots, a_d$ in $R[p_1, \dots, p_\ell]_{\mathfrak{q}}$ is regular in $A_{\mathfrak{p}}$ so that $A_{\mathfrak{p}}$ is Cohen-Macaulay. Hence A_* is Cohen-Macaulay.

(ii) The "only if" part results from (i) and Proposition 2.2. Suppose that A_* is a finite free extension of $R_*[q_1, \dots, q_\ell]$. Let $\mathfrak{p}$ be a minimal prime ideal of A_* containing $q_1, \dots, q_\ell$ and let $\mathfrak{q}$ be its intersection with $R_*[q_1, \dots, q_\ell]$. Then $\mathfrak{q}$ is generated by $q_1, \dots, q_\ell$. In particular it has height ℓ. So $\mathfrak{p}$ has height ℓ since A_* is a finite extension of $R_*[q_1, \dots, q_\ell]$. As a result, $\mathfrak{p} = R_*A_+$ since R_*A_+ is a prime ideal of height ℓ, containing $q_1, \dots, q_\ell$, whence the assertion. □

Recall that B is a graded subalgebra of A. Set $B_* := R_* \otimes_R B$ and for $\mathfrak{p}$ a prime ideal of B, denote by $B_{\mathfrak{p}}$ its localization at $\mathfrak{p}$.

Proposition 2.6. *Suppose that the following conditions are satisfied:*

(1) *B is normal,*

(2) *A_+ is the radical of AB_+,*
(3) *A is Cohen-Macaulay.*

(i) *Let $p_1, \ldots, p_\ell$ be a homogeneous sequence in B_+ such that B_+ is the radical of the ideal of B generated by this sequence. Then for some graded subspace V of A, having finite dimension, the linear morphisms*

$$V \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow A_*, \qquad v \otimes a \longmapsto va,$$

$$(V \cap B) \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow B_*, \qquad v \otimes a \longmapsto va$$

are isomorphisms.
(ii) *If $R = \Bbbk$ or $R = \Bbbk[[t]]$, the algebra B_* is Cohen-Macaulay.*
(iii) *For $\mathfrak{p}$ prime ideal of B, containing t, the local ring $B_\mathfrak{p}$ is Cohen-Macaulay.*

Proof. (i) By Proposition 2.2 and by Condition (2), B is finitely generated and A is a finite extension of B. By Condition (2) and by Lemma 2.1(iii), for some homogeneous sequence $p_1, \ldots, p_\ell$ in B_+, A_+ is the radical of the ideal generated by $p_1, \ldots, p_\ell$.

Let $\underline{p}$ be as in Lemma 2.4. Denote by m the degree of the extension $K(A)$ of $K(B)$. For a in $A_* \subset K(A)$, set:

$$a^\# := \frac{1}{m} \operatorname{tr} a$$

with $\operatorname{tr} := \operatorname{tr}_{K(A)/K(B)}$ the trace map. By Condition (1), B_* is normal and the map $a \mapsto a^\#$ is a projection from A_* onto B_* whose restriction to A is a projection onto B. Moreover, it is a graded morphism of B-modules. Let M be its kernel. Let J_0 and J be the ideals of B and A generated by $\underline{p}$ respectively. Since $t, p_1, \ldots, p_\ell$ are in B, J is the direct sum of J_0 and MJ_0. Let V_0 be a graded complement in B to the $\Bbbk$-space J_0 and let V_1 be a graded complement in M to the $\Bbbk$-space MJ_0. Setting $V := V_0 + V_1$, V is a graded complement in A to the $\Bbbk$-space J. By Condition (3) and Lemma 2.4, V has finite dimension and the linear map

$$V \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow A_*, \qquad v \otimes a \longmapsto va$$

is an isomorphism. So, since $V_0 = V^\#$, the linear map

$$V_0 \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow B_*, \qquad v \otimes a \longmapsto va$$

is an isomorphism, whence the assertion.

(ii) results from (i) and Corollary 2.5.

(iii) By (i) and Corollary 2.5, A_* is Cohen-Macaulay. For $\mathfrak{p}$ a prime ideal of B, containing t, $B_\mathfrak{p}$ is the localization of B_* at the prime ideal $B_*\mathfrak{p}$, whence the assertion by (ii). ∎

2.2. In this subsection $R = \Bbbk[t]$. Then $\widehat{R} = \Bbbk[[t]]$. For M a graded module over R such that $M^{[j]}$ is a free submodule of finite rank for all j, we denote by $P_{M,R}(T)$ its Hilbert series:

$$P_{M,R}(T) := \sum_{j \in \mathbb{N}} \operatorname{rk} M^{[j]} T^j.$$

For V a graded space over $\Bbbk$ such that $V^{[j]}$ has finite dimension, we denote by $P_{V,\Bbbk}(T)$ its Hilbert series:

$$P_{V,\Bbbk}(T) := \sum_{j \in \mathbb{N}} \dim V^{[j]} T^j.$$

Let S be a graded polynomial algebra over $\Bbbk$ such that $S^{[0]} = \Bbbk$ and $S^{[j]}$ has finite dimension for all j. Consider on $S[t]$ and $S[[t]]$ the gradings extending that of S and such that t has degree 0. Consider the following conditions on A:

(C1) A is graded subalgebra of $S[t]$,
(C2) for some homogeneous sequence $a_1, \dots, a_\ell$ in A_+, $A = \Bbbk[t, t^{-1}, a_1, \dots, a_\ell] \cap S[t]$,
(C3) A is Cohen-Macaulay.

If the condition (C2) holds, then $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Moreover, if so, since A has dimension $\ell + 1$, then the elements $t, a_1, \dots, a_\ell$ are algebraically independent over $\Bbbk$. Set $\widehat{A} := \widehat{R} \otimes_R A$.

Lemma 2.7. *Assume that the conditions* (C1) *and* (C2) *hold.*

(i) *The element t is a prime element of A.*
(ii) *The algebra A is a factorial ring.*
(iii) *The Hilbert series of the R-module A is equal to*

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

with $d_1, \dots, d_\ell$ the degrees of $a_1, \dots, a_\ell$ respectively.

Proof. (i) Let a and b be in A such that ab is in tA. Since $tS[t]$ is a prime ideal of $S[t]$, a or b is in $tS[t]$. Suppose $a = ta'$ for some a' in $S[t]$. Then a' is in $A[t^{-1}]$. By Condition (C2), $A[t^{-1}] = R[a_1, \dots, a_\ell][t^{-1}]$. Hence a' is in A by Condition (C2) again. As a result, At is a prime ideal of A.

(ii) Since A is finitely generated, it suffices to prove that all prime ideal of height 1 is principal by [Ma86, Ch. 7, Theorem 20.1]. Let $\mathfrak{p}$ be a prime ideal of height 1. If t is in $\mathfrak{p}$, then $\mathfrak{p} = At$ by (i). Suppose that t is not in $\mathfrak{p}$ and set $\overline{\mathfrak{p}} = A[t^{-1}]\mathfrak{p}$. Then $\overline{\mathfrak{p}}$ is a prime ideal of height 1 of $R[a_1, \dots, a_\ell][t^{-1}]$ by Condition (C2). For a in $\overline{\mathfrak{p}}$, $t^m a$ is in $\mathfrak{p}$ for some nonnegative integer m. Hence

$$\mathfrak{p} = \overline{\mathfrak{p}} \cap A$$

since $\mathfrak{p}$ is prime. As a polynomial ring over the principal ring $\Bbbk[t,t^{-1}]$, the ring $R[a_1,\ldots,a_\ell][t^{-1}]$ is a factorial ring. Then $\overline{\mathfrak{p}}$ is generated by an element a in $\mathfrak{p}$. Since S is a polynomial ring, $S[t]$ is a factorial ring. So, for some nonnegative integer m and for some a' in $S[t]$, prime to t, $a = t^m a'$. By Condition (C2), a' is in A. Then a' is an element of $\mathfrak{p}$, generating $\overline{\mathfrak{p}}$ and not divisible by t in A. Let b and c be in A such that bc is in Aa'. Then b or c is in $A[t^{-1}]a'$. Suppose b in $A[t^{-1}]a'$. So, for some l in $\mathbb{N}$, $t^l b = b'a'$ for some b' in A. We choose l minimal satisfying this condition. By (i), since a' is not divisible by t in A, b' is divisible by t in A if $l > 0$. By minimality of l, $l = 0$ and b is in Aa'. As a result, Aa' is a prime ideal and $\mathfrak{p} = Aa'$ since $\mathfrak{p}$ has height 1.

(iii) By Condition (C2),

$$A[t^{-1}] = \Bbbk[t,t^{-1}] \otimes_{\Bbbk} \Bbbk[a_1,\ldots,a_\ell] \quad \text{whence} \quad \mathrm{rk}\, A^{[d]} = \dim \Bbbk[a_1,\ldots,a_\ell]^{[d]}$$

for all nonnegative integer d. Since $a_1,\ldots,a_\ell$ are algebraically independent over $\Bbbk$,

$$P_{\Bbbk[a_1,\ldots,a_\ell],\Bbbk}(T) = \prod_{i=1}^{\ell} \frac{1}{1-T^{d_i}},$$

whence the assertion. □

Let $p_1,\ldots,p_\ell$ be a homogeneous sequence in A such that A_+ is the radical of the ideal of A generated by this sequence. By Lemma 2.1(ii), such a sequence does exist. Denote by C the integral closure of $\Bbbk[p_1,\ldots,p_\ell]$ in $\Bbbk(t,a_1,\ldots,a_\ell)$.

Lemma 2.8. *Assume that the conditions* (C1)*,* (C2) *and* (C3) *hold.*

(i) *The algebra C is a graded subalgebra of A and t is not algebraic over C.*
(ii) *The algebra C is Cohen-Macaulay. Moreover, C is a finite free extension of $\Bbbk[p_1,\ldots,p_\ell]$.*
(iii) *The algebra $C + tA$ is normal.*

Proof. (i) By Lemma 2.7(ii), A is a normal ring such that $K(A) = \Bbbk(t,a_1,\ldots,a_\ell)$ by Condition (C2). Then C is contained in A since $\Bbbk[p_1,\ldots,p_\ell]$ is contained in A. Moreover, C is a graded algebra since so is $\Bbbk[p_1,\ldots,p_\ell]$. By Proposition 2.2, A is a finite extension of $R[p_1,\ldots,p_\ell]$. So, since A has dimension $\ell+1$, the elements $t,p_1,\ldots,p_\ell$ are algebraically independent over $\Bbbk$. As a result, t is not algebraic over C.

(ii) By (i), $C[[t]] = C \otimes_{\Bbbk} \Bbbk[[t]]$ so that $C[[t]]$ is a flat extension of $\Bbbk[[t]]$. Moreover, C is the quotient of $C[[t]]$ by $tC[[t]]$. As C and $\Bbbk[[t]]$ are normal rings, $C[[t]]$ is a normal ring by [Ma86, Ch. 8, Corollary of Theorem 23.9]. By definition, A_+ is the radical of the ideal of A generated by $p_1,\ldots,p_\ell$. As $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, from the short exact sequence

$$0 \longrightarrow A_+ \longrightarrow A \longrightarrow \Bbbk[t] \longrightarrow 0$$

we deduce the short exact sequence

$$0 \longrightarrow \widehat{A}_+ \longrightarrow \widehat{A} \longrightarrow \Bbbk[[t]] \longrightarrow 0 .$$

Hence $\widehat{A}_+$ is a prime ideal. As A_+ is the radical of the ideal generated by the sequence $p_1, \ldots, p_\ell$, $\widehat{A}_+$ is contained in the radical of $AC[[t]]_+$. Then, by (i), $\widehat{A}_+$ is the radical of $AC[[t]]_+$. Since $\widehat{R}$ is a flat extension of R, the algebra $\widehat{A}$ is Cohen-Macaulay by Condition (C3). Then, by Proposition 2.6(ii), $C[[t]]$ is Cohen-Macaulay. Let V be a graded complement in C to the ideal of C generated by $p_1, \ldots, p_\ell$. Since t is not algebraic over C, the space V is a complement in $C[t]$ to the ideal of $C[t]$ generated by $t, p_1, \ldots, p_\ell$. Then, by Lemma 2.4, V has finite dimension and the linear morphism

$$V \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow R_*C, \qquad v \otimes a \longmapsto va$$

is an isomorphism. As a result, the linear morphism

$$V \otimes_{\Bbbk} \Bbbk[p_1, \ldots, p_\ell] \longrightarrow C, \qquad v \otimes a \longmapsto va$$

is an isomorphism, whence the assertion by Corollary 2.5(ii).

(iii) Set $\tilde{A} := C + tA$. At first, $\tilde{A}$ is a graded subalgebra of A since C is a graded algebra and tA is a graded ideal of A. According to Proposition 2.6(i), for some graded subspace V of A, having finite dimension, the linear morphisms

$$V \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow A_*, \qquad v \otimes a \longmapsto va,$$

$$(V \cap C[t]) \otimes_{\Bbbk} R_*[p_1, \ldots, p_\ell] \longrightarrow R_*C, \qquad v \otimes a \longmapsto va$$

are isomorphisms. Let $v_1, \ldots, v_n$ be a basis of V such that $v_1, \ldots, v_m$ is a basis of $V \cap C[t]$. For a in A_*, the element a has unique expansion

$$a = v_1 a_1 + \cdots + v_n a_n$$

with $a_1, \ldots, a_n$ in $R_*[p_1, \ldots, p_\ell]$. If a is in tA_*, $a_1, \ldots, a_n$ are in $tR_*[p_1, \ldots, p_\ell]$ and if a is in R_*C, $a_1, \ldots, a_m$ are in $\Bbbk[p_1, \ldots, p_\ell]$ and $a_{m+1}, \ldots, a_n$ are equal to 0, whence $R_*C \cap tA_* = tR_*C$ and $C \cap tA = \{0\}$. In particular, C is the quotient of $\tilde{A}$ by $t\tilde{A}$.

For $\mathfrak{p}$ a prime ideal of $\tilde{A}$, denote by $\tilde{A}_{\mathfrak{p}}$ the localization of $\tilde{A}$ at $\mathfrak{p}$. If t is not in $\mathfrak{p}$, then $A[t^{-1}]$ is contained in $\tilde{A}_{\mathfrak{p}}$ so that $\tilde{A}_{\mathfrak{p}}$ is a localization of the regular algebra $R[a_1, \ldots, a_\ell][t^{-1}]$ by Condition (C2). Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra. Suppose that t is in $\mathfrak{p}$. Denote by $\overline{\mathfrak{p}}$ the image of $\mathfrak{p}$ in C by the quotient map. Then $\tilde{A}_{\mathfrak{p}}/t\tilde{A}_{\mathfrak{p}}$ is the localization $C_{\overline{\mathfrak{p}}}$ of C at the prime ideal $\overline{\mathfrak{p}}$. Since C is Cohen-Macaulay, so are $C_{\overline{\mathfrak{p}}}$ and $\tilde{A}_{\mathfrak{p}}$. As a result, $\tilde{A}$ is Cohen-Macaulay.

Let $\mathfrak{p}$ be a prime ideal of height 1 of $\tilde{A}$. If t is not in $\mathfrak{p}$, $\tilde{A}_{\mathfrak{p}}$ is a regular local algebra as it is already mentioned. Suppose that t is in $\mathfrak{p}$. By Lemma 2.7(i), $t\tilde{A} = \mathfrak{p}$ so that all element of $C \setminus \{0\}$ is invertible in $\tilde{A}_{\mathfrak{p}}$, whence

$$\tilde{A}_{\mathfrak{p}} = K(C) + t\tilde{A}_{\mathfrak{p}} \quad \text{and} \quad t\tilde{A}_{\mathfrak{p}} = tK(C) + t^2\tilde{A}_{\mathfrak{p}}.$$

Hence $\tilde{A}_{\mathfrak{p}}$ is a regular local ring of dimension 1. As a result, $\tilde{A}$ is regular in codimension 1. Then, by Serre's normality criterion [B98, §1, n°10, Théorème 4], $\tilde{A}$ is normal since $\tilde{A}$ is Cohen-Macaulay. □

Corollary 2.9. *Assume that the conditions* (C1), (C2) *and* (C3) *hold.*

(i) *The algebra* $\widehat{A}$ *is equal to* $C[[t]]$.

(ii) *For a in* A, *the element* ra *is in* $C[t]$ *for some* r *in* $\Bbbk[t]$ *such that* $r(0) \neq 0$.

Proof. (i) Since tA is contained in A, we have $K(A) = K(\tilde{A})$. Since C_+ is contained in $\tilde{A}_+$, A_+ is the radical of $A\tilde{A}_+$. Then, by Proposition 2.2, A is a finite extension of $\tilde{A}$. So, by Lemma 2.8(iii), $A = \tilde{A}$ and by induction on m,

$$A \subset C[t] + t^m A$$

for all positive integer m. Since A and $C[t]$ are graded and since the R-module $A^{[d]}$ is finitely generated for all d, $\widehat{A} = C[[t]]$.

(ii) The assertion results from (i) and Lemma 2.3. □

Proposition 2.10. *Assume that the conditions* (C1), (C2) *and* (C3) *hold. Then the algebra* A_* *is polynomial over* R_*. *Moreover, for some homogeneous sequence* $q_1, \ldots, q_\ell$ *in* A_+ *such that* $q_1, \ldots, q_\ell$ *have degree* $d_1, \ldots, d_\ell$ *respectively,* $A_* = R_*[q_1, \ldots, q_\ell]$.

Proof. According to Corollary 2.9 and Lemma 2.8(i), it suffices to prove that C is a polynomial algebra over $\Bbbk$ generated by a homogeneous sequence $q_1, \ldots, q_\ell$ such that $q_1, \ldots, q_\ell$ have degree $d_1, \ldots, d_\ell$ respectively. According to Corollary 2.9(i) Lemma 2.8(i) and Lemma 2.7(iii),

$$P_{C,\Bbbk}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}}.$$

By Corollary 2.9(ii), for $i = 1, \ldots, \ell$, for some r_i in R such that $r_i(0) \neq 0$, $r_i a_i$ has an expansion

$$r_i a_i = \sum_{m \in \mathbb{N}} c_{i,m} t^m$$

with $c_{i,m}, m \in \mathbb{N}$ in $C^{[d_i]}$, with finite support. For z in $\Bbbk$ and $i = 1, \ldots, \ell$, set:

$$b_i(z) = \sum_{m \in \mathbb{N}} c_{i,m} z^m$$

so that $b_i(z)$ is in $C^{[d_i]}$ for all z. As already mentioned, $t, a_1, \ldots, a_\ell$ are algebraically independent over $\Bbbk$ by Condition (C2) since A has dimension $\ell + 1$. Then, so are $t, r_1 a_1, \ldots, r_\ell a_\ell$ and for some z in $\Bbbk$, $b_1(z), \ldots, b_\ell(z)$ are algebraically independent over $\Bbbk$. Denoting by C' the subalgebra of C generated by this sequence,

$$P_{C',\Bbbk}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{d_i}},$$

whence $C = C'$ so that C is a polynomial algebra. □

3. Proof of Theorem 1.5

In this section, unless otherwise specified, the grading on $\mathrm{S}(\mathfrak{g}^e)$ is the Slodowy grading.

For m a nonnegative integer, $\mathrm{S}(\mathfrak{g}^e)^{[m]}$ denotes the space of degree m of $\mathrm{S}(\mathfrak{g}^e)$. We retain the notations of the introduction, in particular of Subsection 1.4.

3.1. Let R be the ring $\Bbbk[t]$. As in Section 2, for M a graded subspace of $\mathrm{S}(\mathfrak{g}^e)[t] = R \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g}^e)$, its subspace of degree m is denoted by $M^{[m]}$. In particular, $\mathrm{S}(\mathfrak{g}^e)[t]^{[m]}$ is equal to $\mathrm{S}(\mathfrak{g}^e)^{[m]}[t]$ and it is a free R-module of finite rank. As a result, for all graded R-submodule M of $\mathrm{S}(\mathfrak{g}^e)[t]$, its Hilbert series is well defined.

For m a nonnegative integer, denote by F_m the space of elements of $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$ whose component of minimal standard degree is at least m. Then $F_0, F_1, \ldots$ is a decreasing filtration of the algebra $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$. Let $d_1, \ldots, d_\ell$ be the standard degrees of a homogeneous generating sequence of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. We assume that the sequence $d_1, \ldots, d_\ell$ is increasing.

Recall that A is the intersection of $\mathrm{S}(\mathfrak{g}^e)[t]$ with the sub-$\Bbbk[t, t^{-1}]$-module of $\mathrm{S}(\mathfrak{g}^e)[t, t^{-1}]$ generated by $\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$, and that A_+ is the augmentation ideal of A.

Lemma 3.1. (i) *For p a homogeneous element of standard degree d in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the element $\kappa(p)$ and ${}^e p$ have degree $2d$.*

(ii) *For some homogeneous sequence $a_1, \ldots, a_\ell$ in A_+, the elements $t, a_1, \ldots, a_\ell$ are algebraically independent over $\Bbbk$, and A is the intersection of $\mathrm{S}(\mathfrak{g}^e)[t]$ with $\Bbbk[t, t^{-1}, a_1, \ldots, a_\ell]$.*

(iii) *The Hilbert series of the R-algebra A is equal to*

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

(iv) *The Hilbert series of the $\Bbbk$-algebra $\varepsilon(A)$ is equal to*

$$P_{\varepsilon(A),\Bbbk}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

(v) *The subalgebra $\varepsilon(A)$ is the graded algebra associated with the filtration $F_0, F_1, \ldots$.*

Proof. (i) Let ρ be as in Subsection 1.4. For y in $\mathfrak{g}^f$ and s in $\Bbbk^*$,

$$p(s^{-2}\rho(s)(e + y)) = s^{-2d} p(\rho(s)(e + y)) = s^{-2d} p(e + y)$$

since p is invariant under the one-parameter subgroup ρ. Hence $\kappa(p)$ is homogeneous of degree $2d$. Since the monomials $x^{\mathbf{j}}$ are homogeneous, ${}^e p$ has degree $2d$.

(ii) Let $q_1, \ldots, q_\ell$ be a homogeneous generating sequence of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. By a well known fact (cf. e.g. [CM16, Lemma 4.4(i)]), the morphism

$$G \times (e + \mathfrak{g}^f) \longrightarrow \mathfrak{g}, \qquad (g, x) \longmapsto g(x)$$

is dominant. Then $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$ is a polynomial algebra generated by $\kappa(q_1), \ldots, \kappa(q_\ell)$. So, setting $a_i := \tau\circ\kappa(q_i)$ for $i = 1, \ldots, \ell$, the sequence $a_1, \ldots, a_\ell$ is a homogeneous sequence in A_+ such that

$$\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \Bbbk[t, t^{-1}, a_1, \ldots, a_\ell].$$

Let $\overline{\tau}$ be the automorphism of $\mathrm{S}(\mathfrak{g}^e)[t, t^{-1}]$ extending τ and such that $\overline{\tau}(t) = t$. Then

$$\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \overline{\tau}(\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]).$$

Since $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell+1$, $\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$ has dimension $\ell+1$ too, and $t, a_1, \ldots, a_\ell$ are algebraically independent over $\Bbbk$. By definition, $A = \mathrm{S}(\mathfrak{g}^e)[t] \cap \tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}]$. Hence

$$A[t^{-1}] = \Bbbk[t, t^{-1}, a_1, \ldots, a_\ell] \quad \text{and} \quad A = \mathrm{S}(\mathfrak{g}^e)[t] \cap \Bbbk[t, t^{-1}, a_1, \ldots, a_\ell].$$

(iii) Since t has degree 0, the grading of $\mathrm{S}(\mathfrak{g}^e)[t]$ extends to a grading of $\mathrm{S}(\mathfrak{g}^e)[t, t^{-1}]$ such that for all m, its space of degree m is equal to $\mathrm{S}(\mathfrak{g}^e)^{[m]}[t, t^{-1}]$. Then for all $\Bbbk[t, t^{-1}]$-submodule M of $\mathrm{S}(\mathfrak{g}^e)[t, t^{-1}]$, M has a Hilbert series:

$$P_{M,\Bbbk[t,t^{-1}]}(T) := \sum_{m\in\mathbb{N}} \mathrm{rk}\, M^{[m]} T^m$$

with $M^{[m]}$ the subspace of degree m of M. From the equality $A[t^{-1}] = \Bbbk[t, t^{-1}, a_1, \ldots, a_\ell]$, we deduce

$$P_{A[t^{-1}],\Bbbk[t,t^{-1}]}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}$$

since for $i = 1, \ldots, \ell$, the element a_i has degree $2d_i$ by (i). For all m, the rank of the R-module $A^{[m]}$ is equal to the rank of the $\Bbbk[t, t^{-1}]$-module $A[t^{-1}]^{[m]}$, whence

$$P_{A,R}(T) = \prod_{i=1}^{\ell} \frac{1}{1 - T^{2d_i}}.$$

(iv) Let m be a nonnegative integer. The R-module $A^{[m]}$ is free of finite rank and for $(v_1, \ldots, v_n)$ a basis of this module, $(tv_1, \ldots, tv_n)$ is a basis of the R-module $tA^{[m]}$. Since $\varepsilon(A)^{[m]}$ is the quotient of $A^{[m]}$ by $tA^{[m]}$,

$$\dim \varepsilon(A)^{[m]} = n = \mathrm{rk}\, A^{[m]},$$

whence the assertion by (iii).

(v) Let $\mathrm{gr}_F A$ be the graded algebra associated with the filtration $F_0, F_1, \ldots$ of $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$. Denote by $a \mapsto a(1)$ the evaluation map at $t = 1$ from $\mathrm{S}(\mathfrak{g}^e)[t]$ to $\mathrm{S}(\mathfrak{g}^e)$. For a in A such that $\varepsilon(a) \neq 0$, $a(1)$ is in $\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$ and $\varepsilon(a)$ is the component of minimal degree of $a(1)$ with respect to the standard grading, whence $\varepsilon(A) \subset \mathrm{gr}_F A$. Conversely, let $\overline{a}$ be a homogeneous element of degree m of $\mathrm{gr}_F A$ and let a be a representative of $\overline{a}$ in F_m. Then $\tau(a) = t^m b$ with b in A such that $\varepsilon(b) = \overline{a}$, whence $\mathrm{gr}_F A \subset \varepsilon(A)$ and the assertion. □

Let R_* be the localization of R at the prime ideal tR and set

$$\widehat{R} := \Bbbk[[t]], \qquad A_* := R_* \otimes_R A, \qquad \widehat{A} := \widehat{R} \otimes_R A.$$

The grading of A extends to gradings on A_* and $\widehat{A}$ such that $A_*^{[0]} = R_*$ and $\widehat{A}^{[0]} = \widehat{R}$.

Proposition 3.2. (i) *The algebra $\varepsilon(A)$ is polynomial if and only if for some standard homogeneous generating sequence $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. Moreover, in this case, A is a polynomial algebra.*

(ii) *If A_* is a polynomial algebra over R_*, then for some homogeneous sequence $p_1, \ldots, p_\ell$ in A_+, we have $A_* = R_*[p_1, \ldots, p_\ell]$, the elements $t, p_1, \ldots, p_\ell$ are algebraically independent over $\Bbbk$ and $p_1, \ldots, p_\ell$ have degree $2d_1, \ldots, 2d_\ell$ respectively.*

Proof. (i) Let $q_1, \ldots, q_\ell$ be a homogeneous generating sequence of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. We can assume that for $i = 1, \ldots, \ell$, q_i has standard degree d_i. For $i = 1, \ldots, \ell$, ${}^e q_i$ has degree $2d_i$ by Lemma 3.1(i), and we set

$$Q_i := t^{-2d_i} \tau{\circ}\kappa(q_i).$$

Then Q_i, for $i = 1, \ldots, \ell$, is in A by definition of A. For $\mathbf{i} = (i_1, \ldots, i_\ell)$ in $\mathbb{N}^\ell$, set:

$$q^{\mathbf{i}} := q_1^{i_1} \cdots q_\ell^{i_\ell}, \qquad Q^{\mathbf{i}} := Q_1^{i_1} \cdots Q_\ell^{i_\ell}, \qquad {}^e q^{\mathbf{i}} := {}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell},$$
$$|\mathbf{i}|_{\min} := 2 i_1 d_1 + \cdots + 2 i_\ell d_\ell.$$

Then, for all $\mathbf{i}$ in $\mathbb{N}^\ell$,

$$\tau{\circ}\kappa(q^{\mathbf{i}}) = t^{|\mathbf{i}|_{\min}} Q^{\mathbf{i}}.$$

Moreover,

$$\tau{\circ}\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})[t, t^{-1}] = \Bbbk[t, t^{-1}, Q_1, \ldots, Q_\ell].$$

Let a be in A. For some l in $\mathbb{N}$ and for some sequence $c_{\mathbf{i},m}$, $(\mathbf{i}, m) \in \mathbb{N}^\ell \times \mathbb{N}$ in $\Bbbk$, of finite support,

$$t^l a = \sum_{(\mathbf{i},m) \in \mathbb{N}^\ell \times \mathbb{N}} c_{\mathbf{i},m} t^m Q^{\mathbf{i}} \quad \text{whence} \quad \sum_{\mathbf{i} \in \mathbb{N}^\ell} c_{\mathbf{i},m} \, {}^e q^{\mathbf{i}} = 0$$

for $m < l$. Hence a is in $R[Q_1, \ldots, Q_\ell]$ since the elements ${}^e q^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}^\ell$ are linearly independent over $\Bbbk$. As a result,

$$A = R[Q_1, \ldots, Q_\ell] \quad \text{and} \quad \varepsilon(A) = \Bbbk[{}^e q_1, \ldots, {}^e q_\ell]$$

so that A and $\varepsilon(A)$ are polynomial algebras over $\Bbbk$ since ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$.

Conversely, suppose that $\varepsilon(A)$ is a polynomial algebra. By Lemma 3.1, (i) and (iv), the algebra $\varepsilon(A)$ is graded for both Slodowy grading and standard grading.

Let d be the dimension of $\varepsilon(A)$. As $\varepsilon(A)$ is a polynomial algebra, it is regular so that the $\Bbbk$-space $\varepsilon(A)_+/\varepsilon(A)_+^2$ has dimension d. Moreover, the two gradings on $\varepsilon(A)$ induce gradings on $\varepsilon(A)_+/\varepsilon(A)_+^2$. Hence $\varepsilon(A)_+/\varepsilon(A)_+^2$ has a bihomogeneous basis. Then some bihomogeneous sequence $u_1,\dots,u_d$ in $\varepsilon(A)_+$ represents a basis of $\varepsilon(A)_+/\varepsilon(A)_+^2$. As a result, the $\Bbbk$-algebra $\varepsilon(A)$ is generated by the bihomogeneous sequence $u_1,\dots,u_d$. For $i = 1,\dots,d$, denote by δ_i the Slodowy degree of u_i. As ε is homogeneous with respect to the Slodowy grading, $u_i = \varepsilon(r_i)$ for some homogeneous element r_i of degree δ_i of A. Let m_i be the smallest nonnegative integer such that $t^{m_i}r_i$ is in $\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$. According to Lemma 3.1(i), δ_i is even and for some standard homogeneous element p_i of standard degree $\delta_i/2$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, $t^{m_i}r_i = \tau\circ\kappa(p_i)$. Then $u_i = {}^e p_i$ since p_i is standard homogeneous.

Let $\mathfrak{P}$ be the subalgebra of $\mathrm{S}(\mathfrak{g})$ generated by $p_1,\dots,p_d$. Suppose that $\mathfrak{P}$ is strictly contained in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. A contradiction is expected. For some positive integer m, the space $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}_m$ of standard degree m of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is not contained in $\mathfrak{P}$. Let q be in $(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})_m \setminus \mathfrak{P}$ such that ${}^e q$ has maximal standard degree. By Lemma 3.1(i), ${}^e q$ is a polynomial in $u_1,\dots,u_d$, of degree $2m$. So, for some polynomial q' of degree m in $\mathfrak{P}$, ${}^e(q-q')$ has standard degree bigger than the standard degree of ${}^e q$. So, by maximality of the standard degree of ${}^e q$, the elements $q-q'$ and q are in $\mathfrak{P}$, whence the contradiction. As a result, $\mathfrak{P} = \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and $d = \ell$.

(ii) Suppose that A_* is a polynomial algebra. Denoting by J the ideal of A_* generated by t and A_+, the $\Bbbk$-space J/J^2 is a graded space of dimension $\ell+1$ since A_* is a regular algebra of dimension $\ell+1$. Then for some homogeneous sequence $p_1,\dots,p_\ell$ in A_+, $(t,p_1,\dots,p_\ell)$ is a basis of J modulo J^2. Since $p_1,\dots,p_\ell$ have positive degree, we prove by induction on d that

$$A_*^{[d]} \subset R_*[p_1,\dots,p_\ell]^{[d]} + tA_*^{[d]}.$$

Then by induction on m, we get

$$A_*^{[d]} \subset R_*[p_1,\dots,p_\ell] + t^m A_*^{[d]}.$$

So, since the R_*-module $A_*^{[d]}$ is finitely generated,

$$A_*^{[d]} \subset \widehat{R}[p_1,\dots,p_\ell]^{[d]}.$$

Apply Lemma 2.3 to $N = A$ and $M = \mathrm{S}(\mathfrak{g}^e)[t]$. Since $\widehat{N} = \widehat{R}[p_1,\dots,p_\ell]$, for $a \in N$, there exists $r \in R$ such that $r(0) \neq 0$ and $ra \in R[p_1,\dots,p_\ell]$ by Lemma 2.3. So A_* is contained in $R_*[p_1,\dots,p_\ell]$, whence $A_* = R_*[p_1,\dots,p_\ell]$.

Denote by $\delta_1,\dots,\delta_\ell$ the respective degrees of $p_1,\dots,p_\ell$. We can suppose that $p_1,\dots,p_\ell$ is ordered so that $\delta_1 \leqslant \cdots \leqslant \delta_\ell$. Prove by induction on i that $\delta_j = 2d_j$ for $j = 1,\dots,i$. By Lemma 3.1(iii), $2d_1$ is the smallest positive degree of the elements of A. Moreover, δ_1 is the smallest positive degree of the elements of $R[p_1,\dots,p_\ell]$, whence $\delta_1 = 2d_1$. Suppose $\delta_j = 2d_j$ for $j = 1,\dots,i-1$. Set $A_i := R[p_i,\dots,p_\ell]$.

Then, by induction hypothesis and Lemma 3.1(iii),

$$P_{A_i,R}(T) = \prod_{j=i}^{\ell} \frac{1}{1-T^{\delta_j}} = \prod_{j=i}^{\ell} \frac{1}{1-T^{2d_j}}.$$

By the first equality, δ_i is the smallest positive degree of the elements of A_i and by the second equality, $2d_i$ is the smallest positive degree of the elements of A_i too, whence $\delta_i = 2d_i$. Then with $i = \ell$, we get that $\delta_j = 2d_j$ for $j = 1, \dots, \ell$. □

Recall that $\widehat{R} = \Bbbk[[t]]$.

Corollary 3.3. *Suppose that A_* is a polynomial algebra. Then for some standard homogeneous generating sequence $q_1, \dots, q_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*

$$A_* = R_*[t^{-2d_1}\tau\circ\kappa(q_1), \dots, t^{-2d_\ell}\tau\circ\kappa(q_\ell)].$$

Proof. For m nonnegative integer, denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}_m$ the space of standard degree m of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. By Proposition 3.2(ii), for some homogeneous sequence $p_1, \dots, p_\ell$ in A_+ such that $p_1, \dots, p_\ell$ have degree $2d_1, \dots, 2d_\ell$ respectively,

$$A_* = R_*[p_1, \dots, p_\ell].$$

For $i = 1, \dots, \ell$, let m_i be the smallest integer such that $t^{m_i}p_i$ is in $\tau\circ\kappa(\mathrm{S}(\mathfrak{g})^{\mathfrak{g}})$. By Lemma 3.1(i), $t^{m_i}p_i$ has an expansion

$$t^{m_i}p_i = \sum_{j\in\mathbb{N}} t^j \tau\circ\kappa(q_{i,j})$$

with $q_{i,j}$, $j \in \mathbb{N}$, in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}_{d_i}$ of finite support. Denoting by $\delta_{i,j}$ the standard degree of ${}^e q_{i,j}$, set:

$$J'_i := \{j \in \mathbb{N} \; ; \; m_i = j + \delta_{i,j}\}, \qquad \delta_i := \inf\{\delta_{i,j} \; ; \; j \in J'_i\},$$

$$j_i := m_i - 2d_i, \qquad Q_i := t^{-2d_i}\tau\circ\kappa(q_{i,j_i}).$$

For $i = 1, \dots, \ell$, since p_i is not divisible by t in A,

$$p_i - Q_i \in tA,$$

whence

$$A_* \subset R_*[Q_1, \dots, Q_\ell] + tA_*.$$

Then, by induction m,

$$A_* \subset R_*[Q_1, \dots, Q_m] + t^m A_*$$

for all m. As a result,

$$\widehat{A} = \widehat{R}[Q_1, \dots, Q_\ell],$$

since for all d, the R_*-module $A_*^{[d]}$ is finitely generated. Then, by Lemma 2.3,

$$A_* = R_*[Q_1, \dots, Q_\ell].$$

As a result, since A has dimension $\ell+1$, the elements $t, Q_1, \ldots, Q_\ell$ are algebraically independent over $\Bbbk$ and so are $q_{1,j_1}, \ldots, q_{\ell,j_\ell}$. Moreover the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is generated by $q_{1,j_1}, \ldots, q_{\ell,j_\ell}$ since they have degree $d_1, \ldots, d_\ell$ respectively. □

3.2. Denote by $\mathcal{V}$ the nullvariety of A_+ in $\mathfrak{g}^f \times \Bbbk$. Let $\mathcal{V}_*$ be the union of the irreducible components of $\mathcal{V}$ which are not contained in $\mathfrak{g}^f \times \{0\}$. The following result is proven in [CM16, Corollary 4.4(i)]. Indeed, the proof of this result does not use the assumption of [CM16, Section 4] that for some homogeneous generators $q_1, \ldots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent.

Lemma 3.4 ([CM16, Corollary 4.4(i)])**.** (i) *The variety $\mathcal{V}_*$ is equidimensional of dimension $r + 1 - \ell$.*

(ii) *For all irreducible component X of $\mathcal{V}_*$ and for all z in $\Bbbk$, X is not contained in $\mathfrak{g}^f \times \{z\}$.*

Let $\mathcal{N}$ be the nullvariety of $\varepsilon(A)_+$ in $\mathfrak{g}^f$. Then $\mathcal{V}$ is the union of $\mathcal{V}_*$ and $\mathcal{N} \times \{0\}$.

Lemma 3.5. (i) *All irreducible component of $\mathcal{N}$ have dimension at least $r - \ell$ and all irreducible component of $\mathcal{V}$ have dimension at least $r + 1 - \ell$.*

(ii) *Assume that $\mathcal{N}$ has dimension $r - \ell$. Then for some homogeneous sequence $p_1, \ldots, p_{r-\ell}$ in $\mathrm{S}(\mathfrak{g}^e)_+$, the nullvariety of $t, p_1, \ldots, p_{r-\ell}$ in $\mathcal{V}$ is equal to $\{0\}$.*

Proof. (i) By Lemma 3.1(ii), for some homogeneous sequence $a_1, \ldots, a_\ell$ in A_+, the elements $t, a_1, \ldots, a_\ell$ are algebraically independent over $\Bbbk$. Let $b_1, \ldots, b_m$ be a homogeneous sequence in A_+, generating the ideal $\mathrm{S}(\mathfrak{g}^e)[t]A_+$ of $\mathrm{S}(\mathfrak{g}^e)[t]$. Set:

$$B := \Bbbk[a_1, \ldots, a_\ell, b_1, \ldots, b_m], \qquad B_+ := Ba_1 + \cdots + Ba_\ell + Bb_1 + \cdots + Bb_m,$$
$$C := B[t], \qquad C_{++} := B_+[t] + Ct.$$

Then B and C are graded subalgebras of A and B_+ and C_{++} are maximal ideals of B and C respectively. Moreover, C has dimension $\ell + 1$. We have a commutative diagram

$$\begin{tikzcd} & \mathfrak{g}^f \times \Bbbk \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathrm{Specm}(C) \arrow[rr, "\pi"] & & \mathrm{Specm}(B) \end{tikzcd}$$

with α, β, π the morphisms whose comorphisms are the canonical injections

$$C \hookrightarrow \mathrm{S}(\mathfrak{g}^e)[t], \quad B \hookrightarrow \mathrm{S}(\mathfrak{g}^e)[t], \quad B \hookrightarrow C$$

respectively. Since C has dimension $\ell + 1$, the irreducible components of the fibers of α have dimension at least $r-\ell$, whence the result for $\mathcal{N}$ since $\mathcal{N}\times\{0\} = \alpha^{-1}(C_{++})$. Moreover, $\mathcal{V} = \beta^{-1}(B_+)$ and $\pi^{-1}(B_+)$ is a subvariety of dimension 1 of $\mathrm{Specm}(C)$. Hence all irreducible component of $\mathcal{V}$ has dimension at least $r + 1 - \ell$.

(ii) Prove by induction on i that there exists a homogeneous sequence $p_1, \ldots, p_i$ in $\mathrm{S}(\mathfrak{g}^e)_+$ such that the minimal prime ideals of $\mathrm{S}(\mathfrak{g}^e)$ containing $\varepsilon(A)_+$ and $p_1, \ldots, p_i$

have height $\ell + i$. First of all, $\mathrm{S}(\mathfrak{g}^e)\varepsilon(A)_+$ is graded. Then the minimal prime ideals of $\mathrm{S}(\mathfrak{g}^e)$ containing $\varepsilon(A)_+$ are graded too. By, (i), they have height ℓ since $\mathcal{N}$ has dimension $r - \ell$ by hypothesis. In particular, they are strictly contained in $\mathrm{S}(\mathfrak{g}^e)_+$. Hence, by Lemma 2.1(ii), for some homogeneous element p_1 in $\mathrm{S}(\mathfrak{g}^e)$, p_1 is not in the union of these ideals so that the statement is true for $i = 1$ by [Ma86, Ch. 5, Theorem 13.5]. Suppose that it is true for $i - 1$. Then the minimal prime ideals containing $\varepsilon(A)_+$ and $p_1, \ldots, p_{i-1}$ are graded and strictly contained in $\mathrm{S}(\mathfrak{g}^e)_+$ by the induction hypothesis. So, by Lemma 2.1(ii), for some homogeneous element p_i in $\mathrm{S}(\mathfrak{g}^e)$, p_i is not in the union of these ideals and the sequence $p_1, \ldots, p_i$ satisfy the condition of the statement by [Ma86, Ch. 5, Theorem 13.5]. For $i = r - \ell$, the nullvariety of $p_1, \ldots, p_{r-\ell}$ in $\mathcal{N}$ has dimension 0. Then it is equal to $\{0\}$ as the nullvariety of a graded ideal, whence the assertion since $\mathcal{N} \times \{0\}$ is the nullvariety of t in $\mathcal{V}$. □

3.3. We assume in this subsection that $\mathcal{N}$ has dimension $r-\ell$. Let $p_1, \ldots, p_{r-\ell}$ be as in Lemma 3.5(ii), and set

$$C := A[p_1, \ldots, p_{r-\ell}].$$

Then $p_1, \ldots, p_{r-\ell}$ are algebraically independent over A since $\mathcal{N}$ has dimension $r-\ell$.

Lemma 3.6. *The ideal* $\mathrm{S}(\mathfrak{g}^e)[t]_+$ *of* $\mathrm{S}(\mathfrak{g}^e)[t]$ *is the radical of* $\mathrm{S}(\mathfrak{g}^e)[t]C_+$.

Proof. Let Y be an irreducible component of the nullvariety of C_+ in $\mathfrak{g}^f \times \Bbbk$. Then Y has dimension at least 1. By definition the nullvariety of t in Y is equal to $\{0\}$. Hence Y has dimension 1. The grading on $\mathrm{S}(\mathfrak{g}^e)[t]$ induces an action of the one-dimensional multiplicative group $\mathrm{G_m}$ on $\mathfrak{g}^f \times \Bbbk$ such that for all (x,z) in $\mathfrak{g}^f \times \Bbbk$, $(0,z)$ is in the closure of the orbit of (x,z) under $\mathrm{G_m}$. Since C_+ is graded, Y is invariant under $\mathrm{G_m}$. As a result, $Y = \{0\} \times \Bbbk$ or for some x in $\mathfrak{g}^f \times \Bbbk$, Y is the closure of the orbit of $(x,0)$ under $\mathrm{G_m}$ since 0 is the nullvariety of t in Y. In the last case, x is a zero of $p_1, \ldots, p_{r-\ell}$ in $\mathcal{N}$, that is $x = 0$. Hence $Y = \{0\} \times \Bbbk$. As a result, the nullvariety of C_+ in $\mathfrak{g}^f \times \Bbbk$ is equal to $\{0\} \times \Bbbk$ that is the nullvariety of $\mathrm{S}(\mathfrak{g}^e)[t]_+$, whence the assertion since $\mathrm{S}(\mathfrak{g}^e)[t]_+$ is a prime ideal of $\mathrm{S}(\mathfrak{g}^e)[t]$. □

For $\mathfrak{p}$ a prime ideal of A, denote by $A_{\mathfrak{p}}$ the localization of A at $\mathfrak{p}$ and by $\overline{\mathfrak{p}}$ the ideal of C generated by $\mathfrak{p}$. Since C is a polynomial algebra over A, $\overline{\mathfrak{p}}$ is a prime ideal of C and $A \setminus \mathfrak{p}$ is the intersection of A and $C \setminus \overline{\mathfrak{p}}$. Hence the localization $C_{\overline{\mathfrak{p}}}$ of C at $\overline{\mathfrak{p}}$ is a localization of the polynomial algebra $A_{\mathfrak{p}}[p_1, \ldots, p_{r-\ell}]$. Moreover, $A_{\mathfrak{p}}$ is the quotient of $C_{\overline{\mathfrak{p}}}$ by the ideal generated by $p_1, \ldots, p_{r-\ell}$. According to [Ma86, Ch. 6, Theorem 17.4], if $C_{\overline{\mathfrak{p}}}$ is Cohen-Macaulay, $p_1, \ldots, p_{r-\ell}$ is a regular sequence in $C_{\overline{\mathfrak{p}}}$ since $A_{\mathfrak{p}}$ has dimension $\dim C_{\overline{\mathfrak{p}}} - r + \ell$. Then, again by [Ma86, Ch. 6, Theorem 17.4], $A_{\mathfrak{p}}$ is Cohen-Macaulay if so is $C_{\overline{\mathfrak{p}}}$.

Proof of Theorem 1.5. By Lemma 3.6 and Proposition 2.2, the algebra C is finitely generated. Then A is finitely generated as a quotient of C. Hence by Lemma 2.7(ii),

A is a factorial ring and so is C as a polynomial ring over A. As a result, C is normal so that $\mathrm{S}(\mathfrak{g}^e)[t]$ and C satisfy the conditions (1), (2), (3) of Proposition 2.6. Hence by Proposition 2.6, for all prime ideal $\mathfrak{p}$ of A, containing t, $C_{\overline{\mathfrak{p}}}$ is Cohen-Macaulay, whence $A_{\mathfrak{p}}$ is Cohen-Macaulay. By Lemma 3.1(ii), for $\mathfrak{p}$ a prime ideal of A, not containing t, $A_{\mathfrak{p}}$ is the localization of $\Bbbk[t, t^{-1}, a_1, \ldots, a_\ell]$ at the prime ideal generated by $\mathfrak{p}$. Therefore $A_{\mathfrak{p}}$ is Cohen-Macaulay since the algebra $\Bbbk[t, t^{-1}, a_1, \ldots, a_\ell]$ is regular. As a result A is Cohen-Macaulay. In particular, A satisfies the conditions (1), (2), (3) of Subsection 2.2. So, by Proposition 2.10, A_* is a polynomial algebra over R_*. Then by Corollary 3.3, for some homogeneous generating sequence $q_1, \ldots, q_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,

$$A_* = R_*[t^{-2d_1}\tau{\circ}\kappa(q_1), \ldots, t^{-2d_\ell}\tau{\circ}\kappa(q_\ell)].$$

Form the above equality, we deduce that any element of A is the product of an element of the algebra $R[t^{-2d_1}\tau{\circ}\kappa(q_1), \ldots, t^{-2d_\ell}\tau{\circ}\kappa(q_\ell)]$ by a polynomial in t with nonzero constant term, whence

$$A = R[t^{-2d_1}\tau{\circ}\kappa(q_1), \ldots, t^{-2d_\ell}\tau{\circ}\kappa(q_\ell)] \quad \text{and so} \quad \varepsilon(A) = \Bbbk[{}^e q_1, \ldots, {}^e q_\ell]$$

since for $i = 1, \ldots, \ell$,

$${}^e q_i := \varepsilon(t^{-2d_i}\tau{\circ}\kappa(q_i)).$$

Since $\mathcal{N} \times \{0\}$ is the nullvariety of t and A_+ in $\mathfrak{g}^f \times \Bbbk$, $\mathcal{N}$ is the nullvariety in $\mathfrak{g}^f$ of ${}^e q_1, \ldots, {}^e q_\ell$. Hence ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over $\Bbbk$ since $\mathcal{N}$ has dimension $r - \ell$. □

4. Proof of Theorem 1.4

Let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We use the notations κ and ${}^e p$, $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, as in the introduction. In this section, we use the standard gradings on $\mathrm{S}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{g}^e)$. Let A_0 be the subalgebra of $\mathrm{S}(\mathfrak{g}^e)$ generated by the family ${}^e p$, $p \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, and let $\mathcal{N}_0$ be the nullvariety of $A_{0,+}$ in $\mathfrak{g}^f$ where $A_{0,+}$ denotes the augmentation ideal of A_0.

Let $a_1, \ldots, a_m$ be a homogeneous sequence in $A_{0,+}$ generating the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by $A_{0,+}$. According to [PPY07, Corollary 2.3], A_0 contains homogeneous elements $b_1, \ldots, b_\ell$ algebraically independent over $\Bbbk$.

Lemma 4.1. *Let $\mathfrak{A}$ be the integral closure of $\Bbbk[a_1, \ldots, a_m, b_1, \ldots, b_\ell]$ in the fraction field of $\mathrm{S}(\mathfrak{g}^e)$.*

(i) *The algebra $\mathfrak{A}$ is contained in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ and its fraction field is the fraction field of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.*
(ii) *Let a in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}_+$. If a is equal to 0 on $\mathcal{N}_0$, then a is in $\mathfrak{A}_+$.*
(iii) *The algebra $\mathfrak{A}$ is the integral closure of A_0 in the fraction field of $\mathrm{S}(\mathfrak{g}^e)$.*

Proof. (i) Let K_0 be the field of invariant elements under the adjoint action of $\mathfrak{g}^e$ in the fraction field of $\mathrm{S}(\mathfrak{g}^e)$. According to [CM16, Lemma 3.1], K_0 is the fraction field of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Since $a_1, \ldots, a_m, b_1, \ldots, b_\ell$ are in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, $\mathfrak{A}$ is contained in K_0. Moreover, $\mathfrak{A}$ is contained in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is integrally closed in K_0. Since K_0 has transcendence degree ℓ over $\Bbbk$ and since $b_1, \ldots, b_\ell$ are algebraically independent over $\Bbbk$, K_0 is the fraction field of $\mathfrak{A}$.

(ii) Since $\mathcal{N}_0$ is the nullvariety of $a_1, \ldots, a_m, b_1, \ldots, b_\ell$ in $\mathfrak{g}^f$, $\mathcal{N}_0$ is the nullvariety of $\mathfrak{A}_+$ in $\mathfrak{g}^f$. Let a be in $\mathrm{S}(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$ such that a is equal to 0 on $\mathcal{N}_0$. Since $\mathcal{N}_0$ is a cone, all homgogeneous components of a is equal to 0 on $\mathcal{N}_0$. So it suffices to prove the assertion for a homogeneous. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{g}^f \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & & \mathrm{Specm}(\mathfrak{A}[a]) \arrow[dl, "\beta"] \\ & \mathrm{Specm}(\mathfrak{A}) & \end{tikzcd}$$

with π, α, β the comorphisms of the canonical injections

$$\mathfrak{A}[a] \hookrightarrow \mathrm{S}(\mathfrak{g}^e), \quad \mathfrak{A} \hookrightarrow \mathrm{S}(\mathfrak{g}^e), \quad \mathfrak{A} \hookrightarrow \mathfrak{A}[a].$$

Since $\mathcal{N}_0$ is the nullvariety of $\mathfrak{A}[a]_+$ and $\mathfrak{A}_+$ in $\mathfrak{g}^f$, $\beta^{-1}(\mathfrak{A}_+) = \mathfrak{A}[a]_+$. The gradings of $\mathfrak{A}$ and $\mathfrak{A}[a]$ induce actions of $\mathrm{G_m}$ on $\mathrm{Specm}(\mathfrak{A})$ and $\mathrm{Specm}(\mathfrak{A}[a])$ such that β is equivariant. Moreover, $\mathfrak{A}_+$ is in the closure of all orbit under $\mathrm{G_m}$ in $\mathrm{Specm}(\mathfrak{A})$. Hence β is a quasi finite morphism. Moreover, β is a birational since $\mathfrak{A}$ and $\mathfrak{A}[a]$ have the same fraction field by (i). Hence, by Zariski's main theorem [Mu88], β is an open immersion from $\mathrm{Specm}(\mathfrak{A}[a])$ into $\mathrm{Specm}(\mathfrak{A})$. So, β is surjective since $\mathfrak{A}_+$ is in the image of β and since it is in the closure of all $\mathrm{G_m}$-orbit in $\mathrm{Specm}(\mathfrak{A})$. As a result, β is an isomorphism and a is in $\mathfrak{A}$, whence the assertion.

(iii) By (ii), A_0 is contained in $\mathfrak{A}$. Moreover, since $a_1, \ldots, a_m, b_1, \ldots, b_\ell$ are in A_0, $\mathfrak{A}$ is contained in the integral closure of A_0 in the fraction field of $\mathrm{S}(\mathfrak{g}^e)$, whence the assertion. □

Corollary 4.2. *Suppose that the algebra* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is finitely generated. Then* $\mathfrak{A}$ *is equal to* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Proof. Let C be the quotient of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by the ideal $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$. By hypothesis, C is finitely generated. Then it has finitely many minimal prime ideals. Denote them by $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. For a in the radical of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$, a is equal to 0 on $\mathcal{N}_0$. Moreover, it is in $\mathrm{S}(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$. Then, by Lemma 4.1(ii), a is in $\mathfrak{A}_+$. As a result, C is a reduced algebra and the canonical map

$$C \longrightarrow C/\mathfrak{p}_1 \times \cdots \times C/\mathfrak{p}_m$$

is injective. Since $\mathfrak{A}$ and $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ have the same fraction field, they have the same Krull dimension. Denote by d this dimension and by $\mathfrak{p}'_j$, for $j = 1, \ldots, m$, the inverse image of $\mathfrak{p}_j$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by the quotient map $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e} \rightarrow C$.

Claim 4.3. Let $j = 1,\dots,m$. For $i = 1,\dots,d$, there exists a sequence $c_1,\dots,c_i$ of elements of $\mathfrak{A}_+$ and an increasing sequence

$$\{0\} = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_i \subset \mathfrak{p}'_j$$

of prime ideals of of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that c_i is not $\mathfrak{q}_{i-1}$ and $c_1,\dots,c_j$ are in $\mathfrak{q}_j$ for $j = 1,\dots,i$.

Proof of Claim 4.3. Prove the claim by induction on i. Let c_1 be in $\mathfrak{A}_+ \setminus \{0\}$. As $\mathfrak{A}_+$ is contained in $\mathfrak{p}'_j$, there exists a minimal prime ideal $\mathfrak{q}_1$ of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, contained in $\mathfrak{p}'_j$ and containing c_1. Suppose $i > 1$ and the claim true for $i-1$. As the sequence

$$\{0\} = \mathfrak{A}_+ \cap \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{A}_+ \cap \mathfrak{q}_{i-1} \subset \mathfrak{A}_+$$

is an increasing sequence of prime ideals of $\mathfrak{A}_+$ and $\mathfrak{A}_+$ has height $d > i-1$, $\mathfrak{A}_+$ is not contained in $\mathfrak{q}_{i-1}$. Let c_i be in $\mathfrak{A}_+ \setminus \mathfrak{q}_{i-1}$ and $\mathfrak{q}_i$ the minimal prime ideal of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ contained in $\mathfrak{p}'_j$ and containing c_i and $\mathfrak{q}_{i-1}$. So by the induction hypothesis, the sequence $c_1,\dots,c_i$ satisfies the conditions of the claim. This concludes the proof. □

By the claim, $\mathfrak{p}'_j$ has height at least d for $j = 1,\dots,m$. Hence $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ are maximal ideals of C. As a result, the $\Bbbk$-algebra C is finite dimensional. Let V be a graded complement to $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$ in $\mathrm{S}(\mathfrak{g}^e)$. From the equality $\mathrm{S}(\mathfrak{g}^e) = V + \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+$, we get that $\mathrm{S}(\mathfrak{g}^e) = V\mathfrak{A} + \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}\mathfrak{A}_+^m$ for any nonnegative integer m by induction on m. Hence $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e} = V\mathfrak{A}$ so that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a finite extension of $\mathfrak{A}$. Since $\mathfrak{A}$ is integrally closed in the fraction field of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, $\mathfrak{A} = \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. □

Proof of Theorem 1.4. The "if" part results from [CM16, Theorem 1.5] (or, here, Theorem 1.3).

Suppose that e is good. By Definition 1.1 and Theorem 1.2, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and the nullvariety of $\mathrm{S}(\mathfrak{g}^e)_+^{\mathfrak{g}^e}$ in $\mathfrak{g}^f$ is equidimensional of dimension $r-\ell$. On the other hand, by Lemma 4.1(iii), $\mathfrak{A}$ is the integral closure of A_0 in the fraction field of $\mathrm{S}(\mathfrak{g}^e)$. Hence the nullvarieties of $\mathfrak{A}_+$ and $A_{0,+}$ in $\mathfrak{g}^f$ are the same. But by Corollary 4.2, $\mathfrak{A} = \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, so $\mathcal{N}_0$ has dimension $r-\ell$ since e is good. On the other hand, A_0 is contained in $\varepsilon(A)$ by construction of $\varepsilon(A)$, and $\varepsilon(A)$ is contained in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by [PPY07, Proposition 0.1], whence $\mathcal{N} = \mathcal{N}_0$.

As a result, $\mathcal{N}$ has dimension $r-\ell$ and so by Theorem 1.5, for some homogeneous generating sequence $q_1,\dots,q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the element ${}^eq_1,\dots,{}^eq_\ell$ are algebraically independent over $\Bbbk$. □

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