SATELLITES OF SPHERICAL SUBGROUPS

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Abstract. Let G be a connected reductive group over $\mathbb{C}$. One can associate with every spherical homogeneous space G/H its lattice of weights $\mathcal{X}^*(G/H)$ and a subset $\mathcal{S} := \{s_1, \ldots, s_k\} \subset \mathcal{X}^*(G/H)$ of linearly independent primitive lattice vectors which are called the spherical roots. For any subset $I \subseteq \mathcal{S}$ we define, up to conjugation, a spherical subgroup $H_I \subseteq G$ such that $\dim H_I = \dim H$ and $\mathcal{X}^*(G/H) = \mathcal{X}^*(G/H_I)$. One has $H_{\mathcal{S}} = H$, and the subgroup $H_I \subseteq G$ is horospherical if and only if $I = \varnothing$. We call the subgroups H_I ($I \subseteq \mathcal{S}$) the satellites of the spherical subgroup H. Our interest in satellites H_I is motivated by the space of arcs of the spherical homogeneous space G/H. We show that a satellite can be obtained from values at $t = 0$ of elements in the isotropy subgroup $\mathrm{Iso}(x) \subset G(\mathbb{C}[[t]])$ of an arc $x \in (G/H)(\mathbb{C}((t)))$. We show a close relation between the virtual Poincaré polynomials of the two spherical homogeneous spaces G/H and G/H_I ($I \subseteq \mathcal{S}$).

1. Introduction

Let G be a connected reductive group over $\mathbb{C}$. Denote by B a Borel subgroup in G. An irreducible algebraic G-variety X is said to be *spherical* if X is normal and if B has an open orbit in X. An algebraic subgroup $H \subset G$ is called *spherical* if the homogeneous space G/H is spherical, i.e., there exists an element $g \in G$ such that the subset $BgH \subset G$ is Zariski dense.

From now on, we fix a a spherical subgroup H of G and we assume that BH is Zariski dense in G. A *spherical embedding* of G/H is a normal G-variety X together with a G-equivariant open embedding $j : G/H \hookrightarrow X$ which allows us to identify H with the isotropy subgroup of the point $x := j(H) \in X$ such that $Gx \cong G/H$. Therefore we will always regards G/H as a open G-orbit in X whenever we write (X, x) for a G/H-embedding.

Let $U := BH \subset G/H$ be the open dense B-orbit in G/H. The stabilizer of $U \subset G/H$ is a parabolic subgroup P of G containing B. Then $U \cong B/B \cap H$ is an affine variety isomorphic to $(\mathbb{C}^*)^r \times \mathbb{C}^s$, where s is the dimension of the unipotent radical P_u of P. The number r is called the *rank* of the spherical homogeneous space G/H. We denote by $\mathcal{M}(U)$ the free abelian group of rank r consisting of all invertible regular functions f in the affine coordinate ring $\mathbb{C}[U]$ such that $f(H) = 1$. Any such a regular function $f \in \mathcal{M}(U)$ is a B-eigenfunction with some weight $\omega(f) \in \mathcal{X}^*(B)$, where $\mathcal{X}^*(B)$ is the lattice of characters of the Borel subgroup B. Using the map $f \mapsto \omega(f)$, we obtain a natural embedding of the

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lattice $\mathcal{M}(U)$ into the lattice $\mathcal{X}^*(B)$. The lattice

$$M = \mathcal{X}^*(G/H) := \omega(\mathcal{M}(U))$$

considered as a sublattice of $\mathcal{X}^*(B)$ is called the *weight lattice* of the spherical homogeneous space G/H.

Let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We denote by $\langle -, - \rangle : M \times N \to \mathbb{Z}$ the natural pairing. A spherical embedding X of G/H is called *elementary* if it consists of exactly two G-orbits: a dense orbit X^0 isomorphic to G/H and a closed orbit X' of codimension one. In this case X is always smooth and it is uniquely determined by the restriction of the divisorial valuation $v_{X'} : \mathbb{C}(G/H) \to \mathbb{Z}$ to the free abelian group $\mathcal{M}(U) \subset \mathbb{C}(G/H)$, where $\mathbb{C}(G/H)$ denotes the field of rational functions on G/H, [LV83, BLV86]. Using the isomorphism $\omega : \mathcal{M}(U) \cong M$ we can identify this restriction with an element $n_{X'} \in N$. In this way, we obtain a bijection between the set $\mathcal{V} = \mathcal{V}(G/H)$ of $\mathbb{Q}$-valued discrete G-invariant valuations of $\mathbb{C}(G/H)$ and a convex polyhedral cone in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ which is called the *valuation cone* of the spherical homogeneous spaces G/H, see [LV83, Proposition 7.4] or [Kn91, Corollary 1.8]. So we may identify $\mathcal{V}$ with the valuation cone of G/H. According to the Luna-Vust theory [LV83], any G-equivariant embedding X of the spherical homogeneous space G/H can be described combinatorially by a *colored fan* in the r-dimensional $N_{\mathbb{Q}}$; see [Kn91] or [Pe14] for surveys on the topic. Elementary embeddings correspond to uncolored cones $\mathbb{Q}_{\geqslant 0} v$ with v a nonzero lattice point in $N \cap \mathcal{V}$.

It is known that $\mathcal{V}$ is a cosimplicial cone, i.e., there exists $k \leqslant r$ linearly independent primitive lattice vectors $s_1, \ldots, s_k \in M$ such that

$$\mathcal{V} = \{n \in N_{\mathbb{Q}} \mid \langle s_i, n \rangle \leqslant 0,\ 1 \leqslant i \leqslant k\}.$$

The lattice vectors $s_1, \ldots, s_k$ are called the *spherical roots* of the homogeneous space G/H. We denote by $\mathcal{S} = \mathcal{S}(G/H)$ the set of spherical roots of G/H. The spherical roots $s_1, \ldots, s_k$ are nonnegative integral linear combinations of the positive roots with respect to the chosen Borel subgroup $B \subset G$.

1.1. **Main illustrating example.** In order to illustrate the content of our paper, we consider the simple classical example of the spherical homogeneous space G/H where $G = GL(n) \times GL(n)$ and $H \cong GL(n)$ is embedded diagonally in G. Then G/H can be identified with $GL(n)$ on which G acts by left and right multiplication: $(a, b)c := acb^{-1}$, $(a, b) \in G$, $c \in GL(n)$. Let $T^+(n)$ (resp. $T^-(n)$) be the subgroup of $GL(n)$ consisting of upper (resp. lower) triangular matrices. We denote by B the Borel subgroup $T^-(n) \times T^+(n)$ in G. Then $U := BH$ is Zariski open in G/H, U is the B-orbit of the identity matrix $E \in GL(n)$, and the lattice $\mathcal{M}(U)$ is freely generated by the n first principal minors $\Delta_1(c), \ldots, \Delta_n(c)$ of a matrix $c \in GL(n)$. Let $D(n)$ be the subgroup of $GL(n)$ consisting of diagonal matrices. The maximal diagonal torus $T = D(n) \times D(n) \subset B$ acts on Δ_i as follows:

$$(\mathrm{diag}(\lambda_1, \ldots, \lambda_n), \mathrm{diag}(\mu_1, \ldots, \mu_n))\, \Delta_i = (\lambda_1 \cdots \lambda_i)^{-1}(\mu_1 \cdots \mu_i)\Delta_i.$$

Therefore, the lattice of weights $M = \mathcal{X}^*(G/H)$ as a sublattice of $\mathcal{X}^*(B) = \mathcal{X}^*(D(n)) \oplus \mathcal{X}^*(D(n))$ consists of pairs (ω_1, ω_2) such that $\omega_1, \omega_2 \in \mathcal{X}^*(D(n))$ and $\omega_1 + \omega_2 = 0$. It can be shown that the G-invariant valuations v of the function field $\mathbb{C}(GL(n))$ are characterized by the inequalities:

$$v\left(\frac{\Delta_{i-1}\Delta_{i+1}}{\Delta_i^2}\right) \geqslant 0, \qquad 1 \leqslant i \leqslant n-1,$$

where $\Delta_0 = 1$. Consider the rational functions

$$f_i := \frac{\Delta_i^2}{\Delta_{i-1}\Delta_{i+1}}, \qquad 1 \leqslant i \leqslant n-1.$$

The maximal diagonal torus T in $B \subset G$ acts on f_i as follows:

$$(\mathrm{diag}(\lambda_1, \ldots, \lambda_n), \mathrm{diag}(\mu_1, \ldots, \mu_n))\, f_i = \lambda_{i+1}\lambda_i^{-1}\mu_i\mu_{i+1}^{-1} f_i.$$

So the spherical roots of the homogeneous space G/H are the weights $\omega(f_i) \in \mathcal{X}^*(B)$ which are the sums of two orthogonal simple B-positive roots corresponding to the characters $\lambda_{i+1}\lambda_i^{-1}$ and $\mu_i\mu_{i+1}^{-1}$. The last fact allows to obtain a natural bijection between the set of spherical roots $\mathcal{S}(G/H)$ and the set S of simple positive roots of $GL(n)$, see e.g. [Br97, Example 4.2].

For any subset $I \subset S$ of simple roots of $GL(n)$, we consider two opposite parabolic subgroups P_I^- and P_I^+ containing $T(n)^-$ and $T^+(n)$ respectively such that $L_I = P_I^- \cap P_I^+$ is their common Levi component, with $L_\varnothing = D(n)$ and $L_S = GL(n)$. We denote by L the natural surjective homomorphisms from P_I^- and P_I^+ to their common Levi factor L_I. Now we define the spherical subgroup $H_I \subset G$ by:

$$H_I := \{(a, b) \in G \mid a \in P_I^+,\ b \in P_I^-,\ L(a) = L(b)\}.$$

We notice that BH_I is Zariski dense in G. We call the subgroups H_I, $I \subset S$, the *satellites of* H. All satellites of H have the same dimension n^2. Since $B \cap H = B \cap H_I$ for all $I \subset S$, we can identify the weight lattices of all satellites with M:

$$M = \mathcal{X}^*(G/H) = \mathcal{X}^*(G/H_I), \qquad \forall\, I \subset S.$$

We refer the reader to Example 6.7 for another approach to this example.

Our purpose is to generalize the construction of the spherical subgroups H_I ($I \subseteq S$) in the above example to an arbitrary spherical homogeneous space G/H.

1.2. **Brion subgroups.** Let G/H be an arbitrary spherical homogeneous space. Consider an arbitrary primitive lattice point $v \neq 0$ in the valuation cone $\mathcal{V} \subset N_{\mathbb{Q}}$. Then v defines an elementary embedding $G/H \hookrightarrow X_v$. The smooth variety X_v consists of two G-orbits: the dense orbit $X_v^0 \cong G/H$ and a closed orbit X_v' of codimension one. Denote by $\tilde{X}_v$ the total space of the normal bundle to the G-invariant divisor X_v' in X_v. Then $\tilde{X}_v$ has a natural G-action and we can show that $\tilde{X}_v$ is again a spherical G-variety containing exactly two G-orbits: the zero section of the normal bundle (it is isomorphic to X_v') and its open complement $\tilde{X}_v^0$. We call the isotropy group H_v of a point in the open G-orbit $\tilde{X}_v^0$ a *Brion subgroup* corresponding to $v \in \mathcal{V}$. Such a subgroup was first considered by Michel Brion in

[Br90, §1.1][1]. It is defined up to G-conjugation. We explain in more detail the construction of H_v in Section 2.

We can alternatively construct the Brion subgroup H_v by considering the total space $\tilde{X}_v^\vee$ of the conormal bundle of X'_v in X_v. More precisely, if $(\tilde{X}_v^\vee)^0$ denotes the open complement to the zero section of $\tilde{X}_v^\vee$, then $(\tilde{X}_v^\vee)^0 \cong G/H_v$ and $(\tilde{X}_v^\vee)^0$ is spherical, [Pan99, §2.1]. Furthermore, $\tilde{X}_v^\vee$ is an elementary spherical embedding of G/H_v. Note that $\tilde{X}_v^\vee$ has a natural symplectic structure whose induced G-action is Hamiltonian. If $v = 0$, we define the Brion subgroup H_v to be just H.

There is also an algebraic description of Brion subgroups; see Section 3 about this approach.

Our main observation is the following (see Theorem 4.4 for a more precise formulation):

Theorem 1.1. *The Brion subgroup $H_v \subset G$ only depends, up to conjugation, on the minimal face $\mathcal{V}(v)$ of the valuation cone $\mathcal{V}$ containing v.*

Theorem 1.1 is proven in Section 4. It follows from the description of the homogeneous spherical data of the homogeneous space G/H_v (see Theorem 4.4). Set

$$I(v) := \{s_i \in \mathcal{S} \mid \langle s_i, v\rangle = 0\} \subset \mathcal{S} = \mathcal{S}(G/H).$$

Then the minimal face $\mathcal{V}(v)$ of the valuation cone $\mathcal{V}$ containing v is determined by the equations $\langle s_i, v\rangle = 0$, $\forall\, s_i \in I(v)$. We remark that for any subset $I \subset \mathcal{S}$ there exists a primitive lattice point $v \in \mathcal{V}$ such that $I(v) = I$ unless $I = \mathcal{S}$ and $\mathcal{S}$ is a basis of $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. We obtain a natural reversing bijection between the faces of the valuation cone $\mathcal{V}$ and the subsets $I \subseteq \mathcal{S}$. Therefore the conjugacy class of a Brion subgroup H_v only depends on the subset $I(v) \subset \mathcal{S}$. For a subset $I \subset \mathcal{S}$, $I \neq \mathcal{S}$, we define up to conjugacy a *spherical satellite* H_I of H as a Brion subgroup H_v where $I = I(v)$. For $I = \mathcal{S}$ we simply set $H_{\mathcal{S}} := H$.

In particular, if v is an interior lattice point of the valuation cone $\mathcal{V}$ then $I(v) = \varnothing$ and a Brion subgroup H_v is known to be *horospherical*, i.e., H_v contains a maximal unipotent subgroup of G, [BPa87, Corollary 3.8] (see also Proposition 2.4(2)). The horospherical satellite $H_\varnothing$ of H is closely related to a flat deformation of an arbitrary spherical homogeneous space G/H the a horospherical one $G/H_\varnothing$ (it has appeared already in the paper of Alexeev-Brion [AB04], and Kaveh [Ka05]). If $I(v) = \mathcal{S}$ then a Brion subgroup H_v is known to be conjugate to H, because in this case the elementary embedding X_v is isomorphic to the total space of the normal bundle to its closed divisorial orbit X'_v (see [Br90]).

1.3. **Isotropy subgroups in toroidal compactifications.** A spherical embedding X of G/H is called *simple* if X has a unique closed G-orbit, and it is called *toroidal* if X has no B-invariant divisor which is not G-invariant. If X is simple and toroidal, its fan has no color and contains a unique cone of maximal dimension.

[1]In [Br90], H is assumed to be equal to its normalizer, but this assumption is not needed in the construction of the subgroup H_v.

If the number $k = |\mathcal{S}|$ of spherical roots of G/H is equal to the rank r (i.e., the maximal possible number of spherical roots), then according to Luna-Vust, G/H admits a canonical normal projective G-equivariant toroidal compactification X such that we have a natural bijection $I \leftrightarrow X_I$ between the subsets $I \subset \mathcal{S}$ and the G-orbits $X_I \subset X$. We show that the normalizer of the isotropy subgroup of points in the G-orbit X_I is equal to the normalizer $N_G(H_I)$ of the spherical satellite H_I (cf. Proposition 5.1). This isotropy subgroup is exactly $N_G(H_I)$ if $I = \varnothing$. In particular, the unique closed G-orbit $X_\varnothing$ is a smooth projective homogeneous G-variety G/N where the parabolic subgroup N is the normalizer $N_G(H_\varnothing)$ of the horospherical satellite $H_\varnothing$.

One of our reasons for introducing the spherical satellites is the following. In general we cannot expect that the isotropy groups of points in a spherical embedding of G/H contain H up to a conjugation (even in the case where X is toroidal). However, this statement holds if we consider the conjugacy classes of all spherical satellites corresponding to the different faces of $\mathcal{V}$. In more detail, since all satellites H_I ($I \subset \mathcal{S}$) have the same weight lattice M (cf. Proposition 2.4(1) or Proposition 3.1), we may consider every lattice point $m \in M$ as a character of the normalizer $N_G(H_I)$ of H_I in G (cf. [Br97, Theorem 4.3] or Lemma 5.2), and we obtain (see also Proposition 5.3):

Theorem 1.2. *Let X be a simple toroidal spherical embedding of G/H corresponding to an uncolored cone σ. Denote by $I(\sigma)$ the set all spherical roots in $\mathcal{S}$ that vanish on σ. Let x' be a point in the unique closed G-orbit X' of X, and $\mathrm{Iso}_G(x')$ the isotropy subgroup of x' in G. Then, up to a conjugation, we have the inclusions:*

$$H_{I(\sigma)} \subset \mathrm{Iso}_G(x') \subset N_G(H_{I(\sigma)}).$$

Moreover, there is a homomorphism from $N_G(H_{I(\sigma)})$ to $\mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$ whose kernel is $\mathrm{Iso}_G(x')$.

Since every simple spherical embedding of G/H is dominated by a simple toroidal one, we obtain that the isotropy group of a point in any spherical embedding of G/H contains, up to conjugation, one of the spherical satellites H_I (cf. Corollary 5.4). This result can be viewed as a a generalization of [BaM13, Proposition 2.4] where such isotropy groups are described in the case where H is horospherical.

1.4. **Limits of isotropy groups of points in arc spaces.** Let $\mathcal{O} := \mathbb{C}[[t]]$ be the ring of formal power series and $\mathcal{K} := \mathbb{C}((t))$ its field of fractions. We denote by $(G/H)(\mathcal{K})$ the set of $\mathcal{K}$-valued points of the spherical homogeneous space G/H and by $G(\mathcal{O})$ the group of $\mathcal{O}$-valued points of the reductive group G. We know that the set of spherical roots $\mathcal{S}$ of G/H is determined by the valuation cone $\mathcal{V}$ of G/H. On the other hand, Luna and Vust suggested in [LV83] a description of the valuation cone $\mathcal{V}$ using the natural action of the group $G(\mathcal{O})$ on $(G/H)(\mathcal{K})$. More precisely, the approach of Luna-Vust establishes a natural bijection between lattice points v in $\mathcal{V}$ and $G(\mathcal{O})$-orbits $G(\mathcal{O})\widehat{\lambda}_v \subset (G/H)(\mathcal{K})$. One can choose a representative $\widehat{\lambda}_v$ of such a $G(\mathcal{O})$-orbit using an appropriate one-parameter subgroup λ_v in G. We refer the reader to Section 6 for more details about this topic.

Theorem 1.3. *The Brion subgroup H_v consists of limits when t goes to 0 of elements in the isotropy subgroup $\mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v) \subset G(\mathcal{O})$.*

Theorem 1.3 is proven in Section 6 (see Theorem 6.2). It can be used in practice to compute satellites, and we give a number of examples; cf. Examples 6.5–6.9.

1.5. **Poincaré polynomial of satellites.** To every complex algebraic variety X we associate its *virtual Poincaré polynomial* $P_X(t)$, uniquely determined by the following properties (cf. [BPe02]):

(1) $P_X(t) = P_Y(t) + P_{X\setminus Y}(t)$ for every closed subvariety Y of X,
(2) if X is smooth and complete, then $P_X(t) = \sum_m \dim H^m(X)t^m$ is the usual Poincaré polynomial.

Then $P_X(t) = P_Y(t)P_F(t)$ for every locally trivial fibration $X \to Y$ with a fiber isomorphic to F.

If X is a spherical homogeneous space G/H then $P_X(t)$ is a polynomial in t^2 and so the function $P_X(t^{1/2})$ is a polynomial in t. More generally, if X is a spherical embedding of a spherical homogeneous space G/H, then $P_X(t^{1/2})$ is a polynomial since X is a disjoint finite union of G-orbits which are all spherical homogeneous spaces. Therefore, for our purpose, it will be convenient to set:

$$\tilde{P}_X(t) := P_X(t^{1/2}).$$

For example, $\tilde{P}_X(t) = (t-1)^r t^s$ if $X \cong (\mathbb{C}^*)^r \times \mathbb{C}^s$. Abusing of notations, we continue to call the function $\tilde{P}_X$ the *virtual Poincaré polynomial* of X.

If G/H admits a wonderful compactification X, we observe that we can describe $\tilde{P}_X(t)$ using the Poincaré polynomials $\tilde{P}_{G/H_I}(t)$ for $I \subseteq \mathcal{S}$. More precisely, if we denote by $\mathcal{V}_I$ the face of the valuation cone $\mathcal{V}$ defined by the conditions $\langle x, s_i\rangle = 0$ for all $s_i \in I$ and by $\mathcal{V}_I^\circ$ the relative interior of the face $\mathcal{V}_I$, then

$$\tilde{P}_X(t) = \sum_I \tilde{P}_{G/H_I}(t) \sum_{v\in\mathcal{V}_I^\circ} t^{\kappa(v)}, \tag{1}$$

where κ is a linear function on N which takes value -1 on the dual basis $e_1, \dots, e_r$ of N to the basis of M consisted of spherical roots $s_1, \dots, s_r$. We refer the reader to Section 7 for more details about this example.

We can consider formula (1) as a generalization of the one we obtained for the Poincaré polynomial of a smooth projective horospherical variety (see [BaM13]). The general version of this formula for the stringy E-function of an arbitrary spherical embedding will be proven in [BaM] (in preparation). This formula has motivated us to investigate the ratio of the two virtual Poincaré polynomials,

$$\frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)},$$

for general spherical homogeneous space G/H.

Looking at some series of examples we came to the following conjecture:

Conjecture 1.4. *Let G/H be an arbitrary spherical homogeneous space. Then for any satellite H_I of the spherical subgroup $H \subset G$, the ratio of the two virtual Poincaré polynomials*

$$R_I(t^{-1}) := \frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)}$$

is always a polynomial R_I in t^{-1} with integral coefficients.

We prove this conjecture in several examples, in particular in the case where the spherical group H is connected (cf. Theorem 7.3) and in the case where the spherical homogeneous space G/H is of rank one (cf. Theorem 7.9).

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2. Brion subgroups, first properties and examples

Assume that BH is an open dense subset of G. Let P be the subset of $s \in G$ such that $sBH = BH$. Then P is a parabolic subgroup of G which contains B. We denote by P_u its unipotent radical.

2.1. Elementary embeddings.

We first review some results of [LV83], [BLV86] and [BPa87] on elementary embeddings.

Definition 2.1. *Let L be a Levi subgroup of P, and C the neutral component of the center of L. We say that L is* adapted to H *if the following conditions are satisfied:*

1. $P \cap H = L \cap H$,
2. $P \cap H$ *contains the derived subgroup* (L, L) *of* L,
3. *for any elementary embedding* (X, x) *of* G/H, *the action of* P_u *on* $Y := Px \cup Px'$, *with* Px' *an open* P-*orbit in the closed orbit of* X, *induces an isomorphism of algebraic varieties* $P_u \times (\overline{Cx} \cap Y) \to Y$.

Fix a Levi subgroup L of P, adapted to H, and let C be the neutral component of the center of L. Let $f \in \mathcal{M}(U) \cong M$, where $\mathcal{M}(U)$ is as in the introduction. Then f is a P-eigenvector in $\mathbb{C}(G/H)$. Since P has an open dense orbit in G/H, f is determined by its weight $\omega(f) \in \mathcal{X}^*(P)$. Moreover, the restriction to $P \cap H$ of $\omega(f)$ is trivial. But the sublattice of $\mathcal{X}^*(P)$ consisted of characters whose restriction to $P \cap H$ is trivial identifies with the lattice of characters $\mathcal{X}^*(C/C \cap H)$ of the torus $C/C \cap H$ since $P = P^u L$ and $H \supset (L, L)$. As a result, the weight lattice $M = \mathcal{X}^*(G/H)$ identifies with the character lattice $\mathcal{X}^*(C/C\cap H)$. Hence, by duality, we get an identification of $N_{\mathbb{Q}} \cong \mathrm{Hom}(M,\mathbb{Q})$ with $\mathcal{X}_*(C/C\cap H)\otimes_{\mathbb{Z}}\mathbb{Q}$, where $\mathcal{X}_*(C/C\cap H)$ is the free abelian group of one-parameter subgroups of the torus $C/C\cap H$.

Let (X,x) be an elementary embedding of G/H, with closed G-orbit X'. Then for some one-parameter subgroup λ of C, $\lim_{t\to 0}\lambda(t)x$ exists and belongs to the open P-orbit in X'.

Definition 2.2. *Such a one-parameter subgroup λ of C is said to be* adapted to *the elementary embedding (X,x).*

If μ is another one-parameter subgroup of C, adapted to (X,x), then the images of λ and μ are proportional in $\mathcal{X}_*(C/C\cap H)$. Through the identification $N_{\mathbb{Q}} \cong \mathcal{X}_*(C/C\cap H)\otimes_{\mathbb{Z}}\mathbb{Q}$, the primitive lattice points in $N\cap\mathcal{V}$ are in bijection with the indivisible one-parameter subgroups of $C/C\cap H$, adapted to the different elementary embeddings of G/H. More precisely, if v is a nonzero lattice point in $N\cap\mathcal{V}$, let (X_v,x_v) be the unique, up to G-conjugacy, elementary embedding of G/H corresponding to the one-dimensional uncolored cone $\mathbb{Q}_{\geqslant 0}v$ in $N_{\mathbb{Q}}$. Then for any one-parameter subgroup λ of C, adapted to (X_v,x_v), $\lim_{t\to 0}\lambda(t)x$ exists in the closed G-orbit of X_v, and the image of λ in $\mathcal{X}_*(C/C\cap H)$ corresponds to a point v_λ of $N\cap\mathcal{V}$ which is equivalent to v, cf. [BPa87, §2.10].

2.2. **Brion subgroups.** We give in this paragraph a more explicit construction of the Brion subgroups H_v ($v\in N\cap\mathcal{V}$) considered in the introduction, following [Br90, §1.1].

Let v be in $N\cap\mathcal{V}$. Assume first that v is nonzero. Then consider the corresponding elementary embedding (X_v,x_v) of G/H as above. Denote by X'_v the closed G-orbit of X_v, and choose a one-parameter subgroup λ_v of C adapted to (X_v,x_v). Thus

$$x'_v := \lim_{t\to 0}\lambda_v(t)x_v$$

belongs to X'_v. Denote by H'_v the isotropy group of x'_v in G. The subgroup H'_v acts on the tangent space $T_{x'_v}(X_v)$ at x'_v to X_v and leaves invariant the tangent space $T_{x'_v}(X'_v)$. Hence H'_v acts on the one-dimensional normal space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ via a character χ_v. The character χ_v is nontrivial since H'_v contains the image of λ_v which non trivially acts on $T_{x'_v}(Cx_v\cup Cx'_v)/T_{x'_v}(Cx'_v)$, cf. [Br90, §1.1]. We define H_v to be the kernel of χ_v in H'_v.

If $v=0$, we simply define H_v to be H.

Definition 2.3. *We call the subgroup H_v for $v\in N\cap\mathcal{V}$ a* Brion subgroup *of H.*

Brion subgroups are defined up to G-conjugacy. They have the same dimension as H and they are spherical by [Br90].

Denote by $\mathcal{V}^\circ$ the interior of $\mathcal{V}$. Recall that a subgroup of G is said to be *horospherical* is it contains a maximal unipotent subgroup of G.

Proposition 2.4. *Let $v\in N\cap\mathcal{V}$.*

1. *The weight lattice $\mathcal{X}^*(G/H_v)$ of the spherical homogeneous space G/H_v is equal to the weight lattice $M=\mathcal{X}^*(G/H)$ of the spherical homogeneous space G/H.*

(2) *The valuation cone of the spherical homogeneous space G/H_v is equal to $\mathcal{V} + \mathbb{Q}v$. In particular, $v \in \mathcal{V}^\circ$ if and only if the spherical subgroup H_v is horospherical.*

Proof. The statements are clear for $v = 0$. So we assume that v is nonzero.

(1) Denote by $\tilde{X}_v$ the total space of the normal bundle to the G-invariant divisor X'_v in X_v. We use the deformation of X_v to the normal bundle $\tilde{X}_v$ and we follow the ideas of the proof of [Br90, Proposition 1.2].

Consider the product $X_v \times \mathbb{C}$ as a spherical embedding of $G/H \times \mathbb{C}^*$ endowed with the regular $G \times \mathbb{C}^*$-action. It is a toroidal simple spherical embedding defined by a 2-dimensional cone in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ with the generators $(v, 0)$ and $(0, 1)$.

Let

$$p : \tilde{Y}_v \to X_v \times \mathbb{C}$$

be the blow-up of the 2-codimension closed $G \times \mathbb{C}^*$-orbit $X'_v \times \{0\}$ in $X_v \times \mathbb{C}$. Since $X'_v \times \{0\}$ is the intersection of two $G \times \mathbb{C}^*$-invariant divisors $X_v \times \{0\}$ and $X'_v \times \mathbb{C}$, the p-preimage of $X'_v \times \{0\}$ contains a $G \times \mathbb{C}^*$-orbit (it has codimension 2 in $\tilde{Y}_v$) which is isomorphic to X'_v. Denote by $\tilde{Z}_v$ the complement to this $G \times \mathbb{C}^*$-orbit in $\tilde{Y}_v$ and define $\rho : \tilde{Z}_v \to \mathbb{C}$ to be the restriction of the composition $\mathrm{pr}_2 \circ p$ to $\tilde{Z}_v \subset \tilde{Y}_v$, where pr_2 denotes the projection from $X_v \times \mathbb{C}$ to $\mathbb{C}$. Then ρ defines a fibration over $\mathbb{C}$ such that $\rho^{-1}(\mathbb{C} \smallsetminus \{0\})$ can be identified with $X_v \times \mathbb{C}^*$, and the fiber $\rho^{-1}(0)$ is isomorphic to $\tilde{X}_v$. The fibration ρ deforms the spherical variety X_v to $\tilde{X}_v$. We have a $G \times \mathbb{C}^*$ action on $\tilde{Z}_v$ so that $(\tilde{Z}_v, p^{-1}(x_v, 1))$ can be considered as a smooth elementary embedding of $G \times \mathbb{C}^*/H \times \{1\}$ containing a closed 1-codimension orbit isomorphic to $\tilde{X}_v^0 \cong G/H_v$. Let $\tilde{z}'_v \in \tilde{Z}_v$ be a point in this closed $G \times \mathbb{C}^*$-orbit of $\tilde{Z}_v$, and let $\tilde{H}'_v$ be its isotropy group in $G \times \mathbb{C}^*$. Up to G-conjugacy, $\tilde{H}'_v$ is the image of H'_v by the map

$$j \colon H'_v \to G \times \mathbb{C}^*, \quad h \mapsto (h, \chi_v(h)).$$

As G-varieties, G/H_v is isomorphic to $G \times \mathbb{C}^*/\tilde{H}'_v$, where G acts on $G \times \mathbb{C}^*$ by left multiplication on the left-factor. By the combinatorial interpretation of blow-ups in the Luna-Vust theory, the elementary embedding $\tilde{Z}_v$ is determined by the lattice point $\tilde{v} := (v, 0) + (0, 1) = (v, 1)$ in the lattice $N \oplus \mathbb{Z}$. According to [BPa87, §3.6], we get

$$\mathcal{X}^*(G \times \mathbb{C}^*/\tilde{H}'_v) = (M \oplus \mathbb{Z}) \cap (v, 1)^\perp$$

since $\mathcal{X}^*(G \times \mathbb{C}^*) \cong \mathcal{X}^*(B) \oplus \mathbb{Z}$. Indeed, the elementary embedding $(\tilde{Z}_v, p^{-1}(x_v, 1))$ of $G \times \mathbb{C}^*/H \times \{1\}$ corresponds to the lattice point $v + (0, 1)$ of the valuation cone of $G \times \mathbb{C}^*/H \times \{1\}$. The weight lattice of $G \times \mathbb{C}^*/\tilde{H}'_v$ as a G-variety is then the image of $(M \oplus \mathbb{Z}) \cap (v, 1)^\perp$ by the projection map

$$\mathcal{X}^*(B) \oplus \mathbb{Z} \to \mathcal{X}^*(B).$$

This image is nothing but M. Therefore the weight lattice of G/H_v is M since $G/H_v \cong G \times \mathbb{C}^*/\tilde{H}'_v$ as a G-variety.

(2) In the notations of (1), $\tilde{Z}_v$ is an elementary embedding of $G \times \mathbb{C}^*/H \times \{1\}$ with a 1-codimension closed $G \times \mathbb{C}^*$-orbit isomorphic to $\tilde{X}_v^0 \cong G \times \mathbb{C}^*/\tilde{H}'_v \cong G/H_v$. It is not difficult

to show (see the proof of [BPa87, Corollary 3.7]) that

$$\mathcal{V}(G \times \mathbb{C}^*/H \times \{1\}) \cong \mathcal{V} \oplus \mathbb{Q}(0,1).$$

On the other hand by [BPa87, Theorem 3.6], $\mathcal{V}(G \times \mathbb{C}^*/\tilde{H}_v') \cong \mathcal{V}(G/H_v)$ is a quotient of $\mathcal{V}(G \times \mathbb{C}^*/H)$ by $\mathbb{Q}(v,1)$. But the image of $\mathcal{V} \oplus \mathbb{Q}(0,1)$ by the isomorphism $N_{\mathbb{Q}} \xrightarrow{\sim} (N_{\mathbb{Q}} + \mathbb{Q}(0,1))/\mathbb{Q}(v,1)$ is $\mathcal{V} + \mathbb{Q}v$, whence

$$\mathcal{V}(G/H_v) \cong \mathcal{V} + \mathbb{Q}v.$$

In particular, $v \in \mathcal{V}^\circ$ if and only if $\mathcal{V}(G/H_v) = \mathcal{V} + \mathbb{Q}v = N_{\mathbb{Q}}$. Since the equality $\mathcal{V}(G/H_v) = N_{\mathbb{Q}}$ holds if and only if H_v is horospherical by [Kn91, Corollary 6.2], the statement follows. □

Example 2.5. *Assume that H is horospherical. Then $\mathcal{V} = N_{\mathbb{Q}}$, $N \cap \mathcal{V} = N$ and $\mathcal{V}^\circ = \mathcal{V}$. Let $v, v' \in N \cap \mathcal{V}$. Then by Proposition 2.4(2), H_v and $H_{v'}$ are horospherical. So we can assume that H_v and $H_{v'}$ contains the unipotent radical of a common Borel subgroup of G. By Proposition 2.4(1), H_v and $H_{v'}$ have the same weight lattice $M \subset \mathcal{X}^*(B)$. Therefore they are G-conjugate since they are both horospherical subgroups containing the unipotent radical of the same Borel subgroup of G (cf. e.g.* [Pas07, Proposition 1.6]*). Thus, if H is horospherical, then all Brion subgroups of H are equal to H up to G-conjugacy.*

Example 2.6. *Let $G = SL(2)$ and $H = T$, with T a maximal torus of B where B is the subgroup of G consisted of upper triangular matrices. Then $G/H \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1 \times \mathbb{P}^1)$ where G acts diagonally and linearly on each copy of $\mathbb{P}^1$. The Borel subgroup B has an open orbit, given by the set of $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1$, $x \neq y$ and both x, y are different from $[1,0] \in \mathbb{P}^1$. The homogeneous space G/H admits only one elementary embedding, $X = \mathbb{P}^1 \times \mathbb{P}^1$. The closed G-orbit of X is $\Delta := \mathrm{diag}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{P}^1$. Then the normal bundle N_Δ of Δ in X is the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ where $a = (\Delta, \Delta)$,* [H77, Chapter V, Exercice 1.4.1]*. Since Δ is a curve in the surface $\mathbb{P}^1 \times \mathbb{P}^1$, (Δ, Δ) is equals to the Euler number of $\mathbb{P}^1$ (cf.* [H77, Chapter V, proof of Proposition 1.4]*), that is 2. Since $G/B \cong \mathbb{P}^1$, we deduce that the complement to the zero section in N_Δ is isomorphic to G/U_2 where*

$$U_2 := \{\begin{pmatrix} \lambda & u \\ 0 & \lambda^{-1} \end{pmatrix} \; ; \; \lambda, u \in \mathbb{C}, \; \lambda^2 = 1\}.$$

Indeed, any character of B defines a line bundle of $\mathbb{P}^1$, and any line bundle of $\mathbb{P}^1$ is obtained in this way. In conclusion, $H_0 \cong H = T$ and for $v \neq 0$, $H_v \cong U_2$.

Example 2.7. *Let $G = SL(2)$ and $H = N_G(T)$, the normalizer of T in G. Since $SL(2)/T$ is spherical, G/H is spherical. The homogeneous space G/H admits only one elementary embedding, $X = \mathbb{P}^2 \cong \mathbb{P}(\mathfrak{sl}_2)$ where $\mathfrak{sl}_2$ is the Lie algebra of G. The closed G-orbit of X is isomorphic to the projective conic $C := \{[a,b,c] \mid a^2 + bc = 0\} \subseteq \mathbb{P}^2$, and the normal bundle N_C of C in $\mathbb{P}^2$ is $\mathcal{O}_{\mathbb{P}^2}(a)$ where $a = (C,C)$. But $(C,C) = 4$ since $C = 2L$ in the Picard group of $\mathbb{P}^2$ for some line L in $\mathbb{P}^2$. Arguing as in Example 2.6, we deduce that the complement to* *the zero section in* N_C *is isomorphic to* G/U_4 *where*

$$U_4 := \{\begin{pmatrix} \lambda & u \\ 0 & \lambda^{-1} \end{pmatrix} \; ; \; \lambda, u \in \mathbb{C}, \, \lambda^4 = 1\}.$$

In conclusion, $H_0 \cong H = N_G(T)$ *and for* $v \neq 0$, $H_v \cong U_4$.

3. Algebraic approach to Brion subgroups

We consider in this section an algebraic approach to Brion subgroups and we give another proof of Proposition 2.4(1). Let us fix a primitive lattice point v in $N \cap \mathcal{V}$.

Let $K := \mathbb{C}(G/H)$ be the field of rational functions over G/H. The field K is a B-module and it is the quotient field of the affine coordinate ring $A := \mathbb{C}[U]$ of the open B-orbit $U \subseteq G/H$. Let (X_v, x_v) be the elementary spherical embedding of G/H corresponding to the G-invariant discrete valuation v of K. We consider the restriction of the valuation $v : K \to \mathbb{Z}$ to the subring $A \subset K$ and define the subring

$$A'_v := \{f \in A \mid v(f) \geqslant 0 \text{ or } f = 0\}.$$

By the local structure theorem, the ring A'_v is isomorphic to the coordinate ring of a B-invariant open subset $U' \subset X_v$ containing U. Moreover, U' is isomorphic to $(\mathbb{C}^*)^{r-1} \times \mathbb{C}^{s+1}$, with r, s as in the introduction. Let $f_v \in A'_v$ be the generator of the vanishing ideal I_v in $A'_v = \mathbb{C}[U']$ of the divisor $X'_v \cap U'$ obtained by the intersection of U' with the closed G-orbit X'_v in X_v. One has

$$I_v := \{f \in A \mid v(f) > 0 \text{ or } f = 0\}.$$

Then we set for any $i \in \mathbb{Z}$,

$$I_v^i := \{f \in A \mid v(f) \geqslant i\}.$$

We have a decreasing filtration

$$\cdots \supset I_v^{-j} \supset \cdots \supset I_v^{-2} \supset I_v^{-1} \supset A'_v \supset I_v \supset I_v^2 \supset \cdots \supset I_v^i \supset \cdots$$

such that:

$$A = \bigcup_{i \in \mathbb{Z}} I_v^i.$$

The group B acts on this filtration, and each I_v^i is a free A'_v-module of rank one. We set

$$\widetilde{A'_v} := \bigoplus_{i \geqslant 0} I_v^i / I_v^{i+1}, \qquad \widetilde{A_v} := \bigoplus_{i \in \mathbb{Z}} I_v^i / I_v^{i+1}$$

so that $\widetilde{A'_v} = \operatorname{gr} A'_v$ and $\widetilde{A_v} = \operatorname{gr} A$ with respect to the above filtrations. Denote by $\overline{A'_v}$ the residue field A'_v/I_v. The rings $\widetilde{A'_v}$, A'_v and $\overline{A'_v}$ are naturally endowed with a B-action. Note that $\widetilde{A'_v}$ is the affine coordinate ring of the normal bundle to the divisor $X_v \cap U'$ in U'. Let K_v be the fraction field of $\widetilde{A'_v}$. Then K_v is the field of rational functions on the total space $\tilde{X}_v$ of the normal bundle to the divisor X'_v in X_v [H77, §8.20]. Moreover, $\widetilde{A'_v}$ is the affine coordinate of a B-invariant open subset $\tilde{U}'$ in $\tilde{X}_v$ which has a nonempty intersection with the closed G-orbit in $\tilde{X}_v$ and $\widetilde{A_v}$ is the affine coordinate ring of the open B-orbit $\tilde{U}$ in $\tilde{U}'$.

Proposition 3.1. *There is a natural B-equivariant isomorphism between the rings $\widetilde{A_v}$ and A which induces an isomorphism between the groups of B-eigenfunctions in $\widetilde{A_v}$ and in A. In particular, the lattice of weights of the spherical homogeneous spaces G/H and G/H_v are the same.*

Proof. Denote by $K^{(B)}$ the set of B-eingenvectors of K. Since H is spherical, the dimension of

$$K_m^{(B)} := \{f \in K \mid bf = m(b)f,\ \forall\, b \in B\}, \qquad m \in \mathcal{X}^*(B),$$

is at most one. Choose for each $m \in M$ a generator f_m of $K_m^{(B)}$ and set,

$$\hat{M} := \bigoplus_{m \in M} \mathbb{C} f_m.$$

Then $\hat{M}$ is a $\mathbb{C}$-vector subspace of K. Similarly, let $K_v := \mathbb{C}(G/H_v)$, M_v be the weight lattice of G/H_v, and set

$$\hat{M}_v := \bigoplus_{m \in M_v} \mathbb{C} f_{v,m}$$

where for each $m \in M_v$, $f_{v,m}$ is a generator of

$$K_{v,m}^{(B)} := \{f \in K_v \mid bf = m(b)f,\ \forall\, b \in B\}, \qquad m \in \mathcal{X}^*(B).$$

Let $T \in I_v/I_v^2$ be the class of the generator $f_v \in I_v$. Then we have

$$\begin{aligned} \widetilde{A'_v} \cong \overline{A'_v}[T] &\cong \overline{A'_v} \oplus I_v/I_v^2 \oplus \cdots \oplus I_v^i/I_v^{i+1} \oplus \cdots \\ &\subset \overline{A'_v}[T, T^{-1}] \cong \bigoplus_{i \in \mathbb{Z}} I_v^i/I_v^{i+1} = \widetilde{A_v}. \end{aligned}$$

For $i \in \mathbb{Z}$, let ψ_i be the projection map

$$\psi_i \colon I_v^i \cap \hat{M} \longrightarrow I_v^i/I_v^{i+1}.$$

Since $I_v^i \cap \hat{M} = \bigoplus_{m \in M,\, \langle m,v \rangle \geqslant i} \mathbb{C} f_m$,

$$\operatorname{im} \psi_i \cong \bigoplus_{m \in M,\, \langle m,v \rangle = i} \mathbb{C} f_m.$$

By construction, $\bigoplus_{i \in \mathbb{Z}} \operatorname{im} \psi_i$ is contained in K_v and consists of B-eigenvectors, whence

$$\bigoplus_{i \in \mathbb{Z}} \operatorname{im} \psi_i \subseteq \hat{M}_v \subset K_v.$$

To get the converse inclusion $\hat{M}_v \subseteq \bigoplus_{i \in \mathbb{Z}} \operatorname{im} \psi_i$, let us prove that for each $m \in M_v$, $f_{v,m} \in \bigoplus_{i \in \mathbb{Z}} \operatorname{im} \psi_i$. Note that K_v admits a $\mathbb{C}^*$-action induced from the $\mathbb{Z}_{\geqslant 0}$-grading on A_v which commutes with the G-action. Then for $m \in M_v$, the B-eigenvector $f_{v,m}$ is also an eigenvector for the $\mathbb{C}^*$-action, which means that $f_{v,m}$ is homogeneous with respect to the $\mathbb{Z}$-grading on K_v. Hence, $f_{v,m}$ is in the image of ψ_i for some i.

In conclusion,

$$\hat{M}_v = \bigoplus_{i \in \mathbb{Z}} \operatorname{im} \psi_i \cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{m \in M,\, \langle m,v \rangle = i} \mathbb{C} f_m \cong \hat{M}.$$

So the weight lattice in K_v is M and $\hat{M}_v \cong \hat{M}$, whence the proposition since the above isomorphisms are B-equivariant. □

4. Homogeneous spherical data of satellites

Denote by $\mathcal{D} := \{D_1, \ldots, D_k\}$ the set of irreducible B-invariant divisors in G/H that are called *colors*. Using the finite double coset $B \setminus G/H$, we obtain a natural bijection $D_i \leftrightarrow D_i'$ between the set $\mathcal{D}$ of B-invariant divisors in G/H and the set $\mathcal{D}' := \{D_1', \ldots, D_k'\}$ of H-invariant divisors in the generalized flag manifold $B\backslash G$. The Picard group of the generalized flag manifold $B\backslash G$ is a free group whose base consists of the classes of line bundles $L_1, \ldots, L_n$ such that the space of global sections $H^0(B \setminus G, L_j)$ is the j-th fundamental representation (with the highest weight ϖ_j) of the Lie algebra of G ($1 \leqslant j \leqslant n$). Therefore we can associate with every color $D_i \subset G/H$ a nonnegative linear combination of fundamental weights: $\sum_{j=1}^{n} a_{ij}\varpi_j = [D_i'] \in \mathrm{Pic(B \setminus G)}$. Let $A := (a_{ij})_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant n}}$ and let $C := (c_1, \ldots, c_n)$ be the sum of rows of A, i.e., $c_j = \sum_{i=1}^{k} a_{ij}$ and

$$\sum_{i=1}^{k}[D_i'] = \sum_{j=1}^{n} c_j\varpi_j.$$

It can be shown that all entries c_j of the vector C belong to the set $\{0, 1, 2\}$ (in particular, the same statement holds for all entries of the matrix A). Moreover, the matrix A satisfies certain stronger additional conditions that are used in the classification of colors. Let $i \in \{1, \ldots, k\}$.

(1) if $a_{ij} = 2$ for some j then $a_{il} = 0$ for all $l \neq j$, i.e., $[D_i'] = 2\varpi_j$. In this case the B-invariant divisor $D_i \subset G/H$ is said to be *of type* $2a$,

(2) if the color D_i is not of type $2a$ then for some nonempty finite subset $J(i) \subset \{1, \ldots, n\}$ we have
$$[D_i'] = \sum_{j\in J(i)} \varpi_j,$$
and either $c_j = 2$, $\forall j \in J(i)$ (and the color D_i is called *of type* a), or $c_j = 1$, $\forall j \in J(i)$ (and the color D_i is called *of type* b).

Denote by $\mathcal{D}^{2a}$, $\mathcal{D}^a$ and $\mathcal{D}^b$ the set of colors of type $2a$, type a and type b respectively. For details we refer the reader to [GH15] and the references therein.

We consider two maps

$$\delta\colon \mathcal{D} \to \mathrm{Pic}(B \setminus G), \qquad \rho\colon \mathcal{D} \to N,$$

where $\delta(D_i) := [D_i'] \in \mathrm{Pic}(B \setminus G)$ ($1 \leqslant i \leqslant k$) and $\rho(D_i)$ denotes the lattice point in N corresponding to the restriction of the divisorial valuation ν_D of the field $\mathbb{C}(G/H)$ to the group of B-semiinvariants in $\mathbb{C}(G/H)$ corresponding to the elements of the weight lattice $M = \mathcal{X}^*(G/H)$.

Luna has suggested to describe every spherical homogeneous space G/H, up to a G-isomorphism, by some combinatorial data involving the root data of the algebraic group G, the sublattice $M \subseteq \mathcal{X}^*(B)$, the set of spherical roots $\mathcal{S} \subset M$ and the set of colors $\mathcal{D}$. More precisely, consider the triple $(M, \mathcal{S}, \mathcal{D})$ consisting of the lattice of weights $M \subseteq \mathcal{X}^*(B)$, the set of spherical roots $\mathcal{S} \subset M$ and the set $\mathcal{D}$ of all B-invariant divisors in G/H together with the two maps $\delta\colon \mathcal{D} \to \mathrm{Pic}(B \setminus G)$ and $\rho : \mathcal{D} \to N$. The following statement is equivalent to a conjecture of Luna which was proved by Losev [Lo09, Theorem 1]:

Theorem 4.1. *The triple $(M, \mathcal{S}, \mathcal{D})$ uniquely determines the spherical subgroup $H \subseteq G$ up to conjugation.*

As it is explained above the set of colors $\mathcal{D}$ splits into a disjoint union of three subsets $\mathcal{D} = \mathcal{D}^a \cup \mathcal{D}^{2a} \cup \mathcal{D}^b$. Let S be the set of simple roots of the reductive group G. Using a natural bijection between S and the set of fundamental weights $\alpha \leftrightarrow \varpi_\alpha$ we can assume that the set $J(i)$ in the equation

$$[D_i'] = \sum_{j \in J(i)} \varpi_j$$

is a subset in S. Moreover, the set $\mathcal{D}^a$ of colors of type a can be characterized as the set of those B-invariant divisors $D_i \in \mathcal{D}$ such that the set $J(i)$ contains a spherical root, i.e., $J(i) \cap \mathcal{S} \neq \varnothing$. Set $S^p := \{\alpha \in S \mid c_\alpha = 0\}$, that is S^p consists of those simple roots $\alpha \in S$ such that the fundamental weight ϖ_α is not a summand of $\delta(D_i)$ for all colors $D_i \in \mathcal{D}$. We can show that the triple $(M, \mathcal{S}, \mathcal{D})$ is uniquely determined by the quadruple $(M, \mathcal{S}, S^p, \mathcal{D}^a)$ where the set $\mathcal{D}^a$ of colors of type a is considered together with only one map $\rho\colon \mathcal{D}^a \to N$. Therefore we can uniquely determine the spherical subgroups $H \subseteq G$ up to conjugation by the quadruples $(M, \mathcal{S}, S^p, \mathcal{D}^a)$. This quadruple is called the *homogeneous spherical datum* of the spherical homogeneous space G/H.

The following theorem has been proved by Gagliardi and Hofscheier, [GH15, Theorem 1.1].

Theorem 4.2. *Let $(M, \mathcal{S}, S^p, \mathcal{D}^a)$ be the homogeneous spherical datum of a spherical homogeneous space G/H. Let $(\mathcal{C}, \varnothing)$ be a colored rational polyhedral cone with the empty set of colors (in particular, $\dim(\mathcal{C} \cap \mathcal{V}) = \dim \mathcal{C}$) that defines a simple toroidal spherical embedding $G/H \hookrightarrow X$ (the corresponding colored fan consists of pairs $(\mathcal{C}', \varnothing)$ where $\mathcal{C}'$ is a face of the cone $\mathcal{C}$). Let $X_0 \subset X$ be the unique closed G-orbit in X. We write the spherical homogeneous space X_0 as G/H_0 for some spherical subgroup $H_0 \subset G$. Then the homogeneous spherical datum of X_0 is the quadruple*

$$(M_0, \mathcal{S}_0, S^p, \mathcal{D}_0^a),$$

with

$$M_0 := M \cap \mathcal{C}^\perp, \quad \mathcal{S}_0 := \mathcal{S} \cap \mathcal{C}^\perp, \quad \mathcal{D}_0^a = \{D_i \in \mathcal{D}^a \mid J(i) \cap \mathcal{S}_0 \neq \varnothing\},$$

where the map $\rho_0\colon \mathcal{D}_0^a \to N_0 = N/\langle\mathcal{C}\rangle$ is the the restriction to $\mathcal{D}_0^a$ of the map $\rho\colon \mathcal{D}^a \to N$ composed with the natural homomorphism $N \to N/\langle\mathcal{C}\rangle = N_0$.

In particular, we obtain the following.

Corollary 4.3. *Let $v \in N \cap \mathcal{V}$ be a primitive lattice point in the valuation cone $\mathcal{V}$ of a spherical homogeneous spaces G/H defined by the homogeneous spherical datum $(M, \mathcal{S}, S^p, \mathcal{D}^a)$,* *and let X_v be an elementary spherical embedding of G/H corresponding to v whose closed divisorial G-orbit X'_v is a spherical homogeneous space G/H'_v. Then the homogeneous spherical datum of $X'_v = G/H'_v$ is the quadruple*

$$(M_0, \mathcal{S}_0, S^p, \mathcal{D}_0^a),$$

with

$$M_0 := M \cap v^\perp, \quad \mathcal{S}_0 := \mathcal{S} \cap v^\perp, \quad \mathcal{D}_0^a = \{D_i \in \mathcal{D}^a \mid J(i) \cap \mathcal{S}_0 \neq \emptyset\},$$

where the map $\rho_0 \colon \mathcal{D}_0^a \to N_0 = N/\langle v\rangle$ is the the restriction to $\mathcal{D}_0^a$ of the map $\rho\colon \mathcal{D}^a \to N$ composed with the natural homomorphism $N \to N/\langle v\rangle = N_0$.

Now we compare the spherical homogeneous datum of the spherical homogeneous space G/H and the spherical homogeneous datum of G/H_v corresponding to the Brion subgroup $H_v \subset G$.

Theorem 4.4. *Let $H \subset G$ be a spherical subgroup of G, and $v \in N$ an arbitrary primitive nonzero lattice point in the valuation cone of the spherical homogeneous space G/H defined by the homogeneous spherical datum $(M, \mathcal{S}, S^p, \mathcal{D}^a)$. Then the homogeneous spherical datum of the spherical homogeneous space G/H_v is the quadruple*

$$(M, \mathcal{S} \cap v^\perp, S^p, \mathcal{D}_v^a),$$

where $\mathcal{D}_v^a = \{D_i \in \mathcal{D}^a \mid J(i) \cap (\mathcal{S} \cap v^\perp) \neq \varnothing\}$ and the map $\rho_v \colon \mathcal{D}_v^a \to N$ is the restriction of $\rho\colon \mathcal{D}^a \to N$ to the subset $\mathcal{D}_v^a \subset \mathcal{D}^a$. In particular, the G-variety G/H_v (and hence the conjugacy class of the Brion subgroup $H_v \subset G$) only depends on the minimal face of the valuation cone of G/H containing the lattice point v.

Proof. We consider the spherical homogeneous space $G/H \times \mathbb{C}^*$ together with the $G \times \mathbb{C}^*$-action. Its lattice of weights is equal to $M \oplus \mathbb{Z}$; see the proof of Proposition 2.4. Then $G/H_v \cong (G \times \mathbb{C}^*)/H'_v$ is the closed $G \times \mathbb{C}^*$-orbit in the elementary spherical embedding of $G/H \times \mathbb{C}^*$ corresponding to the primitive lattice vector $(v, 1) \in N \oplus \mathbb{Z}$. Since the generalized flag manifolds of the reductive groups G and $G \times \mathbb{C}^*$ are the same we get a natural bijection between the set of colors in $G/H \times \mathbb{C}^*$ and in G/H as well as a natural bijection between the set of colors in G/H'_v and in $(G \times \mathbb{C}^*)/H'_v$ that both preserve the type of colors. On the other hand, by Corollary 4.3, we obtain a natural bijection between the set of a-colors $D_i \times \mathbb{C}^*$ in $G/H \times \mathbb{C}^*$ such that $J(i) \cap (\mathcal{S} \cap v^\perp) \neq \varnothing$ and the set of a-colors in $G/H_v \cong (G \times \mathbb{C}^*)/H'_v$. Since the composition of the natural embedding $N \hookrightarrow N \oplus \mathbb{Z}$ and the epimorphism $N \oplus \mathbb{Z} \to (N \oplus \mathbb{Z})/\langle (v, 1)\rangle \cong N$ is the indentity map on N we obtain that the ρ-images in N of the a-colors in G/H_v and the ρ-images of the a-colors D_i in G/H such that $J(i) \cap (\mathcal{S} \cap v^\perp) \neq \emptyset$ are the same, i.e., the map $\rho_v \colon \mathcal{D}_v^a \to N$ is the restriction of $\rho\colon \mathcal{D}^a \to N$ to the subset $\mathcal{D}_v^a \subset \mathcal{D}^a$. □

According to Theorem 4.4, the following definition is legitimate:

Definition 4.5. *For $I \subseteq \mathcal{S}$, denote by H_I the spherical subgroup H_v, where v is any point in the interior of $\mathcal{V}_I := \{n \in N_\mathbb{Q} \mid \langle s_i, n\rangle = 0,\ \forall\, s_i \in I\}$. Then H_I is well-defined up to G-conjugation. We call H_I the* spherical satellite of H associated with I *(or* with the face $\mathcal{V}_I$*).*

The spherical satellite $H_\mathcal{S}$ corresponding to the minimal face of $\mathcal{V}$ is G-conjugate to H. On the opposite side, there is a unique, up to a G-conjugation, horospherical satellite $H_\varnothing$ which corresponds to the whole cone $\mathcal{V}$ (cf. Proposition 2.4(2)). The valuation cone $\mathcal{V}$ is cosimplicial and it has 2^k faces. So H has exactly 2^k spherical satellites.

Remark 4.6. *Theorem 4.4 was already known in the case where H is horospherical; see Example 2.5. More generally, for $v \in N \cap \mathcal{V}^\circ$, Theorem 4.4 can be proven easily following the arguments of Example 2.5.*

As a corollary of Theorem 4.4 we obtain the following description of the spherical homogeneous datum of the spherical homogeneous space G/H_I corresponding to a satellite H_I.

Corollary 4.7. *Let $H \subset G$ be a spherical subgroup of G, and I an arbitrary subset of the set of spherical roots $\mathcal{S}$ of the spherical homogeneous space G/H defined by the homogeneous spherical datum $(M, \mathcal{S}, S^p, \mathcal{D}^a)$. Then the homogeneous spherical datum of the spherical homogeneous space G/H_I is the quadruple*

$$(M, I, S^p, \mathcal{D}^a_I),$$

where $\mathcal{D}^a_I = \{D_i \in \mathcal{D}^a \mid J(i) \cap I \neq \varnothing\}$ and the map $\rho_I\colon \mathcal{D}^a_I \to N$ is the restriction of $\rho\colon \mathcal{D}^a \to N$ to the subset $\mathcal{D}^a_I \subset \mathcal{D}^a$.

5. Normalizers of satellites and consequences

Let $N_G(H)$ be the normalizer of H in G. The homogeneous space $G/N_G(H)$ is spherical and $N_G(N_G(H)) = N_G(H)$ by [BPa87, §5] (cf. [T11, Lemma 30.2]). So its valuation cone $\widehat{\mathcal{V}} := \mathcal{V}(G/N_G(H))$ is strictly convex by [BPa87, Corollary 5.3]. More precisely, $\widehat{\mathcal{V}} = \mathcal{V}/(\mathcal{V} \cap -\mathcal{V})$ (see e.g. the proof of [T11, Theorem 29.1]). Let $\widehat{X}$ be the unique, up to a G-equivariant isomorphism, complete and simple embedding of $G/N_G(H)$. It is a toroidal embedding of $G/N_G(H)$. It has no color and the corresponding uncolored cone of $N_\mathbb{Q}$ is $\widehat{\mathcal{V}}$. Moreover, the G-orbits of $\widehat{X}$ are in bijection with the faces of $\widehat{\mathcal{V}}$. Thus we have a natural bijection $I \leftrightarrow \widehat{X}_I$ between the subsets $I \subset \mathcal{S}$ and the G-orbits $\widehat{X}_I \subset \widehat{X}$ such that $\widehat{X}_\mathcal{S} \cong G/N_G(H)$.

Proposition 5.1. *Let I be a subset of the set of spherical roots $\mathcal{S}$ of the spherical homogeneous space G/H. The normalizer of the isotropy subgroup of any point in the G-orbit $\widehat{X}_I$ is equal to the normalizer $N_G(H_I)$ of the spherical satellite H_I.*

Proof. Let v be a point in the interior $\mathcal{V}^\circ_I$ of $\mathcal{V}_I := \{n \in N_\mathbb{Q} \mid \langle s_i, n\rangle = 0,\ \forall\, s_i \in I\}$. Let (X_v, x_v) be the elementary embedding of G/H with uncolored cone $\mathbb{Q}_{\geqslant 0} v$.

For $n \in \mathcal{V}$, denote by $\widehat{n}$ the image of n by the projection map $N_{\mathbb{Q}} \to N_{\mathbb{Q}}/(\mathcal{V} \cap -\mathcal{V})$. Then set

$$\widehat{\mathcal{V}}_I := \{\widehat{n}\ ;\ n \in \mathcal{V}_I\}.$$

According to [T11, Theorem 15.10], the canonical map

$$G/H \to G/N_G(H)$$

extends to a G-equivariant map

$$\pi\colon X_v \to \widehat{X}.$$

Let x'_v be a point in the closed G-orbit X'_v of X_v and set

$$\widehat{x}_v := \pi(x'_v).$$

Then $\widehat{x}_v$ belongs to the G-orbit of $\widehat{X}$ corresponding to the face $\widehat{\mathcal{V}}_I$ of $\widehat{\mathcal{V}}$. Moreover, since π is G-equivariant, the isotropy group H'_v of x'_v in G is contained in the isotropy group $\mathrm{Iso}_G(\widehat{x}_v)$ of $\widehat{x}_v$ in G. Hence we have the following inclusions of spherical subgroups:

$$H_I = H_v \subseteq H'_v \subseteq \mathrm{Iso}_G(\widehat{x}_v). \tag{2}$$

Let us show that $\mathrm{Iso}_G(\widehat{x}_v)$ is contained in $N_G(H'_v)$. To start with, we observe that the fiber of π at x'_v is

$$\pi^{-1}(\pi(x'_v)) = \mathrm{Iso}_G(\widehat{x}_v)x'_v \cong \mathrm{Iso}_G(\widehat{x}_v)/H'_v. \tag{3}$$

Let B a Borel subgroup of G such that BH is dense in G, and P the set of $s \in G$ such that $sBH = BH$. Recall that P is a parabolic subgroup of G containing B. The set $BN_G(H)$ is dense in G and P is the set of $s \in G$ such that $sBN_G(H) = BN_G(H)$. Let L be a Levi subgroup of P adapted to H (cf. Definition 2.1), and denote by C the neutral component of the center of L.

Recall that the *colors* of the spherical homogeneous space G/H are the irreducible B-invariant divisors in G/H. Let $\mathcal{D}$ and $\widehat{\mathcal{D}}$ be the set of colors of G/H and $G/N_G(H)$ respectively. Set

$$X_v^\circ := X_v \setminus \bigcup_{D\in\mathcal{D}} D, \qquad \widehat{X}^\circ := \widehat{X} \setminus \bigcup_{D\in\widehat{\mathcal{D}}} D$$

and apply the local structure theorem to $\widehat{X}$ (cf. [T11, §29.1]). There is a locally closed L-stable subvariety $\widehat{S}$ of $\widehat{X}^\circ$ on which (L, L) trivially acts and such that the map

$$P_u \times \widehat{S} \longrightarrow \widehat{X}^\circ$$

is an isomorphism. Moreover, $\widehat{S}$ is a toric embedding of the torus $C/C \cap H$ and the G-orbits of $\widehat{X}$ intersects $\widehat{S}$ in a unique L-orbit.

Since X_v° is the preimage of $\widehat{X}^\circ$ by π, the local structure for X_v holds with $S := \pi^{-1}(\widehat{S})$ as the L-stable variety. Since the G-orbits of X_v (resp. $\widehat{X}$) intersects S (resp. $\widehat{S}$) in a unique L-orbit, we can assume that $x'_v \in S$ and $\widehat{x}_v \in \widehat{S}$. Then we verify that

$$\pi^{-1}(\pi(x'_v)) \cap X_v^\circ \cong \mathrm{Iso}_C(\widehat{x}_v)/\mathrm{Iso}_C(x'_v),$$

where $\mathrm{Iso}_C(\widehat{x}_v)$ (resp. $\mathrm{Iso}_C(x'_v)$) is the isotropy group of $\widehat{x}_v$ (resp. x'_v) in C. In particular, $\pi^{-1}(\pi(x'_v)) \cap X_v^\circ$ is isomorphic to an algebraic torus. So its closure in X_v is an algebraic torus too. This means that $\pi^{-1}(\pi(x'_v))$ is an algebraic torus. Hence by (3), $\mathrm{Iso}_G(\widehat{x}_v)$ normalizes H'_v, i.e.,

$$\mathrm{Iso}_G(\widehat{x}_v) \subseteq N_G(H'_v).$$

On the other hand, since H_v is a normal subgroup of H'_v, H'_v normalizes H_v. Hence by [T11, Lemma 30.2] and the inclusions (2), we get[2] the expected result:

$$N_G(H_I) = N_G(H'_v) = N_G(\mathrm{Iso}_G(\widehat{x}_v)).$$

It remains to consider the case that $v = 0$. In this case, $I = \mathcal{S}$ and $\widehat{\mathcal{V}_\mathcal{S}} = \{0\}$ so that $\mathrm{Iso}_G(\widehat{x}_v) \cong N_G(H)$, whence the statement since $H_0 = H_\mathcal{S} = H$ up to G-conjugacy. □

We now derive some applications of Proposition 5.1 to the description of isotropy groups of points in spherical embeddings.

Let $\gamma \in N_G(H)$, and f a B-eigenvector of $\mathbb{C}(G/H)$ with weight $m \in M$. The map $g \mapsto f(g\gamma)$ is invariant by the right action of H, and it is a B-eigenvector of $\mathbb{C}(G/H)$ with weight m. Hence for some nonzero complex number $\omega_{H,f}(\gamma)$,

$$f(g\gamma) = \omega_{H,f}(\gamma) f(g), \qquad \forall\, g \in G.$$

The map

$$\omega_{H,f} \colon N_G(H) \to \mathbb{C}^*, \quad \gamma \mapsto \omega_{H,f}(\gamma)$$

is a character whose restriction to H is trivial, and which only depends on m, [Br97, §4.3]. Denote it by $\omega_{H,m}$.

Lemma 5.2 (Knop, Brion)**.** *The spherical subgroup H is the kernel of the homomorphism*

$$\Omega_H \colon N_G(H) \to \mathrm{Hom}(M, \mathbb{C}^*), \quad \gamma \mapsto (m \mapsto \omega_{H,m}(\gamma)).$$

Proof. The action of $N_G(H)$ in G/H by right multiplication allows to identify $N_G(H)/H$ with the group $\mathrm{Aut}^G(G/H)$ of G-equivariant automorphisms of G/H. Since G/H is spherical, any element φ in $\mathrm{Aut}^G(G/H)$ induces a G-equivariant automorphism φ^* of $\mathbb{C}(G/H)$ satisfying $\varphi^*(f) \in \mathbb{C}^* f$ for all $f \in \mathbb{C}(G/H)^{(B)}$. Hence the map

$$\overline{\Omega}_H \colon : N_G(H)/H \to \mathrm{Hom}(M, \mathbb{C}^*), \quad \gamma \mapsto \Omega_H(\widetilde{\gamma}),$$

with $\widetilde{\gamma}$ a representative of γ in $N_G(H)$, identifies with the map λ of [Kn96, Theorem 5.5] through the identification

$$\mathrm{Aut}^G(G/H) \cong N_G(H)/H.$$

So by [Kn96, Theorem 5.5], the map $\overline{\Omega}_H$ is injective.

The injectivity of the map $\overline{\Omega}_H$ is also obtained in the proof of [Br97, Theorem 4.3]. □

[2]The fact that $N_G(H_v) = N_G(H'_v)$ also results from [BPa87, §5.1].

Let $G/H \hookrightarrow X$ be a simple toroidal spherical embedding of G/H corresponding to an uncolored cone $\sigma \subseteq \mathcal{V}$. Let $I(\sigma)$ be the set of all spherical roots in $\mathcal{S}$ that vanish on σ so that $\mathcal{V}_{I(\sigma)}$ is the minimal face of $\mathcal{V}$ containing σ. Fix a point x' in the unique closed G-orbit of X.

Theorem 1.2 will be a consequence of the following proposition.

Proposition 5.3. *There is a homomorphism from $N_G(H_{I(\sigma)})$ to $\operatorname{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$ whose kernel is $\operatorname{Iso}_G(x')$. This homomorphism is given by $\Omega_{\operatorname{Iso}_G(x')}$. Namely, up to a conjugation, we have:*

$$\operatorname{Iso}_G(x') = \{\gamma \in N_G(H_{I(\sigma)}) \mid \omega_{\operatorname{Iso}_G(x'),m}(\gamma) = 1,\ \forall\, m \in M \cap \sigma^\perp\}.$$

Moreover, up to a conjugation, we have the inclusions:

$$H_{I(\sigma)} \subset \operatorname{Iso}_G(x') \subset N_G(H_{I(\sigma)}).$$

Proof. Let $\widehat{X}$ be a complete and simple embedding of $G/N_G(H)$ corresponding to the uncolored cone $\widehat{\mathcal{V}} = \mathcal{V}/(\mathcal{V} \cap -\mathcal{V})$ as in the proof of Theorem 4.4, and retain the notations therein. Let also v be a nonzero element in $\sigma \subset \mathcal{V}_{I(\sigma)}$, and (X_v, x_v) the corresponding elementary embedding of G/H, with closed G-orbit X'_v.

By [T11, Theorem 15.10], the canonical map

$$G/H \to G/N_G(H)$$

extends to a G-equivariant map

$$\pi \colon X \to \widehat{X}.$$

In addition, the identity morphism $G/H \to G/H$ extends to a G-equivariant map

$$\pi_v \colon X_v \to X.$$

Since $\mathbb{Q}_{\geqslant 0} v \subset \sigma$, one can choose a point x'_v in the closed G-orbit X'_v of X_v such that $x' = \pi_v(x'_v)$. Then set $\widehat{x} := \pi(x')$. Since σ is contained in $\mathcal{V}_{I(\sigma)}$, $\widehat{x}$ belongs to the G-orbit of $\widehat{X}$ corresponding to the face $\widehat{\mathcal{V}}_{I(\sigma)}$ of $\widehat{\mathcal{V}}$. On the other hand, since π_v and π are G-equivariant, we get the inclusions of spherical subgroups,

$$H_{I(\sigma)} = H_v \subset H'_v \subset \operatorname{Iso}_G(x') \subset \operatorname{Iso}_G(\widehat{x}), \tag{4}$$

where H'_v is the isotropy group in G of x'_v. By Proposition 5.1, $N_G(H_{I(\sigma)}) = N_G(\operatorname{Iso}_G(\widehat{x}))$. So by [T11, Lemma 30.2] and (4), we get $N_G(\operatorname{Iso}_G(x')) = N_G(H_{I(\sigma)})$.

As a result, by Lemma 5.2 applied to the spherical subgroup $\operatorname{Iso}_G(x')$, we obtain

$$\operatorname{Iso}_G(x') = \{\gamma \in N_G(H_{I(\sigma)}) \mid \omega_{\operatorname{Iso}_G(x'),m}(\gamma) = 1,\ \forall\, m \in M \cap \sigma^\perp\},$$

since the weight lattice of $G/\operatorname{Iso}_G(x')$ is $M \cap \sigma^\perp$. The rest of the proposition follows. ☐

Let X be an arbitrary simple embedding of G/H, and x' in the unique closed G-orbit of X. There exists a simple toroidal G/H embedding $\tilde{X}$ and a proper birational G-equivariant morphism $\pi : \tilde{X} \to X$ such that $\pi(\tilde{x}') = x'$ for some $\tilde{x}'$ in the unique closed G-orbit of $\tilde{X}$ (cf. e.g. [Br97, Proposition 2]). Let $(\sigma, \mathcal{F})$ be the colored cone corresponding to the simple embedding X, and $I(\sigma)$ the set of all spherical roots in $\mathcal{S}$ that vanish on σ. Thus $\mathcal{V}_{I(\sigma)}$ is the minimal face of $\mathcal{V}$ containing $\sigma \cap \mathcal{V}$. Note that $\sigma \cap \mathcal{V}$ is the uncolored cone corresponding to the simple embedding $\tilde{X}$.

The following result is a consequence of Theorem 1.2 (or Proposition 5.3).

Corollary 5.4. *The isotropy group* $\mathrm{Iso}_G(x')$ *of* x' *in* G *contains the satellite* $H_{I(\sigma)}$.

Proof. Since π is G-equivariant, $\mathrm{Iso}_G(x')$ contains the isotropy group $\mathrm{Iso}_G(\tilde{x}')$ of $\tilde{x}'$ in G. But $\mathrm{Iso}_G(\tilde{x}')$ contains $H_{I(\sigma)}$ by Proposition 5.3, whence the statement. □

6. Limits of isotropy groups of points in arc spaces

Let $\mathcal{K} := \mathbb{C}((t))$ be the field of formal Laurent series, and $\mathcal{O} := \mathbb{C}[[t]]$ the ring of formal power series. For X a scheme of finite type over $\mathbb{C}$, denote by $X(\mathcal{K})$ and $X(\mathcal{O})$ the sets of $\mathcal{K}$-valued points and $\mathcal{O}$-valued points of X respectively. If X is a normal variety admitting an action of an algebraic group A, then $X(\mathcal{K})$ and $X(\mathcal{O})$ both admit a canonical action of the group $A(\mathcal{O})$ induced from the A-action on X. For example, the action of the group G on the homogeneous space G/H by left multiplication induces an action of the group $G(\mathcal{O})$ on $(G/H)(\mathcal{K})$.

We have seen that the primitive lattice points of $N \cap \mathcal{V}$ parameterize the elementary embeddings of G/H, as well as the one-parameter subgroups of C, adapted to the different elementary embeddings of G/H. As suggested by Luna and Vust, the set $N \cap \mathcal{V}$ also parameterizes the $G(\mathcal{O})$-orbits in $(G/H)(\mathcal{K})$.

To each $\lambda \in (G/H)(\mathcal{K}) \setminus (G/H)(\mathcal{O})$ one associates a valuation $v_\lambda \colon \mathbb{C}(G/H)^* \to \mathbb{Z}$ as follows (cf. [LV83, §4.2]). The action of G on G/H induces a dominant morphism

$$G \times \operatorname{Spec} \mathcal{K} \to G \times G/H \to G/H$$

where the first map is $1 \times \lambda$, whence an injection of fields $\iota_\lambda \colon \mathbb{C}(G/H) \to \mathbb{C}(G)((t))$. Then we set

$$v_\lambda := v_t \circ \iota_\lambda \colon \mathbb{C}(G/H)^* \to \mathbb{Z}$$

where $v_t : \mathbb{C}(G)((t))^* \to \mathbb{Z}$ is the natural discrete valuation on $\mathbb{C}(G)((t))$ given by the degree in t of formal series. If $\lambda \in (G/H)(\mathcal{O})$, we define v_λ to be the trivial valuation. By [LV83, §§4.4 and 4.6], v_λ is a G-invariant discrete valuation of G/H, i.e., an element of $N \cap \mathcal{V}$. Moreover, for any $\gamma \in G(\mathcal{O})$ and $\lambda \in (G/H)(\mathcal{K})$, $v_{\gamma\lambda} = v_\lambda$. Furthermore, the mapping $(G/H)(\mathcal{K}) \to \mathcal{V}$, $\lambda \mapsto v_\lambda$ induces a one-to-one correspondence between the set of $G(\mathcal{O})$-orbits in $(G/H)(\mathcal{K})$ and the set of lattice points in $N \cap \mathcal{V}$ (cf. Theorem 6.1(1)). More recently, Gaitsgory and Nadler has revisited this statement and has given a new proof in [GN10, Theorem 3.2.1].

Let X be a smooth spherical embedding of G/H associated with a colored fan Σ. Denote by $|\Sigma|$ the *support* of Σ defined as

$$|\Sigma| = \bigcup_{(\sigma,\mathcal{F}) \in \Sigma} \sigma \cap \mathcal{V}.$$

The group $G(\mathcal{O})$ acts on $X(\mathcal{O})$ and $(G/H)(\mathcal{K})$ which are both viewed as subsets of $X(\mathcal{K})$. Hence the group $G(\mathcal{O})$ acts on $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$.

The following theorem, essentially obtained by Luna-Vust [LV83, Proposition 4.10], was reformulated by Docampo in [D09, Theorem 4.1.11 and Remark 4.1.12] (and revisited by Gaitsgory-Nadler in [GN10]).

Theorem 6.1 ([LV83, D09, GN10]). (1) *The mapping*

$$(G/H)(\mathcal{K}) \to N \cap \mathcal{V}, \quad \lambda \mapsto v_\lambda$$

is a surjective map and the fiber over any $v \in N \cap \mathcal{V}$ is precisely one $G(\mathcal{O})$-orbit. In particular, there is a one-to-one correspondence between the set of $G(\mathcal{O})$-orbits in $(G/H)(\mathcal{K})$ and the set $N \cap \mathcal{V}$.

(2) *By restriction to $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$, the map of (1) induces a surjective map*

$$X(\mathcal{O}) \cap (G/H)(\mathcal{K}) \to N \cap |\Sigma|, \quad \lambda \mapsto v_\lambda$$

whose fiber over any $v \in N \cap |\Sigma|$ is precisely one $G(\mathcal{O})$-orbit. In particular, there is a one-to-one correspondence between the set of $G(\mathcal{O})$-orbits in $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$ and the set $N \cap |\Sigma|$.

For $v \in N \cap \mathcal{V}$, we denote by $\mathcal{C}_v := \{\lambda \in (G/H)(\mathcal{K}) \mid v_\lambda = v\}$ the corresponding $G(\mathcal{O})$-orbit in $(G/H)(\mathcal{K})$.

Let B and P as in Section 2 and choose a Levi subgroup L of P adapted to H (cf. Definition 2.1). Let C be the neutral component of the center of L, and $\lambda \in \mathcal{X}_*(C)$ a one-parameter subgroup of C. Set $x := eH$ be the base point in G/H. The one-parameter subgroup λ induces a morphism of algebraic varieties,

$$\widetilde{\lambda}\colon \mathbb{C}^* \to G/H, \quad t \mapsto \lambda(t)x,$$

whose comorphism gives rise to an element of $(G/H)(\mathcal{K})$. We denote by $\widehat{\lambda}$ this element of $(G/H)(\mathcal{K})$.

Let v be a nonzero lattice point in $N \cap \mathcal{V}$, and (X_v, x_v) the corresponding elementary embedding of G/H. Since the open orbit of X_v is G/H, one can assume that $x_v = eH$. Then, λ is adapted to X_v if and only if $\lim_{t\to 0} \lambda(t)x_v$ exists in X_v and lies in the closed G-orbit X'_v of X_v, that is, if and only if $\widehat{\lambda}$ lies in $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$ and $\lim_{t\to 0} \widehat{\lambda}(t) \in X'_v$.

One can choose $\lambda_v \in \mathcal{X}_*(C)$, adapted to (X_v, x_v), such that $v_{\lambda_v} = v$; see Section 2. Moreover, according to [LV83, §5.4],

$$G(\mathcal{O})\widehat{\lambda}_v = \mathcal{C}_v,$$

and so $v = v_{\lambda_v} = v_{\widehat{\lambda}_v}$.

Set $x'_v := \lim_{t\to 0} \lambda_v(t)x_v$. As usual, denote by H'_v the isotropy group of x'_v in G. Recall that the Brion subgroup H_v is by definition the kernel of a certain character χ_v of H'_v (cf. Definition 2.3). The main goal in this section is to prove the following result (see Theorem 1.3).

Theorem 6.2. *We have, up to G-conjugacy,*

$$H_v = \{\lim_{t\to 0} \gamma(t)\ ;\ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)\},$$

where $\mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)$ is the isotropy group of $\widehat{\lambda}_v$ in $G(\mathcal{O})$, that is, the set of $\gamma \in G(\mathcal{O})$ such that $\gamma(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v$ for all $t \in \mathbb{C}^*$.

Theorem 6.2 will be proven after some lemmas.

Setting $\widetilde{\lambda}_v(0) := \lim_{t\to 0} \widetilde{\lambda}_v(t) = x'_v$, one gets a morphism of algebraic varieties

$$\widetilde{\lambda}_v \colon \mathbb{C} \longrightarrow X_v.$$

One can consider the differential $d\widetilde{\lambda}_v(0)$ of $\widetilde{\lambda}_v$ at 0 as an element of $T_{x'_v}(X_v)$.

Lemma 6.3. *The one-dimensional space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ is generated by the class of $d\widetilde{\lambda}_v(0)$ in $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$.*

Proof. According to [LV83, §4.10], one can choose a smooth curve $\mathscr{C}$, a point $c \in \mathscr{C}$ and a morphism $\gamma\colon \mathscr{C} \to X_v$ such that $\gamma(\mathscr{C} \setminus \{c\}) \subset G/H$, $\gamma(c) = x'_v$ and γ is transversal to X'_v. This means that

$$T_{x'_v}(X'_v) \oplus T_{x'_v}(\mathscr{C}) = T_{x'_v}(X_v). \tag{5}$$

In particular, $T_{x'_v}(\mathscr{C})$ has dimension one and it is generated by $d\gamma(c)$. By [LV83, §4.10], using the inclusion of fields $\mathbb{C}(\mathscr{C}) \subset \mathbb{C}((t)) = \mathcal{K}$, one can attach to the morphism γ an element $\widehat{\mu}$ of $(G/H)(\mathcal{K})$ such that $\widehat{\mu} \in X_v(\mathcal{O})$ and $\lim_{t\to 0} \widehat{\mu}(t) = \gamma(c) = x'_v$. So the corresponding one-parameter subgroup of C, denoted by μ, is adapted to (X_v, x_v) and hence the images of λ_v and μ are proportional in $\mathcal{X}_*(C/C\cap H)$. So we can assume that $\lambda_v = \mu$. But by construction of $\widehat{\mu}$, we have:

$$\lim_{t\to 0} d\widetilde{\mu}(0) = d\gamma(c),$$

whence the lemma by (5). ∎

Lemma 6.4. *We have $C \cap H_v = C \cap H$.*

Proof. Let $g \in C \cap H$. For all $t \in \mathbb{C}^*$,

$$g(\lambda(t)x_v) = \lambda(t)(gx_v) = \lambda(t)x_v \tag{6}$$

since g commutes with the image of λ. Taking the limit at 0 on both sides, we get that $gx'_v = x'_v$, whence $g \in H'_v$. Since $g \in H'_v$, g leaves invariant the tangent space $T_{x'_v}(X'_v)$, and acts on it. So by differentiating the equality (6), we get

$$g(d\widetilde{\lambda}_v(0)) = d\widetilde{\lambda}_v(0).$$

The group H'_v acts on the one-dimensional space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ by the character χ_v. So from Lemma 6.3, we deduce that g is in the kernel of χ_v, that is, g is in H_v. This proves the inclusion $C \cap H \subset C \cap H_v$.

Since $Cx_v \cup Cx'_v$ is an elementary embedding of the algebraic torus $C/C \cap H$, $C \cap H_v$ is a Brion subgroup of $C \cap H$. But $C/C \cap H$ is a torus, so $C \cap H$ is equal to each of its Brion subgroups. Therefore we get $C \cap H = C \cap H_v$. □

We are now in position to prove Theorem 6.2.

Proof of Theorem 6.2. Let us first prove the inclusion

$$\{\lim_{t\to 0} \gamma(t) \; ; \; \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)\} \subseteq H_v.$$

Let $\gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)$ and denote by $\gamma(0)$ the limit of γ at 0. For any $t \in \mathbb{C}^*$,

$$\gamma(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v. \tag{7}$$

Taking the limit at 0 on both sides, we get that $\gamma(0)x'_v = x'_v$. Hence $\gamma(0) \in H'_v$. We now argue as in the proof of Lemma 6.4. Since $\gamma(0) \in H'_v$, $\gamma(0)$ leaves invariant the tangent space $T_{x'_v}(X'_v)$. By differentiating the equality (7), we get

$$\gamma(0)(d\widetilde{\lambda}_v(0)) = d\widetilde{\lambda}_v(0) \mod T_{x'_v}(X'_v).$$

Since H'_v acts on the one-dimensional space $T_{x'_v}(X_v)/T_{x'_v}(X'_v)$ by the character χ_v, we deduce from Lemma 6.3 that $\gamma(0)$ is in the kernel H_v of χ_v, whence the expected inclusion.

Let us prove the converse inclusion,

$$H_v \subseteq \{\lim_{t\to 0} \gamma(t) \; ; \; \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)\}.$$

Let $\gamma_0 \in H_v$. We have to prove that $\gamma_0 = \lim_{t\to 0} \gamma(t)$ for some $\gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)$. By [BPa87, Proposition and Corolloray 5.2], $N_G(H_v) = H_v^0(C \cap N_G(H_v))$, with H_v^0 the neutral component of H_v, whence

$$H_v \subset H_v^0(C \cap H_v).$$

Write $\gamma_0 = \tilde{\gamma}_0 c$ with $\tilde{\gamma}_0 \in H_v^0$ and $c \in C \cap H_v$. Since H_v^0 is connected,

$$\tilde{\gamma}_0 = \exp_{H_v}(\eta_0)$$

for some $\eta_0 \in \mathfrak{h}_v$, where $\mathfrak{h}_v := \mathrm{Lie}(H_v)$ and $\exp_{H_v} \colon \mathfrak{h}_v \to H_v$ is the exponential map of H_v. According to [Br90, Proposition 1.2][3], the Lie algebra of H_v is $\lim_{t\to 0} \lambda_v(t)\mathfrak{h}$, with $\mathfrak{h}$ the Lie algebra of H. Therefore $\eta_0 = \lim_{t\to 0} \lambda_v(t)\xi$ for some $\xi \in \mathfrak{h}$. Set for any $t \in \mathbb{C}^*$,

$$\tilde{\gamma}(t) := \lambda_v(t) \exp_H(\xi) \lambda_v(t^{-1}),$$

with $\exp_H \colon \mathfrak{h} \to H$ the exponential map of H. Then for any $t \in \mathbb{C}^*$,

$$\tilde{\gamma}(t)(\lambda_v(t)x_v) = \lambda_v(t) \exp_H(\xi) x_v = \lambda_v(t)x_v$$

since $\exp_H(\xi)$ lies in H, the isotropy group of x_v. As a result $\tilde{\gamma} \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda}_v)$.

[3]The result is stated in [Br90, Proposition 1.2] for H_v equals to its normalizer. However the proof does not use this hypothesis.

We have a natural embedding $G \hookrightarrow G(\mathcal{O})$. So we can view the element c of $C \subset G$ as an element of $G(\mathcal{O})$. Set for any $t \in \mathbb{C}^*$,

$$\gamma(t) := \tilde{\gamma}(t)c.$$

We have

$$\lim_{t\to 0} \tilde{\gamma}(t) = \lim_{t\to 0}(\lambda_v(t)\exp_H(\xi)\lambda_v(t^{-1})) = \exp_{H_v}(\lim_{t\to 0}\lambda_v(t)\xi) = \exp_{H_v}(\eta_0) = \tilde{\gamma}_0,$$

whence

$$\lim_{t\to 0}\gamma(t) = \tilde{\gamma}_0 c = \gamma_0.$$

By Lemma 6.4, the element c of $C \cap H_v$ is in $C \cap H$. In particular, c is in the isotropy group of x_v. Hence,

$$\gamma(t)(\lambda_v(t)x_v) = \tilde{\gamma}(t)c(\lambda_v(t)x_v) = \tilde{\gamma}(t)(\lambda_v(t)cx_v) = \tilde{\gamma}(t)(\lambda_v(t)x_v) = \lambda_v(t)x_v$$

since $\tilde{\gamma} \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda_v})$ and since c commutes with the image of λ_v. This proves that γ is in $\mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda_v})$. So $\gamma_0 \in \{\lim_{t\to 0}\gamma(t)\ ;\ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda_v})\}$ as desired. □

Theorem 6.2 is useful to compute the Brion subgroups H_v, for $v \in N \cap \mathcal{V}$, in practice. According to Theorem 1.1, it suffices to compute finitely many of them to describe the satellites H_I, $I \subseteq \mathcal{S}$, of H. In the remainder of the section, we illustrate the use of Theorem 6.2 and Theorem 1.1 in a number of examples.

In what follows, we identify λ and $\widehat{\lambda}$, and so we write by a slight abuse of notation $\mathrm{Iso}_{G(\mathcal{O})}(\lambda)$ for $\mathrm{Iso}_{G(\mathcal{O})}(\widehat{\lambda})$.

Example 6.5. *Consider the group $G = SL(2)$ together with its natural action on $\mathbb{P}^1 \times \mathbb{P}^1$ which is transitive outside the diagonal $\Delta \cong \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is an elementary spherical embedding of G/H where H is the isotropy group of a point (x, y) in $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. The subgroup H is a maximal torus in G. Let $T \subset G$ be the maximal diagonal torus and $B \supset T$ the Borel subgroup of upper triangular matrices. Then T is a Levi subgroup of B adapted to H if $T \cap H$ is the center of G consisting in two matrices $\pm E$ where E is the identity matrix. This happens, for example, if we choose the point*

$$(x, y) := ([0 : 1], [1 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta.$$

Consider the one-parameter subgroup of T

$$\lambda : \mathbb{C}^* \to T, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Then $\lambda(t)(x, y) = ([0 : t^{-1}], [t : t^{-1}])$ and

$$\lim_{t\to 0}\lambda(t)(x, y) = ([0 : 1], [0 : 1]) \in \Delta \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

A matrix

$$\gamma(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in G(\mathcal{O}), \quad (a, b, c, d \in \mathbb{C}[[t]],\ ad - bc = 1)$$

belongs to the isotropy group $\mathrm{Iso}_{G(\mathcal{O})}(\lambda)$ *if and only if we have*

$$\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} ([0 : t^{-1}], [t : t^{-1}]) = ([0 : t^{-1}], [t : t^{-1}]),$$

i.e., if and only if $b(t) = 0$, $a(t)d(t) = 1$ *and* $a(t) - d(t) = t^2 c(t)$. *The last condition implies that the first two coefficients of the power series* $a(t) = a_0 + a_1 t + \cdots$ *and* $d(t) = d_0 + d_1 t + \cdots$ *are the same* ($a_0 = d_0, a_1 = d_1$). *By* $a(t)d(t) = 1$, *we obtain* $a_0 = d_0 \in \{1, -1\}$. *Therefore, the set of limits* $\{\lim_{t\to 0} \gamma(t)\ ;\ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\}$ *is*

$$\left\{ \begin{pmatrix} a_0 & 0 \\ c_0 & a_0 \end{pmatrix}\ ;\ c_0 \in \mathbb{C},\ a_0^2 = 1 \right\} \cong U_2.$$

We rediscover our result of Example 2.6.

Example 6.6. *The group* $G = SL(2)$ *acts via the adjoint representation on the projective plane* $\mathbb{P}(\mathfrak{sl}_2) \simeq \mathbb{P}^2$. *The closed* G*-orbit is a conic* $Q \subset \mathbb{P}^2$ *represented by the matrices*

$$\begin{pmatrix} X_0 & X_1 \\ X_2 & -X_0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

that are nilpotent, i.e., $X_0^2 + X_1 X_2 = 0$. *The group* G *transitively acts on* $\mathbb{P}^2 \setminus Q$ *and the isotropy group* H *of a point* $[x] \in \mathbb{P}^2 \setminus Q$ *is the normalizer of a maximal torus in* G. *Let* $T \subset G$ *be the maximal diagonal torus and let* $B \supset T$ *be the Borel subgroup of upper triangular matrices. Then* T *is a Levi subgroup of* B *adapted to* H *if* $T \cap H$ *is a cyclic group of order 4 containing the center of* G. *This happens for example if the point* $[x] \in \mathbb{P}^2 \setminus Q$ *is represented by the matrix*

$$x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the one-parameter subgroup of T

$$\lambda : \mathbb{C}^* \to T, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Then

$$[\lambda(t)x] = \left[\begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} \right], \qquad \lim_{t\to 0} [\lambda(t)x] = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \in Q.$$

A matrix

$$\gamma(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in G(\mathcal{O}), \quad (a, b, c, d \in \mathbb{C}[[t]],\ ad - bc = 1)$$

belongs to the isotropy group $\mathrm{Iso}_{G(\mathcal{O})}(\lambda)$ *if and only if for some* $\mu \in \mathbb{C}^*$ *we have*

$$\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu t \\ \mu t^{-1} & 0 \end{pmatrix} \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

that is, if and only if

$$b(t) = \mu c(t) t^2, \quad c(t) t^2 = \mu b(t), \quad a(t) = \mu d(t), \quad d(t) = \mu a(t).$$

This implies that the first two coefficients b_0 *and* b_1 *of the power series* $b(t)$ *are zeros,* $a_0d_0 = 1$*,* $a_0 = \mu d_0$ *and* $\mu^2 = 1$*. Therefore, the set of limits* $\{\lim_{t\to 0}\gamma(t)\ ;\ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\}$ *is*

$$\left\{\begin{pmatrix} a_0 & 0 \\ c_0 & a_0 \end{pmatrix}\ ;\ c_0 \in \mathbb{C},\ a_0^4 = 1\right\} \cong U_4.$$

We rediscover the result of Example 2.7.

We now return in more detail to the illustrating example of the introduction.

Example 6.7. *Let* $n \in \mathbb{N}^*$*. Assume that* $G = GL(n)\times GL(n)$ *and* $B = T^+(n)\times T(n)^-$ *where* $T^+(n)$ *(resp.* $T_n^-(n)$*) is the subgroup of* $GL(n)$ *consisting of upper (resp. lower) triangular matrices. The group* G *transitively acts on* $GL(n)$ *by* $(a,b)c = acb^{-1}$ *for* $a,b,c \in GL(n)$ *and the isotropy group of the identity matrix* E *is* $H \cong GL(n)$ *where* $GL(n)$ *is embedded diagonally in* G*.*

Let $D(n) := T^+(n)\cap T^-(n)$ *be the subgroup of* $GL(n)$ *consisting of diagonal matrices, and set* $T := D(n)\times D(n)$*. Then* T *is a maximal torus of* G *and it is adapted to* G/H*, cf.* [Br97, §2.4, Example 2]*. Hence,*

$$M = \mathcal{X}^*(T/T\cap H) \cong \mathcal{X}^*(D(n)) \cong \mathbb{Z}^n.$$

The dual lattice $N = \mathrm{Hom}(M,\mathbb{Z})$ *identifies with the lattice of one-parameters subgroups of* $D(n)$ *and* $\mathcal{V}$ *identifies with the set of sequences* $(\lambda_1,\dots,\lambda_n) \in \mathbb{Q}^n$ *with* $\lambda_1 \geqslant \dots \geqslant \lambda_n$*.*

Let

$$\lambda\colon \mathbb{C}^* \to T,\quad t \mapsto (\mathrm{diag}(t^{\lambda_1},\dots,t^{\lambda_n}), \mathrm{diag}(t^{-\lambda_1},\dots,t^{-\lambda_n}))\ \in\ T$$

be a one-subgroup parameter of T *with* $(\lambda_1,\dots,\lambda_n)$ *in* $\mathbb{Z}^n$ *and* $\lambda_1 \geqslant \dots \geqslant \lambda_n$*. The identity matrix* E *lies in the open* B*-orbit since* $T^+(n)T(n)^-$ *is open in* $GL(n)$*. In order to describe the isotropy group* $\mathrm{Iso}_{G(\mathcal{O})}(\lambda)$*, let us introduce some more notations.*

Write $(\lambda_1,\dots,\lambda_n) = (\mu_1^{k_1},\dots,\mu_r^{k_r})$ *with* $\mu_1 > \dots > \mu_r$ *and* $k_1+\dots+k_r = n$ *(*$k_i \in \mathbb{N}^*$*). Define a parabolic subgroup* P_λ *and a Levi subgroup* L_λ *of* $GL(n)$ *as follows. Set for* $i \in \{1,\dots,r\}$*,* $v_i := k_1+\dots+k_i$*,* $V_i := \mathrm{span}(e_1,\dots,e_{v_i})$ *and* $W_i := \mathrm{span}(e_{v_{i-1}+1},\dots,e_{v_i})$*, where* $(e_1,\dots,e_n)$ *is the standard basis of* $\mathbb{C}^n$ *and* $v_0 := 0$*. Then* P_λ *is the isotropy group of the flag* $V_1 \subset \dots \subset V_r = \mathbb{C}^n$*, and the group* $L_\lambda := GL(k_1)\times\dots\times GL(k_r)$ *naturally embeds into* P_λ *as the group of endomorphisms of* W_i*. It is a Levi factor of* P_λ*. Clearly, the groups* P_λ *and* L_λ *only depend on the sequence* $(k_1,\dots,k_r)$*.*

Let $(x(t),y(t)) \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)$ *and write* $x(t) = (a_{i,j}(t))_{1\leqslant i,j\leqslant n}$*,* $y(t) = (y_{i,j}(t))_{1\leqslant i,j\leqslant n}$ *with* $a_{i,j}(t), b_{i,j}(t) \in \mathbb{C}[[t]]$*,* $\det x(t) \neq 0$ *and* $\det y(t) \neq 0$*. Then*

$$\lambda(t)E = \mathrm{diag}(\underbrace{t^{2\mu_1},\dots,t^{2\mu_1}}_{k_1\ times},\dots,\underbrace{t^{2\mu_r},\ \dots,t^{2\mu_r}}_{k_r\ times}).$$

Since $(x(t),y(t))$ *stabilizes* $\lambda(t)E$*, we get*

$$\begin{pmatrix} A_{1,1}(t)t^{2\mu_1} & \cdots & A_{1,r}(t)t^{2\mu_r} \\ \vdots & & \vdots \\ A_{r,1}(t)t^{2\mu_1} & \cdots & A_{r,r}(t)t^{2\mu_r} \end{pmatrix} = \begin{pmatrix} B_{1,1}(t)t^{2\mu_1} & \cdots & B_{1,r}(t)t^{2\mu_1} \\ \vdots & & \vdots \\ B_{r,1}(t)t^{2\mu_r} & \cdots & B_{r,r}(t)t^{2\mu_r} \end{pmatrix}$$

where for $i, j \in \{1, \ldots, r\}$,

$$A_{i,j}(t) := \begin{pmatrix} a_{k_{i-1}+1,k_{j-1}+1}(t) & \cdots & a_{k_{i-1}+1,k_{j-1}+k_j}(t) \\ \vdots & & \vdots \\ a_{k_{i-1}+k_i,k_{j-1}+1}(t) & \cdots & a_{k_{i-1}+k_i,k_{j-1}+k_j}(t) \end{pmatrix},$$

$$B_{i,j}(t) := \begin{pmatrix} b_{k_{i-1}+1,k_{j-1}+1}(t) & \cdots & b_{k_{i-1}+1,k_{j-1}+k_j}(t) \\ \vdots & & \vdots \\ b_{k_{i-1}+k_i,k_{j-1}+1}(t) & \cdots & b_{k_{i-1}+k_i,k_{j-1}+k_j}(t) \end{pmatrix}$$

with $k_0 := 0$. Then we obtain that for all $i, j \in \{1, \ldots, r\}$,

$$A_{i,j}(t)t^{2\mu_j} = B_{i,j}(t)t^{2\mu_i}.$$

Since $\mu_1 > \cdots > \mu_r$, we get by Theorem 1.3,

$$H_\lambda \cong \lim_{t\to 0}(x(t), y(t)) \in \{(a, b) \in P_{-\lambda} \times P_\lambda \mid L(a) = L(b)\}$$

where L denotes the natural surjective homomorphism from $P_{-\lambda}$ and P_λ to their common Levi factor L_λ.

Observe that for $\lambda' \in \mathcal{V}$, λ' belongs to $\mathcal{V}(\lambda)^\circ$ if and only if λ and λ' have the same number of different parts with the same multiplicities. In other word, $\lambda' = (\nu_1^{l_1}, \ldots, \nu_s^{l_s})$ belongs to $\mathcal{V}(\lambda)^\circ$ if and only if $s = r$ and $l_1 = k_1, \ldots, l_r = k_r$. As a consequence, the faces of $\mathcal{V}$ are in bijection with the sequences $k = (k_1, \ldots, k_r)$ with $r \in \{1, \ldots, n\}$, $k_i \in \mathbb{N}^*$ and $k_1 + \cdots + k_r = n$. Then P_λ, L_λ, and so H_λ, only depend, up to conjugation, on the minimal face of $\mathcal{V}(\lambda)$ containing λ. We thus rediscover Theorem 4.4 in this particular case.

Example 6.8. *Let $n \in \mathbb{N}$ with $n \geqslant 3$, and assume that $G = SL(n)$ and that $H \cong SL(n-1)$ is the subgroup*

$$H = \{\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; A_1 \in SL(n-1)\} \subset G.$$

The spherical homogeneous space G/H can be described as

$$G/H \cong \{(x, y) \in \mathbb{C}^n \times (\mathbb{C}^n)^* \mid \langle y, x\rangle = 1\}.$$

Choose for B the set of upper triangular matrices of G and let $T \subset B$ be the set of diagonal matrices of G. The valuation cone $\mathcal{V}$ has two faces: the whole face $\mathcal{V}$ and the minimal face which is a line containing 0.

Let us describe the horospherical satellite $H_\varnothing$. The Borel subgroup B has an open orbit in G/H consisted of the elements (x, y) such that $\langle e_n^*, x\rangle \neq 0$ and $\langle y, e_1\rangle \neq 0$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{C}^n$, and $(e_1^*, \ldots, e_n^*)$ its dual basis. The complement to this B-orbit in G/H is the union of two B-stable divisors: $D_1 := \{(x, y) \mid \langle e_n^*, x\rangle = 0\}$ and $D_2 := \{(x, y) \mid \langle y, e_1\rangle = 0\}$. The set of $s \in G$ such that $sD_1 = D_1$ and $sD_2 = D_2$ is the parabolic subgroup

$$P := \{\begin{pmatrix} a_1 & A_{1,2} & A_{1,3} \\ 0 & A_2 & A_{2,3} \\ 0 & 0 & a_3 \end{pmatrix} \in SL(n) \; ; \; a_1, a_3 \in \mathbb{C}^*, A_2 \in M_{n-2}(\mathbb{C})\}.$$

It contains the Borel subgroup B. Let $\tilde{H}$ be the isotropy group of the point

$$((1,0,\dots,0,1),(1,0,\dots,0)).$$

Then $\tilde{H}$ is G-conjugate to H, $B\tilde{H}$ is dense in G and $\tilde{H}$ is the subgroup

$$\tilde{H} = \{\begin{pmatrix} 1 & 0 & 0 \\ A_{2,1} & A_2 & -A_{2,1} \\ A_{3,1} & A_{3,2} & 1-A_{3,1} \end{pmatrix} \in SL(n)\ ;\ A_2 \in M_{n-2}(\mathbb{C})\}.$$

The Levi subgroup

$$L := \{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in SL(n)\ ;\ a_1, a_3 \in \mathbb{C}^*, A_2 \in M_{n-2}(\mathbb{C})\}$$

of P is adapted to $\tilde{H}$. Denote by C its neutral center. Let $\lambda\colon \mathbb{C}^* \to C$ be the one-parameter of C with $\lambda(t) = \operatorname{diag}(t,1,\dots,1,,t^{-1})$ for all $t \in \mathbb{C}^*$. Consider the elementary embedding

$$X := \{[x_1;\dots;x_n;y_1;\dots;y_n;z_0] \mid \sum x_i y_i = z_0^2\} \subset \mathbb{P}^{2n}$$

of G/H. The closed G-orbit of X corresponds to the divisor $\{z_0 = 0\}$. We have

$$\lambda(t)([1;0;\dots,0;1;1;0;\dots;0;1]) = [t;0;\dots;t^{-1};t^{-1};0;\dots,0;1] \in \mathbb{P}^{2n}.$$

Let $t \mapsto \gamma(t) = (a_{i,j}(t))_{1\leqslant i,j\leqslant n}$ be in $G(\mathcal{O})$ and write $\gamma^{-1}(t) =: \gamma'(t) = (a'_{i,j}(t))_{1\leqslant i,j\leqslant n}$. Then γ stabilizes λ if and only if for some $\mu \in \mathbb{C}^*$, we have for all $t \in \mathbb{C}^*$,

$$\begin{pmatrix} a_{1,1}(t) & \cdots & a_{1,n}(t) \\ \vdots & & \vdots \\ a_{n,1}(t) & \cdots & a_{n,n}(t) \end{pmatrix} \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \\ t^{-1} \end{pmatrix} = \begin{pmatrix} \mu t \\ 0 \\ \vdots \\ 0 \\ t^{-1} \end{pmatrix},$$

$$\begin{pmatrix} t^{-1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a'_{1,1}(t) & \cdots & a'_{1,n}(t) \\ \vdots & & \vdots \\ a'_{n,1}(t) & \cdots & a'_{n,n}(t) \end{pmatrix} = \begin{pmatrix} \mu^{-1}t^{-1} & 0 & \cdots & 0 \end{pmatrix},$$

and $\gamma(t)\gamma'(t) = E$, that is, if and only if for all $t \in \mathbb{C}^*$,

$$a_{1,1}(t)t^2 + a_{1,n}(t) = \mu t^2, \qquad a_{j,1}(t)t^2 + a_{j,n}(t) = 0,\ \forall j \in \{2,\dots,n-2\},$$
$$a_{n,1}(t)t^2 + a_{n,n}(t) = \mu, \quad a'_{1,1}(t) = \mu^{-1}, \quad a'_{1,j}(t) = 0,\ \forall j \in \{2,\dots,n\}, \quad \gamma(t)\gamma'(t) = E.$$

Thus $\gamma \in \operatorname{Iso}_{G(\mathcal{O})}(\lambda)$ if and only if

$$\gamma(t) = \begin{pmatrix} a_{1,1}(t) & \cdots & a_{1,n-1}(t) & a_{1,1}(t)t^2 - \mu t^2 \\ a_{2,1}(t) & \cdots & a_{n-1,1}(t) & -a_{2,1}(t)t^2 \\ \vdots & & \vdots & \\ a_{n-1,1}(t) & \cdots & a_{n-1,n-1}(t) & -a_{n-1,1}(t)t^2 \\ a_{n,1}(t) & \cdots & a_{n,n-1}(t) & \mu - a_{n,1}(t)t^2 \end{pmatrix},$$

$$\gamma^{-1}(t) = \begin{pmatrix} \mu^{-1} & 0 & \cdots & 0 \\ a'_{2,1}(t) & a'_{2,2}(t) & \cdots & a'_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ a'_{n-1,1}(t) & a'_{n-1,2}(t) & \cdots & a'_{n-1,n}(t) \\ a'_{n,1}(t) & a'_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix},$$

and $\det \gamma(t) = 1$*. Hence* $\{\lim_{t\to 0} \gamma(t) \ ; \ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\}$ *is the set of matrices* $A = (a_{i,j})_{1\leqslant i,j\leqslant n}$ *in* $SL(n)$ *of the form,*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & 0 \\ \vdots & \vdots & & \vdots & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & 0 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & \mu \end{pmatrix}, \qquad a_{i,j} \in \mathbb{C}$$

such that $A^{-1} = (a'_{i,j})_{1\leqslant i,j\leqslant n}$ *is of the form*

$$A^{-1} = \begin{pmatrix} \mu^{-1} & 0 & \cdots & 0 \\ a'_{2,1} & a'_{2,2} & \cdots & a'_{2,n} \\ \vdots & \vdots & & \vdots \\ a'_{n-1,1} & a'_{n-1,2} & \cdots & a'_{n-1,n} \\ a'_{n,1} & a'_{n,2} & \cdots & a'_{n,n} \end{pmatrix}, \qquad a'_{i,j} \in \mathbb{C}.$$

Such matrices automatically verify $a_{1,2} = \cdots = a_{1,n} = 0$ *(and* $a'_{2,n} = \cdots = a'_{n-1,n} = 0$*) and* $a_{1,1} = \mu$*. In conclusion, by Theorem 1.3, we obtain that*

$$H_\varnothing \cong \{\lim_{t\to 0} \gamma(t) \ ; \ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\} = \{ \begin{pmatrix} a_1 & 0 & 0 \\ A_{2,1} & A_2 & 0 \\ A_{3,1} & A_{3,2} & a_1 \end{pmatrix} \in SL(n) \ ; \ A_2 \in SL(n-2), \ a_1^2 = 1\}.$$

Example 6.9. *The group* $G = SL(2)\times SL(2)\times SL(2)$ *acts on* $SL(2)\times SL(2)$ *via* $(a,b,c).(h,k) = (ahc^{-1}, bkc^{-1})$ *and the isotropy group of* (E,E) *is the group* $H \cong SL(2)$ *embedded diagonally in* G*. The homogeneous space* G/H *is spherical and of rank two. Consider the following maximal torus of* G*:*

$$T := \{(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & c^{-1} - c \\ 0 & c \end{pmatrix}) \ ; \ a,b,c \in \mathbb{C}^*\}.$$

Then $T \cap H \cong \{\pm E\}$ *and* T *is adapted to* H*. There are three spherical roots* $\alpha_1, \alpha_2, \alpha_3$ *and so the valuation cone has* $2^3 = 8$ *faces. By symmetry,*

$$H_{\{\alpha_1\}} \cong H_{\{\alpha_2\}} \cong H_{\{\alpha_2\}} \quad \text{and} \quad H_{\{\alpha_1,\alpha_2\}} \cong H_{\{\alpha_2,\alpha_3\}} \cong H_{\{\alpha_1,\alpha_3\}}.$$

In addition $H_{\mathcal{S}} \cong H$*. Hence we have three satellite subgroups to compute:* $H_\varnothing$*,* $H_{\{\alpha_1\}}$*,* $H_{\{\alpha_1,\alpha_2\}}$*.*

Let us first compute the horospherical satellite $H_\varnothing$*. Let* $\lambda\colon \mathbb{C}^* \to T, t \mapsto (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ *be the one-subgroup parameter of* T *with for all* $t \in \mathbb{C}^*$*,*

$$\lambda_1(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \lambda_2(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \quad \lambda_3(t) = \begin{pmatrix} t^{-1} & t-t^{-1} \\ 0 & t \end{pmatrix}.$$

Then $\lambda(t)(E,E) = (\begin{pmatrix} t^2 & -t^2+1 \\ 0 & t^{-2} \end{pmatrix}, \begin{pmatrix} 1 & t^{-2}-1 \\ 0 & 1 \end{pmatrix})$ *and*

$$\gamma(t) = (\begin{pmatrix} a_1(t) & b_1(t) \\ c_1(t) & d_1(t) \end{pmatrix}, \begin{pmatrix} a_2(t) & b_2(t) \\ c_2(t) & d_2(t) \end{pmatrix}, \begin{pmatrix} a_3(t) & b_3(t) \\ c_3(t) & d_3(t) \end{pmatrix}) \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)$$

if and only if for all $t \in \mathbb{C}^*$*,*

$$\begin{aligned}
b_1(t) &= (a_1(t) - d_3(t))(t^2-1)t^2 + b_3(t)t^4 \\
d_1(t) - d_3(t) &= c_1(t)(t^2-1)t^2, \\
(a_2(t) - d_3(t))(1-t^2) &= (b_3(t) - b_2(t))t^2, \\
c_2(t) &= c_3(t), \\
c_2(t)(1-t^2) &= (d_3(t) - d_2(t))t^2, \\
c_3(t)(1-t^2) &= (a_2(t) - a_3(t))t^2, \\
c_3(t) &= c_1(t)t^4, \\
c_3(t)(1-t^2) &= (a_1(t) - a_3(t))t^2, \\
a_i(t)d_i(t) - b_i(t)d_i(t) &= 1, \text{ for } i = 1,2,3,
\end{aligned}$$

that is, if and only for all $t \in \mathbb{C}^*$*,*

$$\begin{aligned}
a_1(t) - a_3(t) &= -c_1(t)(t^2-1)t^2 \\
b_1(t) &= (a_3(t)(t^2-1) - c_1(t)(t^2-2t+1)t^2 + b_3(t)t^2 - d_3(t)(t^2-1))t^2 \\
d_1(t) - d_3(t) &= c_1(t)(t^2-1)t^2 \\
a_2(t) - a_3(t) &= -c_1(t)(t^2-1)t^2 \\
c_2(t) &= c_1(t)t^4 \\
d_2(t) - d_3(t) &= c_1(t)(t^2-1)t^2 \\
c_3(t) &= c_1(t)t^4 \\
(a_3(t) - d_3(t))(1-t^2) &= (c_1(t)(t^3-2t^2+1) + (b_2(t) - b_3(t)))t^2 \\
a_i(t)d_i(t) - b_i(t)d_i(t) &= 1, \text{ for } i = 1,2,3.
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
H_\varnothing &\cong \{\lim_{t\to 0} \gamma(t) \; ; \; \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\} \\
&= \{(\begin{pmatrix} a & 0 \\ c_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & b_3 \\ 0 & a_1 \end{pmatrix}) \in G \; ; \; c_1, b_2, b_3 \in \mathbb{C} \text{ and } a_1^2 = 1\} \subset G.
\end{aligned}$$

Assume now that $\lambda : \mathbb{C}^* \to T, t \mapsto (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ *is the one-subgroup parameter of* T *with for all* $t \in \mathbb{C}^*$,

$$\lambda_1(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \lambda_2(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \quad \lambda_3(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\lambda(t)(E, E) = (\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix})$ *and*

$$\gamma(t) = (\begin{pmatrix} a_1(t) & b_1(t) \\ c_1(t) & d_1(t) \end{pmatrix}, \begin{pmatrix} a_2(t) & b_2(t) \\ c_2(t) & d_2(t) \end{pmatrix}, \begin{pmatrix} a_3(t) & b_3(t) \\ c_3(t) & d_3(t) \end{pmatrix}) \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)$$

if and only if for all $t \in \mathbb{C}^*$,

$$\begin{aligned} &a_1(t) = a_3(t), \quad b_1(t) = b_3(t)t^2, \quad d_1(t) = d_3(t) \\ &a_2(t) = a_3(t), \quad c_2(t) = c_2(t)t^2, \quad d_2(t) = d_3(t) \\ &b_3(t) = b_2(t)t^2, \quad c_3(t) = c_1(t)t^2 \\ &a_i(t)d_i(t) - b_i(t)d_i(t) = 1, \ \textit{for } i = 1, 2, 3. \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} H_{\{\alpha_1\}} &\cong \{\lim_{t\to 0} \gamma(t) \ ; \ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\} \\ &= \{(\begin{pmatrix} a_1 & 0 \\ c_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}) \ ; \ a_1 \in \mathbb{C}^*, c_1, b_2 \in \mathbb{C}\} \subset G. \end{aligned}$$

Assume that $\lambda : \mathbb{C}^* \to T, t \mapsto (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ *is the one-subgroup parameter of* T *with for all* $t \in \mathbb{C}^*$,

$$\lambda_1(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \lambda_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda_3(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\lambda(t)(E, E) = (\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ *and*

$$\gamma(t) = (\begin{pmatrix} a_1(t) & b_1(t) \\ c_1(t) & d_1(t) \end{pmatrix}, \begin{pmatrix} a_2(t) & b_2(t) \\ c_2(t) & d_2(t) \end{pmatrix}, \begin{pmatrix} a_3(t) & b_3(t) \\ c_3(t) & d_3(t) \end{pmatrix}) \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)$$

if and only if for all $t \in \mathbb{C}^*$,

$$\begin{aligned} &a_1(t) = a_3(t), \quad b_1(t) = b_3(t)t^2, \quad d_1(t) = d_3(t) \\ &a_2(t) = a_3(t), \quad b_2(t) = b_3(t), \quad c_2(t) = c_3(t), \quad d_2(t) = d_3(t) \\ &c_3(t) = c_1(t)t^2 \\ &a_i(t)d_i(t) - b_i(t)d_i(t) = 1, \ \textit{for } i = 1, 2, 3. \end{aligned}$$

Therefore we obtain that

$$H_{\{\alpha_1,\alpha_2\}} \cong \{\lim_{t\to 0}\gamma(t)\ ;\ \gamma \in \mathrm{Iso}_{G(\mathcal{O})}(\lambda)\}$$
$$= \{(\begin{pmatrix} a_1 & 0 \\ c_1 & a_1^{-1}\end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1}\end{pmatrix}, \begin{pmatrix} a_1 & b_2 \\ 0 & a_1^{-1}\end{pmatrix})\ ;\ a_1 \in \mathbb{C}^*, c_1, b_2 \in \mathbb{C}\} \subset G.$$

7. Virtual Poincaré polynomials of G/H_I

Let G/H be a spherical homogeneous space of rank r that admits a wonderful embedding $G/H \hookrightarrow X$ into a smooth projective algebraic variety X such that the complement $X \setminus G/H$ is a union of r smooth irreducible divsors $Z_1, \dots, Z_r$ with transversal intersections. Then X contains exactly 2^r G-orbits X_I that are parametrized by all possible subsets $I \subset \{1, \dots, r\}$ such that the closure $\overline{X_I}$ is the intersection of all divisors Z_i with $i \notin I$. The wonderful embedding X is the disjoint union $\bigcup_I X_I$. Therefore one has

$$\tilde{P}_X(t) = \sum_I \tilde{P}_{X_I}(t).$$

On the other hand, the total space of the normal bundle $\tilde{X}_I$ to the smooth subvariety $\overline{X_I} \subset X$ (it is a direct sum of line bundles) is a simple spherical embedding of the spherical homogeneous space G/H_I corresponding to the satellite H_I such that one has a natural locally trivial torus fibration $f_I : G/H_I \to X_I$ whose fiber is isomorphic to $(\mathbb{C}^*)^{r-|I|}$. Therefore one has

$$\tilde{P}_X(t) = \sum_I \frac{\tilde{P}_{G/H_I}(t)}{(t-1)^{r-|I|}}.$$

Since the valuation cone $\mathcal{V} \subset N_{\mathbb{Q}}$ of G/H is generated by a basis $e_1, \dots, e_r$ of the lattice N, every lattice point $v \in N \cap \mathcal{V}$ can be written as a nonnegative integral linear combination $v = l_1e_1 + \cdots + l_re_r$. Let $\{s_1, \dots, s_r\} \subset M$ be the set of spherical roots which forms a dual basis to the basis $\{e_1, \dots, e_r\}$ of the dual lattice N. Denote by $\mathcal{V}_I$ the face of the valuation cone $\mathcal{V}$ defined by the conditions $\langle x, s_i\rangle = 0$ for all $i \in I$. Let $\mathcal{V}_I^\circ$ be the relative interior of the face $\mathcal{V}_I$. Using the power expansion

$$\frac{1}{t-1} = \frac{t^{-1}}{(1-t^{-1})} = \sum_{j\geqslant 1} t^{-j},$$

we can rewrite

$$\tilde{P}_X(t) = \sum_I \frac{\tilde{P}_{G/H_I}(t)}{(t-1)^{r-|I|}} = \sum_I \tilde{P}_{G/H_I}(t) \sum_{v\in\mathcal{V}_I^\circ} t^{\kappa(v)}, \tag{8}$$

where κ is a linear function on N which takes value -1 on the lattice vectors $e_1, \dots, e_r$.

As it is explained in the introduction, the formula (8) has motivated us to investigate the ratio of two virtual Poincaré polynomials

$$R_I(t^{-1}) := \frac{\tilde{P}_{G/H_I}(t)}{\tilde{P}_{G/H}(t)},$$

and, looking at some series of examples, to formulate Conjecture 1.4. We will give below several series of examples that verify the statement of Conjecture 1.4.

To prove our first result (cf. Theorem 7.3), we start with some preliminary results due to Brion and Peyre, [BPe02].

Assume for awhile that G is any complex connected linear algebraic group. For H a closed subgroup of G, we denote by r_H the rank of H and by u_H the dimension of a maximal unipotent subgroup of H. Choose maximal reductive subgroups $H^{red} \subseteq H$ and $G^{red} \subseteq G$ such that $H^{red} \subseteq G^{red}$.

Theorem 7.1 (Brion-Peyre, [BPe02, Theorem 1(b)])**.** *Let G be an arbitray complex connected linear algebraic group, and H a closed subgroup of G. There exists a polynomial $Q_{G/H}$ with nonnegative integer coefficients such that*

$$\tilde{P}_{G/H}(t) = t^{u_G - u_H}(t-1)^{r_G - r_H} Q_{G/H}(t),$$

and

$$Q_{G/H}(t) = Q_{G^{red}/H^{red}}(t).$$

In particular, if G is reductive, then

$$\tilde{P}_{G/H}(t) = t^{u_{H^{red}} - u_H} \tilde{P}_{G/H^{red}}(t).$$

Moreover, $Q_{G/H}(0) = 1$ if H is connected.

Assume now that G is reductive, and let H be a spherical subgroup of G. For each subset $I \subseteq \mathcal{S}$ of the set of spherical roots $\mathcal{S}$ of G/H, choose a maximal reductive subgroup $H_I^{red} \subseteq H_I$ of the satellite H_I of H. We simply denote by r_I and u_I the integers r_{H_I} and u_{H_I} respectively. Note that $r_I = r_{H_I^{red}}$.

Lemma 7.2 (Brion-Peyre)**.** *Let $I \subseteq \mathcal{S}$. We can assume that H_I^{red} is contained in H.*

Proof. If $I = \mathcal{S}$, the statement is clear. Assume that $I \neq \mathcal{S}$ and let v be a lattice point in $N \cap \mathcal{V}_I^\circ$. Let (X_v, x_v) be the elementary embedding of G/H associated with the uncolored cone $\mathbb{Q}_{\geqslant 0} v$. Let H'_v be the isotropy group in G of a point in the closed G-orbit X'_v. Recall that $H_I \cong H_v$ is by definition the kernel of a certain character χ_v of H'_v (cf. Definition 2.3). We may assume that $H_I^{red} = H_v^{red}$ is contained in H_v.

We now resume the arguments of [BPe02, Section 2]. Consider the action of H_v^{red} on the tangent space $T_{x'_v}(X_v)$ and choose a H_v^{red}-invariant complement N to the tangent space $T_{x'_v}(X'_v)$. By construction, H_v^{red} fixes N pointwise. Then we can choose an H_v^{red}-invariant subvariety Z of X_v such that Z is smooth at x'_v and $T_{x'_v}(Z) = N$. Therefore H_v^{red} fixes pointwise a neighborhood of x'_v in Z, and this neighborhood meets the open G-orbit G/H. Thus we may assume that $H_v^{red} = H_I^{red}$ is contained in H, as expected. □

Lemma 7.2 allows us from now on to assume that H_I^{red} is contained in H for each subset $I \subseteq \mathcal{S}$.

Theorem 7.3. *Assume that H is connected, and let $I \subseteq \mathcal{S}$.*

(1) *We have:*

$$\tilde{P}_{G/H_I}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{red}}(t)t^{-(u_I - u_I^{red})}.$$

In particular, $R_I(t^{-1})$ is a polynomial in t^{-1} with integer coefficients and constant term 1.

(2) *The degree of $R_\varnothing$ is $u_G - u_H = u_\varnothing - u_H$.*

(3) *If H_I is connected, the degree of R_I is $u_I - u_H$.*

Proof. (1) Since H is connected, we can apply [DL97, Theorem 6.1] to the fibration $G/H_I^{red} \to G/H$ with fiber H/H_I^{red} to obtain:

$$\tilde{P}_{G/H_I^{red}}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{red}}(t).$$

By Theorem 7.1, we get

$$\tilde{P}_{G/H_I}(t) = \tilde{P}_{G/H}(t)\tilde{P}_{H/H_I^{red}}(t)t^{-(u_I - u_I^{red})}.$$

Moreover, $\tilde{P}_{H/H_I^{red}}(t)$ is a polynomial with integer coefficients. Since $\dim G/H_I = \dim G/H$, $\tilde{P}_{G/H_I}(t)$ and $\tilde{P}_{G/H}(t)$ have the same degree, and so the degree of $\tilde{P}_{H/H_I^{red}}(t)$ must be equal to

$$d := u_I - u_I^{red}.$$

Hence for some integers $a_0, \ldots, a_{d-1}, a_d$, we have

$$\begin{aligned} P_{H/H_I^{red}}(t)t^{-(u_I - u_I^{red})} &= (a_0 + a_1 t + \cdots + a_{d-1}t^{d-1} + a_d t^d)t^{-d} \\ &= a_0 t^{-d} + a_1 t^{-d+1} + \cdots + a_{d-1}t^{-1} + a_d = R_I(t^{-1}). \end{aligned}$$

It remains to show that the constant term a_d of R_I is 1. Since G/H_I and G/H are irreducible of the same dimension, the leading term of $\tilde{P}_{G/H_I}(t)$ and $\tilde{P}_{G/H}(t)$ is $t^{\dim G/H}$, whence $a_d = 1$.

(2) Since $H_\varnothing$ is horospherical, its normalizer $N_G(H_\varnothing)$ in G is a parabolic subgroup of G, and we have a locally trivial fibration

$$G/H_\varnothing \to G/N_G(H_\varnothing)$$

with a fiber isomorphic to $N_G(H_\varnothing)/H_\varnothing$. The algebraic torus $N_G(H_\varnothing)/H_\varnothing$ has dimension the rank r of G/H. Therefore,

$$\tilde{P}_{G/H_\varnothing}(t) = (t-1)^r \tilde{P}_{G/N_G(H_\varnothing)}(t). \qquad (9)$$

Note that $\tilde{P}_{G/N_G(H_\varnothing)}(0) = 1$. On the other hand, by Theorem 7.1,

$$\tilde{P}_{G/H}(t) \underset{t\to 0}{\sim} t^{u_G - u_H}(-1)^{r_G - r_H}$$

since $Q_{G/H}(0) = 1$ for connected H. So, according to (9) and the assertion (1), we obtain that

$$(-1)^r = t^{u_G - u_H}(-1)^{r_G - r_H} t^{-\deg R_\varnothing}$$

since $R_\varnothing(0) = 1$, whence

$$\deg R_\varnothing = u_G - u_H.$$

At last, notice that $u_G = u_\varnothing$ because $H_\varnothing$ is horospherical.

(3) Assume that H_I is connected. Then by Theorem 7.1,

$$\tilde{P}_{G/H_v}(t) \underset{t\to 0}{\sim} t^{u_G-u_I}(-1)^{r_G-r_I}.$$

So, according to the assertion (1), we obtain that

$$t^{u_G-u_I}(-1)^{r_G-r_I} = t^{u_G-u_H}(-1)^{r_G-r_H}t^{-\deg R_I}$$

since $R_I(0) = 1$, whence

$$\deg R_I = u_I - u_H.$$

□

Example 7.4. *Let $G := GL(n)\times GL(n)$ and $H := diag\, GL(n)$. We use the natural bijection between the set of spherical roots $\mathcal{S}$ of the homogeneous space G/H and the set S of simple roots of $GL(n)$. For a subset $I \subseteq S$ we denote by Δ_I the root system generated by I. Let Δ_I^+ be the corresponding set of positive roots, and P_I the parabolic subgroup in $GL(n)$ corresponding to $I \subset S$. Notice that $\dim GL(n)/P_I = \dim(P_I)_u = |\Delta_S^+ \setminus \Delta_I^+|$. Simple computations show that*

$$R_I(t^{-1}) = \tilde{P}_{GL(n)/P_I}(t)t^{-|\Delta_S^+\setminus\Delta_I^+|} = \prod_{\alpha\in\Delta_S^+\setminus\Delta_I^+}\left(\frac{t^{\mathrm{ht}(\alpha)+1}-1}{t^{\mathrm{ht}(\alpha)}-1}\right)t^{-|\Delta_S^+\setminus\Delta_I^+|}$$

where $\mathrm{ht}(\alpha)$ denotes the height of α for $\alpha \in \Delta_S^+$.

Remark 7.5. *When R_I verifies the statement of Conjecture 1.4, it would be interesting to find an algorithm for computing the polynomial R_I using some combinatorial properties of the spherical roots in the subset $I \subseteq \mathcal{S}$, as in Example 7.4.*

Example 7.6. *Assume that $G = SL(2) \times SL(2) \times SL(2)$ and that H is $SL(2)$, embedded diagonally in G. The homogeneous spherical space G/H has rank three. The satellites of H have been computed in Example 6.9. We observe that $G/H_\varnothing$ is a locally trivial fibration over $(\mathbb{P}^1)^3$ with fiber $(\mathbb{C}^*)^3$. So we get*

$$\tilde{P}_{G/H_\varnothing}(t) = (t-1)^3(t+1)^3.$$

On the other hand, $H_{\{\alpha_1\}} \cong H_{\{\alpha_2\}} = H_{\{\alpha_2\}}$ and $H_{\{\alpha_1,\alpha_2\}} \cong H_{\{\alpha_2,\alpha_3\}} \cong H_{\{\alpha_1,\alpha_3\}}$ are connected. So, using Theorem 7.1, we get:

$$\tilde{P}_{G/H_{\{\alpha_i\}}}(t) = \tilde{P}_{G/H_{\{\alpha_i,\alpha_j\}}}(t) = t(t-1)^2(t+1)^3, \qquad i,j \in \{1,2,3\},\ i \neq j.$$

In conclusion,

$$R_\varnothing = 1 - t^{-2} \quad \textit{and} \quad R_{\{\alpha_i\}} = R_{\{\alpha_i,\alpha_j\}} = 1 + t^{-1}, \qquad i,j \in \{1,2,3\},\ i \neq j.$$

Example 7.7. *Let $G = SL(n)$, and H a maximal standard Levi factor of semisimple type $SL(n-1)\times\mathbb{C}^*$. The homogeneous space G/H is spherical and admits only one elementary embedding, up to isomorphism, $X := \mathbb{P}(\mathbb{C}^n)\times\mathbb{P}((\mathbb{C}^n)^*) \cong \mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$. So there is a unique satellite $H_\varnothing$ which is horospherical. The unique closed G-orbit of X is*

$$X' = \{([x_1:\cdots:x_n],[y_1:\cdots:y_n]) \in \mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \mid \sum_{i=1}^n x_iy_i = 0\}.$$

The variety X' is a locally trivial fibration over $\mathbb{P}^{n-1}$ with a fiber isomorphic to $\mathbb{P}^{n-2}$. In addition, denoting by H' the isotropy group of a point in X', we know that $G/H_\varnothing$ is a locally trivial fibration over $G/H' \cong X'$ with fiber $\mathbb{C}^*$. Hence

$$\tilde{P}_{G/H_\varnothing}(t) = (1 + t + \cdots + t^{n-1})(1 + t + \cdots + t^{n-2})(t - 1).$$

On the other hand, $\tilde{P}_{G/H} = \tilde{P}_X(t) - \tilde{P}_{X'}(t)$, whence

$$\tilde{P}_{G/H}(t) = (1 + t + \cdots + t^{n-1})t^{n-1}.$$

In conclusion we get

$$R_\varnothing(t^{-1}) = \frac{\tilde{P}_{G/H_\varnothing}(t)}{\tilde{P}_{G/H}(t)} = 1 - t^{-(n-1)}.$$

Example 7.8. *Assume that $G = SL(2)$ and $H = N_G(T)$. The spherical homogeneous space G/H admits a unique, up to isomorphism, elementary embedding $X = \mathbb{P}(\mathfrak{sl}_2) \cong \mathbb{P}^2$ and $H_\varnothing \cong U_4$; see Example 2.7. We have $\tilde{P}_X(t) = 1 + t + t^2$, $\tilde{P}_{G/H_\varnothing}(t) = (t-1)(t+1) = t^2 - 1$ and $\tilde{P}_{G/H}(t) = P_X(t) - \frac{\tilde{P}_{G/H_\varnothing}(t)}{t-1} = t^2$, whence*

$$R_\varnothing(t^{-1}) = 1 - t^{-2}.$$

Examples 7.7 and 7.8 are particular cases of the following results.

Theorem 7.9. *Assume that the spherical homogeneous space G/H is of rank one. Then $R_\varnothing$ is a polynomial with integer coefficients.*

Proof. Spherical homogeneous spaces G/H of rank one were classified by Akhiezer [Ak83] and Brion [Br89]. Such a homogeneous space G/H is either horospherical or has a wonderful embedding. Theorem 7.9 is obvious if G/H is horospherical. So we are interested only in those homogeneous spaces G/H of rank one that have a wonderful embedding. These spaces are listed in [T11, Table 30.1]. In this case the spherical subgroup H has a unique satellite subgroup different from H: this is the horospherical subgroup $H_\varnothing$. For each homogeneous space G/H from the list in [T11, Table 30.1], we compute the Poincaré polynomials $\tilde{P}_{G/H}(t)$ and $\tilde{P}_{G/H_\varnothing}(t)$, and then the ratio

$$R_\varnothing(t^{-1}) := \frac{\tilde{P}_{G/H_\varnothing}(t)}{\tilde{P}_{G/H}(t)}.$$

The obtained results are described in Table 1. Our calculations show that the ratio $R_\varnothing(t^{-1})$ is a polynomial in t^{-1} containing only two terms, with integer coefficients.

Let us explain our computations. Let $G/H \hookrightarrow X$ be a wonderful embedding of G/H with closed G-orbit X'. We have:

$$\tilde{P}_X(t) = \tilde{P}_{G/H}(t) + \tilde{P}_{X'}(t) = \tilde{P}_{G/H}(t) + \frac{\tilde{P}_{G/H_\varnothing}(t)}{(t-1)}. \tag{10}$$

Note that X' is a projective homogeneous space G/P where P is certain parabolic subgroup of G such that $\dim G/P = \dim G/H - 1$.

In some cases, like in Example 7.7, from the knowledge of X and X', we compute $\tilde{P}_X(t)$, $\tilde{P}_{X'}(t)$ and so $\tilde{P}_{G/H}(t)$ and $\tilde{P}_{G/H_\varnothing}(t)$ by (10). We can argue in this way for cases 1, 3 (which corresponds to Example 7.7), 7a, 7b, 10 of Table 1.

If H is connected, it is sometimes easier to compute directly $\tilde{P}_{G/H}(t)$ using Theorem 7.1 and [BPe02, Theorem 1(c)], instead of computing $\tilde{P}_{X'}(t)$. Then we get $\tilde{P}_{G/H_\varnothing}(t)$ from $\tilde{P}_X(t)$ and $\tilde{P}_{G/H}(t)$ by (10). We can argue in this way for cases 2, 5, 6, 7a, 7b, 9, 10, 11, 13.

If H is connected, it is sometimes possible to compute $\tilde{P}_{G/H_\varnothing}(t)$ even without the knowledge of X. Indeed, it is sometimes possible to deduce the conjugacy class of the parabolic isotropy group P of a point in X' by dimension reasons. Then we get $\tilde{P}_{G/H_\varnothing}(t)$ since

$$\tilde{P}_{G/H_\varnothing}(t) = \tilde{P}_{G/P}(t)(t-1).$$

Consider for example the case 12 of Table 1 where $G = \mathbf{F}_4$ and $H = \mathbf{B}_4$. Then $\dim G/H = 52 - 36 = 16$. So $\dim G/P = 15$ and $\dim P = 37$. Hence, for dimension reasons, P is conjugate to a parabolic subgroup whose semisimple Levi part is either of type either $\mathbf{B}_3$, or of type $\mathbf{C}_3$. In both cases, we get (assuming that P contains the standard Borel subgroup B) that

$$\tilde{P}_{G/P}(t) = \frac{\tilde{P}_{G/B}(t)}{\tilde{P}_{P/B}(t)} = \frac{(t^2-1)(t^6-1)(t^8-1)(t^{12}-1)}{(t-1)(t^2-1)(t^4-1)(t^6-1)}$$

since $\mathbf{B}_3$ and $\mathbf{C}_3$ have the same exponents $1,3,5$. Hence

$$\tilde{P}_{G/H_\varnothing}(t) = \tilde{P}_{G/P}(t)(t-1) = (t^4+1)(t^{12}-1).$$

On the other hand, since H is connected we have by [BPe02]:

$$\tilde{P}_{G/H} = \frac{(t^2-1)(t^6-1)(t^8-1)(t^{12}-1)t^{24}}{(t^2-1)(t^4-1)(t^6-1)(t^8-1)t^{16}} = \frac{(t^{12}-1)t^8}{t^4-1}.$$

In conclusion,

$$R_\varnothing(t^{-1}) = \frac{t^8-1}{t^8} = 1 - t^{-8}.$$

We can argue similarly also for cases 4 (which corresponds to Example 7.8), 8a, 8b, 14, 15.

□

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	G	H	$\tilde{P}_{G/H}(t)$	$\tilde{P}_{G/H_\varnothing}(t)$	$R_\varnothing(t^{-1})$
1	$SL(2)\times SL(2)$	$SL(2)$	$t(t^2-1)$	$(t-1)(1+t)^2$	$1+t^{-1}$
2	$PSL(2)\times PSL(2)$	$PSL(2)$	$t(t^2-1)$	$(t-1)(1+t)^2$	$1+t^{-1}$
3	$SL(n)$	$S(L(1)\times L(n-1))$	$\frac{t^{n-1}(t^n-1)}{t-1}$	$\frac{(t^{n-1}-1)(t^n-1)}{t-1}$	$1-t^{-(n-1)}$
4	$PSL(2)$	$PO(2)$	t^2	t^2-1	$1-t^{-2}$
5	$Sp(2n)$	$Sp(2)\times Sp(2n-2)$	$\frac{t^{2n-2}(t^{2n}-1)}{t^2-1}$	$\frac{(t^{2n-2}-1)(t^{2n}-1)}{t^2-1}$	$1-t^{-(2n-2)}$
6	$Sp(2n)$	$B(Sp(2))\times Sp(2n-2)$	$\frac{t^{2n-1}(t^{2n}-1)}{t-1}$	$\frac{(t^{2n-1}-1)(t^{2n}-1)}{t-1}$	$1-t^{-(2n-1)}$
7a	$SO(2n+1)$	$SO(2n)$	$t^n(t^n+1)$	$t^{2n}-1$	$1-t^{-n}$
7b	$SO(2n)$	$SO(2n-1)$	$t^{n-1}(t^n-1)$	$(t^{n-1}+1)(t^n-1)$	$1+t^{-(n-1)}$
8a	$SO(2n+1)$	$S(O(1)\times O(2n))$	t^{2n}	$t^{2n}-1$	$1-t^{-2n}$
8b	$SO(2n)$	$S(O(1)\times O(2n-1))$	$t^{n-1}(t^n-1)$	$(t^{n-1}+1)(t^n-1)$	$1+t^{-(n-1)}$
9	$SO(2n+1)$	$GL(n)\ltimes\wedge^2\mathbb{C}^n$	$t\prod_{i=1}^n(t^i+1)$	$(t-1)\prod_{i=1}^n(t^i+1)$	$1-t^{-1}$
10	$Spin(7)$	$\mathbf{G}_2$	$t^3(t^4-1)$	$(t^3+1)(t^4-1)$	$1+t^{-3}$
11	$SO(7)$	$\mathbf{G}_2$	$t^3(t^4-1)$	$(t^3+1)(t^4-1)$	$1+t^{-3}$
12	$\mathbf{F}_4$	$\mathbf{B}_4$	$\frac{t^8(t^{12}-1)}{t^4-1}$	$(t^4+1)(t^{12}-1)$	$1-t^{-8}$
13	$\mathbf{G}_2$	$SL(3)$	$t^3(t^3+1)$	$(t^3+1)(t^3-1)$	$1-t^{-3}$
14	$\mathbf{G}_2$	$N(SL(3))$	t^6	t^6-1	$1-t^{-6}$
15	$\mathbf{G}_2$	$GL(2)\ltimes(\mathbb{C}\oplus\mathbb{C}^2)\otimes\wedge^2\mathbb{C}^2$	$\frac{t^2(t^6-1)}{t-1}$	$\frac{(t^2-1)(t^6-1)}{t-1}$	$1-t^{-2}$

TABLE 1. Data for homogeneous spherical spaces of rank one

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