Lectures on W-algebras

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Abstract. These notes are written in preparation for the workshop "W-algebras" which will take place in Melbourne from November 20 to December 2, 2016.

Contents

Introduction

The goal of this workshop is to introduce the theory of vertex algebras and affine W-algebras, which are certain vertex algebras, with emphasis on their geometrical aspects.

Overview of theory, and goals of the workshop

Roughly speaking, a vertex algebra is a vector space V, endowed with a distinguished vector, the vacuum vector, and the vertex operator map from V to the space of formal Laurent series with linear operators on V as coefficients. These data satisfy a number of axioms and have some fundamental properties as, for example, an analogue to the Jacobi identity, locality and associativity. Although the definition is purely algebraic, the above axioms have deep geometric meaning. They reflect the fact that vertex algebras give an algebraic framework of the two-dimensional conformal field theory. The connections of this topic with other branches of mathematics and physics include algebraic geometry (moduli spaces), representation theory (modular representation theory, geometric Langlands correspondence), two dimensional conformal field theory, string theory (mirror symmetry) and four dimensional gauge theory (AGT conjecture).

The (affine) W-algebras were first introduced by Zamolodchikov in the 1980s in physics and then developed by Fateev-Lukyanov, Feigin-Frenkel, Kac-Roan-Wakimoto and others. The finite W-algebras, the finite dimensional analogs of W-algebras, were introduced by Premet. They go back to Kostant's works in the 1970s who studied some particular cases. The W-algebras were extensively studied by physicists in 1990s in connection with two dimensional conformal field theory. More recently, the appearance of the AGT conjecture in physics led many researchers in mathematics towards W-algebras. In the meantime, the finite W-algebras have caught attention for different reasons that are mostly related with more classical problems of representation theory.

The nicest vertex algebras are those which are both rational and lisse (or C_2-cofinite). The rationality means the completely irreducibility of modules. The lisse condition is a certain finiteness condition as explained next paragraph. If a vertex algebra V is rational and lisse, then it gives rise to a rational conformal field theory. In particular, the characters of simple V-modules form vector valued modular functions, and moreover, the category of V-modules forms a modular tensor category, so that one can associate with it an invariant of knots.

To each vertex algebra V one can naturally attach a certain Poisson variety X_V called the associated variety of V. A vertex algebra V is called lisse if $\dim X_V = 0$. Lisse vertex algebras are natural generalizations of finite-dimensional algebras and possess remarkable properties. For instance, the modular invariance of characters still holds without the rationality assumption.

In fact the geometry of the associated variety often reflects some algebraic properties of the vertex algebras V. More generally, vertex algebras whose associated variety has only finitely symplectic leaves, are also of great interest for several reasons that will be addressed in the workshop.

Important examples of vertex algebras are those coming from affine Kac-Moody algebra, which are called affine vertex algebras. They play a crucial role in the representation theory of affine Kac-Moody algebras, and of W-algebras. In the case that V is a simple affine vertex algebra, its associated variety is an invariant and conic subvariety of the corresponding simple Lie algebra. It plays an analog role to the associated variety of primitive ideals of the enveloping algebra of simple Lie algebras. However, associated varieties of affine vertex algebras are not necessarily contained in the nilpotent cone and it is difficult to describe them in general.

In fact, although associated varieties seem to be significant also in connection with the recent study of four dimensional superconformal field theory, their general description is fairly open, except in a few cases.

The affine W-algebras are certain vertex algebras associated with nilpotent elements of simple Lie algebras. They can be regarded as affinizations of finite W-algebras, and can also be considered as generalizations of affine Kac-Moody algebras and Virasoro algebras. They quantize the arc space of the Slodowy slices associated with nilpotent elements. The study of affine W-algebras began with the work of Zamolodchikov in 1985. Mathematically, affine W-algebras are defined by the method of quantized Drinfeld-Sokolov reduction that was discovered by Feigin and Frenkel in the 1990s. The general definition of affine W-algebras were given by Kac, Roan and Wakimoto in 2003. Affine W-algebras are related with integrable systems, the two-dimensional conformal field theory and the geometric Langlands program. The most recent developments in representation theory of affine W-algebras were done by Kac-Wakimoto and Arakawa.

Since they are not finitely generated by Lie algebras, the formalism of vertex algebras is necessary to study them. The study of affine W-algebras will be the ultimate goal of the workshop. In this context, associated varieties of W-algebras, and their quotients, are important tools to understand some properties, such as the lisse condition and even the rationality condition.

It is only quite recently that the study of associated varieties of vertex algebras and their arc spaces, has been more intensively developed. In this workshop we wish to highlight this aspect of the theory of vertex algebras which seems to be very promising. In particular, the workshop will include open problems on associated varieties of W-algebras raised by recent works of Tomoyuki Arakawa and Anne Moreau.

Overview of lectures

One of the first interesting examples of non-commutative vertex algebras are the affine vertex algebras associated with affine Kac-Moody algebras which play a crucial role in the representation theory of affine Kac-Moody algebras, and of W-algebras. For this reason the note will start with an introduction to affine Kac-Moody algebras and their representations (see Part 1).

We will introduce the notion of vertex algebras in Part 2, and discuss some important related objects as Zhu's C_2 algebras, Zhu's algebras and Zhu's functors. The Zhu's functor gives a correspondence between the theory of modules over a vertex algebra and the representation theory of its Zhu's algebra. This correspondence is particularly well-understood in the case of the universal affine vertex algebras, where Zhu's algebras are enveloping algebras of the corresponding finite-dimensional simple Lie algebras.

The W-algebras are certain vertex algebras associated with nilpotent elements of a simple Lie algebra. Zhu's algebras of W-algebras are finite W-algebras. The later are certain generalizations of the enveloping algebra of a simple Lie algebra. They can be defined through the BRST cohomology associated with nilpotent elements. So the definition and properties of (finite and affine) W-algebras are deeply related to the geometry of of nilpotent orbits. We will explain in Part 3 the definition of finite W-algebras by BRST reduction (= a form of quantized Hamiltonian reduction) after outlining basics on nilpotent orbits.

Any vertex algebra is naturally filtered and the corresponding graded algebra is a Poisson vertex algebra. Moreover, the spectrum of the Zhu's C_2 algebra, which is a generating ring of this graded algebra, is what we call the associated variety. Its geometry gives important information on the vertex algebra as we wish to illustrate in this workshop. A nice way to construct Poisson vertex algebras is to consider the coordinate ring of the arc space of a Poisson variety. Actually, strong relations exists, at least conjecturally, between the arc space of the associated variety and the singular support of a vertex algebra, that is, the spectrum of the corresponding graded algebra. All these aspects will be discussed in Part 4.

Part 5 will be about affine W-algebras. They are defined by a certain BRST reduction, called the quantum Drinfeld-Sokolov reduction, associated with nilpotent elements. Rational W-algebras and lisse W-algebras are particularly interesting classes of W-algebras. The rationality and the lisse conditions, and some other properties will be considered. Associated varieties of affine W-algebras, and their quotients, will be also discussed.

We assume the reader is familiar with basics on semisimple Lie algebras and their representations (although we give a short review), commutative algebras, algebraic geometry and algebraic groups.

PART 1

Introduction to affine Lie algebras and their representations

One of the first interesting examples of non-commutative vertex algebras are the affine vertex algebras associated with affine Kac-Moody algebras which play a crucial role in the representation theory of affine Kac-Moody algebras, and of W-algebras. For this reason we start with an introduction to affine Kac-Moody algebras and their representations.

1.1. Quick review on semisimple Lie algebras, main notations

Let $\mathfrak{g}$ be a complex finite dimensional *semisimple* Lie algebra, i.e., $\{0\}$ is the only abelian ideal of $\mathfrak{g}$. Let $\mathbf{G}$ be the *adjoint group* of $\mathfrak{g}$: it is the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra contains $\operatorname{ad}\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $\mathbf{G} = \mathrm{Aut}_e(\mathfrak{g})$, where $\mathrm{Aut}_e(\mathfrak{g})$ is the subgroup of elementary elements, that is, the elements $\exp(\operatorname{ad} x)$ with x a *nilpotent element* of $\mathfrak{g}$ (i.e., $(\operatorname{ad} x)^n = 0$ for $n \gg 0$). Hence

$$\mathrm{Lie}(\mathbf{G}) = \operatorname{ad}\mathfrak{g} \cong \mathfrak{g}$$

since the adjoint representation $\operatorname{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto (\operatorname{ad} x)(y) = [x,y]$ is faithful, $\mathfrak{g}$ being semisimple.

1.1.1. Main notations. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$, we shall denote by $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$ and by $U(\mathfrak{a})$ its enveloping algebra which are the quotient of the tensor algebra of $\mathfrak{a}$ by the two-sided ideal generated by the elements $x\otimes y - y \otimes x$ and the two-sided ideal generated by the elements $x \otimes y - y \otimes x - [x,y]$ respectively, with $x, y \in \mathfrak{a}$.

For $x \in \mathfrak{g}$, we shall denote by $\mathfrak{a}^x$ the centralizer of x in $\mathfrak{a}$, that is,

$$\mathfrak{a}^x = \{y \in \mathfrak{a} \mid [x,y] = 0\},$$

which is also the intersection of $\mathfrak{a}$ with the kernel of the map

$$\operatorname{ad} x\colon \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x,y].$$

Let $\kappa_{\mathfrak{g}}$ be the *Killing form* of $\mathfrak{g}$,

$$\kappa_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (x,y) \mapsto \mathrm{tr}(\operatorname{ad} x \operatorname{ad} y).$$

It is a nondegenerate symmetric bilinear form of $\mathfrak{g}$ which is $\mathbf{G}$-invariant, that is,

$$\kappa_{\mathfrak{g}}(g.x, g.y) = \kappa_{\mathfrak{g}}(x,y) \quad \text{for all } x, y \in \mathfrak{g},$$

or else,

$$\kappa_{\mathfrak{g}}([x,y],z) = \kappa_{\mathfrak{g}}(x,[y,z]) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is semisimple, any other such bilinear form is a nonzero multiple of the Killing form.

EXAMPLE 1.1. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$, $n \geqslant 2$, which is the set of traceless complex n-size square matrices, with bracket $[A, B] = AB - BA$. The Lie algebra $\mathfrak{sl}_n$ is actually *simple*, that is, $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$. Its Killing form is given by

$$(A, B) \mapsto 2n \operatorname{tr}(AB).$$

The bilinear form $(A, B) \mapsto \operatorname{tr}(AB)$ looks more natural. In fact, for our purpose, we will prefer a certain normalization $(\ |\ )$ of the Killing form which will coincide with this bilinear form for $\mathfrak{sl}_n$ (see Section 1.2).

1.1.2. Cartan matrix and Chevalley generators. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} \mid [x, y] = \alpha(x)y \text{ for all } x \in \mathfrak{h}\},$$

be the corresponding root decomposition of $(\mathfrak{g}, \mathfrak{h})$, where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a basis of Δ, with r the *rank* of $\mathfrak{g}$, and let $\alpha_1^\vee, \dots, \alpha_r^\vee$ be the coroots of $\alpha_1, \dots, \alpha_r$ respectively. The element $\alpha_i^\vee$, $i = 1, \dots, r$, viewed as an element of $(\mathfrak{h}^*)^* \cong \mathfrak{h}$, will be often denoted it by h_i.

Recall that the *Cartan matrix* of Δ is the matrix $C = (C_{i,j})_{1 \leqslant i,j \leqslant r}$ where $C_{i,j} := \alpha_j(h_i)$. The Cartan matrix C does not depend on the choice of the basis Π. It verifies the following properties:

$$C_{i,j} \in \mathbb{Z} \text{ for all } i, j, \tag{1}$$

$$C_{i,i} = 2 \text{ for all } i, \tag{2}$$

$$C_{i,j} \leqslant 0 \text{ if } i \neq j, \tag{3}$$

$$C_{i,j} = 0 \text{ if and only if } C_{j,i} = 0. \tag{4}$$

Moreover, all principal minors of C are strictly positive,

$$\det\left((C_{i,j})_{1 \leqslant i,j \leqslant s}\right) > 0 \quad \text{for} \quad 1 \leqslant s \leqslant r.$$

The semsimple Lie algebra $\mathfrak{g}$ has a presentation in term of Chevalley generators. Namely, consider the generators $(e_i)_{1 \leqslant i \leqslant r}$, $(f_i)_{1 \leqslant i \leqslant r}$, $(h_i)_{1 \leqslant i \leqslant r}$ with relations

$$[h_i, h_j] = 0, \tag{5}$$

$$[e_i, f_j] = \delta_{i,j} h_i, \tag{6}$$

$$[h_i, e_j] = C_{i,j} e_j, \tag{7}$$

$$[h_i, f_j] = -C_{i,j} f_j, \tag{8}$$

$$(\operatorname{ad} e_i)^{1 - C_{i,j}} e_j = 0 \text{ for } i \neq j, \tag{9}$$

$$(\operatorname{ad} f_i)^{1 - C_{i,j}} f_j = 0 \text{ for } i \neq j, \tag{10}$$

where $\delta_{i,j}$ is the Kronecker symbol. The last two relations are called the *Serre relations*. By (3) and (4), $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ for all i.

It is well-known that $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta$. One can choose nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$ for all α such that $(h_i; i = 1, \dots, r) \cup (e_\alpha; \alpha \in \Delta)$ forms a Chevalley basis of $\mathfrak{g}$. This means, apart from the above relations, that:

$$[e_\beta, e_\gamma] = \pm(p + 1) e_{\beta + \gamma} \tag{11}$$

for all $\beta, \gamma \in \Delta$, where p is the greatest positive integer such that $\gamma - p\beta$ is a root. Here we consider that $e_{\beta+\gamma} = 0$ if $\beta + \gamma$ is not a root, and that $e_{\alpha_i} = e_i$, $e_{-\alpha_i} = f_i$ for $i = 1, \dots, r$.

Let Δ_+ be the positive root system corresponding to Π, and let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{12}$$

be the corresponding triangular decomposition. Thus $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$ are both nilpotent Lie subalgebras of $\mathfrak{g}$.

1.1.3. Verma modules. Let $\lambda \in \mathfrak{h}^*$. Set

$$K_\mathfrak{g}(\lambda) := U(\mathfrak{g})\mathfrak{n}_+ + \sum_{x\in\mathfrak{h}} U(\mathfrak{g})(x - \lambda(x)).$$

Since $K_\mathfrak{g}(\lambda)$ is a left $U(\mathfrak{g})$-module,

$$M_\mathfrak{g}(\lambda) := U(\mathfrak{g})/K_\mathfrak{g}(\lambda)$$

is naturally a left $U(\mathfrak{g})$-module, called a *Verma module*.

THEOREM 1.2 ([Carter, Theorem 10.6]). (1) *Each element of $M_\mathfrak{g}(\lambda)$ is uniquely written in the form um_λ for some $u \in U(\mathfrak{g})$ where $m_\lambda := 1 + K_\mathfrak{g}(\lambda)$.*
(2) *The elements $f_{\beta_1}^{n_1} \dots f_{\beta_s}^{n_s} m_\lambda$ for all $n_i \geqslant 0$ form a basis of $M_\mathfrak{g}(\lambda)$.*

Note that $M_\mathfrak{g}(\lambda)$ can also be described as follows (up to isomorphism of $U(\mathfrak{g})$-modules):

$$M_\mathfrak{g}(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =: \mathrm{Ind}_\mathfrak{b}^\mathfrak{g}(\mathbb{C}_\lambda),$$

where $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathbb{C}_\lambda$ is a 1-dimensional $\mathfrak{b}$-module whose $\mathfrak{b}$-action is given by: $(x + n).z = \lambda(x)z$ for $x \in \mathfrak{h}$, $n \in \mathfrak{n}_+$ and $z \in \mathbb{C}_\lambda$. Then, up to scalars, $m_\lambda = 1 \otimes 1$.

For each $\mu \in \mathfrak{h}^*$, set

$$M_\mathfrak{g}(\lambda)_\mu := \{m \in M_\mathfrak{g}(\lambda) \mid xm = \mu(x)m \text{ for all } x \in \mathfrak{h}\}.$$

For $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \preccurlyeq \lambda$ if $\lambda - \mu = \sum_{i=1}^r m_i\alpha_i$ where $m_i \in \mathbb{Z}$, $m_i \geqslant 0$. This defines a partial order on $\mathfrak{h}^*$.

THEOREM 1.3 ([Carter, Theorem 10.7]). (1) $M_\mathfrak{g}(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mathfrak{g}(\lambda)_\mu$.
(2) $M_\mathfrak{g}(\lambda)_\mu \neq 0$ *if and only if* $\mu \preccurlyeq \lambda$, *and* $\dim M_\mathfrak{g}(\lambda)_\mu$ *is the number of ways of expressing* $\lambda - \mu$ *as a sum of positive roots. In particular,* $\dim M_\mathfrak{g}(\lambda)_\lambda = 1$.

If $M_\mathfrak{g}(\lambda)_\mu \neq 0$, then μ called a *weight of* $M_\mathfrak{g}(\lambda)$, and $M_\mathfrak{g}(\lambda)_\mu$ is called the *weight space of* $M_\mathfrak{g}(\lambda)$ *with weight* μ.

Theorem 1.3 says that the weights of $M_\mathfrak{g}(\lambda)$ are precisely the elements $\mu \in \mathfrak{h}^*$ such that $\mu \preccurlyeq \lambda$. Thus λ is the *highest weight* of $M_\mathfrak{g}(\lambda)$ with respect to the partial order $\preccurlyeq$. We say that $M_\mathfrak{g}(\lambda)$ is the *Verma module with highest weight* λ.

One of the important fact about $M_\mathfrak{g}(\lambda)$ is that it has a unique proper submodule $N_\mathfrak{g}(\lambda)$. It is constructed as follows: since $M_\mathfrak{g}(\lambda)_\lambda = \mathbb{C}m_\lambda$ and that $M_\mathfrak{g}(\lambda)$ is generated by m_λ, any proper submodule N of $M_\mathfrak{g}(\lambda)$ satisfy $N_\lambda = 0$. In particular the sum $N_{\max}$ of all proper submodules of M satisfies $(N_{\max})_\lambda = 0$. This proves the existence and the unicity of the maximal proper submodule of $M_\mathfrak{g}(\lambda)$: just set $N_\mathfrak{g}(\lambda) := N_{\max}$.

Since $N_\mathfrak{g}(\lambda)$ is a maximal submodule of $M_\mathfrak{g}(\lambda)$,

$$L_\mathfrak{g}(\lambda) := M_\mathfrak{g}(\lambda)/N_\mathfrak{g}(\lambda).$$

is a simple $U(\mathfrak{g})$-module (that is, an irreducible representation of $\mathfrak{g}$). There is $v_\lambda \in L_{\mathfrak{g}}(\lambda) \setminus \{0\}$ such that

$$h_i v = \lambda(h_i) v \text{ for all } i = 1, \dots, r, \tag{13}$$

$$e_i v = 0 \text{ for all } i = 1, \dots, r, \text{ that is, } \mathfrak{n}_+ v = 0, \tag{14}$$

$$L_{\mathfrak{g}}(\lambda) = U(\mathfrak{n}_-) v_\lambda, \tag{15}$$

$$\lambda \text{ is the highest weight of } L_{\mathfrak{g}}(\lambda). \tag{16}$$

Let

$$P := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i = 1, \dots, r\},$$
$$P^+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geqslant 0} \text{ for all } i = 1, \dots, r\},$$

be the *weight lattice* of $\mathfrak{h}^*$ and the set of *dominant integral weights* respectively. The elements $\varpi_i \in \mathfrak{h}^*$, $i = 1, \dots, r$, satisfying $\varpi_i(h_j) = \delta_{i,j}$ for all j are called the *fundamental weights*. We denote by $\varpi_1^\vee, \dots, \varpi_r^\vee$ the fundamental coweights. They are the elements of $\mathfrak{h}$ such that $(\varpi_1^\vee, \dots, \varpi_r^\vee)$ is the dual basis of $(\alpha_1, \dots, \alpha_r)$.

We conclude this section by the following crucial result.

THEOREM 1.4 ([**Carter**, Theorem 10.21]). *The simple $U(\mathfrak{g})$-module $L_{\mathfrak{g}}(\lambda)$ is finite dimensional if and only if $\lambda \in P^+$. Moreover, all simple finite dimensional $U(\mathfrak{g})$-modules are of the form $L_{\mathfrak{g}}(\lambda)$ for some $\lambda \in P^+$. These modules are pairwise non-isomorphic.*

The highest weight modules $M_{\mathfrak{g}}(\lambda)$ and $L_{\mathfrak{g}}(\lambda)$ are both elements of the *category* $\mathcal{O}$ of $\mathfrak{g}$. To avoid repetitions, we will define the category $\mathcal{O}$ only for affine Kac-Moody algebras (see Section 1.4); the definition and properties are very similar.

For more about semisimple Lie algebras and their representations, possible references are [**Tauvel-Yu, Carter**]; see [**Humphreys**] about the category $\mathcal{O}$.

For the category $\mathcal{O}$ in the affine Kac-Moody algebras setting, we refer to Moody-Pianzola's book [**Moody-Pianzola**].

1.2. Affine Kac-Moody algebras

We assume from now on that $\mathfrak{g}$ is simple, that is, the only ideals of $\mathfrak{g}$ are $\{0\}$ or $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$.

1.2.1. The loop algebra. Consider the *loop algebra of* $\mathfrak{g}$ which is the Lie algebra

$$\mathcal{L}\mathfrak{g} := \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z},$$

where xt^m stands for $x \otimes t^m$.

REMARK 1.5. The Lie algebra $\mathcal{L}\mathfrak{g}$ is the Lie algebra of polynomial functions from the unit circle to $\mathfrak{g}$. This is the reason why it is called the loop algebra.

1.2.2. Definition of affine Kac-Moody algebras. Define the bilinear form $(\ |\ )$ on $\mathfrak{g}$ by:

$$(\ |\ ) = \frac{1}{2h^\vee}\kappa_{\mathfrak{g}},$$

where $h^\vee$ is the *dual Coxeter number* (see §1.3.3 for the definition). For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then $h^\vee = n$. Thus with respect to the induced bilinear form on $\mathfrak{h}^*$, $(\theta|\theta) = 2$ where θ is the *highest root* of $\mathfrak{g}$, that is, the unique (positive) root $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i = 1, \dots, r$.

DEFINITION 1.6. We define a bilinear map ν on $\mathcal{L}\mathfrak{g}$ by setting:

$$\nu(x \otimes f, y \otimes g) := (x|y)\mathrm{Res}_{t=0}(\frac{df}{dt}g),$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$, where the linear map $\mathrm{Res}_{t=0}\colon \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is defined by $\mathrm{Res}_{t=0}(t^m) = \delta_{m,-1}$ for $m \in \mathbb{Z}$.

The bilinear ν is a *2-cocycle on* $\mathcal{L}\mathfrak{g}$, that is, for any $a, b, c \in \mathcal{L}\mathfrak{g}$,

$$\nu(a, b) = -\nu(b, a), \tag{17}$$

$$\nu([a, b], c) + \nu([b, c], a) + \nu([c, a], b) = 0. \tag{18}$$

DEFINITION 1.7. We define the *affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$ as the vector space $\widehat{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relations $[K, \widehat{\mathfrak{g}}] = 0$ (so K is a *central element*), and

$$[x \otimes f, y \otimes g] = [x, y]_{\mathcal{L}\mathfrak{g}} + \nu(x \otimes f, y \otimes g)K, \quad x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}], \tag{19}$$

where $[\ ,\ ]_{\mathcal{L}\mathfrak{g}}$ is the Lie bracket on $\mathcal{L}\mathfrak{g}$. In other words the commutation relations are given by:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K,$$
$$[K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

EXERCISE 1.8. Verify that the identifies (17) and (18) are true, and then that the above commutation relations indeed define a Lie bracket on $\widehat{\mathfrak{g}}$.

1.2.3. Chevalley generators. The following result shows that affine Kac-Moody algebras are natural generalizations of finite dimensional semisimple Lie algebras.

THEOREM 1.9. *The Lie algebra* $\widehat{\mathfrak{g}}$ *can be presented by generators* $(E_i)_{0 \leqslant i \leqslant r}$, $(F_i)_{0 \leqslant i \leqslant r}$, $(H_i)_{0 \leqslant i \leqslant r}$, *and relations*

$$[H_i, H_j] = 0, \tag{20}$$

$$[E_i, F_j] = \delta_{i,j}H_i, \tag{21}$$

$$[H_i, E_j] = C_{i,j}E_j, \tag{22}$$

$$[H_i, F_j] = -C_{i,j}F_j, \tag{23}$$

$$(\mathrm{ad}\, E_i)^{1-C_{i,j}}E_j = 0 \textit{ for } i \neq j, \tag{24}$$

$$(\mathrm{ad}\, F_i)^{1-C_{i,j}}F_j = 0 \textit{ for } i \neq j, \tag{25}$$

where $\widehat{C} = (C_{i,j})_{0\leqslant i\leqslant r}$ *is an* affine Cartan matrix*, that is,* $\widehat{C}$ *satisfies the relations (1)–(4) of a Cartan matrix, all proper principal minors are strictly positive,*

$$\det\left((C_{i,j})_{1\leqslant i,j\leqslant s}\right) > 0 \quad \textit{for} \quad 0 \leqslant s \leqslant r-1,$$

and $\det(\widehat{C}) = 0$*.*

Moreover, we can choose the labeling $\{0,\dots,r\}$ so that the subalgebra generated by $(E_i)_{1\leqslant i\leqslant r}$, $(F_i)_{1\leqslant i\leqslant r}$, $(H_i)_{1\leqslant i\leqslant r}$ is isomorphic to $\mathfrak{g}$, that is, $(C_{i,j})_{1\leqslant i\leqslant r}$ is the Cartan matrix C of $\mathfrak{g}$.

Let us give the general idea of the construction of the Chevalley generators of $\widehat{\mathfrak{g}}$ (see [**Hernandez-lectures**][1]). Set for $i = 1,\dots,r$,

$$E_i := e_i = e_i \otimes 1, \quad F_i := f_i = f_i \otimes 1, \quad H_i := h_i = h_i \otimes 1.$$

The point is to define E_0, F_0, H_0. Recall that θ is the highest root of Δ. Consider the *Chevalley involution* ω which is the linear involution map of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h_i) = -h_i$ for $i = 1,\dots,r$. Then pick $f_0 \in \mathfrak{g}_\theta$ and $e_0 \in \mathfrak{g}_{-\theta}$ such that

$$(f_0|\omega(f_0)) = -\frac{2}{(\theta|\theta)} = -1.$$

Then we set $e_0 := -\omega(f_0) \in \mathfrak{g}_{-\theta}$ and,

$$E_0 := e_0 t = e_0 \otimes t, \quad F_0 := f_0 t^{-1} = f_0 \otimes t^{-1}, \quad H_0 := [E_0, F_0].$$

Example 1.10. Assume that $\mathfrak{g} = \mathfrak{sl}_2$. Then the Cartan matrix C is $C = (2)$. Let us check that the affine Cartan matrix of $\widehat{\mathfrak{sl}}_2$ is $\widehat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We have

$$\widehat{\mathfrak{sl}}_2 = e \otimes \mathbb{C}[t,t^{-1}] \oplus f \otimes \mathbb{C}[t,t^{-1}] \oplus h \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K,$$

where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We follow the above construction. We set $E_1 := e$, $F_1 := f$ and $H_1 := h$. We have $h^\vee = 2$ and $\Delta = \{\alpha, -\alpha\}$ with $\alpha(h) = 2$. The highest root is $\theta = \alpha$ and $(\mathfrak{sl}_2)_\theta = \mathbb{C}e$. So f_0 is of the form $f_0 = \lambda e$, $\lambda \in \mathbb{C}^*$ and verifies:

$$-1 = (f_0, \omega(f_0)) = -\lambda^2,$$

whence $\lambda^2 = \pm 1$. Let us fix $\lambda = 1$. So we have

$$E_0 = ft \quad \text{and} \quad F_0 = et^{-1}.$$

Then

$$H_0 = [E_0, F_0] = [f,e] + (f|e)K = K - H_1.$$

We can verify the relations of Chevalley generators. In particular, $[H_1, E_0] = -2E_0$ and $[H_0, E_1] = -2E_1$ whence the expected affine Cartan matrix $\widehat{C}$.

1.3. Root systems and triangular decomposition

In order to construct analogs of highest weight representations, we need a triangular decomposition for $\widehat{\mathfrak{g}}$ and the corresponding combinatoric, that is, a system of roots.

[1] Since we don't have exactly the same normalization, we give the details here.

1.3.1. Triangular decomposition. Recall the triangular decomposition (12) of $\mathfrak{g}$, and consider the following subspaces of $\widehat{\mathfrak{g}}$:

$$\begin{aligned}
\widehat{\mathfrak{n}}_+ &:= (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t] = \mathfrak{n}_+ + t\mathfrak{g}[t],\\
\widehat{\mathfrak{n}}_- &:= (\mathfrak{n}_+ \oplus \mathfrak{h}) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}],\\
\widehat{\mathfrak{h}} &:= (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K = \mathfrak{h} + \mathbb{C}K.
\end{aligned}$$

They are Lie subalgebras of $\widehat{\mathfrak{g}}$ and we have

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+. \tag{26}$$

In fact, $\widehat{\mathfrak{n}}_+$ (resp. $\widehat{\mathfrak{n}}_-, \widehat{\mathfrak{h}}$) is generated by the E_i (resp. F_i, H_i), for $i = 0, \dots, r$. The verifications are left to the reader.

1.3.2. Extended affine Kac-Moody algebras. We now intend to define a corresponding root system, and simple roots. The simple roots $\alpha_i \in \widehat{\mathfrak{h}}^*$ are defined by $\alpha_j(H_i) = C_{i,j}$ for $0 \leqslant i, j \leqslant r$. As $\det(\widehat{C}) = 0$, the simple roots $\alpha_0, \dots, \alpha_r$ are not linearly independent. For example, for $\widehat{\mathfrak{sl}}_2$, $\alpha_0 + \alpha_1 = 0$.

For the following constructions, we need linearly independent simple roots. That is why we consider the *extended affine Lie algebra*:

$$\widetilde{\mathfrak{g}} := \widehat{\mathfrak{g}} \oplus \mathbb{C}D,$$

with commutation relations (apart from those of $\widehat{\mathfrak{g}}$),

$$[D, x \otimes f] = x \otimes t\frac{df}{dt}, \qquad [D, K] = 0, \quad x \in \mathfrak{g},\ f \in \mathbb{C}[t, t^{-1}],$$

that is,

$$[D, xt^m] = mxt^m, \qquad [D, K] = 0, \qquad x \in \mathfrak{g},\ m \in \mathbb{Z}.$$

We have the new Cartan subalgebra

$$\widetilde{\mathfrak{h}} := \widehat{\mathfrak{h}} \oplus \mathbb{C}D.$$

It is a commutative Lie subalgebra of $\widetilde{\mathfrak{g}}$ of dimension $r + 2$, and we have the corresponding triangular decomposition :

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+.$$

Let us define the new simple roots $\alpha_i \in \widetilde{\mathfrak{h}}^*$ for $i = 0, \dots, r$. The action of α_i on $\widehat{\mathfrak{h}}$ has already been defined, and so we only have to specify $\alpha_i(D)$ for $i = 0, \dots, r$. From the relations

$$\alpha_i(D)E_i = [D, E_i] = [D, e_i] = 0, \qquad i = 1, \dots, r,$$

we deduce that $\alpha_i(D) = 0$ for $i = 1, \dots, r$. From the relation

$$\alpha_0(D)E_0 = [D, E_0] = [D, e_0 t] = E_0,$$

we deduce that $\alpha_0(D) = 1$.

1.3.3. Root system. The bilinear form $(\,|\,)$ extends from $\mathfrak{g}$ to a symmetric bilinear form on $\widetilde{\mathfrak{g}}$ by setting for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$:

$$(xt^m|yt^n) = \delta_{m+n,0}(x|y), \qquad (\mathcal{L}\mathfrak{g}|\mathbb{C}K \oplus \mathbb{C}D) = 0,$$
$$(K|K) = (D|D) = 0, \qquad (K|D) = 1.$$

Since the restriction of the bilinear form $(\,|\,)$ to $\widetilde{\mathfrak{h}}$ is nondegenerate, we can identify $\widetilde{\mathfrak{h}}^*$ with $\widetilde{\mathfrak{h}}$ using this form. Through this identification, $\alpha_0 = K - \theta$. For $\alpha \in \widetilde{\mathfrak{h}}^*$ such that $(\alpha|\alpha) \neq 0$, we set $\alpha^\vee = \dfrac{2\alpha}{(\alpha|\alpha)}$. Note that $\alpha^\vee$ obviously corresponds to $\alpha_i^\vee = h_i$ for $\alpha = \alpha_i$, $i = 1, \ldots, r$.

The set of roots $\widehat{\Delta}$ of $\widetilde{\mathfrak{g}}$ with basis $\widehat{\Pi} := \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ is

$$\widehat{\Delta} = \widehat{\Delta}^{\mathrm{re}} \cup \widehat{\Delta}^{\mathrm{im}},$$

where the set of *real roots* is

$$\widehat{\Delta}^{\mathrm{re}} := \{\alpha + nK \mid \alpha \in \Delta,\, n \in \mathbb{Z}\},$$

and the set of *imaginary roots* is

$$\widehat{\Delta}^{\mathrm{im}} := \{nK \mid n \in \mathbb{Z},\, n \neq 0\}.$$

Then we set $\widehat{\Delta}^\vee := \widehat{\Delta}^{\vee,\mathrm{re}} \cup \widehat{\Delta}^{\vee,\mathrm{im}}$, with

$$\widehat{\Delta}^{\vee,\mathrm{re}} := \{\alpha^\vee \mid \alpha \in \widehat{\Delta}^{\mathrm{re}}\}, \qquad \widehat{\Delta}^{\vee,\mathrm{im}} := \{\alpha^\vee \mid \alpha \in \widehat{\Delta}^{\mathrm{im}}\}.$$

The positive integers

$$h := (\rho^\vee|\theta) + 1 \quad \text{and} \quad h^\vee = (\rho|\theta^\vee) + 1$$

are called the *Coxeter number* and the *dual Coxeter number* of $\mathfrak{g}$ respectively, where ρ (resp. $\rho^\vee$) is as usual the half sum of positive roots (resp. coroots), that is, it is defined by $(\rho|\alpha_i^\vee) = 1$ (resp. $(\rho^\vee|\alpha_i) = 1$), for $i = 1, \ldots, r$. Defining $\widehat{\rho} := h^\vee D + \rho \in \widetilde{\mathfrak{h}}$ and $\widehat{\rho}^\vee := hD + \rho^\vee \in \widetilde{\mathfrak{h}}$ we have the following formulas: $(\widehat{\rho}|\alpha_i^\vee) = 1$ and $(\widehat{\rho}^\vee|\alpha_i) = 1$, for $i = 0, \ldots, r$.

1.4. Representations of affine Kac-Moody algebras, category $\mathcal{O}$

We extend some notations and definitions of Section 1.1 to $\widetilde{\mathfrak{g}}$. For example, for M a $\widetilde{\mathfrak{g}}$-module and $\lambda \in \widetilde{\mathfrak{h}}^*$, we set

$$M_\lambda := \{m \in M \mid xm = \lambda(x)m \text{ for all } x \in \widetilde{\mathfrak{h}}^*\}.$$

The space M_λ is called the *weight space of weight λ of M*. The set of weights of M is

$$\mathrm{wt}(M) := \{\lambda \in \widetilde{\mathfrak{h}}^* \mid M_\lambda \neq 0\}.$$

The partial order $\preccurlyeq$ is extended to $\widetilde{\mathfrak{h}}^*$ as follows: we write $\mu \preccurlyeq \lambda$ if $\lambda - \mu = \sum_{i=0}^r m_i\alpha_i$ with $m_i \in \mathbb{Z}$, $m_i \geqslant 0$. For $\lambda \in \widetilde{\mathfrak{h}}^*$, we set $D(\lambda) := \{\mu \in \widetilde{\mathfrak{h}}^* \mid \mu \preccurlyeq \lambda\}$.

1.4.1. The category $\mathcal{O}$. Let $U(\widehat{\mathfrak{g}})$ and $U(\widetilde{\mathfrak{g}})$ be the enveloping algebras of $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ respectively. Let $U(\widetilde{\mathfrak{g}})$-Mod be the category of left $U(\widetilde{\mathfrak{g}})$-modules.

DEFINITION 1.11. The *catgeory* $\mathcal{O}$ is defined to be the full subcategory of $U(\widetilde{\mathfrak{g}})$-Mod whose objects are the modules M satisfying the following conditions:

($\mathcal{O}1$) M is $\widetilde{\mathfrak{h}}$-diagonalizable, that is, $M = \oplus_{\lambda \in \widetilde{\mathfrak{h}}^*} M_\lambda$,
($\mathcal{O}2$) all weight spaces of M are finite dimensional,
($\mathcal{O}3$) there exists a finite number of $\lambda_1, \dots, \lambda_s \in \widetilde{\mathfrak{h}}^*$ such that

$$\mathrm{wt}(M) \subset \bigcup_{1 \leqslant i \leqslant s} D(\lambda_i).$$

The category $\mathcal{O}$ is stable by submodules and quotients. For M_1, M_2 two representations of $\widetilde{\mathfrak{g}}$ we can define a structure of $\widetilde{\mathfrak{g}}$-module on $M_1 \otimes M_2$ by using the coproduct $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$, $x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in \widetilde{\mathfrak{g}}$. Then if M_1 and M_2 are objects of $\mathcal{O}$, then so are $M_1 \oplus M_2$ and $M_1 \otimes M_2$.

EXERCISE 1.12. Check the last assertion.

1.4.2. Verma modules. We now give important examples of modules in the category $\mathcal{O}$. For $\lambda \in \widetilde{\mathfrak{h}}^*$, set:

$$K(\lambda) := U(\widetilde{\mathfrak{g}})\widehat{\mathfrak{n}}_+ + \sum_{x \in \widetilde{\mathfrak{h}}^*} U(\widetilde{\mathfrak{g}})(x - \lambda(x)) \subset U(\widetilde{\mathfrak{g}}).$$

As it is a left ideal of $U(\widetilde{\mathfrak{g}})$,

$$M(\lambda) := U(\widetilde{\mathfrak{g}})/K(\lambda)$$

has a natural structure of a left $U(\widetilde{\mathfrak{g}})$-module. It is called a *Verma module*.

PROPOSITION 1.13. *The $U(\widetilde{\mathfrak{g}})$-module $M(\lambda)$ is in the category $\mathcal{O}$ and has a unique proper submodule $N(\lambda)$.*

We construct $N(\lambda)$ in the same way as $N_{\mathfrak{g}}(\lambda)$ for $\mathfrak{g}$ (see §1.1.3).

As a consequence of the proposition, $M(\lambda)$ has a unique simple quotient

$$L(\lambda) := M(\lambda)/N(\lambda).$$

PROPOSITION 1.14. *The simple module $L(\lambda)$ is in the category $\mathcal{O}$ and all simple modules of the category $\mathcal{O}$ are of the form $L(\lambda)$ for some $\lambda \in \widetilde{\mathfrak{h}}^*$.*

The *character* of a module M in the category $\mathcal{O}$ is by definition

$$\mathrm{ch}(M) = \sum_{\lambda \in \widetilde{\mathfrak{h}}^*} (\dim M_\lambda) e(\lambda)$$

where the $e(\lambda)$ are formal elements.

In general a representation M in $\mathcal{O}$ does not have a finite composition series. However, the multiplicity $[M : L(\lambda)]$ of $L(\lambda)$ in M makes sense ([**KK79**]). As a consequence, we have

$$\mathrm{ch}\, M = \sum_{\lambda} [M : L(\lambda)]\, \mathrm{ch}\, L(\lambda), \quad [M : L(\lambda)] \in \mathbb{Z}_{\geqslant 0},$$

1.5. Integrable and admissible representations

1.5.1. Integrable representations. The representation $L(\lambda)$, $\lambda \in \widetilde{\mathfrak{h}}^*$, is finite dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation. The notion of finite dimensional representations has to be replaced by the notion of *integrable representations* in the category $\mathcal{O}$.

Definition 1.15. A representation M of $\widetilde{\mathfrak{g}}$ is said to be *integrable* if

(1) M is $\widetilde{\mathfrak{h}}$-diagonalizable,
(2) for $\lambda \in \widetilde{\mathfrak{h}}^*$, M_λ is finite dimensional,
(3) for all $\lambda \in \mathrm{wt}(M)$, for all $i = 0, \dots, r$, there is $N \geqslant 0$ such that for $m \geqslant N$, $\lambda + m\alpha_i \notin \mathrm{wt}(M)$ and $\lambda - m\alpha_i \notin \mathrm{wt}(M)$.

As a $\mathfrak{a}_i$-module, $i = 0, \dots, r$, an integrable representation M decomposes into a direct sum of finite dimensional irreducible $\widetilde{\mathfrak{h}}$-invariant modules, where $\mathfrak{a}_i \cong \mathfrak{sl}_2$ is the Lie algebra generated by E_i, F_i, H_i. Hence the action of $\mathfrak{a}_i$ on M can be "integrated" to the action of the group $SL_2(\mathbb{C})$.

The character of the simple integrable representations in the category $\mathcal{O}$ satisfy remarkable combinatorial identities (related to MacDonald identities).

1.5.2. Level of a representation. According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation L. As the Schur Lemma extends to a representation with countable dimension, the result holds for highest weight $\widetilde{\mathfrak{g}}$-modules. In particular, $K \in \widetilde{\mathfrak{g}}$ acts as a scalar $k \in \mathbb{C}$ on the simple representations of the category $\mathcal{O}$.

Definition 1.16. A representation M is said to be *of level* k if K acts as $k\mathrm{Id}$ on M.

All simple representations of the category $\mathcal{O}$ have a level. Namely, $L(\lambda)$ has level $k = \lambda(K) \in \mathbb{C}$, and so $k = \mu(K)$ for all $\mu \in \mathrm{wt}(L(\lambda))$. Note that

$$k = \lambda(K) = \sum_{i=0}^{r} a_i \lambda(\alpha_i^\vee)$$

where the a_i are defined by $K = \sum_{i=0}^{r} a_i \alpha_i^\vee$.

Lemma 1.17. *The simple representation* $L(\lambda)$ *is integrable if and only if* λ *is dominant and integrable, that is,* $\lambda(H_i) \in \mathbb{Z}_{\geqslant 0}$ *for all* $i = 0, \dots, r$*. It has level 0 if and only if* $\dim L(\lambda) = 1$.

Recall that $\widetilde{\mathfrak{h}}^*$ is identified with $\widetilde{\mathfrak{h}}$ through $(\mid)$, and that through this identification the dual of K is D. Then, as a particular case of Lemma 1.17, $L(kD)$ is integrable if and only if $k \in \mathbb{Z}_{\geqslant 0}$.

The category of modules of the category $\mathcal{O}$ of level k will be denoted by $\mathcal{O}_k$ ([**Kac74**]).

The level $k = -h^\vee$ is particular since the center of $\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}(K-k)$ is large and the representation theory changes drastically at this level. This level is called the *critical level.* It is of particular importance for applications to Conformal Field Theory and the Geometric Langlands Program.

Unless the category $\mathcal{O}$ is stable by tensor product, the category $\mathcal{O}_k$ is not stable by tensor product (except for $k = 0$). Indeed from the coproduct, we get that for M_1, M_2 representations in $\mathcal{O}_{k_1}$, $\mathcal{O}_{k_2}$ respectively, the module $M_1 \otimes M_2$ is in $\mathcal{O}_{k_1+k_2}$.

This is one motivation to study the *fusion product*; see [**Bakalov-Kirillov01**], [**Hernandez-lectures**, Section 5] for more details on this topic.

1.5.3. Admissible representations. We now introduce a class of representations, called *admissible representations*, which includes the class of integrable representations. The definition goes back to Kac and Wakimoto [**Kac-Wakimoto89**]. While the notion of integrable representations has a geometrical meaning, the notion of admissible representations is purely combinatorial. However, conjecturally, admissible representations are precisely the representations which satisfy a certain modular invariant property (see below).

Recall the definition of the affine and extended affine Weyl groups (see e.g., [**Kac-Wakimoto08**]). Let $\mathbf{W}$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and extend it to $\widehat{\mathfrak{h}}$ by setting $w(K) = K$, $w(D) = D$ for all $w \in \mathbf{W}$. Let $Q^\vee = \sum_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}$, define the translation ([**Kac1**]),

$$t_\alpha(v) = v + (v|K)\alpha - \left(\frac{1}{2}|\alpha|^2(v|K) + (v|\alpha)\right) K, \qquad v \in \widehat{\mathfrak{h}},$$

and for a subset $L \subset \mathfrak{h}$, let

$$t_L := \{t_\alpha \mid \alpha \in L\}.$$

The *affine Weyl groups* $\widehat{\mathbf{W}}$ and the *extended affine Weyl group* $\widetilde{\mathbf{W}}$ are then defined by:

$$\widehat{\mathbf{W}} := \mathbf{W} \ltimes t_{Q^\vee}, \qquad \widetilde{\mathbf{W}} := \mathbf{W} \ltimes t_{P^\vee},$$

so that $\widehat{\mathbf{W}} \subset \widetilde{\mathbf{W}}$. Here $P^\vee = \{\lambda \in \mathfrak{h} \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in Q\}$, with $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ the root lattice.

The group $\widetilde{\mathbf{W}}_+ := \{w \in \widetilde{\mathbf{W}} \mid w(\widehat{\Pi}^\vee) = \widehat{\Pi}^\vee\}$ acts transitively on orbits of $\operatorname{Aut} \widehat{\Pi}^\vee$ and simply transitively acts on the orbit of $\alpha_0^\vee$. Moreover $\widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}_+ \ltimes \widehat{\mathbf{W}}$. Here, $\widehat{\Pi}^\vee := \{\alpha^\vee \mid \alpha \in \widehat{\Pi}\}$.

Definition 1.18 ([**Kac-Wakimoto89, Kac-Wakimoto08**]). A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if

(1) λ is *regular dominant*, that is,

$$\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geqslant 0} \quad \text{for all} \quad \alpha \in \widehat{\Delta}_+^{re},$$

(2) the $\mathbb{Q}$-span of $\widehat{\Delta}_\lambda$ contains $\widehat{\Delta}^{re}$ where $\widehat{\Delta}_\lambda := \{\alpha \in \widehat{\Delta}^{re} \mid (\lambda|\alpha^\vee) \in \mathbb{Z}\}$.

The irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. Note that an irreducible integrable representation of $\widehat{\mathfrak{g}}$ is admissible.

Proposition 1.19 ([**Kac-Wakimoto08**, Proposition 1.2]). *For $k \in \mathbb{C}$, the weight $\lambda = kD$ is admissible if and only if k satisfies one of the following conditions:*

(1) $k = -h^\vee + \dfrac{p}{q}$ *where $p, q \in \mathbb{Z}_{>0}$, $(p,q) = 1$, and $p \geqslant h^\vee$,*

(2) $k = -h^\vee + \dfrac{p}{r^\vee q}$ *where $p, q \in \mathbb{Z}_{>0}$, $(p,q) = 1$, $(p, r^\vee) = 1$ and $p \geqslant h$.*

Here $r^\vee$ is the lacety *of $\mathfrak{g}$ (i.e., $r^\vee = 1$ for the types A, D, E, $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), h and $h^\vee$ are the Coxeter and dual Coxeter numbers.*

DEFINITION 1.20. If k satisfies one of the conditions of Proposition 1.19, we say that k is an *admissible level*.

For an admissible representation $L(\lambda)$ we have [**Kac-Wakimoto88**]

$$\mathrm{ch}(L(\lambda)) = \sum_{w \in \widehat{W}(\lambda)} (-1)^{\ell_\lambda(w)} \mathrm{ch}(M(w \circ \lambda)) \tag{27}$$

since λ is regular dominant, where $\widehat{\mathbf{W}}(\lambda)$ is the *integral Weyl group* ([**Kashiwara-Tanisaki98, Moody-Pianzola**]) of λ, that is, the subgroup of $\widehat{\mathbf{W}}$ generated by the reflections s_α associated with $\alpha \in \widehat{\Delta}_\lambda$, $w \circ \lambda = w(\lambda + \rho) - \rho$ and ℓ_λ is the length function of the Coxeter group $\widehat{\mathbf{W}}(\lambda)$. Further, Condition (2) of Proposition 1.19 implies that $\mathrm{ch}(L(\lambda))$ is written in terms of certain *theta functions* [**Kac1**, Ch 13]. Kac and Wakimoto [**Kac-Wakimoto89**] showed that admissible representations are *modular invariant*, that is, the characters of admissible representations form an $SL_2(\mathbb{Z})$ invariant subspace.

Let λ, μ be distinct admissible weights. Then Condition (1) of Proposition 1.19 implies that

$$\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\mu)) = 0.$$

Further, the following fact is known by Gorelik and Kac [**Gorelik-Kac11**].

THEOREM 1.21 ([**Gorelik-Kac11**]). *Let λ be admissible. Then* $\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\lambda)) = 0$.

Therefore admissible representations form a semisimple full subcategory of the category of $\hat{\mathfrak{g}}$-modules.

PART 2

Vertex algebras and Zhu functors, the canonical filtration and Zhu's C_2-algebras

Our main references for this part are [**Frenkel-BenZvi**, **Kac2**].

2.1. Definition of vertex algebras, first properties

A *field* on a vector space V is a formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in \operatorname{End} V[[z, z^{-1}]]$$

such that for any $v \in V$, $a_{(n)}v = 0$ for large enough n. Denote by $\mathscr{F}(V)$ the space of all fields on V.

2.1.1. Definition. A *vertex algebra* is a vector space V equipped with the following data:

- (*the vertex operators*) a linear map
$$Y(?, z) \colon V \to \mathscr{F}(V), \quad a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$
- (*the vacuum vector*) a vector $|0\rangle \in V$,
- (*the translation operator*) a linear map $T \colon V \to V$.

These data are subject to the following axioms:

- (*the vacuum axiom*) $|0\rangle(z) = \mathrm{id}_V$. Furthermore, for all $a \in V$,
$$a(z)|0\rangle \in V[[z]]$$
and $\lim_{z \to 0} a(z)|0\rangle = a$. In other words, $a_{(n)}|0\rangle = 0$ for $n \geqslant 0$ and $a_{(-1)}|0\rangle = a$.
- (*the translation axiom*) for any $a \in V$,
$$[T, a(z)] = \partial_z a(z),$$
and $T|0\rangle = 0$.
- (*the locality axiom*) for all $a, b \in V$, $(z - w)^{N_{a,b}}[a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.

The locality axiom is equivalent to the fact that

$$[a(z), b(w)] = \sum_{n=0}^{N_{a,b}-1} (a_{(n)}b)(w) \frac{1}{n!} \partial_w^n \delta(z - w), \tag{28}$$

where $\delta(z - w) := \sum_{n \in \mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z, w, z^{-1}, w^{-1}]]$. Note that the translation axiom says that

$$[T, a_{(n)}] = -n a_{(n-1)}, \qquad n \in \mathbb{Z},$$

and together with the vacuum axiom we get that

$$Ta = a_{(-2)}|0\rangle.$$

2.1.2. Goddard's uniqueness theorem. The Goddard uniqueness theorem is important since it says that one can reconstruct a vertex algebra from the knowledge of how it acts in the vaccum vector.

THEOREM 2.1. *Let V be a vertex algebra, and $\widetilde{Y}(?, z)\colon V \to \mathscr{F}(V)$ a (new) field on V. Suppose there exists $a \in V$ such that*

$$\widetilde{Y}(a, z)|0\rangle = Y(a, z)|0\rangle$$

and $\widetilde{Y}(a, z)$ is local with respect to $Y(b, z)$, that is, $(z-w)^N[\widetilde{Y}(a, z), Y(b, z)] = 0$ for $N \gg 0$, for any $b \in V$. Then $\widetilde{Y}(a, z) = Y(a, z)$.

EXERCISE 2.2. Using the Goddard uniqueness theorem, verify that for all $a \in V$,

$$Y(Ta, z) = \partial_z Y(a, z). \tag{29}$$

Note that the above property is different from the translation axiom.

EXERCISE 2.3. Using the Goddard uniqueness theorem, verify that for all $a, b \in V$,

$$Y(a, z)b = e^{zT}Y(b, -z)a. \tag{30}$$

2.1.3. Borcherds identities and λ-bracket. A consequence of the definition is the following relations, called *Borcherds identities*:

$$[a_{(m)}, b_{(n)}] = \sum_{i \geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \tag{31}$$

$$(a_{(m)}b)_{(n)} = \sum_{j \geqslant 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}), \tag{32}$$

for $m, n \in \mathbb{Z}$. In the above formulas, the notation $\binom{m}{i}$ for $i \geqslant 0$ and $m \in \mathbb{Z}$ means

$$\binom{m}{i} = \frac{m(m-1) \times \cdots \times (m-i+1)}{i(i-1) \times \cdots \times 1}.$$

The vertex algebra V is also endowed with a λ-*bracket* structure. Set for $a, b \in V$:

$$[a_\lambda b] = \mathrm{Res}_{z=0} e^{\lambda z} Y(a, z)b = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)}b \in V[\lambda].$$

The λ-bracket satisfies properties similar to the axioms of a Lie algebra:

$$[(Ta)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda (Tb)] = (\lambda + T)[a_\lambda b], \tag{33}$$

$$[b_\lambda a] = -[a_{-\lambda - T} b], \tag{34}$$

$$[a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \tag{35}$$

In the above, we have extended the λ-bracket $(a, b) \mapsto [a_\lambda b]$ to $V[\lambda] \times V[\lambda]$ by bilinearity.

2.1.4. Normally ordered product. The *normally ordered product* on V is defined as $:ab:= a_{(-1)}b$. We also write $:ab:(z) =: a(z)b(z):$. We have

$$:a(z)b(z): = a(z)_+ b(z) + b(z)a(z)_-,$$

where

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}, \qquad a(z)_- = \sum_{n\geqslant 0} a_{(n)} z^{-n-1}.$$

We have the following *non-commutative Wick formulas*:

$$[a_\lambda :bc:] =: [a_\lambda b]c: + :[a_\lambda c]b: + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu, \tag{36}$$

$$[:ab:_\lambda c] =: (e^{T\partial_\lambda} a)[b_\lambda c]: + :(e^{T\partial_\lambda} b)[a_\lambda c]: + \int_0^\lambda [b_\mu [a_{\lambda-\mu} c]] d\mu. \tag{37}$$

2.2. First examples of vertex algebras

2.2.1. Commutative vertex algebras. A vertex algebra V is called *commutative* if all vertex operators $Y(a,z)$, $a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0, \qquad \forall a,b \in \mathbb{Z},\ m,n \in \mathbb{Z}$$

by (31). It is also equivalent to that

$$[a_\lambda b] = 0, \qquad \forall a,b \in V,$$

or else that, $a_{(n)} = 0$ for $n \geqslant 0$ in $\operatorname{End} V$ for all $a \in V$.

Hence if V is a commutative vertex algebra, then $a(z) \in \operatorname{End} V[[z]]$ for all $a \in V$. Then a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b =: ab := a_{(-1)} b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a\cdot b) = (Ta)\cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation. Conversely, there is a unique vertex algebra structure on a differential algebra R with derivation ∂ such that:

$$Y(a,z)b = \left(e^{z\partial} a\right) b = \sum_{n\geqslant 0} \frac{z^n}{n!} (\partial^n a) b,$$

for all $a, b \in R$. We take the unit as the vacuum vector. This correspondence gives the following result.

Theorem 2.4 ([**Borcherds86**]). *The category of commutative vertex algebras is the same as that of differential algebras.*

2.2.2. Universal affine vertex algebras. Let $\mathfrak{a}$ be a Lie algebra endowed with a symmetric invariant bilinear form κ. Let

$$\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

be the *Kac-Moody affinization* of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \quad x, y \in \mathfrak{a},\ m, n \in \mathbb{Z}, \qquad [\mathbf{1}, \widehat{\mathfrak{a}}] = 0.$$

Let

$$V^\kappa(\mathfrak{a}) = U(\widehat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C}, \tag{38}$$

where $\mathbb{C}$ is a one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity. By the PBW Theorem, we have the following isomorphism of vector spaces:

$$V^\kappa(\mathfrak{a}) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

The space $V^\kappa(\mathfrak{a})$ is naturally graded: $V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta\in\mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta$, where the grading is defined by setting $\deg(xt^n) = -n$, $\deg|0\rangle = 0$. Here $|0\rangle = 1 \otimes 1$. We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C}|0\rangle$. We identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

There is a unique vertex algebra structure on $V^\kappa(\mathfrak{a})$, such that $|0\rangle$ is the vacuum vector and

$$Y(x, z) = x(z) := \sum_{n\in\mathbb{Z}} (xt^n)z^{-n-1}, \quad x \in \mathfrak{a}.$$

(So $x_{(n)} = xt^n$ for $x \in \mathfrak{a} = V^\kappa(\mathfrak{a})_1$, $n \in \mathbb{Z}$). The vertex algebra $V^\kappa(\mathfrak{a})$ is called the *universal affine vertex algebra associated with* $(\mathfrak{a}, \kappa)$.

Let us describe the vertex algebra structure in more details. Set

$$x_{(n)} = xt^n, \qquad \forall\, x \in \mathfrak{a},\ n \in \mathbb{Z},$$

and let $|0\rangle$ be the image of $1 \otimes 1 \in U(\hat{\mathfrak{a}}) \otimes \mathbb{C}$ in $V^\kappa(\mathfrak{a})$. Let $(x^i\,; i = 1\dots, \dim \mathfrak{a})$ be an ordered basis of $\mathfrak{a}$. By the PBW Theorem, $V^\kappa(\mathfrak{a})$ has a basis of the form

$$x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle,$$

where $n_1 \leqslant n_2 \leqslant \dots \leqslant n_m < 0$, and if $n_j = n_{j+1}$, then $i_j \leqslant i_{j+1}$.

Then $(V^\kappa(\mathfrak{a}), |0\rangle, T, Y)$ is a vertex algebra where the translation operator T is given by

$$T|0\rangle = 0, \qquad [T, x^i_{(n)}] = -nx^i_{(n-1)},$$

for $n \in \mathbb{Z}$, and the vertex operators $Y(?, z)$ are given by:

$$Y(|0\rangle, z) = \mathrm{Id}_{V^\kappa(\mathfrak{a})}, \qquad Y(x^i_{-1}|0\rangle, z) = x^i(z) = \sum_{n\in\mathbb{Z}} x^i_{(n)}z^{-n-1},$$

$$Y(x^{i_1}_{(n_1)} \dots x^{i_m}_{(n_m)}|0\rangle, z) = \frac{1}{(-n_1 - 1)! \dots (-n_m - 1)!} : \partial_z^{-n_1-1}x^{i_1}(z) \dots \partial_z^{-n_m-1}x^{i_m}(z) :$$

When $\mathfrak{a} = \mathfrak{g}$ is the simple Lie algebra as in Part 1, so that $\widehat{\mathfrak{a}} = \widehat{\mathfrak{g}}$ is the affine Kac-Moody algebra as in Section 1.2, and and

$$\kappa = k(\ |\ ) = \frac{k}{2h^\vee} \times \kappa_{\mathfrak{g}}, \quad \text{for } k \in \mathbb{C},$$

with $\kappa_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$, then we write $V^k(\mathfrak{g})$ for the universal affine vertex algebra vertex algebra $V^\kappa(\mathfrak{a})$. We call it the *universal vacuum representation of* *level* k of $\widehat{\mathfrak{g}}$. By what foregoes, $V^k(\mathfrak{g})$ has a natural vertex algebra structure, and it is called the *universal affine vertex algebra associated with* $\widehat{\mathfrak{g}}$ *of level* k.

2.2.3. The Virasoro vertex algebra. Let $Vir = \mathbb{C}((t))\partial_t \oplus \mathbb{C}C$ be the *Virasoro Lie algebra*, with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}C, \tag{39}$$

$$[C, Vir] = 0, \tag{40}$$

where $L_n := -t^{n+1}\partial_t$ for $n \in \mathbb{Z}$.

Let $c \in \mathbb{C}$ and define the induced representation

$$\mathrm{Vir}^c = \mathrm{Ind}^{Vir}_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c = U(Vir) \otimes_{\mathbb{C}[[t]]\partial_t \oplus \mathbb{C}C} \mathbb{C}_c,$$

where C acts as multiplication by c and $\mathbb{C}[[t]]\partial_t$ acts by 0 on the 1-dimensional module $\mathbb{C}_c$.

By the PBW Theorem, Vir^c has a basis of the form

$$L_{j_1} \dots L_{j_m}|0\rangle, \qquad j_1 \leqslant \dots \leqslant j_l \leqslant -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c. Then $(\mathrm{Vir}^c, |0\rangle, T, Y)$ is a vertex algebra, called the *universal Virasoro vertex algebra with central charge* c, such that $T = L_{-1}$ and:

$$Y(|0\rangle, z) = \mathrm{Id}_{\mathrm{Virc}_c}, \qquad Y(L_{-2}|0\rangle, z) =: L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

$$Y(L_{j_1} \dots L_{j_m}|0\rangle, z)$$
$$= \frac{1}{(-j_1-2)! \dots (-j_m-2)!} : \partial_z^{-j_1-2}T(z) \dots \partial_z^{-j_m-2}T(z) :$$

Moreover, Vir^c is $\mathbb{Z}_{\geqslant 0}$-graded by $\deg|0\rangle = 0$ and $\deg L_n|0\rangle = -n$.

2.2.4. Conformal vertex algebras. A *Hamiltonian* of V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)}$$

for all $a \in V$, $n \in \mathbb{Z}$.

DEFINITION 2.5. A vertex algebra equipped with a Hamiltonian H is called *graded*. Let $V_\Delta = \{a \in V \mid Ha = \Delta a\}$ for $\Delta \in \mathbb{C}$, so that $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$. For $a \in V_\Delta$, Δ is called the *conformal weight* of a and it is denoted by Δ_a. We have

$$a_{(n)}b \in V_{\Delta_a + \Delta_b - n - 1} \tag{41}$$

for homogeneous elements $a, b \in V$.

For example, the universal affine vertex algebra $V^k(\mathfrak{g})$ is $\mathbb{Z}_{\geqslant 0}$-graded (that is, $V^k(\mathfrak{g})_\Delta = 0$ for $\Delta \notin \mathbb{Z}_{\geqslant 0}$) and the Hamiltonian is given by $H = -D$.

DEFINITION 2.6. A graded vertex algebra $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$ is called *conformal* of central charge $c \in \mathbb{C}$ if there is a *conformal vector* $\omega \in V_2$ such that the *Fourier coefficients* L_n of the corresponding vertex operators

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfy the defining relations (39) of the Virasoro algebra with central charge c, and if in addition we have

$$\omega_{(0)} = L_{-1} = T,$$
$$\omega_{(1)} = L_0 = H \quad \text{i.e.,} \quad L_0|_{V_\Delta} = \Delta \mathrm{Id}_{V_\Delta} \quad \forall\, \Delta \in \mathbb{Z}.$$

A $\mathbb{Z}$-graded conformal vertex algebra is also called a *vertex operator algebra*.

EXAMPLE 2.7. The Virasoro vertex algebra Vir^c is clearly conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

EXAMPLE 2.8. The universal affine vertex algebra $V^k(\mathfrak{g})$ has a natural conformal vector, provided that $k \neq -h^\vee$. Set

$$S = \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{g}} x_{i,-1}x^i_{(-1)}|0\rangle,$$

where $(x_i;\, i = 1, \ldots, \dim \mathfrak{g})$ is the dual basis of $(x^i;\, i = 1, \ldots, \dim \mathfrak{g})$ with respect to the bilinear form $(\ |\ )$, and

$$x^i(z) = \sum_{n\in\mathbb{Z}} x^i_{(n)} z^{-n-1}, \qquad x_i(z) = \sum_{n\in\mathbb{Z}} x_{i,(n)} z^{-n-1}.$$

Then for $k \neq -h^\vee$, $L = \dfrac{S}{k + h^\vee}$ is a conformal vector of $V^k(\mathfrak{g})$, called the *Segal-Sugawara vector*, with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

We have

$$[L_m, x_{(n)}] = (m-n)x_{(m+n)} \quad x \in \mathfrak{g},\ m.n \in \mathbb{Z}. \tag{42}$$

2.3. Modules over vertex algebras

2.3.1. Definition. A *module* over the vertex algebra V is a vector space M together with a linear map

$$Y^M(?, z)\colon V \to \mathscr{F}(M), \quad a \mapsto a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1},$$

which satisfies the following axioms:

$$|0\rangle(z) = \mathrm{Id}_M, \tag{43}$$

$$Y^M(Ta, z) = \partial_z Y^M(a, z), \tag{44}$$

$$\sum_{j\leqslant 0} \binom{m}{j} (a_{(n+j)}b)^M_{(m+k-j)} = \sum_{j\geqslant 0} (-1)^j \binom{n}{j} (a^M_{(m+n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)} a^M_{(m+j)}). \tag{45}$$

Notice that (45) is equivalent to (31) and (32) for $M = V$.

Suppose in addition V is graded (cf. Definition 2.5). A V-module M is called *graded* if there is a compatible semisimple action of H on M, that is, $M = \bigoplus_{d\in\mathbb{C}} M_d$, where $M_d = \{m \in M \mid Hm = dm\}$ and $[H, a^M_{(n)}] = -(n+1)a^M_{(n)} + (Ha)^M_{(n)}$ for all $a \in V$. We have

$$a^M_{(n)} M_d \subset M_{d+\Delta_a-n-1} \tag{46}$$

for homogeneous $a \in V$.

When there is no ambiguity, we will often denote by $a_{(n)}$ the element $a^M_{(n)}$ of $\mathrm{End}(M)$.

The axioms imply that V is a module over itself (called the *adjoint module*). We have naturally the notions of *submodules*, *quotient module* and *vertex ideals*. Note that vertex ideals are the same as submodules of adjoint modules by (30). A module whose only submodules are 0 and itself is called *simple*.

2.3.2. Modules of the universal affine vertex algebra. In the case that V is the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with $\widehat{\mathfrak{g}}$ at level $k \in \mathbb{C}$, V-modules play a crucial important role in the representation theory of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$.

A $\widehat{\mathfrak{g}}$-module M of level k is called *smooth* if $x(z)$ is a field on M for $x \in \mathfrak{g}$, that is, for any $m \in M$ there is $N > 0$ such that $(xt^n)m = 0$ for all $x \in \mathfrak{g}$ and $n \geqslant N$. Any $V^k(\mathfrak{g})$-module M is naturally a smooth $\widehat{\mathfrak{g}}$-module of level k. Conversely, any smooth $\widehat{\mathfrak{g}}$-module of level k can be regarded as a $V^k(\mathfrak{g})$-module. It follows that a $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k.

Namely, we have the following.

PROPOSITION 2.9 (See [**Frenkel-BenZvi**, §5.1.18] for a proof). *There is an equivalence of category between the category of $V^k(\mathfrak{g})$-modules and the category of smooth $\widehat{\mathfrak{g}}$-modules of level k.*

REMARK 2.10. Suppose that k is not critical, that is, $k \neq -h^\vee$, so that $V^k(\mathfrak{g})$ is conformal. Then, by (42), any smooth $\widehat{\mathfrak{g}}$-module of level k module can be regarded as a $\widetilde{\mathfrak{g}}$-module by letting D act by $-L_0$. Thus, the representation theory of $\widetilde{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$ are essentially the same. To put it another way, one can define the generalized Casimir operator [**Kac1**] of $\widetilde{\mathfrak{g}}$ as the sum $S_0 + 2(k + h^\vee)D$, where $S(z) = \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$ and S is as in Example 2.8.

The vertex algebra $V^k(\mathfrak{g})$, as a module over itself, has a unique proper graded submodule $N_k(\mathfrak{g})$ (it is a maximal vertex ideal of $V^k(\mathfrak{g})$), and so the quotient

$$V_k(\mathfrak{g}) := V^k(\mathfrak{g})/N_k(\mathfrak{g})$$

is a simple $V^k(\mathfrak{g})$-module, that is, an irreducible $\widehat{\mathfrak{g}}$-representation of level k. Moreover, as a $\widehat{\mathfrak{g}}$-module, it is isomorphic to $L(kD)$,

$$V_k(\mathfrak{g}) \cong L(kD),$$

in the notations of Part 1 (Sections 1.4 and 1.5). Note that D, the dual of the central element K with respect to $(\ |\ )$, is the highest weight of the basic representation of $\widehat{\mathfrak{g}}$ (i.e., obtained for $k = 1$). As a quotient of $V^k(\mathfrak{g})$, $V_k(\mathfrak{g})$ has a natural vertex algebra structure induced from that of $V_k(\mathfrak{g})$.

2.3.3. C_2-cofinite condition. For a V-module M, set

$$C_2(M) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}m \,; a \in V,\, m \in M\}.$$

Then

$$C_2(M) = \mathrm{span}_{\mathbb{C}}\{a_{(-n)}m \,; a \in V,\, m \in M,\, n \geqslant 2\}$$

by the property (44).

A V-module M is called *C_2-cofinite* if $\dim M/C_2(M) < \infty$, and V is called *C_2-cofinite* if it is C_2-cofinite as a module over itself. Such vertex algebras are also called *lisse*. Later we shall see a more geometrical interpetation of the lisse condition (cf. Lemma 4.21) in term of the *associated variety of V* (cf. Definition 4.17).

Lisse vertex algebras may be regarded as an analogue of finite-dimensional algebras. One of remarkable properties of a lisse vertex algebra V is the modular invariance of characters of modules [**Zhu96, Miyamoto04**]. Further, if it is also *rational*, it is known [**Huang08**] that under some mild assumptions, the category of V-modules forms a modular tensor category, which for instance yields an invariant of 3-manifolds, see [**Bakalov-Kirillov01**].

2.3.4. Rational vertex algebras.

DEFINITION 2.11. A conformal vertex algebra V is called *rational* if every $\mathbb{Z}_{\geqslant 0}$-graded V-modules is completely reducible (i.e., isomorphic to a direct sum of simple V-modules).

It is known ([**Dong-Li-Mason98**]) that this condition implies that V has finitely many simple $\mathbb{Z}_{\geqslant 0}$-graded modules and that the graded components of each of these $\mathbb{Z}_{\geqslant 0}$-graded modules are finite dimensional.

In fact lisse vertex algebras also verify this property (see Theorem 2.26). It is actually conjectured by [**Zhu96**] that rational vertex algebras must be lisse (this conjecture is still open).

2.4. The canonical filtration, Zhu's C_2 algebras, and Zhu's functors

2.4.1. The canonical filtration and the Zhu's C_2-algebra. Haisheng Li [**Li05**] has shown that *every* vertex algebra is canonically filtered: For a vertex algebra V, let F^pV be the subspace of V spanned by the elements

$$a^1_{(-n_1-1)}a^2_{(-n_2-1)}\cdots a^r_{(-n_r-1)}|0\rangle$$

with $a^1, a^2, \cdots, a^r \in V$, $n_i \geqslant 0$, $n_1+n_2+\cdots+n_r \geqslant p$. Then

$$V = F^0V \supset F^1V \supset \dots .$$

It is clear that $TF^pV \subset F^{p+1}V$.

Set

$$(F^pV)_{(n)}F^qV := \operatorname{span}_{\mathbb{C}}\{a_{(n)}b\,;\ a \in F^pV,\, b \in F^qV\}.$$

Note that $F^1V = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\} = C_2(V)$.

LEMMA 2.12. *We have*

$$F^pV = \sum_{j\geqslant 0}(F^0V)_{(-j-1)}F^{p-j}V.$$

PROPOSITION 2.13. (1) $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n-1}V$. *Moreover, if* $n \geqslant 0$, *we have* $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n}V$. *Here we have set* $F^pV = V$ *for* $p<0$.

(2) *The filtration* $F^\bullet V$ *is* separated, *that is,* $\bigcap_{p\geqslant 0} F^pV = \{0\}$, *if V is a positive energy representation, i.e., positively graded over itself.*

EXERCISE 2.14. The verifications are straightforward and are left as an exercise.

In this note we always assume that the filtration $F^\bullet V$ is separated.

Set

$$\mathrm{gr}^F V = \bigoplus_{p\geqslant 0} F^p V/F^{p+1}V.$$

We denote by $\sigma_p\colon F^pV \mapsto F^pV/F^{p+1}V$, for $p \geqslant 0$, the canonical quotient map.

Recall that a commutative vertex algebra is the same as a differential algebra (Theorem 2.4). Then Proposition 2.13 gives the following.

PROPOSITION 2.15 ([**Li05**]). *The space* $\mathrm{gr}^F V$ *is a commutative vertex algebra by*

$$\begin{aligned} \sigma_p(a)\cdot\sigma_q(b) &:= \sigma_{p+q}(a_{(-1)}b),\\ T\sigma_p(a) &:= \sigma_{p+1}(Ta), \end{aligned}$$

for $a \in F^pV$, $b \in F^qV$, $n \geqslant 0$.

When the filtration F if obvious, we often denote simply by $\mathrm{gr}\, V$ the space $\mathrm{gr}^F V$.

Set

$$R_V := F^0V/F^1V = V/C_2(V) \subset \mathrm{gr}\, V.$$

DEFINITION 2.16. The algebra R_V is called the *Zhu's C_2-algebra* of V. The algebra structure is given by:

$$\bar{a}\cdot\bar{b} := \overline{a_{(-1)}b}, \tag{47}$$

where $\bar{a} = \sigma_0(a)$.

We will see in Part 4 that $\mathrm{gr}\, V$ is actually a *Poisson vertex algebra* (see Subsection 4.2.1) and that R_V inherits a *Poisson algebra structure* (cf. §3.3.1) from the Poisson vertex algebra structure on V (see Propositions 4.15 and 4.16).

DEFINITION 2.17. A vertex algebra V is called *finitely strongly generated* if there exist finitely many elements $a^1,\dots,a^r$ in V such that V is spanned by the elements of the form

$$a^{i_1}_{(-n_1)}\dots a^{i_s}_{(-n_s)}|0\rangle$$

with $s \geqslant 0$, $n_i \geqslant 1$.

For example, the universal affine vertex algebra and the Virasoro vertex algebra are strongly finitely generated.

From now we always assume that a vertex algebra V is finitely strongly generated.

2.5. The Zhu's algebra

Let V be a $\mathbb{Z}$-graded vertex algebra. The *Zhu's algebra* $\mathrm{Zhu}(V)$ of V ([**Frenkel-Zhu92, Zhu96**]) is defined as

$$\mathrm{Zhu}(V) := V/V\circ V$$

where $V\circ V := \mathrm{span}\{a\circ b\ ;\ a,b\in V\}$, and

$$a\circ b := \sum_{i\geqslant 0}\binom{\Delta_a}{i} a_{(i-2)}b$$

for homogeneous elements a, b, is extended linearly. It is an associative algebra with multiplication defined as

$$a * b := \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} b$$

for homogeneous elements $a, b \in V$.

For a simple positive energy representation $M = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} M_{\lambda+n}$, $M_\lambda \neq 0$, of V, let M_{top} be *the top degree component* M_λ of M. Also, for a homogeneous vector $a \in V$, let $o(a) = a_{(\Delta_a - 1)} = a^M_{(\Delta_a - 1)}$, so that $o(a)$ preserves the homogeneous components of any graded representation of V by (46).

The importance of Zhu's algebra in vertex algebra theory comes from the following fact that was established by Yonchang Zhu.

Theorem 2.18 ([**Zhu96**]). *For any positive energy representation M of V, $[a] \mapsto o(a)$ defines a well-defined representation of $\mathrm{Zhu}(V)$ on M_{top}, where $[a]$ is the image of a in $\mathrm{Zhu}(V)$. Moreover, the correspondence $M \mapsto M_{top}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of V and that of simple $\mathrm{Zhu}(V)$-modules.*

A vertex algebra V is called a *chiralization* of an algebra A if $\mathrm{Zhu}(V) \cong A$.

Now we define an increasing filtration of the Zhu's algebra. For this, we assume that V is $\mathbb{Z}_{\geqslant 0}$-graded, $V = \bigoplus_{\Delta \geqslant 0} V_\Delta$. Then $V_{\leqslant p} := \bigoplus_{\Delta=0}^{p} V_\Delta$ gives an increasing filtration of V. Define

$$\mathrm{Zhu}_p(V) := \mathrm{im}(V_{\leqslant p} \to \mathrm{Zhu}(V)).$$

Obviously, we have

$$0 = \mathrm{Zhu}_{-1}(V) \subset \mathrm{Zhu}_0(V) \subset \mathrm{Zhu}_1(V) \subset \cdots, \quad \text{and} \quad \mathrm{Zhu}(V) = \bigcup_{p \geqslant -1} \mathrm{Zhu}_p(V).$$

Also, since $a_{(n)} b \in V_{\Delta_a + \Delta_b - n - 1}$ for $a \in V_{\Delta_a}$, $b \in V_{\Delta_b}$, we have

$$\mathrm{Zhu}_p(V) * \mathrm{Zhu}_q(V) \subset \mathrm{Zhu}_{p+q}(V). \tag{48}$$

The following assertion follows from the skew symmetry.

Lemma 2.19 ([**Zhu96**]). *We have*

$$b * a \equiv \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)} b \qquad (\mathrm{mod}\ V \circ V),$$

and hence,

$$a * b - b * a \equiv \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i)} b \qquad (\mathrm{mod}\ V \circ V).$$

By Lemma 2.19, we have

$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(V)] \subset \mathrm{Zhu}_{p+q-1}(V). \tag{49}$$

By (48) and (49), the associated graded $\mathrm{gr}\,\mathrm{Zhu}(V) = \bigoplus_p \mathrm{Zhu}_p(V)/\mathrm{Zhu}_{p-1}(V)$ is naturally a graded Poisson algebra.

Note that $a \circ b \equiv a_{(-2)} b \ (\mathrm{mod}\ \bigoplus_{\Delta \leqslant \Delta_a + \Delta_b} V_\Delta)$ for homogeneous elements a, b in V.

LEMMA 2.20 (Zhu; see [**DeSole-Kac06**, Proposition 2.17(c)], [**Arakawa-Lam-Yamada14**, Proposition 3.3]). *The following map defines a well-defined surjective algebra homomorphism:*

$$\eta_V : R_V \longrightarrow \operatorname{gr} \operatorname{Zhu}(V)$$
$$\bar{a} \longmapsto a \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta}.$$

REMARK 2.21. Later we shall see (cf. Lemma 4.42) that η_V is actually a surjective homomorphism of Poisson algebras.

REMARK 2.22. The map η_V is not an isomorphism in general. For example, let $\mathfrak{g}$ be the simple Lie algebra of type E_8 and $V = V_1(\mathfrak{g})$. Then $\dim R_V > \dim \operatorname{Zhu}(V) = 1$.

CONJECTURE 1 ([**Arakawa15b**]). *If V is a simple $\mathbb{Z}_{\geqslant 0}$-graded conformal vertex algebra, then*
$$(X_V)_{\mathrm{red}} \cong \operatorname{Specm}(\operatorname{gr} \operatorname{Zhu} V).$$

REMARK 2.23. One may also ask wether the following diagram is commutative.

$$\begin{tikzcd}
\operatorname{gr}^F V \arrow[d, "\operatorname{Zhu}(?)"] & V \arrow[l, "\operatorname{gr}^F(?)"'] \arrow[d, "\operatorname{Zhu}(?)"] \\
? & \operatorname{Zhu}(V) \arrow[l, "\operatorname{gr}(?)"]
\end{tikzcd}$$

In other words, one may ask wether one has $\operatorname{Zhu}(\operatorname{gr}^F V) \cong \operatorname{gr} \operatorname{Zhu}(V)$. Note that R_V is not isomorphic to $\operatorname{Zhu}(\operatorname{gr}^F V)$ in general since $C_2(V) \neq V \circ V$ even for commutative vertex algebras.

Although the above diagram is known to be commutative in several examples, e.g., the unviversal affine vertex algebra $V^k(\mathfrak{g})$, the fermion Fock space (cf. §**??**), the *W-algebra* $W^k(\mathfrak{g}, f)$ (cf. Section **??**), etc., it is not true in general.

EXERCISE 2.24. Verify that the example in Remark 2.22 provides a counterexample, that is, $\operatorname{Zhu}(\operatorname{gr}^F V) \ncong \operatorname{gr} \operatorname{Zhu}(V)$ in this case.

COROLLARY 2.25. *If V is lisse then $\operatorname{Zhu}(V)$ is finite dimensional. Hence the number of isomorphic classes of simple positive energy representations of V is finite.*

In fact the following stronger facts are known.

THEOREM 2.26 ([**Abe-Buhl-Dong04**]). *Let V be lisse. Then any simple V-module is a positive energy representation. Therefore the number of isomorphic classes of simple V-modules is finite.*

THEOREM 2.27 ([**Dong-Li-Mason98**, **Matsuo-Nagatomo-Tsuchiya10**]). *Le V be lisse. Then the abelian category of V-modules is equivalent to the module category of a finite-dimensional associative algebra.*

To give examples of computations of Zhu's algebras, one needs more ingredients. So this will be done in Part 4.

PART 3

BRST cohomology, quantum Hamiltonian reduction, geometry of nilpotent orbits and finite W-algebras

This part is independent from Part 2, and use only notations of Section 1.1 in Part 1.

It will be important for Part 5. Also, some geometrical aspects on nilpotent orbits and Poisson algebras are useful for Part 4.

Recall that $\mathfrak{g}$ is assumed to be simple.

3.1. Nilpotent orbits and nilpotent elements

Our main references for the results of this section are [**Jantzen**, **Collingwood-McGovern**, **Tauvel-Yu**].

3.1.1. Nilpotent cone. Let $\mathcal{N}$ be the *nilpotent cone of* $\mathfrak{g}$, that is, the set of all nilpotent elements of $\mathfrak{g}$. If $\mathfrak{g}$ is a simple Lie algebra of matrices, note that $\mathcal{N}$ coincides with the set of nilpotent matrices. For $e \in \mathfrak{g}$, we denote by $\mathbf{G}.e$ its adjoint $\mathbf{G}$-orbit. The nilpotent cone is a finite union of nilpotent $\mathbf{G}$-orbits and it is itself the closure of the *regular nilpotent orbit*, denoted by $\mathbb{O}_{reg}$. It is the unique nilpotent orbit of codimension the rank r of $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is *regular* if its centralizer $\mathfrak{g}^x$ has the minimal dimension, that is, the rank r of $\mathfrak{g}$. Thus, $\mathbb{O}_{reg}$ is the set of all regular nilpotent elements of $\mathfrak{g}$.

EXAMPLE 3.1. If $\mathfrak{g} = \mathfrak{sl}_n$, then the rank of $\mathfrak{g}$ is $n-1$ and $\mathbb{O}_{reg}$ is the conjugacy class of the n-size Jordan block J_n, i.e., $\mathbb{O}_{reg} = SL_n.J_n$ with

$$J_n := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \sum_{i=1}^{n-1} e_{i,i+1},$$

where $e_{i,j}$ is the elementary matrix whose entries are all zero, except the one in position (i,j) which equals 1.

Next, there is a unique dense open orbit in $\mathcal{N} \setminus \mathbb{O}_{reg}$ which is called the *subregular nilpotent orbit of* $\mathfrak{g}$, and denoted by $\mathbb{O}_{subreg}$. Its codimension in $\mathfrak{g}$ is the rank of $\mathfrak{g}$ plus two. At the extreme opposite, there is a unique nilpotent orbit of smallest positive dimension called the *minimal nilpotent orbit of* $\mathfrak{g}$, and denoted by $\mathbb{O}_{min}$. Its dimension is $2h^\vee - 2$ ([**Wang99**]).

3.1.2. Chevalley order. The set of nilpotent orbits in $\mathfrak{g}$ is naturally a poset $\mathscr{P}$ with partial order $\leqslant$, called the *Cheveally order*, defined as follows: $\mathbb{O}' \leqslant \mathbb{O}$ if and only if $\mathbb{O}' \subseteq \overline{\mathbb{O}}$. The regular nilpotent orbit $\mathbb{O}_{reg}$ is maximal and the zero orbit is the minimal with respect to this order. Moreover, $\mathbb{O}_{subreg}$ is maximal in the poset $\mathscr{P} \setminus \mathbb{O}_{reg}$ and $\mathbb{O}_{min}$ is minimal in the poset $\mathscr{P} \setminus \{0\}$.

The Chevalley order on $\mathscr{P}$ corresponds to a partial order on the set $\mathscr{P}(n)$ of partitions of n, $n > 1$, for $\mathfrak{g} = \mathfrak{sl}_n$, first described by Gerstenhaber. More generally, the Chevalley order corresponds to a partial order on some subset of $\mathscr{P}(n)$ when $\mathfrak{g}$ is of classical type as we explain below.

Let $n \in \mathbb{Z}_{>0}$. As a rule, unless otherwise specified, we write an element $\boldsymbol{\lambda}$ of $\mathscr{P}(n)$ as a decreasing sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ omitting the zeroes. Thus,

$$\lambda_1 \geqslant \dots \geqslant \lambda_s \geqslant 1 \quad \text{and} \quad \lambda_1 + \dots + \lambda_s = n.$$

We shall denote the dual partition of a partition $\boldsymbol{\lambda} \in \mathscr{P}(n)$ by ${}^t\boldsymbol{\lambda}$.

Let us denote by $\geqslant$ the partial order on $\mathscr{P}(n)$ relative to dominance. More precisely, given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_s), \boldsymbol{\eta} = (\mu_1, \dots, \mu_t) \in \mathscr{P}(n)$, we have $\boldsymbol{\lambda} \ \geqslant \boldsymbol{\eta}$ if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i \quad \text{for} \quad 1 \leqslant k \leqslant \min(s,t).$$

Case $\mathfrak{sl}_n$. Every nilpotent matrix in $\mathfrak{sl}_n$ is conjugate to a Jordan block diagonal matrix. Therefore, the nilpotent orbits in $\mathfrak{g}$ are parameterized by $\mathscr{P}(n)$. We shall denote by $\mathbb{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{sl}_n$. Then $\mathbb{O}_{\boldsymbol{\lambda}}$ is represented by the standard Jordan form $\mathrm{diag}(J_{\lambda_1}, \dots, J_{\lambda_s})$, where J_k is the k-size Jordan block. If we write ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}_{\boldsymbol{\lambda}} = n^2 - \sum_{i=1}^{t} d_i^2.$$

If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}(n)$, then $\mathbb{O}_{\boldsymbol{\eta}} \subset \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

The regular, subregular, minimal and zero nilpotent orbits of $\mathfrak{sl}_n$ correspond to the partitions (n), $(n-1,1)$, $(2, 1^{n-2})$ and (1^n) of n respectively.

We give in Figure 1 the description of the poset $\mathscr{P}(n)$ for $n = 6$. The column on the right indicates the dimension of the orbits appearing in the same row. Such diagram is called a *Hasse diagram*.

Cases $\mathfrak{o}_n$ *and* $\mathfrak{so}_n$. For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_1(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) \ ; \ \text{number of parts of each even number is even}\}.$$

The nilpotent orbits of $\mathfrak{so}_n$ are parametrized by $\mathscr{P}_1(n)$, with the exception that each *very even* partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ (i.e., $\boldsymbol{\lambda}$ has only even parts) corresponds to two nilpotent orbits. For $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, not very even, we shall denote by $\mathbb{O}_{\mathbf{1},\boldsymbol{\lambda}}$, or simply by $\mathbb{O}_{\boldsymbol{\lambda}}$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{so}_n$. For very even $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, we shall denote by $\mathbb{O}^{I}_{1,\boldsymbol{\lambda}}$ and $\mathbb{O}^{II}_{1,\boldsymbol{\lambda}}$ the two corresponding nilpotent orbits of $\mathfrak{so}_n$. In fact, their union forms a single $O(n)$-orbit. Thus nilpotent orbits of $\mathfrak{o}_n$ are parametrized by $\mathscr{P}_1(n)$.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s) \in \mathscr{P}_1(n)$ and ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}^{\bullet}_{1,\boldsymbol{\lambda}} = \frac{n(n-1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{t} d_i^2 - \#\{i; \lambda_i \text{ odd}\}\right),$$

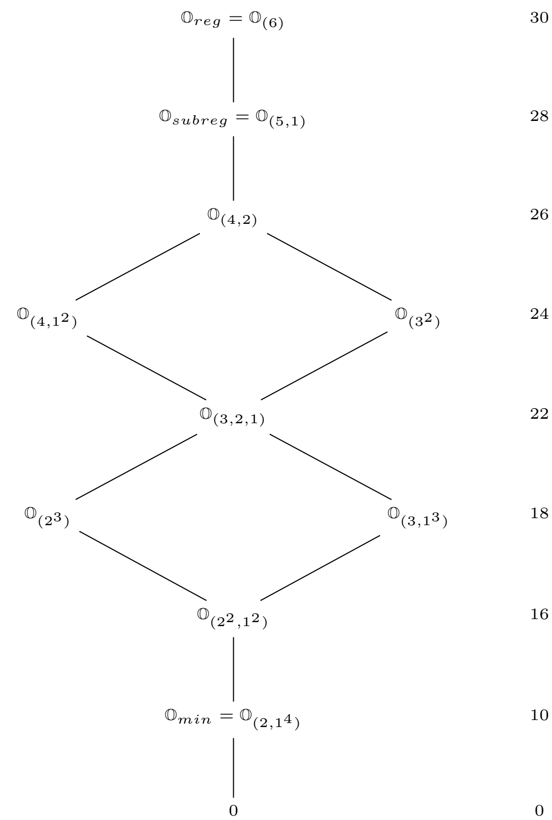


FIGURE 1. Hasse diagram for $\mathfrak{sl}_6$

where $\mathbb{O}^{\bullet}_{1,\boldsymbol{\lambda}}$ is either $\mathbb{O}_{1,\boldsymbol{\lambda}}$, $\mathbb{O}^{I}_{1,\boldsymbol{\lambda}}$ or $\mathbb{O}^{II}_{1,\boldsymbol{\lambda}}$ according to whether $\boldsymbol{\lambda}$ is very even or not. Using the same notations, If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_1(n)$, then $\overline{\mathbb{O}^{\bullet}_{1,\boldsymbol{\eta}}} \subsetneq \overline{\mathbb{O}^{\bullet}_{1,\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\eta} < \boldsymbol{\lambda}$, where $\mathbb{O}^{\bullet}_{1,\boldsymbol{\lambda}}$ is either $\mathbb{O}_{1,\boldsymbol{\lambda}}$, $\mathbb{O}^{I}_{1,\boldsymbol{\lambda}}$ or $\mathbb{O}^{II}_{1,\boldsymbol{\lambda}}$ according to whether $\boldsymbol{\lambda}$ is very even or not.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^+ \in \mathscr{P}_1(n)$ such that $\boldsymbol{\lambda}^+ \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_1(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^+$. More precisely, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (adding zeroes if necessary). If $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, then $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}$. Otherwise if $\boldsymbol{\lambda} \notin \mathscr{P}_1(n)$, set

$$\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_s, \lambda_{s+1} - 1, \lambda_{s+2}, \ldots, \lambda_{t-1}, \lambda_t + 1, \lambda_{t+1}, \ldots, \lambda_n),$$

where s is maximum such that $(\lambda_1, \ldots, \lambda_s) \in \mathscr{P}_1(\lambda_1 + \cdots + \lambda_s)$, and t is the index of the first even part in $(\lambda_{s+2}, \ldots, \lambda_n)$. Note that $s = 0$ if such a maximum does not exist, while t is always defined. If $\boldsymbol{\lambda}'$ is not in $\mathscr{P}_1(n)$, then we repeat the process until we obtain an element of $\mathscr{P}_1(n)$ which will be our $\boldsymbol{\lambda}^+$.

Case $\mathfrak{sp}_n$. For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_{-1}(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) \; ; \; \text{number of parts of each odd number is even}\}.$$

The nilpotent orbits of $\mathfrak{sp}_n$ are parametrized by $\mathscr{P}_{-1}(n)$. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathscr{P}_{-1}(n)$, we shall denote by $\mathbb{O}_{-1,\boldsymbol{\lambda}}$, or simply by $\mathbb{O}_{\boldsymbol{\lambda}}$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{sp}_n$, and if we write ${}^t\boldsymbol{\lambda} = (d_1, \ldots, d_t)$, then

$$\dim \mathbb{O}_{-1,\boldsymbol{\lambda}} = \frac{n(n+1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{s} d_i^2 + \#\{i; \lambda_i \text{ odd}\}\right).$$

As in the case of $\mathfrak{sl}_n$, if $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$, then $\mathbb{O}_{-1,\boldsymbol{\eta}} \subset \overline{\mathbb{O}_{-1,\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^- \in \mathscr{P}_{-1}(n)$ such that $\boldsymbol{\lambda}^- \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^-$. The construction of $\boldsymbol{\lambda}^-$ is the same as in the orthogonal case except that t is the index of the first odd part in $(\lambda_{s+2}, \dots, \lambda_n)$.

3.1.3. Jacobson-Morosov Theorem and Dynkin grading. A $\frac{1}{2}\mathbb{Z}$*-grading* of the Lie algebra $\mathfrak{g}$ is a decomposition $\Gamma\colon \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ which verifies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j.

LEMMA 3.2. *If Γ is a $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$, then for some semisimple element h_Γ of $\mathfrak{g}$,*

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h_\Gamma, x] = 2jx\}.$$

PROOF. See [**Tauvel-Yu**, Proposition 20.1.5]. □

Since the bilinear form $(\ |\ )$ of $\mathfrak{g}$ is $\operatorname{ad} h_\Gamma$-invariant and nondegenerate, we get

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \iff i + j \neq 0.$$

Hence $\mathfrak{g}_j$ and $\mathfrak{g}_{-j}$ are in pairing. In particular, they have the same dimension.

Fix a nonzero nilpotent element $e \in \mathfrak{g}$. By the Jacobson-Morosov Theorem (cf. e.g., [**Collingwood-McGovern**, §3.3]), there exist $h, f \in \mathfrak{g}$ such that the triple (e, h, f) verifies the $\mathfrak{sl}_2$*-triple relations*:

$$[h, e] = 2e, \qquad [e, f] = h, \qquad [h, f] = -2f.$$

In particular, h is semisimple and the eigenvalues of $\operatorname{ad} h$ are integers. Moreover, e and f belongs to the same nilpotent $\mathbf{G}$-orbit.

EXAMPLE 3.3. Let $\mathfrak{g} = \mathfrak{sl}_n$, and set,

$$e := J_n, \qquad h := \sum_{i=1}^{n} (n + 1 - 2i) e_{i,i}, \qquad f := \sum_{i=1}^{n-1} i(n - i) e_{i+1,i}.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple. From this observation, we readily construct $\mathfrak{sl}_2$-triples for any standard Jordan form $\operatorname{diag}(J_{\lambda_1}, \dots, J_{\lambda_n})$ with $(\lambda_1, \dots, \lambda_n) \in \mathscr{P}(n)$.

The group $\mathbf{G}$ acts on the collection of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ by simultaneous conjugation. This defines a natural map:

$$\Omega : \{\mathfrak{sl}_2\text{-triples}\}/\mathbf{G} \longrightarrow \{\text{nonzero nilpotent orbits}\}, \quad (e, h, f) \mapsto \mathbf{G}.e.$$

THEOREM 3.4 ([**Collingwood-McGovern**, Theorem 3.2.10]). *The map Ω is bijective.*

The map Ω is surjective according to Jacobson-Morosov Theorem. The injectivity is a result of Kostant ([**Collingwood-McGovern**, Theorem 3.4.10]); see [**Wang-lectures**, §2.6] for a sketch of proof.

Since h is semisimple and since the eigenvalues of $\operatorname{ad} h$ are integers, we get a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ defined by h, called the *Dynkin grading associated with h*:

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{g}_j := \{x \in \mathfrak{g} \mid [h, x] = 2jx\}. \tag{50}$$

We have $e \in \mathfrak{g}_1$. Moreover, it follows from the representation theory of $\mathfrak{sl}_2$ that $\mathfrak{g}^e \subset \bigoplus_{j \geqslant 0} \mathfrak{g}_j$ and that $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}$.

REMARK 3.5. One can draw a picture to visualize the above properties. Decompose $\mathfrak{g}$ into simple $\mathfrak{sl}_2$-modules $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ and denote by d_k the dimension of V_k for $k = 1, \dots, s$. We can assume that $d_1 \geqslant \cdots \geqslant d_s \geqslant 1$. We have $\dim V_k \cap \mathfrak{g}_j \leqslant 1$ for any $j \in \frac{1}{2}\mathbb{Z}$. We represent the module V_k on the kth row with d_k boxes, each box corresponding to a nonzero element of $V_k \cap \mathfrak{g}_j$ for j such that $V_k \cap \mathfrak{g}_j \neq \{0\}$. We organize the rows so that the $2j$th column corresponds to a generator of $V_k \cap \mathfrak{g}_j$. Then the boxes appearing on the right position of each row lie in $\mathfrak{g}^e$.

EXAMPLE 3.6. Consider the element $e = \mathrm{diag}(J_3, J_1)$ of $\mathfrak{sl}_4$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 0$, $\dim \mathfrak{g}_1 = 4$ and $\dim \mathfrak{g}_2 = 1$.

The picture here gives:

0 1 2
□□□□■
□□■
□□■
□□■
■

In the picture, the boxes □ correspond to nonzero elements lying in $[f, \mathfrak{g}]$. The boxes ■ correspond to nonzero elements lying in $\mathfrak{g}^e$.

This is an example of *even nilpotent element*, which means that $\mathfrak{g}_i = \{0\}$ for all half-integers i. The nilpotent orbit of an even nilpotent element is called an *even nilpotent orbit*. Note that the regular nilpotent orbit is always even.

EXAMPLE 3.7. Consider the element $e = \mathrm{diag}(J_2, J_1, J_1)$ of $\mathfrak{sl}_4$ which lies in the minimal nilpotent orbit of $\mathfrak{sl}_4$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 4$, $\dim \mathfrak{g}_1 = 1$.

The picture here gives:

$0\frac{1}{2}1$
□□■
□■
□■
□■
□■
■
■
■
■

We observe that $\bigoplus_{i \geqslant 1} \mathfrak{g}_i$ equals $\mathfrak{g}_1$ and has dimension 1. This is actually a general fact: if e lies in the minimal nilpotent orbit of any simple $\mathfrak{g}$, then $\oplus_{i \geqslant 1} \mathfrak{g}_i = \mathfrak{g}_1 = \mathbb{C}e$ and thus $\bigoplus_{i \geqslant 1} \mathfrak{g}_i$ has dimension 1.

One can assume that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is also a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_0$.

LEMMA 3.8. (1) *For any $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_j$ for some $j \in \frac{1}{2}\mathbb{Z}$.*
(2) *Fix a root system Δ_0 of $(\mathfrak{g}_0, \mathfrak{h})$, and set $\Delta_{0,+} = \Delta_+ \cap \Delta_0$. Then $\Delta_+ = \Delta_{0,+} \cup \{\alpha \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_{>0}\}$.*

Denoting by Π the set of simple roots of Δ_+, we get

$$\Pi = \bigcup_{j \in \frac{1}{2}\mathbb{Z}} \Pi_j \quad \text{with} \quad \Pi_j := \{\alpha \in \Pi \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}.$$

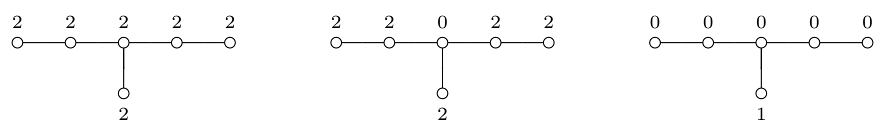
LEMMA 3.9. *We have* $\Pi = \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$.

PROOF. Assume that there exists $\beta \in \Pi_s$ for $s > 1$. A contradiction is expected. Since $e \in \mathfrak{g}_1$ and since $\mathfrak{g}_1$ is contained in the subalgebra generated by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$, we get $[e, \mathfrak{g}_{-\beta}] = \{0\}$. In other words, $\mathfrak{g}_{-\beta} \subset \mathfrak{g}^e$. This contradicts the fact that $\mathfrak{g}^e \subset \mathfrak{g}_{\geqslant 0}$. □

From Lemma 3.9 we define the *weighted Dynkin diagram*, or *characteristic*, of the nilpotent orbit $\mathbf{G}.e$ when $\mathfrak{g}$ is simple as follows. Consider the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$. Each node of this diagram corresponds to a simple root $\alpha \in \Pi$. Then the weighted Dynkin diagram is obtained by labeling the node corresponding to α with the value $\alpha(h) \in \{0, 1, 2\}$.

By convention, the zero orbit has a weighted Dynkin diagram with every node labeled with 0.

EXAMPLE 3.10. In type E_6, the characteristics of the regular, subregular and minimal nilpotent orbits are respectively:



An important consequence of Lemma 3.9 is that there are only finitely many nilpotent orbits, namely at most $3^{\operatorname{rank}\mathfrak{g}}$. Also, the weighted Dynkin diagram is a complete invariant, i.e., two such diagrams are equal if and only if the corresponding nilpotent orbits are equal, [**Collingwood-McGovern**, Theorem 3.5.4].

The regular nilpotent orbit always corresponds to the weighted Dynkin diagram with only 2's (this result is not obvious, cf. e.g., [**Collingwood-McGovern**, Theorem 4.1.6]). More generally, a nilpotent orbit is even if and only if the weighted Dynkin diagram have only 2's or 0's (see Example 3.6 for the definition of *even*).

3.2. Kirillov form and Slodowy slice

Below we will often identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the nondenegerate bilinear form $(\ |\ )$.

3.2.1. Kirillov form. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ and let $\chi = (f|\cdot)$ be the linear form associated with f.

The restriction of the antisymmetric bilinear form,

$$\omega_\chi \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad (x, y) \mapsto (f|[x, y]),$$

to $\mathfrak{g}_{\frac{1}{2}} \times \mathfrak{g}_{\frac{1}{2}}$ is nondegenerate. This results from the paring between $\mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{-\frac{1}{2}}$, and from the injectivity of the map $\operatorname{ad} f \colon \mathfrak{g}_{\frac{1}{2}} \to \mathfrak{g}_{-\frac{1}{2}}$. It is called the *Kirillov form associated with* f. Let $\mathfrak{L}$ be a Lagrangian subspace of $\mathfrak{g}_{\frac{1}{2}}$, that is, $\omega_\chi(\mathfrak{L}, \mathfrak{L}) = 0$ and $\dim \mathfrak{L} = \frac{1}{2} \dim \mathfrak{g}_{\frac{1}{2}}$, and set

$$\mathfrak{m} = \mathfrak{m}_{\chi,\mathfrak{L}} := \mathfrak{L} \oplus \bigoplus_{j > \frac{1}{2}} \mathfrak{g}_j.$$

Then $\mathfrak{m}$ is an ad-nilpotent[1], $\operatorname{ad} h$-graded subalgebra, of $\mathfrak{g}$. Moreover, the algebra $\mathfrak{m}$ verifies the following properties:

[1] i.e., $\mathfrak{m}$ only consists of nilpotent elements of $\mathfrak{g}$.

(χ1) $\chi([\mathfrak{m},\mathfrak{m}]) = (f|[\mathfrak{m},\mathfrak{m}]) = 0$,
(χ2) $\mathfrak{m} \cap \mathfrak{g}^f = \{0\}$;
(χ3) $\dim \mathfrak{m} = \frac{1}{2} \dim \mathbf{G}.f$.

Consider a slightly more general situation. Let $\mathfrak{L}$ be an isotropic subspace of $\mathfrak{g}_{\frac{1}{2}}$, that is, $\omega_\chi(\mathfrak{L},\mathfrak{L}) = 0$, and set

$$\mathfrak{m}_{\chi,\mathfrak{L}} := \mathfrak{l} \oplus \bigoplus_{j>\frac{1}{2}} \mathfrak{g}_j, \qquad \mathfrak{n}_{\chi,\mathfrak{L}} := \mathfrak{L}^{\perp_{\omega_\chi}} \oplus \bigoplus_{j>\frac{1}{2}} \mathfrak{g}_j$$

where

$$\mathfrak{L}^{\perp_{\omega_\chi}} = \{x \in \mathfrak{g}_{\frac{1}{2}a} \mid \omega_\chi(x,\mathfrak{L}) = 0\}$$

is the orthogonal complement of $\mathfrak{L}$ in $\mathfrak{g}_{\frac{1}{2}}$ with respect to the bilinear form ω_χ. Then $\mathfrak{m} := \mathfrak{m}_{\chi,\mathfrak{L}}$ and $\mathfrak{n} := \mathfrak{n}_{\chi,\mathfrak{L}}$ satisfy the following properties:

(1) $\mathfrak{m}$ and $\mathfrak{n}$ are $\operatorname{ad} h$-graded and $\mathfrak{g}_{\geqslant 1} \subset \mathfrak{m} \subset \mathfrak{n} \subset \mathfrak{g}_{>0}$,
(2) $\mathfrak{m}^\perp \cap [\mathfrak{g}, f] = [\mathfrak{n}, f]$,
(3) $\mathfrak{n} \cap \mathfrak{g}^f = 0$,
(4) $[\mathfrak{n},\mathfrak{m}] \subset \mathfrak{m}$,
(5) $\dim \mathfrak{m} + \dim \mathfrak{n} = \dim \mathfrak{g} - \dim \mathfrak{g}^f$.

Here, $\perp$ refers to the orthogonal with respect to the bilinear form $(\ |\ )$.

EXAMPLE 3.11. If f is regular, that is, $\dim \mathfrak{g}^f = r$, then $\mathfrak{m} = \mathfrak{n} = \mathfrak{n}_+$.

3.2.2. Slodowy slice. Let

$$\mathscr{S}_f := \chi + (\mathfrak{g}^f)^* \subset \mathfrak{g}^*$$

be the *Slodowy slice associated with* f, or *with* (e,h,f). The affine space $\mathscr{S}_f$ is identified with $f + \mathfrak{g}^e$ through $(\ |\ )$ by the theory of $\mathfrak{sl}_2$-triples.

Let us introduce a $\mathbb{C}^*$-action on $\mathfrak{g}$ which stabilizes $\mathscr{S}_f \cong f + \mathfrak{g}^e$. The embedding $\operatorname{span}_\mathbb{C}\{e,h,f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \to \mathbf{G}$. By restriction to the 1-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho\colon \mathbb{C}^* \to \mathbf{G}$. Thus $\rho(t)x = t^{2j}x$ for any $x \in \mathfrak{g}_j$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\tilde{\rho}(t)x := t^2\rho(t)(x). \tag{51}$$

So, for any $x \in \mathfrak{g}_j$, $\tilde{\rho}(t)x = t^{2+2j}x$. In particular, $\tilde{\rho}(t)f = f$ and the $\mathbb{C}^*$-action of $\tilde{\rho}$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is,

$$\lim_{t\to 0} \tilde{\rho}(t)(f + x) = f$$

for any $x \in \mathfrak{g}^e$, because $\mathfrak{g}^e \subset \mathfrak{m}^\perp \subseteq \mathfrak{g}_{>-1}$. The same lines of arguments show that the action $\tilde{\rho}$ stabilizes $f + \mathfrak{m}^\perp$ and it is contracting to f on $f + \mathfrak{m}^\perp$, too.

The affine space $\mathscr{S}_f$ is a "slice" according to the following result.

THEOREM 3.12. *The affine space $\mathscr{S}_f$ is transversal to the coadjoint orbits of $\mathfrak{g}^*$. More precisely, for any $\xi \in \mathscr{S}_f$, $T_\xi(\mathbf{G}.\xi) + T_\xi(\mathscr{S}_f) = \mathfrak{g}^*$. An analogue statement holds for the affine variety $\chi + \mathfrak{m}^\perp$.*

SKETCH OF PROOF. We have to prove that $[\mathfrak{g}, x] + \mathfrak{g}^e = \mathfrak{g}$ for any $x \in f + \mathfrak{g}^e$ since $T_x(\mathbf{G}.x) = [\mathfrak{g}, x]$ and $T_x(e + \mathfrak{g}^e) = \mathfrak{g}^e$. First, we verify that the map

$$\eta\colon \mathbf{G} \times (f + \mathfrak{g}^e) \to \mathfrak{g}$$

is a submersion[2] at any point of $\mathbf{G} \times \Omega$ where Ω is an open neighborhood of f in $f + \mathfrak{g}^e$. In particular, for any $x \in \Omega$,

$$\mathfrak{g} = [\mathfrak{g}, x] + \mathfrak{g}^e$$

Next, we use the contracting $\mathbb{C}^*$-action ρ on $f + \mathfrak{g}^e$ to show that η is actually a submersion at any point of $\mathbf{G} \times (f + \mathfrak{g}^e)$. □

Let $\mathbf{N}$ be the unipotent subgroup (hence it is connected) of $\mathbf{G}$ with Lie algebra $\mathfrak{n}$. When $\mathfrak{m} = \mathfrak{n}$, that is, $\mathfrak{L}$ is Lagrangian, we also write $\mathbf{M}$ the closed connected subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{m}$.

Consider the adjoint map

$$\mathbf{N} \times (f + \mathfrak{m}^{\perp}) \to \mathfrak{g}, \quad (g, x) \mapsto g.x$$

It image is contained in $f + \mathfrak{m}^{\perp}$. Indeed, for any $x \in \mathfrak{n}$ and any $y \in \mathfrak{m}^{\perp}$, $\exp(\operatorname{ad} x)(f + y) \in f + \mathfrak{m}^{\perp}$ since $[\mathfrak{n}, \mathfrak{m}] \subset \mathfrak{m}$ and $\chi([\mathfrak{n}, \mathfrak{m}]) = 0$, and this is enough to conclude because, $\mathfrak{n}$ being ad-nilpotent, $\mathbf{N}$ is generated by the elements $\exp(\operatorname{ad} x)$ for x running through $\mathfrak{n}$. As a result, by restriction, we get a map

$$\alpha \colon \mathbf{N} \times \mathscr{S}_f \to f + \mathfrak{m}^{\perp}.$$

THEOREM 3.13 ([**Gan-Ginzburg02**, §2.3]). *The map α is an isomorphism of affine varieties.*

PROOF. We have a contracting $\mathbb{C}^*$-action on $\mathbf{N} \times \mathscr{S}_f$ defined by:

$$\forall\, t \in \mathbb{C}^*,\, g \in \mathbf{N},\, x \in \mathscr{S}_f, \qquad t.(g, x) := (\rho(t^{-1}) g \rho(t), \tilde{\rho}(t) x).$$

The morphism α is $\mathbb{C}^*$-equivariant with respect to this contracting $\mathbb{C}^*$-action, and the $\mathbb{C}^*$-action $\tilde{\rho}$ on $f + \mathfrak{m}^{\perp}$.

Then we conclude thanks to the following result, formulated in [**Gan-Ginzburg02**, Proof of Lemma 2.1]:

"*a $\mathbb{C}^*$-equivariant morphism $\alpha \colon X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points is an isomorphism.*" □

As a consequence of this result, we get the isomorphism:

$$\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[f + \mathfrak{m}^{\perp}]^{\mathbf{N}}.$$

3.3. Poisson algebras, Poisson varieties and Hamiltonian reduction

We want to show that the Slodowy slice $\mathscr{S}_f$ inherits a Poisson structure from that of $\mathfrak{g}^*$. To explain this, we start with some recalls on Poisson algebras and Poisson varieties.

3.3.1. Poisson algebras and Poisson varieties. Let A be a commutative associative $\mathbb{C}$-algebra with unit.

DEFINITION 3.14. Suppose that A is endowed with an additional $\mathbb{C}$-bilinear bracket $\{\ ,\ \} : A \times A \to A$. Then A is called a *Poisson algebra* if the following conditions holds:

(1) A is a Lie algebra with respect to $\{\ ,\ \}$,
(2) (*Leibniz rule*) $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$, for all $a, b, c \in A$.

---

[2] η is a submersion at a point $(g, x) \in \mathbf{G} \times (f + \mathfrak{g}^e)$ if the differential of η at (g, x), that is, the linear map $\mathfrak{g} \times \mathfrak{g}^e \to \mathfrak{g}$, $(v, w) \mapsto g([v, x]) + g(w)$, is surjective.

The Lie bracket $\{\ ,\ \}$ is called a *Poisson bracket* on A.

EXAMPLE 3.15. Let (X, ω) be a symplectic variety. Then the algebra $(\mathcal{O}(X), \{\ ,\ \})$ of regular functions, with pointwise multiplication, is a Poisson algebra.

As an example, let $\mathbb{O} = \mathbf{G}.\xi$ be a coadjoint orbit of $\mathfrak{g}^*$. Then $\mathbb{O}$ has a natural structure of symplectic structure, see e.g. [**Chriss-Ginzburg**, Proposition 1.1.5]; for $\xi \in \mathfrak{g}^*$, we have

$$T_\xi(\mathbb{O}) = T_\xi(\mathbf{G}/\mathbf{G}^\xi) \simeq \mathfrak{g}/\mathfrak{g}^\xi$$

and the bilinear form $\omega_\xi \colon (x, y) \mapsto \xi([x, y])$ descends to $\mathfrak{g}/\mathfrak{g}_\xi$. This gives the symplectic structure. Hence, together with a coadjoint orbit in $\mathfrak{g}^*$, we have a natural Poisson algebra.

3.3.2. Almost commutative algebras. In another direction, we have examples of Poisson algebras coming from some noncommutative algebras. Let B be an associative filtered (noncommutative) algebra with unit,

$$0 = B_{-1} \subset B_0 \subset B_1 \subset \cdots\ ,\ \bigcup_{i \geqslant 0} B_i = B,$$

such that $B_i.B_j \subset B_{i+j}$ for any $i, j \geqslant 0$. Let

$$A := \operatorname{gr} B = \bigoplus_i B_i/B_{i-1}$$

be its graded algebra (the multiplication in B gives rise a well-defined product $B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j}/B_{i+j-1}$, making A an associative algebra). We said that B is *almost commutative* if A is commutative: this means that $a_i b_j - b_j a_i \in B_{i+j-1}$ for $a_i \in B_i, b_j \in B_j$.

Assume that B is almost commutative. Then $\operatorname{gr} B$ has a natural structure of Poisson algebra. We define the Poisson bracket

$$\{\ ,\ \} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

as follows: for $a_1 \in B_i/B_{i-1}$ and $a_2 \in B_j/B_{j-1}$, let b_1 (resp. b_2) be a representative of a_1 in B_i (resp. B_j) and set

$$\{a_1, a_2\} := b_1 b_2 - b_2 b_1 \mod B_{i+j-2}\ .$$

Then we can check the required properties.

EXAMPLE 3.16. Let $\{U_i(\mathfrak{g})\}$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, that is, $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$, and $U(\mathfrak{g})_0 = \mathbb{C}1$. Then

$$0 = U_{-1}(\mathfrak{g}) \subset U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \ldots, \quad U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}),$$

$$U_i(\mathfrak{g}) \cdot U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}), \quad [U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g}).$$

Then the associated graded space $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i \geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})$ is naturally a Poisson algebra, and the PBW Theorem states that

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$$

as Poisson algebras, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

Let us describe explicitly the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ (see [**Chriss-Ginzburg**, Proposition 1.3.18]). Let $(x_1, \ldots, x_n)$ be a basis of $\mathfrak{g}$, with structure constants $c_{i,j}^k$, that is, $[x_i, x_j] = \sum\limits_k c_{i,j}^k x_k$. Through the canonical isomorphism $(\mathfrak{g}^*)^* \cong \mathfrak{g}$, any element of $\mathfrak{g}$ is regarded as a linear functions on $\mathfrak{g}^*$, and thus as an element of $\mathbb{C}[\mathfrak{g}^*]$. We get for $f, g \in \mathbb{C}[\mathfrak{g}^*]$,

$$\{f, g\} = \sum_{i,j,k} c_{i,j}^k x_k \frac{\partial f}{\partial x_i^*} \frac{\partial g}{\partial x_j^*}.$$

In a more concise way, we have:

$$\{f, g\} \colon \mathfrak{g}^* \to \mathbb{C}, \quad \xi \mapsto \langle \xi, [d_\xi f, d_\xi g] \rangle$$

where $d_\xi f, d_\xi g \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$ denote the differentials of f and g at ξ. In particular, if $x, y \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*]$, then

$$\{x, y\} = [x, y].$$

Moreover, if $\mathbb{O}$ is a coadjoint orbit of $\mathfrak{g}^*$,

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{\text{symplectic}}.$$

A *affine Poisson scheme* (resp. *affine Poisson variety)* is an affine scheme $X = \operatorname{Spec} A$ (resp. $X = \operatorname{Specm} A$) such that A is a Poisson algebra. A *Poisson scheme* (resp. *Poisson variety)* is a scheme (resp. reduced scheme) such that the structure sheaf $\mathcal{O}_X$ is a sheaf of Poisson algebras.

For example, let B be as above and continue to assume that B is almost commutative, that is, $A = \operatorname{gr} B$ is commutative. Assume furthermore that A is a finitely generated commutative algebra without zero-divisors. In other words, $A = \mathbb{C}[X]$ is the coordinate ring of a (reduced) irreducible affine algebraic variety X. So the Poisson structure on A makes X a Poisson variety.

3.3.3. Symplectic leaves. If X is smooth, then one may view X as a complex-analytic manifold equipped with a holomorphic Poisson structure. For each point $x \in X$ one defines the *symplectic leaf* $\mathcal{S}_x$ *through* x to be the set of points that could be reached from x by going along Hamiltonian flows[3].

If X is not necessarily smooth, let $\operatorname{Sing}(X)$ be the singular locus of X, and for any $k \geqslant 1$ define inductively $\operatorname{Sing}^k(X) := \operatorname{Sing}(\operatorname{Sing}^{k-1}(X))$. We get a finite partition $X = \bigsqcup_k X^k$, where the strata $X^k := \operatorname{Sing}^{k-1}(X) \setminus \operatorname{Sing}^k(X)$ are smooth analytic varieties (by definition we put $X^0 = X \setminus \operatorname{Sing}(X)$). It is known (cf. e.g., [**Brown-Gordon03**]) that each X^k inherits a Poisson structure. So for any point $x \in X^k$ there is a well defined symplectic leaf $\mathcal{S}_x \subset X^k$. In this way one defines symplectic leaves on an arbitrary Poisson variety. In general, each symplectic leaf is a connected smooth analytic (but not necessarily algebraic) subset in X. However, if the algebraic variety X consists of finitely many symplectic leaves only, then it was shown in [**Brown-Gordon03**] that each leaf is a smooth irreducible locally-closed algebraic subvariety in X, and the partition into symplectic leaves gives an algebraic stratification of X.

Example 3.17. The space $\mathfrak{g}^*$ is a (smooth) Poisson variety and the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$, cf. [**Vaisman**, Proposition 3.1]. The Poisson structure on the coadjoint orbits of $\mathfrak{g}^*$ is known as the *Kirillov-Kostant Poisson structure.* The nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$, which is the (reduced) subscheme

[3]A *Hamiltonian flow* in X from x to x' is a curve γ defined on an open neighborhood of $[0, 1]$ in $\mathbb{C}$, with $\gamma(0) = x$ and $\gamma(1) = x'$, which is an integral curve of a Hamiltonian vector field ξ_f, for some $f \in \mathcal{O}(X)$, defined on an open neighborhood of $\gamma([0, 1])$. See for example [**LaurentGengoux-Pichereau-Vanhaecke**, Chapter 1] for more details.

of $\mathfrak{g}^*$ associated with the augmentation ideal $\mathbb{C}[\mathfrak{g}^*]_+^{\mathbf{G}}$ of the ring of invariants $\mathbb{C}[\mathfrak{g}^*]$, is an example of Poisson variety with finitely many symplectic leaves. These are precisely the nilpotent orbits of $\mathfrak{g}^* \cong \mathfrak{g}$.

3.3.4. Induced Poisson structures. Recall a result of Weinstein; see [**Vaisman**, Proposition 3.10, p 39]:

PROPOSITION 3.18 (Weinstein, 83). *Let Y be a submanifold of a Poisson manifold X such that:*

(1) *Y is transversal to the symplectic leaves, i.e., for any symplectic leaf S and any $x \in Y \cap S$, $T_xY + T_xS = T_xX$;*
(2) *For any $x \in Y$, $T_xY \cap T_xS$ is a symplectic subspace of T_xS, where S is the leaf of X containing x.*

Then, there is a natural induced Poisson structure on Y and the symplectic leaf of Y through $x \in Y$ is $Y \cap S$ if S is the symplectic leaf through x in X.

We aim to apply the result to $\mathscr{S}_f \subset \mathfrak{g}^*$. Part (1) is known (cf. Theorem 3.12). For the part (2), it suffices to prove that for any coadjoint orbit $\mathbb{O}$ in $\mathfrak{g}^*$ and any $\xi \in \mathbb{O} \cap \mathscr{S}_f$, the restriction of the symplectic form on $T_\xi(\mathbb{O})$ to $T_\xi(\mathscr{S}_f) \cap T_\xi(\mathbb{O})$ is nondegenerate. Remember that the symplectic form on $T_\xi(\mathbb{O})$ was described in Example 3.15. Since the annihilator of $T_\xi(\mathscr{S}_f) \simeq \mathfrak{g}^e$ in $\mathfrak{g}$ is $(\mathfrak{g}^e)^\perp = [e, \mathfrak{g}]$, it suffices to verify that for any $\xi \in \mathscr{S}_f$,

$$[\xi, [e, \mathfrak{g}]] \cap T_\xi(\mathscr{S}_f) = [\xi, [e, \mathfrak{g}]] \cap \mathfrak{g}^e = 0.$$

The result is a consequence of:

LEMMA 3.19. *Let $\xi \in \mathscr{S}_f$. Then $[\xi, [e, \mathfrak{g}]] \cap \mathfrak{g}^e = \{0\}$.*

PROOF. Let Y be the set of $y \in f + \mathfrak{g}^e$ such that $[y, [e, \mathfrak{g}]] \cap \mathfrak{g}^e \neq 0$. Since $\mathfrak{g}^e$ and $[e, \mathfrak{g}]$ are $\operatorname{ad} h$-stable, we have for any $t \in \mathbb{C}^*$,

$$\rho(t^{-1})([y, [e, \mathfrak{g}]]) \cap \mathfrak{g}^e = [\rho(t^{-1})y, [e, \mathfrak{g}]] \cap \mathfrak{g}^e$$

whence

$$\tilde{\rho}(t)([y, [e, \mathfrak{g}]] \cap \mathfrak{g}^e) = [\tilde{\rho}(t)y, [e, \mathfrak{g}]] \cap \mathfrak{g}^e.$$

Therefore, $\tilde{\rho}$ stabilizes Y. In addition, since $\mathfrak{g} = [f, \mathfrak{g}] \oplus \mathfrak{g}^e$,

$$f \in (f + \mathfrak{g}^e) \setminus Y$$

Hence, for any y is an open neighborhood U of f in $f + \mathfrak{g}^e$, $y \in (f + \mathfrak{g}^e) \setminus Y$. Assume that $Y \neq \varnothing$ and let $y \in Y$. Since ρ stabilizes Y, we get $\rho(t)y \in Y$ for any $t \in \mathbb{C}^*$. But for t sufficiently small, $\tilde{\rho}(t)y$ lies in U because $\tilde{\rho}$ is contracting, whence the contradiction. $\square$

In conclusion, according to Proposition 3.18, $\mathscr{S}_f \subset \mathfrak{g}^*$ has a Poisson structure induced by the Kirillov-Kostant structure on $\mathfrak{g}^*$ (see Example 3.17). In other words, the Poisson bracket $\{\ ,\ \}_{\mathscr{S}_f}$ on $\mathbb{C}[\mathscr{S}_f]$ is given by,

$$\{f, g\}_{\mathscr{S}_f}(\xi) = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{\text{symplectic}}(\xi),$$

for any $f, g \in \mathbb{C}[\mathscr{S}_f]$ and $\xi \in \mathscr{S}_f$, if $\mathbb{O}$ denotes the coadjoint orbit through ξ in $\mathfrak{g}^*$.

3.3.5. Hamiltonian reduction. The Poisson structure on $\mathscr{S}_f$ can also be described via Hamiltonian reduction in the case where $\mathfrak{m} = \mathfrak{n}$, that is, $\mathfrak{L}$ is Lagrangian.

Let us first recall the classical Hamiltonian reduction in a more general setting. Let $\mathbf{A}$ be a Lie group, with Lie algebra $\mathfrak{a}$, acting on a Poisson variety $(X, \{\,,\,\})$.

DEFINITION 3.20. The action of $\mathbf{A}$ in X is said to be *Hamiltonian* if there is a Lie algebra homomorphism

$$\tilde{\mu}\colon \mathfrak{a} \longrightarrow \mathcal{O}_X(X), \quad x \mapsto \tilde{\mu}_x$$

such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[dr, dashed, "\tilde{\mu}"'] & \mathscr{X}(X) \\ & \mathcal{O}_X(X) \arrow[u] \end{tikzcd}$$

where $\mathscr{X}(X)$ is the Lie algebra of (symplectic) vector fields on X and where the vertical map is the natural map from $\mathcal{O}_X(X)$ to $\mathscr{X}(X)$. As for the horizontal map, it comes from the $\mathbf{A}$-action on X. Namely, it is the map

$$\mathfrak{a} \to \mathscr{X}(X), \quad a \mapsto (x \mapsto \frac{d}{dt}(\exp(t \operatorname{ad} a).x)|_{t=0} \in T_x X).$$

We call the map $\tilde{\mu}$ the *co-moment map* of the action, or the *Hamiltonian* of the action. Its dual map

$$\mu\colon X \longrightarrow \mathfrak{a}^*, \quad x \longmapsto \mu(x),$$

with $\mu(x) \in \mathfrak{a}^*$ the linear map $a \mapsto \tilde{\mu}_a(x)$, is called the *moment map* of the action.

REMARK 3.21. If the group $\mathbf{A}$ is connected, then μ is $\mathbf{A}$-equivariant with respect to the coadjoint action on $\mathfrak{a}^*$.

We refer to [**Vaisman**] or [**LaurentGengoux-Pichereau-Vanhaecke**, Proposition 5.39 and Definition 5.9] for the following result.

THEOREM 3.22 (Marsden-Weinstein). *Assume that $\mathbf{A}$ is connected and that the action of $\mathbf{A}$ in X is Hamiltonian. Let $\gamma \in \mathfrak{a}^*$. Assume that γ is a regular value*[4] *of μ, that $\mu^{-1}(\gamma)$ is $\mathbf{A}$-stable and that $\mu^{-1}(\gamma)/\mathbf{A}$ is a variety. Let $\iota\colon \mu^{-1}(\gamma) \hookrightarrow X$ and $\pi\colon \mu^{-1}(\gamma) \twoheadrightarrow \mu^{-1}(\gamma)/\mathbf{A}$ be the natural maps: ι is the inclusion and π is the quotient map. Then the triple*

$$(X, \mu^{-1}(\gamma), \mu^{-1}(\gamma)/\mathbf{A})$$

is Poisson-reducible, *i.e., there exists a Poisson structure $\{\,,\}'$ on $\mu^{-1}(\gamma)/\mathbf{A}$ such that for all open subset $U \subset X$ and for all $f, g \in \mathcal{O}_X(\pi(U \cap \mu^{-1}(\gamma))$, on has*

$$\{f, g\}' \circ \pi(u) = \{\tilde{f}, \tilde{g}\} \circ \iota(u)$$

at any point $u \in U \cap \mu^{-1}(\gamma)$, where $\tilde{f}, \tilde{g} \in \mathcal{O}_X(U)$ are arbitrary extensions of $f \circ \pi|_{U \cap \mu^{-1}(\gamma)}, g \circ \pi|_{U \cap \mu^{-1}(\gamma)}$ to U.

[4] If $f\colon X \to Y$ is a smooth map between varieties, we say that a point y is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $d_x f\colon T_x(X) \to T_y(Y)$ is surjective. If so, then $f^{-1}(y)$ is a subvariety of X and the codimension of this variety in X is equal to the dimension of Y.

Assume that $\mathfrak{m} = \mathfrak{n}$ and let in this case $\mathbf{M}$ be the unipotent subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{m}$.

We intend to apply the theorem to the connected Lie group $\mathbf{M}$ acting on the Poisson variety $\mathfrak{g}^*$ by the coadjoint action. The action is Hamiltonian and the moment map

$$\mu\colon \mathfrak{g}^* \to \mathfrak{m}^* \tag{52}$$

is the restriction of functions from $\mathfrak{g}$ to $\mathfrak{m}$. Recall that $\chi = (f|\cdot)$. Since $\chi|_{\mathfrak{m}}$ is a character on $\mathfrak{m}$, it is fixed by the coadjoint action of $\mathbf{M}$. As a consequence, the set

$$\mu^{-1}(\chi|_{\mathfrak{m}}) = \{\xi \in \mathfrak{g}^* \mid \mu(\xi) = \chi|_{\mathfrak{m}}\}$$

is $\mathbf{M}$-stable. Moreover, we have the following lemma:

LEMMA 3.23. *$\chi|_{\mathfrak{m}}$ is a regular value for the restriction of μ to each symplectic leaf of $\mathfrak{g}^*$.*

PROOF. Note that $\mu^{-1}(\chi|_{\mathfrak{m}}) = \chi + \mathfrak{m}^{\perp}$. Then we have to prove that for any $\xi \in \chi + \mathfrak{m}^{\perp}$, the map

$$d_{\xi}\mu\colon T_{\xi}(\mathbf{G}.\xi) \to T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*)$$

is surjective. But $T_{\xi}(\mathbf{G}.\xi) \simeq [\mathfrak{g}, \xi]$ while $T_{\chi|_{\mathfrak{m}}}(\mathfrak{m}^*) = \mathfrak{m}^*$. Since $\chi+\mathfrak{m}^{\perp}$ is transversal to the coadjoint orbits in $\mathfrak{g}^*$ (cf. Theorem 3.12), we have

$$\mathfrak{g} = [\mathfrak{g}, \xi] + \mathfrak{m}^{\perp}.$$

Let $\gamma \in \mathfrak{m}^*$ and write $\gamma = x + x'$, with $x \in [\mathfrak{g}, \xi]$ and $x' \in \mathfrak{m}^{\perp}$, according to the above decomposition of $\mathfrak{g}$. Then $\mu(x) = \gamma$. □

Since the map

$$\mathbf{M} \times \mathscr{S}_f \longrightarrow \chi + \mathfrak{m}^{\perp}$$

is an isomorphism of affine varieties (cf. Theorem 3.13),

$$\mathscr{S}_f \cong (\chi + \mathfrak{m}^{\perp})/\mathbf{M}.$$

Therefore, the conditions of Theorem 3.22 are fulfilled and we get a symplectic structure on $\mathscr{S}_f$.

In fact, thanks to Lemma 3.23, we have shown that the symplectic form on each leaf on $\mathscr{S}_f$ is obtained by symplectic reduction from the symplectic form of the corresponding leaf of $\mathfrak{g}^*$.

From this, one can see that the latter Poisson structure defined on $\mathscr{S}_f$ is the same as that defined in §3.3.4. It is described as follows. Let $\pi\colon \chi + \mathfrak{m}^{\perp} \twoheadrightarrow (\chi + \mathfrak{m}^{\perp})/\mathbf{M} \simeq \mathscr{S}_f$ be the natural projection map, and $\iota\colon \chi + \mathfrak{m}^{\perp} \hookrightarrow \mathfrak{g}^*$ be the natural inclusion. Then for any $f, g \in \mathbb{C}[\mathscr{S}_f]$,

$$\{f, g\}_{\mathscr{S}_f} \circ \pi = \{\tilde{f}, \tilde{g}\} \circ \iota$$

where $\tilde{f}, \tilde{g}$ are arbitrary extensions of $f \circ \pi, g \circ \pi$ to $\mathfrak{g}^*$.

The Poisson structure of $\mathscr{S}_f$ is described as follows. Let

$$I_{\chi} = \mathbb{C}[\mathfrak{g}^*] \sum_{x \in \mathfrak{m}} (x - \chi(x)),$$

so that

$$\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/I_{\chi}.$$

Then $\mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\mu^{-1}(\chi)]^{\mathbf{M}}$ can be identified as the subspace of $\mathbb{C}[\mathfrak{g}^*]/I_\chi$ consisting of all cosets $\phi + I_\chi$ such that $\{x, \phi\} \in I_\chi$ for all $x \in \mathfrak{m}$. In this realization, the Poisson structure on $\mathbb{C}[\mathscr{S}_f]$ is defined by the formula

$$\{\phi + I_\chi, \phi' + I_\chi\} = \{\phi, \phi'\} + I_\chi$$

for ϕ, ϕ' such that $\{x, \phi\}, \{x, \phi'\} \in I_\chi$ for all $x \in \mathfrak{m}$.

3.4. BRST cohomology, quantum Hamiltonian reduction and definition of finite W-algebras

We introduce in this part the finite W-algebras. We first describe the Hamiltonian reduction in terms of BRST cohomology, essentially following Kostant and Sternberg [**Kostant-Sternberg87**]. Then we describe its natural quantization, and define finite W-algebras. With this definition, finite W-algebras will naturally appear as finite dimensional analogs of (affine) W-algebras.

3.4.1. BRST reduction. In this subsection we shall describe the Hamiltonian reduction of §3.3.5 in a more factorial way, in terms of the *BRST cohomology* (where BRST refers to the physicists Becchi, Rouet, Stora and Tyutin) for later purpose.

We refer the reader to Appendix A for backgrounds on superspaces, superalgebras, Lie superalgebras, etc. and Clifford algebras.

Let G be any connected affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$ (we don't assume G is semisimple). Let $\{x_i\}_{1 \leqslant i \leqslant d}$ a basis of $\mathfrak{g}$, and let $\{x_i^*\}_{1 \leqslant i \leqslant d}$ be the corresponding dual basis of $\mathfrak{g}^*$. Denote by $c_{i,j}^k$ the structure constants of $\mathfrak{g}$, that is, $[x_i, x_j] = \sum_{k=1}^d c_{i,j}^k x_k$ for $i, j = 1, \ldots, d$.

Let $Cl(\mathfrak{g})$ be the *Clifford algebra* associated with the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ and the nondegenerate bilinear form $\langle \cdot | \cdot \rangle$ defined by $\langle f + x | g + y \rangle = f(y) + g(x)$ for $f, g \in \mathfrak{g}^*$, $x, y \in \mathfrak{g}$. Namely, $Cl(\mathfrak{g})$ is the unital superalgebra isomorphic to $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*) \cong \wedge(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*)$ as $\mathbb{C}$-vector spaces, the natural embeddings $\wedge(\mathfrak{g}) \hookrightarrow Cl(\mathfrak{g})$, $\wedge(\mathfrak{g}^*) \hookrightarrow Cl(\mathfrak{g})$ are homogeneous homomorphism of superalgebras, and

$$[x, f] = f(x) \quad x \in \mathfrak{g} \subset \wedge(\mathfrak{g}), \; f \in \mathfrak{g}^* \subset \wedge(\mathfrak{g}^*).$$

(Note that $[x, f] = xf + fx$ since x, f are odd.)

Lemma 3.24. *The following map gives a Lie algebra homomorphism.*

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow Cl(\mathfrak{g}) \\ x_i &\longmapsto \sum_{1 \leqslant j,k \leqslant d} c_{i,j}^k x_k x_j^* \end{aligned}$$

We have

$$[\rho(x), y] = [x, y] \in \mathfrak{g} \subset Cl(\mathfrak{g}),$$

for $x, y \in \mathfrak{g}$ where the first bracket is in $Cl(\mathfrak{g})$ while the second is in $\mathfrak{g}$.

Define an increasing filtration on $Cl(\mathfrak{g})$ by setting $Cl_p(\mathfrak{g}) := \wedge^{\leqslant p}(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*)$ where $\wedge(\mathfrak{g}) = \bigoplus_{i \geqslant 0} \wedge^i(\mathfrak{g})$ is the natural grading. We have

$$0 = Cl_{-1}(\mathfrak{g}) \subset Cl_0(\mathfrak{g}) \subset Cl_1(\mathfrak{g}) \cdots \subset Cl_d(\mathfrak{g}) = Cl(\mathfrak{g}),$$

and

$$Cl_p(\mathfrak{g}) \cdot Cl_q(\mathfrak{g}) \subset Cl_{p+q}(\mathfrak{g}), \quad [Cl_p(\mathfrak{g}), Cl_q(\mathfrak{g})] \subset Cl_{p+q-1}(\mathfrak{g}). \tag{53}$$

Let $\overline{Cl}(\mathfrak{g})$ be its associated graded algebra:

$$\overline{Cl}(\mathfrak{g}) := \operatorname{gr} Cl(\mathfrak{g}) = \bigoplus_{p\geqslant 0} \frac{Cl_p(\mathfrak{g})}{Cl_{p-1}(\mathfrak{g})}.$$

By (53), $\overline{Cl}(\mathfrak{g})$ is naturally a graded Poisson superalgebra, called the *classical Clifford algebra* associated with $\mathfrak{g}\oplus\mathfrak{g}^*$.

We have $\overline{Cl}(\mathfrak{g}) = \wedge(\mathfrak{g}) \otimes \wedge(\mathfrak{g}^*)$ as a commutative superalgebra. Its Poisson (super)bracket is given by

$$\begin{aligned}
&\{x, f\} = f(x), \quad x \in \mathfrak{g} \subset \wedge(\mathfrak{g}),\ f \in \mathfrak{g}^* \subset \wedge(\mathfrak{g}^*),\\
&\{x, y\} = 0, \quad x, y \in \mathfrak{g} \subset \wedge(\mathfrak{g}), \quad \{f, g\} = 0,\ f, g \in \mathfrak{g}^* \subset \wedge(\mathfrak{g}^*).
\end{aligned}$$

Lemma 3.25. *We have $\overline{Cl}(\mathfrak{g})^{\mathfrak{g}} = \wedge(\mathfrak{g})$, where $\overline{Cl}(\mathfrak{g})^{\mathfrak{g}} := \{w \in \overline{Cl} \mid \{x, w\} = 0$ for all $x \in \mathfrak{g}\}$.*

The Lie algebra homomorphism $\rho\colon \mathfrak{g} \longrightarrow Cl_1(\mathfrak{g}) \subset Cl(\mathfrak{g})$ induces a Lie algebra homomorphism

$$\bar{\rho} := \sigma_1 \circ \rho\colon \mathfrak{g} \longrightarrow \overline{Cl}(\mathfrak{g}), \tag{54}$$

where σ_1 is the projection $Cl_1(\mathfrak{g}) \to Cl_1(\mathfrak{g})/Cl_0(\mathfrak{g}) \subset \operatorname{gr} Cl(\mathfrak{g})$. We have for $x, y \in \mathfrak{g}$,

$$\{\bar{\rho}(x), y\} = [x, y].$$

Set

$$\overline{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}(\mathfrak{g}).$$

Since it is a tensor product of Poisson superalgebras, $\overline{C}(\mathfrak{g})$ is naturally a Poisson superalgebra.

Lemma 3.26. *For any character χ of $\mathfrak{g}$, the following map gives a Lie algebra homomorphism:*

$$\begin{aligned}
\bar{\theta}_\chi\colon \mathfrak{g} &\longrightarrow \overline{C}(\mathfrak{g})\\
x &\longmapsto (x - \chi(x)) \otimes 1 + 1 \otimes \bar{\rho}(x),
\end{aligned}$$

that is, $\{\bar{\theta}_\chi(x), \bar{\theta}_\chi(y)\} = \bar{\theta}_\chi([x, y])$ for $x, y \in \mathfrak{g}$.

Let $\overline{C}(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} \overline{C}^n(\mathfrak{g})$ be the $\mathbb{Z}$-grading defined by $\deg \phi\otimes 1 = 0$ for $\phi \in \mathbb{C}[\mathfrak{g}^*]$, $\deg 1 \otimes f = 1$ for $f \in \mathfrak{g}^*$, and $\deg 1 \otimes x = -1$ for $x \in \mathfrak{g}$. We have

$$\overline{C}^n(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \Big(\bigoplus_{j-i=n} \wedge^i(\mathfrak{g}) \otimes \wedge^j(\mathfrak{g}^*)\Big).$$

The following result is due to Beilinson and Drinfeld ([**Beilinson-Drinfeld96**], Lemma 7.13.3]).

Lemma 3.27. *There exists a unique element $\bar{Q} \in \overline{C}^1(\mathfrak{g})$ such that*

$$\{\bar{Q}, 1 \otimes x\} = \bar{\theta}_\chi(x) \quad \textit{for} \quad x \in \mathfrak{g}.$$

We have $\{\bar{Q}, \bar{Q}\} = 0$.

PROOF. Existence. It is straightforward to see that the element

$$\bar{Q} = \sum_i (x_i - \chi(x_i)) \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* x_k$$

satisfies the condition.

Uniqueness. Suppose that $\bar{Q}_1, \bar{Q}_2 \in \overline{C}^1(\mathfrak{g})$ satisfy the condition. Set $R = Q_1 - Q_2 \in \overline{C}^1(\mathfrak{g})$. Then $\{R, 1 \otimes x\} = 0$, and so, $R \in \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}(\mathfrak{g})^{\mathfrak{g}}$. But by Lemma 3.25, $\overline{Cl}(\mathfrak{g})^{\mathfrak{g}} \cap \overline{Cl}^1 = 0$, where $\overline{Cl}^1$ is Cl^1/Cl^0. Thus $R = 0$ as required.

To show that $\{\bar{Q}, \bar{Q}\} = 0$, observe that

$$\{1 \otimes x, \{1 \otimes y, \{\bar{Q}, \bar{Q}\}\}\} = 0, \quad \forall x, y \in \mathfrak{g}$$

(note that $\bar{Q}$ is odd). Applying Lemma 3.25 twice, we get that $\{\bar{Q}, \bar{Q}\} = 0$. □

Since $\bar{Q}$ is odd, Lemma 3.27 implies that

$$\{\bar{Q}, \{\bar{Q}, a\}\} = \frac{1}{2}\{\{\bar{Q}, \bar{Q}\}, a\} = 0$$

for any $a \in \overline{C}(\mathfrak{g})$. That is, $\operatorname{ad} \bar{Q} := \{\bar{Q}, \cdot\}$ satisfies that

$$(\operatorname{ad} \bar{Q})^2 = 0.$$

Thus, $(\overline{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$ is a *differential graded Poisson superalgebra*. Its cohomology

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*]) := H^\bullet(\overline{C}(\mathfrak{g}), \operatorname{ad} \bar{Q}) = \bigoplus_{i \in \mathbb{Z}} H^i(\overline{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$$

inherits a graded Poisson superalgebra structure from $\overline{C}(\mathfrak{g})$.

More generally, let R be a Poisson algebra equipped with a Poisson algebra homomorphism $\mu^* \colon \mathbb{C}[\mathfrak{g}^*] \to R$. View $\bar{Q}$ as an element of $R \otimes \overline{Cl}(\mathfrak{g})$ through the map $\mu^* \otimes \mathrm{id}$. Then $\mu^* \otimes \mathrm{id} \colon \overline{C}(\mathfrak{g}^*) \to R \otimes \overline{Cl}(\mathfrak{g})$ is a Poisson superalgebra homomorphism, and $(R \otimes \overline{Cl}(\mathfrak{g}), \operatorname{ad} \bar{Q})$ is a differential graded Poisson superalgebra, where the image of $\bar{Q}$ is also denoted by $\bar{Q}$. Therefore, its cohomology

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, R) := H^\bullet(R \otimes \overline{Cl}(\mathfrak{g}), \operatorname{ad} \bar{Q}) \tag{55}$$

inherits a graded Poisson superalgebra structure from $R \otimes \overline{Cl}(\mathfrak{g})$.

Let X be any affine Poisson scheme[5] equipped with a Hamiltonian G-action, $\chi \in \mathfrak{g}^*$ a one-point G-orbit, that is, χ is a character of $\mathfrak{g}$. The above construction gives the Poisson algebra $H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) \subset H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$. (Note that the degree zero part is purely even.) The affine Poisson scheme

$$X//_{BRST,\chi} G := \operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]))$$

is called the *BRST reduction* of X. We write $X//_{BRST} G$ for $X//_{BRST,\chi} G$ if $\chi = 0$.

The BRST reduction coincides with the geometric Hamiltonian reduction in §3.3.5 in some "nice" cases.

THEOREM 3.28. *Let $X = \operatorname{Spec} R$ be an affine Poisson scheme equipped with a Hamiltonian G-action, $\chi \in \mathfrak{g}^*$ a one-point G-orbit. Suppose that*

(1) *the moment map $\mu : X \to \mathfrak{g}^*$ is flat,*
(2) *there exists a subscheme $\mathscr{S}$ of $\mu^{-1}(\chi)$ such that the action map gives the isomorphism $G \times \mathscr{S} \xrightarrow{\simeq} \mu^{-1}(\chi)$.*

[5]In this note we assume that all Poisson schemes are of finite type unless otherwise stated.

Then

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, R) \cong \mathbb{C}[\mathscr{S}] \otimes H^\bullet_{DR}(G).$$

Here $H^\bullet_{DR}(G)$ denotes the de Rham cohomology of G equipped with the trivial Poisson structure. In particular, $H^0_{DR}(G) = 0$ since G is connected.

LEMMA 3.29. *Let A be a commutative $\mathbb{C}$-algebra with a regular sequence $x_1, x_2, \dots,$ that is, $A/(x_1, \dots, x_r) \neq 0$ and x_{r+1} is not a zero devisor of $A/(x_1, \dots, x_r)$ for all r. If M is a flat A-module, that is, $M \otimes_A ?$ is an exact functor, then*

$$H_i^{Kos}(A, M) = \delta_{i,0} M/(x_1, x_2, \dots,)M.$$

Here $H_i^{Kos}(A, M)$ denotes the homology of the Koszul complex with respect to the sequence $x_1, x_2, \dots, .$

PROOF OF THEOREM 3.28. Give a bigrading on $\overline{C} := R \otimes \bar{C}l$ by setting

$$\overline{C}^{i,j} = R \otimes \wedge^i(\mathfrak{g}^*) \otimes \wedge^{-j}(\mathfrak{g}),$$

so that $\overline{C} = \bigoplus_{i \geqslant 0, j \leqslant 0} \overline{C}^{i,j}$.

Observe that $\operatorname{ad} \bar{Q}$ decomposes as $\operatorname{ad} \bar{Q} = d_+ + d_-$ such that

$$d_-(\overline{C}^{i,j}) \subset \overline{C}^{i+1,j}, \qquad d_+(\overline{C}^{i,j}) \subset \overline{C}^{i,j+1}. \tag{56}$$

Explicitly, we have

$$\begin{aligned} d_- &= \sum_i (x_i - \chi(x_i)) \otimes \operatorname{ad} x_i^*, \\ d_+ &= \sum_i \operatorname{ad} x_i \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* \operatorname{ad} x_k + \sum_i 1 \otimes \bar{\rho}(x_i) \operatorname{ad} x_i^*. \end{aligned}$$

Since $(\operatorname{ad} \bar{Q})^2 = 0$, (56) implies that

$$d_-^2 = d_+^2 = [d_-, d_+] = 0.$$

It follows that there exists a spectral sequence

$$E_r \Longrightarrow H^\bullet_{BRST,\chi}(\mathfrak{g}, R) = H^\bullet(\bar{C}, \operatorname{ad} \bar{Q})$$

such that

$$\begin{aligned} E_1^{\bullet,q} &= H^q(\overline{C}, d_-) = H^q(R \otimes \wedge(\mathfrak{g}), d_-) \otimes \wedge^\bullet(\mathfrak{g}^*), \\ E_2^{p,q} &= H^p(H^q(\overline{C}, d_-), d_+). \end{aligned}$$

Observe that $(\overline{C}, d_-)$ is identical to the Koszul complex $\mathbb{C}[\mathfrak{g}^*]$ associated with the sequence $x_1 - \chi(x_1), x_2 - \chi(x_2.) \dots, x_d - \chi(x_d)$ tensorized with $\wedge(\mathfrak{g}^*)$. Hence by Lemma 3.29 we get that

$$H^i(\overline{C}, d_-) = \begin{cases} \mathbb{C}[\mu^{-1}(\chi)] \otimes \wedge(\mathfrak{g}^*), & \text{if} \quad i = 0 \\ 0, & \text{if} \quad i \neq 0. \end{cases}$$

Next, notice that $(H^\bullet(\overline{C}, d_-), d_+)$ is identical to the Chevalley complex for the Lie algebra cohomology $H^\bullet(\mathfrak{g}, \mathbb{C}[\mu^{-1}(\chi)])$. Since $\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathscr{S}] \otimes \mathbb{C}[G]$ by the assumption as G-modules, where G acts only on $\mathbb{C}[G]$ on the right-hand-side, we get that

$$H^i(H^j(\overline{C}, d_-), d_+) = \begin{cases} \mathbb{C}[\mathscr{S}] \otimes H^i(\mathfrak{g}, \mathbb{C}[G]), & \text{if} \quad j = 0 \\ 0, & \text{if} \quad j \neq 0. \end{cases}$$

Hence the spectral sequence collapses at $E_2 = E_\infty$. Thus there is an isomorphism

$$H^\bullet(\overline{C}, \operatorname{ad}\bar{Q}) \xrightarrow{\simeq} H^\bullet(H^0(\overline{C}, d_-), d_+) = \mathbb{C}[\mathscr{S}] \otimes H^\bullet(\mathfrak{g}, \mathbb{C}[G]), \quad [c] \mapsto [c],$$

of Poisson algebras. The assertion follows noting that $H^i(\mathfrak{g}, \mathbb{C}[G]) = H^i_{DR}(G)$ as G is affine. □

We write $H^\bullet_{BRST}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*])$ for $H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*])$ if $\chi = 0$.

3.4.2. BRST realization of Slodowy slices. We now apply Theorem 3.28 to $X = \mathfrak{g}^*$, $G = \mathbf{M}$, the projection $\mu : X = \mathfrak{g}^* \to \mathfrak{m}^*$ and $\chi = (f|\cdot)$. Observe that $\mu^{-1}(\chi) = f + \mathfrak{m}^\perp$. Clearly μ is flat, and the second assumption is satisfied by Theorem 3.13.

THEOREM 3.30. *We have* $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) = 0$ *for* $i \neq 0$ *and*

$$H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) \cong \mathbb{C}[\mathscr{S}_f]$$

as Poisson algebras.

PROOF. Since $\mathbf{M}$ is unipotent, we have $H^i_{DR}(\mathbf{M}) = \delta_{i,0}\mathbb{C}$. Therefore the assertion follows immediately from Theorem 3.28. □

3.4.3. BRST realization of equivariant Slodowy slices. Consider the cotangent bundle $T^*\mathbf{G}$ of $\mathbf{G}$. We have $T^*\mathbf{G} = \mathbf{G} \times \mathfrak{g}^*$, and there are the following two commuting Hamiltonian $\mathbf{G}$ action $g \mapsto g_L$ and $g \mapsto g_R$ on $T^*\mathbf{G}$, where

$$g_L(a, x) = (ag^{-1}, g.x), \quad g_R(a, x) = (ga, x). \tag{57}$$

The moment map corresponding to the former is just the projection

$$\mu_L : T^*\mathbf{G} \ni (a, x) \mapsto x \in \mathfrak{g}^*. \tag{58}$$

The moment map corresponding to the latter is given by

$$\mu_R : T^*\mathbf{G} \ni (a, x) \mapsto a.x \in \mathfrak{g}^*. \tag{59}$$

The action of $\mathfrak{g}$ on $\mathbb{C}[T^*\mathbf{G}] = \mathbb{C}[\mathbf{G}] \otimes \mathbb{C}[\mathfrak{g}^*]$ obtained by differentiating these actions are

$$\pi_L(x) = x_L + \operatorname{ad} x \quad , \quad \pi_R(x) = x_R,$$

where x_L and x_R denote the action of x on $\mathbb{C}[\mathbf{G}]$ as a left invariant vector field and a right invariant vector field respectively, and $\operatorname{ad} x$ denotes the action $f \mapsto \{x, f\}$ on $\mathbb{C}[\mathfrak{g}^*]$.

Now consider the composition

$$\mu : T^*\mathbf{G} \xrightarrow{\mu_L} \mathfrak{g}^* \xrightarrow{\text{projection}} \mathfrak{m}^*. \tag{60}$$

Then μ is the moment map for the $\mathbf{M}$-action by restriction to g_L. We have $\mu^{-1}(\chi) = \mathbf{G} \times (\chi + \mathfrak{m}^\perp)$. Clearly the action of $\mathbf{M}$ on $\mu^{-1}(\chi)$ is free and χ is the regular value of μ. Thus,

$$\widetilde{\mathscr{S}_f} := \mu^{-1}(\chi)/\mathbf{M} = \mathbf{G} \times_{\mathbf{M}} (\chi + \mathfrak{m}^\perp)$$

is a symplectic variety. We have

$$\widetilde{\mathscr{S}_f} \cong \mathbf{G} \times \mathscr{S}_f$$

and $\widetilde{\mathscr{S}_f}$ is called the *equivariant Slodowy slice* [**Los10**].

As μ is clearly flat we can apply Theorem 3.28 to obtain the following.

PROPOSITION 3.31. *$H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*\mathbf{G}]) = 0$ for $i \neq 0$ and*

$$H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*\mathbf{G}]) = \mathbb{C}[\widetilde{\mathscr{S}_f}]$$

as Poisson algebras.

REMARK 3.32. The equivariant Slodowy slice $\widetilde{\mathscr{S}_f} = \mathbf{G} \times_{\mathbf{M}} (\chi + \mathfrak{m}^\perp)$ is naturally a vector bundle over $\mathbf{G}/\mathbf{M}$. As a bundle over $\mathbf{G}/\mathbf{M}$, $\widetilde{\mathscr{S}_f}$ is called a *twisted cotangent bundle* [**Chriss-Ginzburg**, 1.4.13].

The relationship between the Slodowy slice $\mathscr{S}_f$ and the equivariant Slodowy slice $\widetilde{\mathscr{S}_f}$ is described as follows. There is an action of $\mathbf{G}$ on $\widetilde{\mathscr{S}_f}$ defined by

$$g(a, x) = (ga, x). \tag{61}$$

PROPOSITION 3.33. *We have $\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[\widetilde{\mathscr{S}_f}]^{\mathbf{G}}$ as Poisson algebras.*

The $\mathbf{G}$-action (61) is Hamiltonian and the corresponding moment map is given by

$$\mu : \widetilde{\mathscr{S}_f} = \mathbf{G} \times_{\mathbf{M}} (\chi + \mathfrak{m}^\perp) \ni (g, x) \mapsto g.x \in \mathfrak{g}^*. \tag{62}$$

THEOREM 3.34 ([**Slodowy80**],[**Premet02**, Theorem 5.4],[**Charbonnel-Moreau16**, Lemma 4.3]). *The moment map $\mu : \widetilde{\mathscr{S}_f} \to \mathfrak{g}^*$ given by(62) is smooth onto a dense open subset of $\mathfrak{g}^*$ containing $\mathbf{G}.\chi$. In particular, μ is flat.*

PROOF. Since the proof is short, we give the argument.
It suffices to prove that the morphism

$$\theta_f \colon \mathbf{G} \times (f + \mathfrak{g}^e) \to \mathfrak{g}^*, \quad (g, x) \mapsto g.x$$

is smooth onto a dense open subset of $\mathfrak{g}^*$ containing $\mathbf{G}.f$. Since $\mathfrak{g} = \mathfrak{g}^e + [f, \mathfrak{g}]$, θ_f is a submersion at $(1_{\mathbf{G}}, f)$. Then θ_f is a submersion at all points of $\mathbf{G} \times (f + \mathfrak{g}^e)$ since it is $\mathbf{G}$-equivariant for the left multiplication in $\mathbf{G}$ and since

$$\lim_{t \to \infty} \tilde{\rho}(t).x = f$$

for all x in $f + \mathfrak{g}^e$. So, by [**Hartshorne77**, Ch. III, Proposition 10.4], θ_f is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $\mathbf{G}.f$. $\square$

THEOREM 3.35. *The natural map $\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}} \to \mathbb{C}[\mathscr{S}_f] = H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*])$ defined by sending p to $p \otimes 1$ induces an isomorphism from $\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}$ to the Poisson center of $\mathbb{C}[\mathscr{S}_f]$.*

PROOF. By Theorem 3.34, the moment map $\mu : \widetilde{\mathscr{S}_f} \to \mathfrak{g}^*$ induces an embedding $\mu^* : \mathbb{C}[\mathfrak{g}^*] \hookrightarrow \mathbb{C}[\widetilde{\mathscr{S}_f}]$ of Poisson algebras. By taking $\mathbf{G}$-invariants, we get the embedding $\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}} \hookrightarrow \mathbb{C}[\widetilde{\mathscr{S}_f}]^{\mathbf{G}} = \mathbb{C}[\mathscr{S}_f]$.

Consider the morphism $\varphi \colon \mathscr{S}_f \to \mathfrak{g}^*/\!/\mathbf{G}$. Each of the fibers is a finite union of symplectic leaves for $\mathscr{S}_f$. Remember that the symplectic leaves of $\mathscr{S}_f$ are the intersections $\mathscr{S}_f \cap \mathbf{G}.\xi$ with $\xi \in \mathfrak{g}^*$. On the other hand, by [**?**, §§5.4 & 6.4], all scheme-theoretic fibers of φ are reduced and irreducible. Hence each fiber of φ is the closure of some symplectic leaf of $\mathscr{S}_f$. Let now z be in the Poisson center of $\mathbb{C}[\mathscr{S}_f]$. It is constant on each symplectic leaf by definition of the Hamiltonian flow: If σ_x is an integral curve of $\{H, .\}$, with $H \in \mathbb{C}[\mathscr{S}_f]$ and $\sigma_x(0) = x \in \mathscr{S}_f$, then $\dfrac{d}{dt}(z \circ \sigma_x) = \{H, z\} \circ \sigma_x = 0$, and so z is constant on all flows through x, that is, on the symplectic leaf of x. As a result, z is constant on all fibers of the morphism φ.

LEMMA 3.36 ([**Goodman-Wallach09**, Theorem A.2.9]). *Let X, Y, Z be irreducible affine varieties. Assume that $f\colon X \to Y$ and $h\colon X \to Z$ are dominant morphisms such that h is constant on the fibers of f. There there exists a rational map $g\colon Y \to Z$ making the following diagram commutative:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[dl, dashed, "g"] \\ Z & \end{tikzcd}$$

If z is constant, then clearly z lies in the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$. In addition, one can assume that z is homogeneous for the *Slodowy grading* on $\mathbb{C}[\mathscr{S}_f]$ induced from the $\mathbb{C}^*$-action of $\tilde{\rho}$ on $\mathscr{S}_f$ since the Poisson center of $\mathbb{C}[\mathscr{S}_f]$ is Slodowy invariant. So for any $t \in \mathbb{C}^*$, $t.z = t^k z$ if k is the Slodowy degree of z. Hence one can assume that $z\colon \mathscr{S}_f \to \mathbb{C}$ is a dominant (and even surjective) morphism.

So by Lemma 3.36, z induces a rational morphism on $\mathfrak{g}^*/\!/\mathbf{G}$ since z is constant on the fibers of the dominant morphism φ. Then it remains to prove the following:

$$\mathbb{C}(\mathfrak{g}^*/\!/\mathbf{G}) \cap \mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}], \tag{63}$$

since $\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}] \cong \mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}$.

To prove (63), we follow the arguments of [**Bass-Connel-Wright82**]. Write $z = p/q$, with p, q relatively prime elements of $\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}]$. Since p, q are relatively prime, the multiplication by p induces an injective homomorphism

$$\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}/q\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}] \to \mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}]/q\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}].$$

Since $\mathbb{C}[\mathscr{S}_f]$ is flat over $\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}]$, the base change $\mathbb{C}[\mathscr{S}_f] \otimes_{\mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}]} -$ yields an injective homomorphism

$$\mathbb{C}[\mathscr{S}_f]/q\mathbb{C}[\mathscr{S}_f] \to \mathbb{C}[\mathscr{S}_f]/q\mathbb{C}[\mathscr{S}_f].$$

The image of 1 is 0 because z is regular. Hence p and q are relatively prime in $\mathbb{C}[\mathscr{S}_f]$. Since $z = p/q \in \mathbb{C}[\mathscr{S}_f]$ we deduce that $q \in \mathbb{C}^*$ and so $z \in \mathbb{C}[\mathfrak{g}^*/\!/\mathbf{G}]$. $\square$

3.4.4. Moore-Tachikawa operation. Let X, Y be (any) affine Poisson schemes equipped with Hamiltonian $\mathbf{G}$-action, $\mu_X : X \to \mathfrak{g}^*$, $\mu_Y : Y \to \mathfrak{g}^*$ the corresponding moment maps. Then the diagonal action of G on $X \times Y$ is Hamiltonian, with the moment map $\mu_{X\times Y} : X \times Y \ni (x, y) \mapsto \mu_X(x) + \mu_Y(y) \in \mathfrak{g}^*$. Motivated by [**MT12**], we define the affine Poisson scheme $X \circ Y$ by

$$X \circ Y := (X \times Y)/\!/_{BRST}\mathbf{G} = \mathrm{Spec}(H^0_{BRST}(\mathfrak{g}, \mathbb{C}[X] \otimes \mathbb{C}[Y]).$$

Clearly, $X \circ Y \cong Y \circ X$.

PROPOSITION 3.37. *$T^*\mathbf{G} \circ X \cong X$ for any affine Poisson scheme X equipped with a Hamiltonian $\mathbf{G}$-action.*

PROOF. From Exercise 3.38 below, it follows that it is enough to show that $H^0_{BRST}(\mathfrak{g}, \mathbb{C}[T^*\mathbf{G}]) = \mathbb{C}$. But this is easy to see. $\square$

EXERCISE 3.38. Let X be an affine Poisson schemes equipped with a Hamiltonian $\mathbf{G}$-action. There are the following four Hamiltonian $\mathbf{G}$-actions on $T^*\mathbf{G}\times X$:

$$\pi_{1,L}(g)(a,f,x)=(ag^{-1},gf,gx),\quad \pi_{1,R}(g)(a,f,x)=(ga,f,x),$$
$$\pi_{2,L}(g)(a,f,x)=(ag^{-1},gf,x),\quad \pi_{2,R}(g)(a,f,x)=(ga,f,gx).$$

Clearly the actions $\pi_{1,L}$ and $\pi_{1,R}$ (resp. $\pi_{2,L}$ and $\pi_{2,R}$) mutually commute.

Consider the morphism

$$\Phi : T^*\mathbf{G}\times X\to T^*\mathbf{G}\times X$$

defined by $(g,f,x)\mapsto (g,f,gx)$ for $g\in\mathbf{G}$, $f\in\mathfrak{g}^*$, $x\in X$.

Check that Φ is an isomorphism of Poisson schemes such that

$$\Phi\circ\pi_{1,L}=\pi_{2,L}\circ\Phi,\quad \Phi\circ\pi_{1,R}=\pi_{2,R}\circ\Phi.$$

THEOREM 3.39. *Let X be an affine Poisson scheme equipped with a Hamiltonian $\mathbf{G}$-action, and $\mu_X\colon X\to\mathfrak{g}^*$ the corresponding moment map. Then $\widetilde{\mathscr{S}_f}\circ X$ is isomorphic to the scheme theoretic intersection $X\times_{\mathfrak{g}^*}\mathscr{S}_f=X\cap\mu_X^{-1}(\mathscr{S}_f)$, where $\mathscr{S}_f\to\mathfrak{g}^*$ is given by the inclusion $x\mapsto -x$.*

PROOF. Let

$$\mu\colon \widetilde{\mathscr{S}_f}\times X=\mathbf{G}\times\mathscr{S}_f\times X\to\mathfrak{g}^*,\quad (g,s,x)\mapsto g.s+\mu_X(x),$$

be the moment map that is the sum of the moment maps. By Theorem 3.34, μ is flat. Further, the action map gives the isomorphism

$$\mathbf{G}\times(\mathscr{S}_f\times_{\mathfrak{g}^*}X)\xrightarrow{\simeq}\mu^{-1}(0).$$

Thus, Theorem 3.28 gives that

$$H^{\bullet}_{BRST}(\mathfrak{g},\mathbb{C}[X\times\widetilde{\mathscr{S}_f}])\cong\mathbb{C}[X\times_{\mathfrak{g}^*}\mathscr{S}_f]\otimes H^{\bullet}_{DR}(\mathbf{G}).$$

$\square$

EXERCISE 3.40. Show that $\mathfrak{g}^*\circ\mathfrak{g}^*\cong\mathfrak{g}^*/\!/\mathbf{G}$, that is, $\mathbb{C}[\mathfrak{g}^*\circ\mathfrak{g}^*]\cong\mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}$.

3.4.5. Drinfeld-Sokolov reduction in the Poisson setting. Let X be an affine Poisson scheme equipped with a Hamiltonian $\mathbf{G}$-action. The composition of the moment map μ_X with the projection $\mathfrak{g}^*\to\mathfrak{m}^*$ is the moment map for the $\mathbf{M}$-action on X. We define the affine Poisson scheme $DS_f(X)$ by

$$DS_f(X):=X/\!/_{BRST,\chi}\mathbf{M}=\operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])).$$

Here $\chi=(f|\cdot)$ as before.

Note that Theorem 3.30 and Proposition 3.31 say that

$$DS_f(\mathfrak{g}^*)=\mathscr{S}_f,\quad DS_f(T^*\mathbf{G})=\widetilde{\mathscr{S}_f}.$$

THEOREM 3.41 (Ginzburg[**Gan-Ginzburg02**]). *For any affine Poisson scheme X equipped with a Hamiltonian $\mathbf{G}$-action, we have*

$$DS_f(X)\cong\widetilde{\mathscr{S}_f}\circ X\cong X\times_{\mathfrak{g}^*}\mathscr{S}_f.$$

Moreover, $H^i_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])=0$ for $i\neq 0$.

Proof. Recall the isomorphism $\Phi\colon T^*\mathbf{G}\times X\to T^*\mathbf{G}\times X$ in Exercise 3.38. We will show that the following diagram commutes:

$$\begin{tikzcd} T^*\mathbf{G}\times X \arrow[r, "//_{BRST}\mathbf{G}"] \arrow[d, "DS_f"] & X \arrow[d, "DS_f"] \\ \widetilde{\mathscr{S}_f}\times X \arrow[r, "//_{BRST}\mathbf{G}"] & DS_f(X). \end{tikzcd}$$

Set

$$C:=\mathbb{C}[X]\otimes\mathbb{C}[T^*\mathbf{G}]\otimes\overline{Cl}(\mathfrak{m})\otimes\overline{Cl}(\mathfrak{g}).$$

Let $\bar{Q}_{\mathfrak{m},\chi}\in\overline{C}(\mathfrak{m})$ and $\bar{Q}_{\mathfrak{g}}\in\overline{C}(\mathfrak{g})$ be the elements that give the differentials of the BRST complex for $H^\bullet_{BRST,\chi}(\mathfrak{m},\mathbb{C}[\mathfrak{m}])$ and $H^\bullet_{BRST}(\mathfrak{g},\mathbb{C}[\mathfrak{g}])$, respectively. Let $\bar{Q}_{1,\mathfrak{m}}\in C$ be the image of $\bar{Q}_{\mathfrak{m},\chi}$ by the embedding $\overline{C}(\mathfrak{m})\hookrightarrow C$ given by the moment map with respect to the action $\pi_{1,L}$ of $\mathbf{M}$ (this corresponds to the right vertical arrow). Let $\bar{Q}_{1,\mathfrak{g}}\in C$ be the image of $\bar{Q}_{\mathfrak{g}}$, by the embedding $\bar{C}(\mathfrak{g})\hookrightarrow C$ given by the moment map with respect to the action $\pi_{1,R}$ of $\mathbf{G}$ (this corresponds to the upper horizontal arrow). Let $\bar{Q}_{2,\mathfrak{m}}\in C$ be the image of $\bar{Q}_{\mathfrak{m},\chi}$ by the embedding $\overline{C}(\mathfrak{m})\hookrightarrow C$ given by the moment map with respect to the action $\pi_{2,L}$ of $\mathbf{M}$ (this corresponds to the left vertical arrow). Let $\bar{Q}_{2,\mathfrak{g}}\in C$ be the image of $\bar{Q}_{\mathfrak{g}}$, by the embedding $\bar{C}(\mathfrak{g})\hookrightarrow C$ given by the moment map with respect to the action $\pi_{2,R}$ of $\mathbf{G}$ (this corresponds to the lower horizontal arrow).

Define

$$Q_1=\bar{Q}_{1,\mathfrak{m}}+\bar{Q}_{1,\mathfrak{g}},\quad Q_2=\bar{Q}_{2,\mathfrak{m}}+\bar{Q}_{2,\mathfrak{g}}.$$

Since $\operatorname{ad}\bar{Q}_{i,\mathfrak{m}}$ and $\operatorname{ad}\bar{Q}_{i,\mathfrak{g}}$ obviously commute each other, $(\operatorname{ad}Q_i)^2=0$ for $i=1,2$. Moreover Φ induces the isomorphism $(C,\operatorname{ad}Q_1)\xrightarrow{\simeq}(C,\operatorname{ad}Q_2)$ of differential graded Poisson algebras. In particular

$$H^\bullet(C,\operatorname{ad}Q_1)\xrightarrow{\simeq}H^\bullet(C,\operatorname{ad}Q_2).\tag{64}$$

To compute $H^\bullet(C,\operatorname{ad}Q_1)$ one can use the spectral sequence (E_r,d_r) whose d_0 is $\operatorname{ad}\bar{Q}_{1,\mathfrak{g}}$ and d_1 is $\operatorname{ad}\bar{Q}_{1,\mathfrak{m}}$. We have

$$\begin{aligned}E_1^{\bullet,q}=H^q(C,\operatorname{ad}\bar{Q}_{1,\mathfrak{g}})&=\mathbb{C}[X]\otimes\overline{Cl}(\mathfrak{m})\otimes H^q_{BRST}(\mathfrak{g},\mathbb{C}[T^*\mathbf{G}])\\&\cong\mathbb{C}[X]\otimes\overline{Cl}(\mathfrak{m})\otimes H^q_{DR}(\mathbf{G}).\end{aligned}$$

It follows that the complex $(E_1^{\bullet,q},d_1)$ is the BRST complex for $H^\bullet_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])$ tensorized with $H^q_{DR}(\mathbf{G})$. Hence

$$E_2^{p,q}\cong H^p_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])\otimes H^q_{DR}(\mathbf{G}).\tag{65}$$

We can therefore represent classes in $E_2^{p,q}$ as tensor products $\omega_1\otimes\omega_2$ of a cocyle ω_1 in $\mathbb{C}[X]\otimes\overline{Cl}(\mathfrak{m})$ representing a class in $H^p_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])$ and a cocycle ω_2 in $\Lambda^q\mathfrak{g}^*\subset\mathbb{C}[T^*\mathbf{G}]\otimes\overline{Cl}(\mathfrak{g})$ representing a class in $H^q_{DR}(\mathbf{G})=H^q(\mathfrak{g},\mathbb{C})$. Applying the differential $\operatorname{ad}Q_1$ to this class, we find that it is identically equal to zero. Therefore all the classes in E_2 survive. Moreover, all of the elements of E_2 in the decomposition (65) lifts canonically to the cohomology $H^\bullet(C,\operatorname{ad}Q_1)$, and thus, we get that

$$H^\bullet(C,\operatorname{ad}Q_1)\cong H^\bullet_{BRST,\chi}(\mathfrak{m},\mathbb{C}[X])\otimes H^\bullet_{DR}(\mathbf{G}).\tag{66}$$

Similarly, to compute $H^{\bullet}(C, \operatorname{ad} Q_2)$ one can use the spectral sequence (E'_r, d'_r) such that $d'_0 = \operatorname{ad} \bar{Q}_{2,\mathfrak{m}}$ and $d'_1 = \operatorname{ad} \bar{Q}_{2,\mathfrak{g}}$. We have

$$\begin{aligned} E_1'^{\bullet,q} = H^q(C, \operatorname{ad} \bar{Q}_{2,\mathfrak{m}}) &= \mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{g}) \otimes H^p_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*\mathbf{G}]) \\ &\cong \delta_{q,0}\mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{g}) \otimes \mathbb{C}[\widetilde{\mathscr{S}_f}] \end{aligned}$$

It follows that the complex $(E_1'^{\bullet,q}, d_1)$ is the BRST complex for the operation $X \circ \widetilde{\mathscr{S}_f}$. Hence

$$E_2'^{p,q} \cong \delta_{q,0}\mathbb{C}[X \circ \widetilde{\mathscr{S}_f}] \otimes H^p_{DR}(\mathbf{G}).$$

We conclude that the spectral sequence collapses at $E_2 = E_\infty$, and we get that

$$H^{\bullet}(C, \operatorname{ad} Q_2) \cong \mathbb{C}[X \circ \widetilde{\mathscr{S}_f}] \otimes H^{\bullet}_{DR}(\mathbf{G}). \tag{67}$$

Finally (64), (66) and (67) give that

$$H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]) \cong \delta_{i,0}\mathbb{C}[X \circ \widetilde{\mathscr{S}_f}].$$

□

Let I be an $\operatorname{ad}\mathfrak{g}$-invariant graded ideal of $\mathbb{C}[\mathfrak{g}^*]$. Then I is a Poisson ideal, so that $\mathbb{C}[\mathfrak{g}^*]/I$ is a Poisson algebra. Set

$$\widetilde{\mathscr{V}}(I) := \operatorname{Spec}(\mathbb{C}[\mathfrak{g}^*]/I), \quad \mathscr{V}(I) := \operatorname{Specm}(\mathbb{C}[\mathfrak{g}^*]/I). \tag{68}$$

Thus, $\mathscr{V}(I)$ is the zero locus of I in $\mathfrak{g}^*$. The action of $\mathbf{G}$ on $\mathfrak{g}^*$ restricts to a Hamiltonian action on $\widetilde{\mathscr{V}}(I)$.

Corollary 3.42. *Let I be an $\operatorname{ad}\mathfrak{g}$-invariant graded ideal of $\mathbb{C}[\mathfrak{g}^*]$.*

(1) *$DS_f(\widetilde{\mathscr{V}}(I)) \neq 0$ if and only if $\mathscr{V}(I) \supset \overline{\mathbf{G}.f}$.*
(2) *The Poisson algebra $\mathbb{C}[DS_f(\widetilde{\mathscr{V}}(I))]$ is finite-dimensional if $\mathscr{V}(I) = \overline{\mathbf{G}.f}$.*

Proof. By applying Theorem 3.41, we get that

$$DS_f(\widetilde{\mathscr{V}}(I)) = \widetilde{\mathscr{V}}(I) \times_{\mathfrak{g}^*} \mathscr{S}_f,$$

which is isomorphic to $\mathscr{V}(I) \cap \mathscr{S}_f \subset \mathscr{S}_f$ as topological spaces.

(1) Since it is stable under the $\mathbb{C}^*$-acton on $\mathscr{S}_f$, $DS_f(\widetilde{\mathscr{V}}(I))$ is nonzero if and only it contains the point $\{f\}$. As $\mathscr{V}(I)$ is $\mathbf{G}$-invariant and closed, this is equivalent to that $\mathscr{V}(I) \supset \overline{\mathbf{G}.f}$.

(2) Clearly, $\mathbb{C}[DS_f(\widetilde{\mathscr{V}}(I))]$ is finite-dimensional if and only if $\dim DS_f(\widetilde{\mathscr{V}}(I)) = 0$, which is equivalent to that $\mathscr{V}(I) \cap \mathscr{S}_f = \{f\}$. On the other hand we have $\overline{\mathbf{G}.f} \cap \mathscr{S}_f = \{f\}$ by the trasversality of $\mathscr{S}_f$ to $\mathbf{G}$-orbits. □

3.4.6. BRST reduction of Poisson modules. The above results can be generalized to *Poisson modules*.

Let R be a Poisson algebra. A *Poisson R-module* is a R-module N in the usual associative sense equipped with a bilinear map

$$R \times N \to N, \quad (r, n) \mapsto \operatorname{ad} r(n) = \{r, n\},$$

which makes N a Lie algebra module over R satisfying

$$\{r_1, r_2 n\} = \{r_1, r_2\}n + r_2\{r_1, n\}, \quad \{r_1 r_2, n\} = r_1\{r_2, n\} + r_2\{r_1, n\}$$

for $r_1, r_2 \in R$, $n \in N$.

Let R-PMod denote the category of Poisson modules over R.

LEMMA 3.43. *For any Lie algebra $\mathfrak{g}$, a Poisson module over $\mathbb{C}[\mathfrak{g}^*]$ is the same as a $\mathbb{C}[\mathfrak{g}^*]$-module N in the usual associative sense equipped with a Lie algebra module structure $\mathfrak{g} \to \operatorname{End} N$, $x \mapsto \operatorname{ad}(x)$, such that*

$$\operatorname{ad}(x)(fn) = \{x, f\}.n + f.\operatorname{ad}(x)(n)$$

for $x \in \mathfrak{g}$, $f \in \mathbb{C}[\mathfrak{g}^*]$, $n \in N$.

Let G be any connected affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, χ a one-point orbit in $\mathfrak{g}^*$. Recall the differential graded algebra $(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$ defined in §3.4.1.

For $N \in \mathbb{C}[\mathfrak{g}^*]$-Pmod, $N \otimes \overline{Cl}$ is naturally a Poisson module over $\overline{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}(\mathfrak{g})$. (The notation of Poisson modules natural extends to the Poisson superalgebras modules.) Thus, $(N \otimes \overline{Cl}(\mathfrak{g}), \operatorname{ad}\bar{Q})$ is a differential graded Poisson module over the differential graded Poisson module $(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$. In particular its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, N) := H^{\bullet}(N \otimes \overline{Cl}, \operatorname{ad}\bar{Q})$$

is a Poisson module over $H^{\bullet}(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$, and thus over $H^{0}(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})$. So we get a functor

$$\mathbb{C}[\mathfrak{g}^*]\text{-Pmod} \to H^0(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})\text{-PMod}, \quad N \mapsto H^0_{BRST,\chi}(\mathfrak{g}, N).$$

More generally, let R be a Poisson algebra equipped with a Poisson algebra homomorphism $\mu^* : \mathbb{C}[\mathfrak{g}^*] \to R$. Then for a Poisson R-module M, $H^0_{BRST,\chi}(\mathfrak{g}, M)$ is a Poisson module over $H^0_{BRST,\chi}(\mathfrak{g}, R)$. Thus we et a functor

$$R\text{-Pmod} \to H^0(\overline{C}(\mathfrak{g}), \operatorname{ad}\bar{Q})\text{-PMod}, \quad N \mapsto H^0_{BRST,\chi}(\mathfrak{g}, N).$$

3.4.7. Results for Poisson modules. Let $\mathfrak{g} = \operatorname{Lie}(\mathbf{G})$ be simple.

Let $\overline{\mathcal{HC}}(\mathfrak{g})$ be the full subcategory of the category of Poisson $\mathbb{C}[\mathfrak{g}^*]$-modules on which the Lie algebra $\mathfrak{g}$-action is *integrable*, that is, locally finite.

If X is an affine Poisson scheme equipped with a Hamiltonian G-action then $\mathbb{C}[X]$ is an object of $\overline{\mathcal{HC}}(\mathfrak{g})$.

For $M, N \in \overline{\mathcal{HC}}(\mathfrak{g})$, define

$$M \circ N := H^0_{BRST}(\mathfrak{g}, M \otimes N),$$

where $\mathfrak{g}$ acts on $M \otimes N$ diagonally. Then $M \otimes N$ is a Poisson module over the trivial Poisson algebra $\mathbb{C}[\mathfrak{g}^* \circ \mathfrak{g}^*] = \mathbb{C}[\mathfrak{g}^*]^{\mathbf{G}}$.

The proof of the following assertion is similar to that of Proposition 3.37.

PROPOSITION 3.44. *For $M \in \overline{\mathcal{HC}}(\mathfrak{g})$,*

$$T^*\mathbf{G} \circ M \cong M$$

as a Poisson module over $\mathbb{C}[T^*\mathbf{G} \circ \mathfrak{g}^*] = \mathbb{C}[\mathfrak{g}^*]$.

For $M \in \overline{\mathcal{HC}}(\mathfrak{g})$, define

$$DS_f(M) = H^0_{BRST,\chi}(\mathfrak{m}, M),$$

which is a Poisson module over $\mathbb{C}[\mathscr{S}_f]$.

The following assertion can be proved in the same way as Theorem 3.41.

THEOREM 3.45. *For $M \in \overline{\mathcal{HC}}(\mathfrak{g})$, we have $H^i_{BRST,\chi}(\mathfrak{m}, M) = 0$ for $i \neq 0$. Therefore, the functor*

$$\overline{\mathcal{HC}}(\mathfrak{g}) \to \mathbb{C}[\mathscr{S}_f]\text{-PMod}, \quad M \mapsto DS_f(M),$$

is exact. We have

$$DS_f(M) \cong \mathbb{C}[\widetilde{\mathscr{S}_f}] \circ M$$

as a Poisson module over $\mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\widetilde{\mathscr{S}_f} \circ \mathfrak{g}^*]$.

COROLLARY 3.46. *Let I be an $\operatorname{ad}\mathfrak{g}$-invariant graded ideal of $\mathbb{C}[\mathfrak{g}^*]$ such that $\mathscr{V}(I) = \overline{\mathbf{G}.f}$ Then*

$$\dim DS_f(\widetilde{\mathscr{V}}(I)) = \operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I),$$

where the integer $\operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I)$ is defined in the below proof.

PROOF. There is a filtration

$$\mathbb{C}[\mathfrak{g}^*]/I = M_0 \supset M_1 \supset \cdots \supset M_r = 0$$

of $\mathbb{C}[\mathfrak{g}^*]$-modules such that $M_i/M_{i+1} = \mathbb{C}[\mathfrak{g}^*]/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal in $\mathbb{C}[\mathfrak{g}^*]$. The integer $\operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I)$ is by definition the number of indexes i such that $\mathfrak{p}_i$ coincides with the prime ideal corresponding to $\overline{G.f}$. As $\mathscr{V}(\mathfrak{p}_i) \subset \mathscr{V}(I) = \overline{\mathbf{G}.f}$,

$$DS_f(M_i) = \begin{cases} \mathbb{C} & \text{if } \mathscr{V}(\mathfrak{p}_i) = \overline{\mathbf{G}.f}, \\ 0 & \text{otherwise.} \end{cases}$$

The assertion follows from the exactness of $DS_f(?)$. □

3.4.8. Quantized BRST cohomology. We shall now quantize the above construction essentially following [**Kostant-Sternberg87**].

Again, let G be any connected affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, and let $\chi : \mathfrak{g} \to \mathbb{C}$ be a character.

Let $\{U_i(\mathfrak{g})\}$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The PBW theorem gives isomorphisms of Poisson algebras (see Example 3.16):

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i \geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*].$$

Set

$$C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl(\mathfrak{g}).$$

It is naturally a $\mathbb{C}$-superalgebra, where $U(\mathfrak{g})$ is considered as a purely even subsuperalgebra. The filtrations of $U(\mathfrak{g})$ and $Cl(\mathfrak{g})$ induce a PBW filtration of $C(\mathfrak{g})$,

$$C_p(\mathfrak{g}) = \sum_{i+j \leqslant p} U_i(\mathfrak{g}) \otimes Cl_j(\mathfrak{g}),$$

and we have

$$\operatorname{gr} C(\mathfrak{g}) \cong \overline{C}(\mathfrak{g})$$

as Poisson superalgebras. Therefore, $C(\mathfrak{g})$ is a quantization of $\overline{C}(\mathfrak{g})$.

Define a $\mathbb{Z}$-grading $C(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} C^n(\mathfrak{g})$ by setting $\deg u \otimes 1 = 1$ for $u \in U(\mathfrak{g})$, $\deg 1 \otimes f = 1$ for $f \in \mathfrak{g}^*$, $\deg 1 \otimes x = -1$ for $x \in \mathfrak{g}$. Then

$$C^n(\mathfrak{g}) = U(\mathfrak{g}) \otimes \Big(\bigoplus_{j-i=n} \wedge^i(\mathfrak{g}) \otimes \wedge^j(\mathfrak{g}^*)\Big).$$

LEMMA 3.47. *The following map defines a Lie algebra homomorphism:*

$$\begin{aligned}\theta_\chi : \mathfrak{g} &\longrightarrow C(\mathfrak{g})\\ x &\longmapsto (x - \chi(x)) \otimes 1 + 1 \otimes \rho(x)\end{aligned}$$

LEMMA 3.48 ([**Beilinson-Drinfeld96**, Lemma 7.13.7]). *There exists a unique element $Q \in C^1(\mathfrak{g})$ such that*

$$[Q, 1 \otimes x] = \theta_\chi(x) \quad \textit{for all} \quad x \in \mathfrak{g}.$$

We have $Q^2 = 0$.

PROOF. The proof is similar to that of Lemma 3.27. In fact the element Q is explicitly given by the same formula as $\bar{Q}$:

$$Q = \sum_i (x_i - \chi(x_i)) \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* x_k$$

□

Since Q is odd, Lemma 3.48 implies that

$$(\operatorname{ad} Q)^2 = 0.$$

Thus, $(C(\mathfrak{g}), \operatorname{ad} Q)$ is a *differential graded algebra*, and its cohomology

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, U(\mathfrak{g})) := H^\bullet(C(\mathfrak{g}), \operatorname{ad} Q)$$

is a graded superalgebra.

More generally, let A be a $U(\mathfrak{g})$-algebra. Then $A \otimes Cl(\mathfrak{g})$ is naturally a $C(\mathfrak{g})$-algebra, and $(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q)$ is naturally a differential graded algebra, where the image of Q is also denote by Q. Therefore, its cohomology

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, A) := H^\bullet(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q) \tag{69}$$

inherits a graded Poisson superalgebra structure from $A \otimes Cl(\mathfrak{g})$. We write $H^\bullet_{BRST}(\mathfrak{g}, A)$ for $H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ if $\chi = 0$.

3.4.9. Kazhdan filtration. In order to discuss the quantization of the BRST reduction, we need to modify the PBW filtration of $C(\mathfrak{g})$ when the character χ of $\mathfrak{g}$ is nonzero. We assume that there is a grading

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{70}$$

of $\mathfrak{g}$ (compatible with the Lie algebra structure) such that such that $\chi(\mathfrak{g}_j) = 0$ unless $j = 1$. (If $\chi = 0$, we can choose the trivial grading.) The grading (70) extends to $C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl(\mathfrak{g})$ by setting $\deg x = j$, $\deg x^* = -j$ for $x, x^* \in Cl(\mathfrak{g})$ if $x \in \mathfrak{g}_j$. Here and after, we omit the tensor product sign. Let

$$C(\mathfrak{g}) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} C(\mathfrak{g})[j]$$

be the corresponding grading.

Put $C_i(\mathfrak{g})[j] = C_i(\mathfrak{g}) \cap C(\mathfrak{g})[j]$. Note that $\mathfrak{g}_{j_0} (= \mathfrak{g}_{j_0} \otimes \mathbb{C}) \subset C_1(\mathfrak{g})[j_0]$. Define

$$K_p C(\mathfrak{g}) = \sum_{i - j \leqslant p} C_i(\mathfrak{g})[j]$$

for $p \in \mathbb{Z}$ (so that $\mathfrak{g}_{j_0} \subset K_0C(\mathfrak{g})$). Then $K_\bullet C(\mathfrak{g})$ defines an increasing, exhaustive, separated filtration of $C(\mathfrak{g})$ such that

$$K_pC(\mathfrak{g}) \cdot K_qC(\mathfrak{g}) \subset K_{p+q}C(\mathfrak{g}), \qquad [K_pC(\mathfrak{g}), K_qC(\mathfrak{g})] \subset K_{p+q-1}C(\mathfrak{g}),$$

and $\operatorname{gr}_K C(\mathfrak{g}) = \bigoplus_p K_pC(\mathfrak{g})/K_{p-1}C(\mathfrak{g})$ is isomorphic to $\overline{C}(\mathfrak{g})$ as Poisson superalgebras. Moreover,

$$\operatorname{ad} Q.K_pC(\mathfrak{g}) \subset K_pC(\mathfrak{g}),$$

and the associated graded complex $(\operatorname{gr}_K C(\mathfrak{g}), \operatorname{ad} Q)$ is identical to $(\overline{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$.

Let $K_\bullet U(\mathfrak{g})$ and $K_\bullet Cl(\mathfrak{g})$ be the restrictions of $K_\bullet C(\mathfrak{g})$ to $U(\mathfrak{g})$ and $Cl(\mathfrak{g})$, respectively, so that $\operatorname{gr}_K U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$, $\operatorname{gr}_K Cl(\mathfrak{g}) \cong \bar{C}l[\mathfrak{g}^*]$.

3.4.10. Quantized BRST reduction. Let G be any connected affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, $\chi \in \mathfrak{g}^*$ a one-point G-orbit.

Let $X = \operatorname{Spec}(R)$ be an affine, Hamiltonian Poisson G-scheme, and $\mu_X : X \to \mathfrak{g}^*$ the moment map.

We wish to quantized the BRST reduction

$$X \rightsquigarrow X/\!/_{BRST,\chi}G = \operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])).$$

A quantization of the Hamiltonian G-scheme X is an almost commutative filtered $U(\mathfrak{g})$-algebra $(A, F_\bullet A)$ equipped with an action of G such that

(1) $\operatorname{gr}_F A \cong \mathbb{C}[X]$ as Poisson algebra,
(2) $g(a.b) = (ga).(gb)$ for $g \in G$, $a, b \in A$,
(3) the action of $\mathfrak{g}$ obtained by differentiating the action of G coincides with the adjoint action of $\mathfrak{g}$,
(4) if we denote by $\tilde{\mu}^*$ the the natural algebra homomorphism $U(\mathfrak{g}) \to A$, $\tilde{\mu}^*(U_p(\mathfrak{g})) = F_pA \cap \tilde{\mu}^*(U(\mathfrak{g}))$, and the induced homomorphism $\mathbb{C}[\mathfrak{g}^*] = \operatorname{gr} U(\mathfrak{g}) \to \operatorname{gr}_F A = \mathbb{C}[X]$ coincides with μ_X^*.

Let $(A, F_\bullet A)$ be a quantization of the Hamiltonian G-scheme X, $\chi \in \mathfrak{g}^*$. Let $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}} \mathfrak{g}_j$ the grading of $\mathfrak{g}$ such that $\chi(\mathfrak{g}_j) = 0$ unless $j = 1$. A compatible grading on A is the grading $A = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} A[j]$ such that $\tilde{\mu}^*(\mathfrak{g}_j) \subset A[j]$ and $F_pA = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} F_pA[j]$, where $F_pA[j] = A[j] \cap F_pA$. With such a grading we can define the Kazudan filtration $K_\bullet A$ on A by

$$K_pA = \sum_{i-j \leqslant p} F_iA(\mathfrak{g})[j]$$

We have $\operatorname{gr}_K A \cong \mathbb{C}[X]$.

Note that by definition the image of the left ideal $A \sum_{x \in \mathfrak{g}} (\tilde{\mu}^*(x) - \chi(x))$ of A in $\operatorname{gr}_K A = \mathbb{C}[X]$ coincides with the defining ideal $\sum_{x \in \mathfrak{g}} (\mu_X^*(x) - \chi(x))\mathbb{C}[X]$ of $\mu_X^{-1}(\chi)$ in X.

We have a natural algebra homomorphism $C(\mathfrak{g}) \to A \otimes Cl(\mathfrak{g})$, and so, $(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q)$ is a differential graded (super)algebra. Set

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, A) := H^\bullet(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q).$$

Note that the filtrations $K_\bullet A$, $K_\bullet Cl$ induce a filtration $K_\bullet(A \otimes Cl(\mathfrak{g}))$ on $A \otimes Cl(\mathfrak{g})$ that is compatible with the action of $\operatorname{ad} Q$, and $(\operatorname{gr}_K(A \otimes Cl(\mathfrak{g})), \operatorname{ad} Q)$ is identical to the complex $(\mathbb{C}[X] \otimes \bar{C}l(\mathfrak{g}), \operatorname{ad} \bar{Q})$.

Let $K_\bullet H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ be the filtration of $H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ induced by $K_\bullet(A \otimes Cl(\mathfrak{g}))$, and $\operatorname{gr}_K H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ the associated graded.

THEOREM 3.49. *Assume that the condition (2) of Theorem 3.28 is verified, that is, there exists a subscheme $\mathscr{S}$ of $\mu_X^{-1}(\chi)$ such that the action map gives the isomorphism $G \times \mathscr{S} \xrightarrow{\simeq} \mu_X^{-1}(\chi)$.*

Under the above setting, there is an isomorphism of Poisson algebras

$$\mathrm{gr}_K\, H^\bullet_{BRST,\chi}(\mathfrak{g}, A) \cong H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) = \mathbb{C}[X/\!/_{BRST,\chi} G] \otimes H^\bullet_{DR}(G).$$

In particular, $H^0_{BRST,\chi}(\mathfrak{g}, A)$ is a quantization of $\mathbb{C}[X/\!/_{BRST,\chi} G]$.

PROOF. Consider the spectral sequence $E_r \Rightarrow H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ such that $E_1 = H^\bullet(\mathrm{gr}_K(A \otimes Cl(\mathfrak{g})), \mathrm{ad}\, Q) = H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$. By Theorem 3.28, $H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) = \mathbb{C}[\mathscr{S}] \otimes H^\bullet_{DR}(G)$. We can therefore represent classes in $E_1^{p,q}$ as tensor products $\omega_1 \otimes \omega_2$ of a cocyle ω_1 in $\mathbb{C}[X] \otimes \Lambda^p(\mathfrak{g})$ representing a class in $H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$ and a cocycle $\omega_2 \in \Lambda^q \subset Cl(\mathfrak{g})$ representing a class in $H^q_{DR}(\mathbf{G}) = H^q(\mathfrak{g}, \mathbb{C})$. Applying the differential $\mathrm{ad}\, Q$ to this class, we find that it is identically equal to zero. It follows that the spectral sequence collapses at $E_1 = E_\infty$. □

3.4.11. Finite W-algebras. We will apply the above construction for $(\mathfrak{g}^*, \mu, \chi)$, where μ is the moment map (52), $\chi = (f|\cdot) \in \mathfrak{m}^*$.

Clearly $U(\mathfrak{g})$ with the PBW filtration is a quantization of the Hamiltonian $\mathbf{G}$-scheme $\mathfrak{g}^*$. By restricting to the $\mathbf{M}$-action, we may regard $U(\mathfrak{g})$ as a quantization of the Hamiltonian $\mathbf{M}$-scheme $\mathfrak{g}^*$. The Dynkin grading (50) satisfies the condition of §3.4.9, as well as its restriction to $\mathfrak{m}$. So we have the corresponding Kazhdan filtrations $K_\bullet U(\mathfrak{m})$ and $K_\bullet U(\mathfrak{g})$.

By Theorem 3.49,

$$H^i_{BRST,\chi}(\mathfrak{m}, U(\mathfrak{g})) = 0 \quad \text{for} \quad i \neq 0, \tag{71}$$

and

$$U(\mathfrak{g}, f) := H^0_{BRST,\chi}(\mathfrak{m}, U(\mathfrak{g}))$$

is a quantization of $\mathbb{C}[\mathscr{S}_f]$. The algebra $U(\mathfrak{g}, f)$ is is called the *finite W-algebra associated with f*.

3.4.12. Definition via Whittaker models. Let $\chi = (f|\cdot) \in \mathfrak{g}^*$. It extends to a representation

$$\chi \colon U(\mathfrak{m}) \longrightarrow \mathbb{C}$$

and we denote by $\mathbb{C}_\chi$ the corresponding left $U(\mathfrak{m})$-module. The right multiplication by an element of $\mathfrak{m}$ induces a right $U(\mathfrak{m})$-module on $U(\mathfrak{g})$. Denote by I_χ the left ideal of $U(\mathfrak{g})$ generated by the elements $x - \chi(x)$, for $x \in \mathfrak{m}$,

$$I_\chi := \sum_{x \in \mathfrak{m}} U(\mathfrak{g})(x - \chi(x)),$$

and set

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi.$$

It is an $U(\mathfrak{g})$-module called a *generalized Gelfand-Graev module.*

The adjoint action of $\mathfrak{n}$ in $\mathfrak{g}$ uniquely extends to an action of $\mathfrak{n}$ in $U(\mathfrak{g})$ and the ideal I_χ is $\mathfrak{n}$-stable. Thus Q_χ is endowed with an $\mathfrak{n}$-module structure.

DEFINITION 3.50. The algebra

$$Q_\chi^{\mathfrak{m}} = \{\bar{u} \in Q_\chi \mid [y, u] \in I_\chi \text{ for any } y \in \mathfrak{m}\},$$

where $\bar{u}$ denotes the coset $u+I_\chi$ of $u \in U(\mathfrak{g})$, is called the *finite W-algebra associated with f*.

We refer the above definition of $U(\mathfrak{g}, f)$ as the *Whittaker model realization* of $U(\mathfrak{g}, f)$.

REMARK 3.51. The algebra $Q_\chi^{\mathfrak{m}}$ is actually the space of *Whittaker vectors* of Q_χ,

$$Q_\chi^{\mathfrak{m}} = \mathrm{Wh}(Q_\chi) = \{u \in Q_\chi \mid xu = \chi(x)u \text{ for any } x \in \mathfrak{m}\}.$$

EXAMPLE 3.52. Assume $e = 0$. Then $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{m} = 0$, $Q_\chi = U(\mathfrak{g})$ and $U(\mathfrak{g}, f) = U(\mathfrak{g})$.

The next result was obtained by Kostant ([**Kostant78**]) for the regular case. For the general case, see [**DeSole-Kac06**] or [**Arakawa07**]. The proof of Arakawa [**Arakawa07**] only concerns the regular case, but can be easily adapted to the general case. We follow here his proof.

PROPOSITION 3.53. *We have*

$$U(\mathfrak{g}, f) \cong \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{\mathrm{op}} \cong Q_\chi^{\mathfrak{m}}$$

where the symbol "op" means that we consider the ring $\mathrm{End}_{U(\mathfrak{g})}(Q_\chi)$ *with "reversed" composition operation $u.v := v \circ u$.*

PROOF. As in the case of $\overline{C}(\mathfrak{g})$, $C(\mathfrak{g})$ is also bigraded, so we can also write $\mathrm{ad}\, Q = d_+ + d_-$ such that $d_+(C^{ij}) \subset C^{i+1,j}, d_-(C^{ij}) \subset C^{i,j+1}$ and get a spectral sequence

$$E_r \Longrightarrow H^\bullet(C(\mathfrak{g}), \mathrm{ad}\, Q)$$

such that

$$\begin{aligned} E_2^{p,q} = H^p(H^q(C(\mathfrak{g}), d_-), d_+) &\cong \delta_{q,0} H^p(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi) \\ \cong \delta_{p,0}\delta_{q,0} H^0(\mathfrak{m}, U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi) &\cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{op}, \end{aligned}$$

where $\mathbb{C}_\chi$ is the one-dimensional representation of $\mathfrak{m}$ as in §3.4.12. Thus we get the Whittaker model isomorphism

$$U(\mathfrak{g}, f) = H^0(C(\mathfrak{g}), \mathrm{ad}\, Q) \cong Q_\chi^{\mathfrak{m}} \cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{op},$$

whence the statement by Proposition 3.53. □

Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The restriction to $\mathcal{Z}(\mathfrak{g})$ of the representation $U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(Q_\chi)$ is injective. So we get an inclusion map,

$$\mathcal{Z}(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}, f).$$

By Theorem 3.35, the above map is surjective onto the center $\mathcal{Z}(U(\mathfrak{g}, f))$ of $U(\mathfrak{g}, f)$ so that we get an algebra isomorphism

$$\mathcal{Z}(\mathfrak{g}) \cong \mathcal{Z}(U(\mathfrak{g}, f)).$$

According to a result of Kostant, if e is regular then $U(\mathfrak{g}, f)$ is isomorphic to $Z(\mathfrak{g})$, which is known to be a polynomial algebra in rank of $\mathfrak{g}$ variables. In particular, $U(\mathfrak{g}, f)$ is commutative in this case.

REMARK 3.54. The finite W-algebra $Q_\chi^{\mathfrak{m}}$ ($\cong U(\mathfrak{g}, f)$) a priori depends on the Lagrangian subspace $\mathfrak{L} \subset \mathfrak{g}_{\frac{1}{2}}$. But by [**Gan-Ginzburg02**], the algebra $Q_\chi^{\mathfrak{m}}$ does not depend, up to isomorphism, on the choice of the Lagrangian subspace $\mathfrak{L}$ in $\mathfrak{g}_{\frac{1}{2}}$. In fact, Gan and Ginzburg proved the following stronger fact:

$$Q_\chi^{\mathfrak{m}} \cong Q_\chi^{\mathfrak{n}} := \{\bar{u} \in Q_\chi \mid [y, u] \in I_\chi \text{ for any } y \in \mathfrak{n}\}$$

for any isotropic subspace $\mathfrak{L} \subset \mathfrak{g}_{\frac{1}{2}}$, where $\mathfrak{n} = \mathfrak{n}_{\chi, \mathfrak{L}}$ is as in §3.2.1. Furthermore, according to the main result of [**Brundan-Goodwin05**], the algebra $U(\mathfrak{g}, f)$ does not depend, up to isomorphism, on the choice of the *good grading adapted to* f ([**Elashvili-Kac05**]). Dynkin gradings are typical example of good grading. More generally, for (optimal) admissible gradings, the result is due to Sakada ([**Sadaka16**]).

3.4.13. Equivariant finite W-algebras. We can also apply the above construction to quantize the equivariant Slodowy slices. Consider the ring $\mathcal{D}_{\mathbf{G}}$ of the (global) differential operators on $\mathbf{G}$. We have

$$\mathcal{D}_{\mathbf{G}} = U(\mathfrak{g}) \otimes \mathbb{C}[\mathbf{G}]$$

as vector spaces, the natural maps

$$\mathbb{C}[\mathbf{G}] \hookrightarrow \mathcal{D}_{\mathbf{G}}, \quad \tilde{\mu}_L : U(\mathfrak{g}) \to \mathcal{D}_{\mathbf{G}}$$

are embeddings of algebras, and we have

$$[x, f] = x_L(f) \qquad (x \in \mathfrak{g},\ f \in \mathbb{C}[\mathfrak{g}^*])$$

The algebra $\mathcal{D}_{\mathbf{G}}$ is almost commutative by the standard filtration $G_\bullet \mathcal{D}_{\mathbf{G}}$ define by $G_p \mathcal{D}_{\mathbf{G}} = U_p(\mathfrak{g}) \otimes \mathbb{C}[\mathbf{G}]$, and we have $\operatorname{gr}_G \mathcal{D}_{\mathbf{G}} \cong \mathbb{C}[T^*\mathbf{G}]$.

The embedding $\tilde{\mu}_L : U(\mathfrak{g}) \to \mathcal{D}_{\mathbf{G}}$ quantizes the comorphism $\mu_L^* \colon \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*\mathbf{G}]$ of moment map μ_L (see (58)). This map is induced by the Lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}(\mathbf{G})$, $x \mapsto x_L$, where $\mathscr{X}(\mathbf{G})$ is the Lie algebra of the vector fields on $\mathbf{G}$ and

$$(x_L f)(g) = \frac{d}{dt} f(g \exp(tx))|_{t=0}.$$

Similarly, the Lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}(\mathbf{G})$, $x \mapsto x_R$, where

$$(x_R f)(g) = \frac{d}{dt} f(\exp(-tx) g)|_{t=0},$$

induces the algebra homomorphism

$$\tilde{\mu}_R \colon U(\mathfrak{g}) \to \mathcal{D}_{\mathbf{G}},$$

which quantizes the moment map $\mu_R \colon T^*\mathbf{G} \to \mathfrak{g}^*$. By definition, the two actions $\tilde{\mu}_L(x)$, $\tilde{\mu}_R(y)$ commute each other.

Thus, $\mathcal{D}_{\mathbf{G}}$ is a quantization of the Hamiltonian $\mathbf{G}$-scheme $T^*\mathbf{G}$ (with respect to both actions).

The grading

$$\mathcal{D}_{\mathbf{G}} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathcal{D}_{\mathbf{G}}[j], \quad \mathcal{D}_{\mathbf{G}}[j] = \{\nabla \in \mathcal{D}_D \mid [\tilde{\mu}_L(h), \nabla] = 2j\nabla\}$$

is compatible with the grading (50), Here recall that (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Thus, we have the corresponding Kazhdan filtration $K_\bullet \mathcal{D}_G$.

By Theorem 3.49, we have $H^i_{BRST,\chi}(\mathfrak{m}, \mathcal{D}_{\mathbf{G}}) = 0$ for $i \neq 0$, and

$$\widetilde{U}(\mathfrak{g}, f) := H^0_{BRST,\chi}(\mathfrak{m}, \mathcal{D}_{\mathbf{G}})$$

is a quantization of the equivariant Slodowy slice $\mathscr{S}_f$, that is, the Kazudan filtration on $\mathcal{D}_{\mathbf{G}}$ induces the filtration $K_\bullet \widetilde{U}(\mathfrak{g}, f)$ and we have $\operatorname{gr}_K \widetilde{U}(\mathfrak{g}, f) \cong \mathbb{C}[\widetilde{\mathscr{S}_f}]$. The algebra $\widetilde{U}(\mathfrak{g}, f)$ is called the *equivariant finite W-algebra* ([**Los10**]).

Exercise 3.55. Show that $\widetilde{U}(\mathfrak{g}, f)$ is a simple algebra.

In the definition of $\widetilde{U}(\mathfrak{g}, f)$, the BRST reduction is taken with respect to, say, the action $\tilde{\mu}_L$. So $\widetilde{U}(\mathfrak{g}, f)$ is a $\mathbf{G}$-module with respect to the action $g \mapsto g_R$, and $\tilde{\mu}_R$ gives the algebra homomorphism

$$\tilde{\mu}_R : U(\mathfrak{g}) \to \widetilde{U}(\mathfrak{g}, f), \quad u \mapsto \tilde{\mu}_R(u).$$

The adjoint action of $\mathfrak{g}$ on $\widetilde{U}(\mathfrak{g}, f)$ is the same as the action pf $\mathfrak{g}$ obtained by differentiating $\mathbf{G}$-action. Thus, $\widetilde{U}(\mathfrak{g}, f)$ is a quantization of the Hamiltonian $\mathbf{G}$-scheme $\widetilde{\mathscr{S}_f}$.

Proposition 3.56. *We have an algebra isomorphism*

$$U(\mathfrak{g}, f) \xrightarrow{\simeq} \widetilde{U}(\mathfrak{g}, f)^{\mathbf{G}} = \widetilde{U}(\mathfrak{g}, f)^{\operatorname{ad}\mathfrak{g}}.$$

Proof. The map $\tilde{\mu}_L$ induces the algebra homomorphism $U(\mathfrak{g}, f) \to \widetilde{U}(\mathfrak{g}, f)$, and the image is contained in $\widetilde{U}(\mathfrak{g}, f)^G$. Moreover this is an isomorphism since it induces an isomorphism

$$\operatorname{gr} U(\mathfrak{g}, f) = \mathbb{C}[\mathscr{S}_f] \xrightarrow{\simeq} \mathbb{C}[\tilde{\mathscr{S}}_f]^{\mathbf{G}} \cong (\operatorname{gr} \widetilde{U}(\mathfrak{g}, f))^{\mathbf{G}} = \operatorname{gr} \widetilde{U}(\mathfrak{g}, f)^{\mathfrak{g}}$$

by Proposition 3.33. In the above, the last equality is true since $\mathfrak{g}$ is simple and $\mathbf{G}$ connected. □

Remark 3.57. The algebra $\widetilde{U}(\mathfrak{g}, f)$ is the *twisted differential operators* (tdo) [**Hotta-Takeuchi-Tanisaki**] that quantizes the twisted contangent bundle $\mathbf{G} \times_{\mathbf{M}} (\chi + \mathfrak{m}^\perp)$. Thus, the finite W-algebra can be defined as the $\mathbf{G}$-invariant subalgebra of this tdo.

3.4.14. Quantized Moore-Tachikawa operation. A *Harish-Chandra* $U(\mathfrak{g})$-*algebra* is a $U(\mathfrak{g})$-algebra A equipped with an action of $\mathbf{G}$ such that $(ga).(gb) = g(ab)$ for $g \in \mathbf{G}$, $a, b \in A$, and the $\mathfrak{g}$-action on A obtained by differentiating the action of $\mathbf{G}$ coincides with the adjoint action of $\mathfrak{g}$. A quantization A of a Hamiltonian $\mathbf{G}$-scheme X is a Harish-Chandra $U(\mathfrak{g})$-algebra.

Let A, B be Harish-Chandra $U(\mathfrak{g})$-algebras. We define an algebra $A \circ B$ by

$$A \circ B := H^0_{BRST}(\mathfrak{g}, A \otimes B),$$

where $A \otimes B$ is considered as a diagonal $\mathfrak{g}$-module.

Exercise 3.58. Show that $U(\mathfrak{g}) \circ U(\mathfrak{g}) \cong U(\mathfrak{g})^{\mathbf{G}} = \mathcal{Z}(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Proposition 3.59. $\mathcal{D}_{\mathbf{G}} \circ A \cong A$ *for any Harish-Chandra* $U(\mathfrak{g})$-*algebra* A.

Proof. From Exercise 3.60 below, it follows that it is enough to show that $H^0_{BRST}(\mathfrak{g}, \mathcal{D}_G) = \mathbb{C}$. But this is easy to see. □

EXERCISE 3.60. Let A be a Harish-Chandra $U(\mathfrak{g})$-algebra, and let $\rho^* : A \to \mathbb{C}[\mathbf{G}] \otimes A$ be the comodule map, so that $\rho^*(a) = \sum_i f_i \otimes a_i$ if $ga = \sum_i f_i(g)a_i$, $f_i \in \mathbb{C}[\mathbf{G}]$, $a_i \in A$, for all $g \in \mathbf{G}$.

(1) Check that ρ^* is an algebra homomorphism.
(2) Define the algebra homomorphism $\phi : \mathbb{C}[\mathbf{G}] \otimes A \to \mathbb{C}[\mathbf{G}] \otimes A$ as the composition
$$\mathbb{C}[\mathbf{G}] \otimes A \xrightarrow{1\otimes\rho^*} \mathbb{C}[\mathbf{G}] \otimes \mathbb{C}[\mathbf{G}] \otimes A \xrightarrow{m^*\otimes 1} \mathbb{C}[\mathbf{G}] \otimes A,$$
where $m : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is the multiplication map. Show that ϕ is an isomorphism such that
$$(g_L \otimes g) \circ \phi = \phi \circ (g_L \otimes 1), \quad (g_R \otimes 1) \circ \phi = \phi \circ (g_R \otimes g)$$
where $(g_L)(f)(g_1) = f(g^{-1}g_1)$, $(g_R)(f)(g_1) = f(g_1 g)$.
(3) Define the algebra homomorphism
$$\Phi \colon \mathcal{D}_{\mathbf{G}} \otimes A \to \mathcal{D}_{\mathbf{G}} \otimes A$$
by
$$\Phi((\tilde{\mu}_R(u)f) \otimes a) = (\tilde{\mu}_R(u) \otimes 1)\phi^{-1}(f \otimes a)$$
for $u \in U(\mathfrak{g})$, $f \in \mathbb{C}[\mathbf{G}]$, $a \in A$. Show that Φ is an isomorphism such that
$$(g_L \otimes g) \circ \Phi = \Phi \circ (g_L \otimes 1), \quad (g_R \otimes 1) \circ \Phi = \Phi \circ (g_R \otimes g).$$

3.4.15. Drinfeld-Sokolov reduction in the algebra setting. For a Harish-Chandra $U(\mathfrak{g})$-algebra A, define the algebra $DS_f(A)$ by
$$DS_f(A) := H^0_{BRST,\chi}(\mathfrak{m}, A).$$
If A is a quantization of a Hamiltonian $\mathbf{G}$-scheme X, one can define the Kazhdan filtration $K_\bullet A$ using the grading $A = \bigoplus_j A[j]$, $A[j] = \{a \in A \mid [h, a] = 2ja\}$. This induces a filtration $K_\bullet DS_f(A)$ of $DS_f(A)$.

Theorem 3.41 gives the following.

THEOREM 3.61. *Let A be a quantization of a Hamiltonian $\mathbf{G}$-scheme X. Then $H^i_{BRST,\chi}(\mathfrak{m}, A) = 0$ for $i \neq 0$, and*
$$DS_f(A) \cong \widetilde{U}(\mathfrak{g}, f) \circ A.$$
Moreover, we have the Poisson algebra isomorphism
$$\operatorname{gr}_K DS_f(A) \cong \mathbb{C}[DS_f(X)] = \mathbb{C}[X \times_{\mathfrak{g}^*} \mathscr{S}_f].$$

3.4.16. Drinfeld-Sokolov reduction for modules. Let $\mathcal{HC}(\mathfrak{g})$ be the category of Harish-Chandra bimodules, that is, the full subcategory of the cagegory of $U(\mathfrak{g})$-bimodules M consisting of objects M on which the adjoint action of $\mathfrak{g}$ is integrable, that is, locally finite.

A good filtration of $M \in \mathcal{HC}(\mathfrak{g})$ is an increasing, separated, exhaustive filtration $F_\bullet M$ of M such that $U_i(\mathfrak{g})F_pMU_j(\mathfrak{g}) \subset F_{p+i+j}M$, $[U_i(\mathfrak{g}), F_jM] \subset F_{i+j-1}M$, and the associated graded $\operatorname{gr}_F M = \bigoplus_p F_pM/F_{p-1}M$ is *finitely generated* as a $\mathbb{C}[\mathfrak{g}^*]$-module. Note that $\operatorname{gr}_F M \in \overline{\mathcal{HC}}(\mathfrak{g})$.

A good filtration exists if M is finitely generated.

For $M, N \in \mathcal{HC}(\mathfrak{g})$, $M \otimes N \otimes Cl(\mathfrak{g})$ is naturally a module over $C(\mathfrak{g}) \otimes Cl(\mathfrak{g})$, where $U(\mathfrak{g})$ acts on $M \otimes N$ diagonally. So we can define
$$H^\bullet_{BRST}(\mathfrak{g}, M \otimes N) = H^\bullet(M \otimes N \otimes Cl(\mathfrak{g}), \operatorname{ad} Q)$$

Let

$$M \circ N := H^0_{BRST}(\mathfrak{g}, M \otimes N).$$

Note that if M and N are bimodules over Harish-Chandra $U(\mathfrak{g})$-algebras A and B respectively, then $M \circ N$ is naturally a bimodule over $A \circ B$. In particular it is a module over $U(\mathfrak{g}) \circ U(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$.

PROPOSITION 3.62. *For any $M \in \mathcal{HC}(\mathfrak{g})$ we have*

$$\mathcal{D}_{\mathbf{G}} \circ M \cong M$$

as a bimodule over $\mathcal{D}_{\mathbf{G}} \circ U(\mathfrak{g}) = U(\mathfrak{g})$.

For $M \in \mathcal{HC}(\mathfrak{g})$, $M \otimes Cl(\mathfrak{m})$ is naturally module over $C(\mathfrak{m}) = U(\mathfrak{m}) \otimes Cl(\mathfrak{m})$, and so we can define the cohomology $H^{\bullet}_{BRST,\chi}(\mathfrak{m}, M)$. Define

$$DS_f(M) := H^0_{BRST,\chi}(\mathfrak{m}, M),$$

which is a bimodule over $DS_f(U(\mathfrak{g}) = U(\mathfrak{g}, f)$.

Let $M \in \mathcal{HC}(\mathfrak{g})$ be finitely generated, and let $F_{\bullet}M$ a good filtration of M. Then we can define the corresponding Kazhdan filtration $K_{\bullet}M$ by

$$K_pM = \sum_{i-j \leqslant p} F_iM[j],$$

where $F_iM[j] = F_iM \cap M[j]$, with $M[j] = \{m \in M \mid [h, m] = 2jm\}$. Then $K_{\bullet}M$ is also good, and $\operatorname{gr}_K M \in \overline{HC}(\mathfrak{g})$. It induces a filtration $K_{\bullet}DS_f(M)$ of $DS_f(M)$, and $\operatorname{gr}_K DS_f(M)$ is naturally a Poisson module over $\operatorname{gr}_K U(\mathfrak{g}, f) = \mathbb{C}[\mathscr{S}_f]$.

The following assertion follows from Theorem 3.45.

THEOREM 3.63. *For $M \in \mathcal{HC}(\mathfrak{g})$, we have $H^i_{BRST,\chi}(\mathfrak{m}, M) = 0$ for $i \neq 0$. Therefore, the functor*

$$\mathcal{HC}(\mathfrak{g}) \to U(\mathfrak{g}, f)\text{-biMod}, \quad M \mapsto DS_f(M),$$

is exact. We have

$$DS_f(M) \cong \widetilde{U}(\mathfrak{g}, f) \circ M$$

as a bimodule over $\widetilde{U}(\mathfrak{g}, f) \circ U(\mathfrak{g}) = U(\mathfrak{g}, f)$. Moreover if $M \in \mathcal{HC}(\mathfrak{g})$ is finitely generated and $K_{\bullet}M$ is a good Kazudan filtration then

$$\operatorname{gr}_K DS_f(M) \cong DS_f(\operatorname{gr}_K M)$$

as a Poisson module over $\mathbb{C}[\mathscr{S}_f]$.

3.4.17. Primitive ideals and representation theory of finite W-algebras. Let I be a two-sided ideal of $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I, so that $\operatorname{gr} I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$. Thus, $U(\mathfrak{g})/I$ is a quantization of the Hamiltonian $\mathbf{G}$-scheme $\widetilde{\mathscr{V}}(\operatorname{gr} I) = \operatorname{Spec} \mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I$.

The variety

$$\mathscr{V}(\operatorname{gr} I) = \operatorname{Specm} \mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I = (\widetilde{\mathscr{V}}(\operatorname{gr} I))_{\mathrm{red}} \subset \mathfrak{g}^*$$

is usually referred to as the *associated variety* of I.

Consider the exact sequence $0 \to I \to U(\mathfrak{g}) \to U(\mathfrak{g})/I \to 0$ in $\mathcal{HC}(\mathfrak{g})$. By Theorem 3.63, applying to the exact functor $DS_f(?)$ we obtain the exact sequence

$$0 \to DS_f(I) \to U(\mathfrak{g}, f) \to DS_f(U(\mathfrak{g})/I) \to 0.$$

Following Losev [**Losev11**], we set

$$I_{\dagger} := DS_f(I),$$

which is a two-sided ideal of $U(\mathfrak{g}, f)$, so that

$$DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_{\dagger}.$$

By Theorem 3.63,

$$\operatorname{gr}_K DS_f(U(\mathfrak{g})/I) \cong DS_f(\operatorname{gr}_K U(\mathfrak{g})/I) = \mathbb{C}[\widetilde{\mathscr{V}}(\operatorname{gr} I) \times_{\mathfrak{g}^*} \mathscr{S}_f].$$

Recall that a proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of a simple left $U(\mathfrak{g})$-module. There are two important results on primitive ideals of $U(\mathfrak{g})$. The first result is the Duflo Theorem [**Duflo77**], stating that any primitive ideal in $U(\mathfrak{g})$ is the annihilator $\operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$ of some simple highest weight module $L_{\mathfrak{g}}(\lambda)$, $\lambda \in \mathfrak{h}^*$.

The second result is the *Irreducibility Theorem*. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(\ |\ )$, we shall often view associated varieties of ideals of $U(\mathfrak{g})$ as subsets of $\mathfrak{g}$. The Irreducibility Theorem says that the associated variety $\mathscr{V}(\operatorname{gr} I)$ of a primitive ideal I in $U(\mathfrak{g})$ is irreducible, specifically, it is the closure $\overline{\mathbb{O}}$ of some nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$. The latter theorem was first partially proved (by a case-by-case argument) in [**Borho-Brylinski82**], and in a more conceptual way in [**Kashiwara-Tanisaki84**] and [**Joseph85**] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others.

It is possible that different primitive ideals share the same associated variety. In addition, not all nilpotent orbit closures appear as associated variety of some primitive ideal of $U(\mathfrak{g})$.

Given a nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$, we denote by $\operatorname{Prim}_{\mathbb{O}} U(\mathfrak{g})$ the set of all primitive ideal of $U(\mathfrak{g})$ such that $\mathscr{V}(\operatorname{gr} I) = \overline{\mathbb{O}}$.

The following assertion follows immediately from Theorem 3.63 and Corollary 3.46.

THEOREM 3.64 ([**Los10**]). *Let $I \in \operatorname{Prim}_{\mathbb{O}} U(\mathfrak{g})$. Then*

(1) *$DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_{\dagger}$ is nonzero if and only if $\mathbf{G}.f \subset \overline{\mathbb{O}}$.*
(2) *$DS_f(U(\mathfrak{g})/I)$ is finite-dimensional if and only if $\overline{\mathbb{O}} = \overline{\mathbf{G}.f}$. Moreover, if this is the case,* $\dim DS_f(U(\mathfrak{g})/I)) = \operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(\operatorname{gr} I)$.

In fact, the following much stronger result is known by I. Losev.

THEOREM 3.65 ([**Losev11**]). *Let $I \in \operatorname{Prim}_{\mathbf{G}.f} U(\mathfrak{g})$. Then $DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_{\dagger}$ is a (finite-dimensional) semisimple algebra.*

3.4.18. Skryabin equivalence. A $\mathfrak{g}$-module E is called a *Whittaker module* if for all $x \in \mathfrak{m}$, $x - \chi(x)$ acts on E locally nilpotently. A *Whittaker vector* in a Whittaker $\mathfrak{g}$-module E is a vector $v \in E$ which satisfies $(x - \chi(x)v = 0$ for any $x \in \mathfrak{m}$, i.e., $xv = \chi(x)v$ for any $x \in \mathfrak{m}$.

Let $\operatorname{Wh}(\mathfrak{g})$ be the category of finitely generated Whittaker $\mathfrak{g}$-modules and set for E an object of this category,

$$\operatorname{Wh}(E) := \{v \in E \mid (x - \chi(x))v = 0 \text{ for any } x \in \mathfrak{m}\}.$$

Observe that $\operatorname{Wh}(E) = 0$ implies that $E = 0$. Let $U(\mathfrak{g}, f)$-Mod be the category of finitely generated $U(\mathfrak{g}, f)$-modules and introduce the *Whittaker functor*:

$$\operatorname{Wh}\colon \operatorname{Wh}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}, f)\text{-Mod}, \qquad E \longmapsto \operatorname{Wh}(E)$$

with $\mathrm{Wh}(\psi)(x) = \psi(x)$ for $E, F \in \mathrm{Ob}(\mathrm{Wh}(\mathfrak{g}))$, $\psi \in \mathrm{Hom}_{\mathrm{Wh}(\mathfrak{g})}(E, F)$ and $x \in \mathrm{Wh}(E)$. Given a Whittaker $\mathfrak{g}$-module E, $\mathrm{Wh}(E)$ is indeed naturally a $U(\mathfrak{g}, f)$-module by setting

$$\bar{y}.v = y.v, \qquad v \in \mathrm{Wh}(E),\ \bar{y} \in U(\mathfrak{g}, f) = (U(\mathfrak{g})/I_\chi)^{\mathfrak{n}}.$$

We define another functor by:

$$Q_\chi \otimes_{U(\mathfrak{g},f)} - \colon U(\mathfrak{g}, f)\text{-Mod} \longrightarrow \mathrm{Wh}(\mathfrak{g}), \qquad V \longmapsto Q_\chi \otimes_{U(\mathfrak{g},f)} V$$

with

$$Q_\chi \otimes_{U(\mathfrak{g},f)} (\varphi)(q \otimes v) = q \otimes \varphi(v)$$

for $V, W \in \mathrm{Ob}(U(\mathfrak{g}, f)\text{-Mod})$, $\varphi \in \mathrm{Hom}_{U(\mathfrak{g},f)\text{-Mod}}(V, W)$, $q \in Q_\chi$ and $v \in V$. For $V \in U(\mathfrak{g}, f)\text{-Mod}$, $Q_\chi \otimes_{U(\mathfrak{g},f)} V$ is a Whittaker $\mathfrak{g}$-module by setting

$$y.(q \otimes v) = (y.q) \otimes v, \qquad y \in U(\mathfrak{g}),\ q \in Q_\chi = U(\mathfrak{g})/I_\chi,\ v \in V.$$

Theorem 3.66 ([**Premet02**, Appendix], [**Gan-Ginzburg02**, Theorem 6.1]). *The functor* $Q_\chi \otimes_{U(\mathfrak{g},f)} - \colon U(\mathfrak{g}, f)\text{-Mod} \longrightarrow \mathrm{Wh}(\mathfrak{g})$ *is an equivalence of categories, with* $\mathrm{Wh}\colon \mathrm{Wh}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}, f)\text{-Mod}$ *as inverse.*

Corollary 3.67. *Let I be a two-sided ideal of $U(\mathfrak{g})$. Then* $\mathrm{Wh}\colon \mathrm{Wh}(\mathfrak{g}) \xrightarrow{\simeq} U(\mathfrak{g}, f)\text{-Mod}$ *restricts to the equivalence*

$$\mathrm{Wh}(\mathfrak{g})^I \xrightarrow{\simeq} U(\mathfrak{g}, f)/I_\dagger\text{-Mod},$$

where $\mathrm{Wh}(\mathfrak{g})^I$ *is the full subcategory of* $\mathrm{Wh}(\mathfrak{g})$ *consisting of objects M that is annihilated by I.*

There is a ramification of the Skryabin's equivalence. It is an equivalence between the *category* $\mathcal{O}$ (see [**Brundan-Goodwin-Kleshchev08**]) for a finite W-algebra and the category of *generalized Whittaker* $U(\mathfrak{g})$*-modules*. This was conjectured in [**Brundan-Goodwin-Kleshchev08**] and proved by Losev [**Losev12**].

3.4.19. Classification of finite-dimensional representation of finite W-algebras and primitive ideals. By Theorem 3.65, any $I \in \mathrm{Prim}_{\mathbf{G}.f} U(\mathfrak{g})$ gives rise to an irreducible finite-dimensional representation of $U(\mathfrak{g}, f)$. Conversely, let E be a finite-dimensional irreducible representation of $U(\mathfrak{g}, f)$. Then, by Theorem 3.66, $Q_\chi \otimes_{U(\mathfrak{g},f)} E$ is simple, and thus, $I = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},f)} E)$ is a primitive ideal of $U(\mathfrak{g})$. Moreover, $I \in \mathrm{Prim}_{\mathbf{G}.f} U(\mathfrak{g})$ by [**?**]. In fact, E is a $DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_\dagger$-module ([**Losev10b, Ginzburg09**]). In other words, the map

$$E \mapsto \mathrm{Ann}_{U(\mathfrak{g},f)}(Q_\chi \otimes_{U(\mathfrak{g},f} E)$$

from the set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, f)$-modules to the set $\mathrm{Prim}_{\mathbf{G}.f} U(\mathfrak{g})$ is surjective. Moreover, any fiber of this map is a single $\mathbf{C}(e)$-fibers, where $\mathbf{C}(e) = \mathbf{Q}/\mathbf{Q}^\circ$ is the component group of the stabilizer $\mathbf{Q} = Z_{\mathbf{G}}(e, h, f)$ of the $\mathfrak{sl}_2$-triple (e, h, f). This was partially proved by Losev in [**Losev12**], and then by Losev and Ostrik [**Losev-Ostrik14**].

3.4.20. Multiplicity free primitive ideals and one-dimensional representations of finite W-algebras. A primitive ideal I of $U(\mathfrak{g})$ is called *multiplicity free* if $\mathrm{mult}_{\overline{\mathbb{O}}}\widetilde{\mathscr{V}}(\mathrm{gr}\, I) = 1$, where $\mathbb{O}$ is the nilpotent orbit such that $\mathscr{V}(\mathrm{gr}\, I) = \overline{\mathbb{O}}$. A multiplicity free primitive ideal I is *completely prime*, that is, $U(\mathfrak{g})/I$ is a domain.

By Theorem 3.65, if I is a multiplicity free primitive ideal such that $\mathscr{V}(\mathrm{gr}\, I) = \overline{G.f}$, then $DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_\dagger$ is one-dimensional. In particular, $U(\mathfrak{g}, f)$ admits a one-dimensional representation. Conversely, it is known that if E is a one-dimensional representation then $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},f)} E)$ is multiplicity free.

THEOREM 3.68. *Any finite W-algebra admits a one-dimensional representation (equivalently, a two-sided ideal of codimension 1).*

3.4.21. The Joseph ideal. If $\mathfrak{g}$ is not of type A, it is known [**Joseph76, Gan-Savin04**] that there exists a unique completely prime ideal, that is, the graded ideal is prime, in $U(\mathfrak{g})$ whose associated variety is $\overline{\mathbb{O}_{min}}$.

DEFINITION 3.69. This ideal is denoted by $\mathcal{J}_0$ and referred to as the *Joseph ideal* of $U(\mathfrak{g})$.

For $\mathfrak{g}$ of type A, the completely prime primitive ideals I of $U(\mathfrak{g})$ with $\mathscr{V}(\mathrm{gr}\, I) = \overline{\mathbb{O}_{min}}$ form a single family parametrized by the elements of $\mathbb{C}$ ([**Joseph76, ?**]).

In [**Joseph76**], Joseph has also computed the *infinitesimal character of* $\mathcal{J}_0$, that is, the algebra homomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ through which the centre $\mathcal{Z}(\mathfrak{g})$ acts on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. In fact, Joseph has described the set of $\lambda \in \mathfrak{h}^*$ such that such that $\mathcal{J}_0 = \mathrm{Ann}_{U(\mathfrak{g})}(L_\mathfrak{g}(\lambda))$ (see [**Joseph76**, Table p.15] or [**Arakawa-Moreau15**, Table 1]).

Let us recall how to get the infinitesimal character of $L_\mathfrak{g}(\lambda)$ (or of $\mathrm{Ann}_{U(\mathfrak{g})}(L_\mathfrak{g}(\lambda))$) from the knowledge of $\lambda \in \mathfrak{h}^*$.

Identify $\mathrm{Spec}\, \mathcal{Z}(\mathfrak{g})$ with the set of all homomorphisms $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. Such morphisms are called *infinitesimal characters*. Consider the projection map from $U(\mathfrak{g})$ to $U(\mathfrak{h}) = S(\mathfrak{h})$ with respect to the decomposition

$$U(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+).$$

It is not a morphism of algebras in general, but its restriction to $U(\mathfrak{g})^\mathfrak{h} = \{u \in U(\mathfrak{g}) \mid (\mathrm{ad}\, x)u = 0 \text{ for all } x \in \mathfrak{h}\}$ is. In particular, we get a morphism

$$p\colon \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$$

since $S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$, usually refers to as the *Harish-Chandra morphism*. Its comorphism gives a map

$$\chi\colon \mathfrak{h}^* \to \mathrm{Spec}(\mathcal{Z}(\mathfrak{g})), \quad \lambda \mapsto \chi_\lambda,$$

where $\chi_\lambda(z) = p(z)(\lambda + \rho)$ for $z \in \mathcal{Z}(\mathfrak{g})$. An important consequence of the Harish-Chandra Theorem is that the map χ induces a bijection

$$\mathfrak{h}^*/\mathbf{W} \xrightarrow{\simeq} \mathrm{Spec}(\mathcal{Z}(\mathfrak{g})).$$

Here the Weyl group $\mathbf{W}$ acts on $\mathfrak{h}^*$ with respect to the twisted action of $\mathbf{W}$:

$$w \circ \lambda = w.(\lambda + \rho) - \rho, \qquad w \in \mathbf{W},\ \lambda \in \mathfrak{h}^*,$$

where $\cdot$ is the usual action of $\mathbf{W}$ on $\mathfrak{h}^*$.

Returning to our subject, the infinitesimal character associated with the irreducible representation $L_\mathfrak{g}(\lambda)$ is χ_λ. In particular, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same $\mathbf{W}$-orbit with respect to the twisted action of $\mathbf{W}$.

Outside type A the orbit $\mathbb{O}_{min}$ is *rigid*, that is, forms a single *sheet* in $\mathfrak{g}^* \cong \mathfrak{g}$ (see §B.0.5). Hence $\mathcal{J}_0$ cannot be obtained by parabolic induction from a primitive ideal of a proper Levi subalgebra of $\mathfrak{g}$.

Different realizations of $\mathcal{J}_0$ can be found in the literature for various types of $\mathfrak{g}$. Joseph's original proof of the uniqueness of $\mathcal{J}_0$ was incomplete. This leads Gan and Savin [**Gan-Savin04**] to give another description of the Joseph ideal $\mathcal{J}_0$. Their argument relies on some invariant theory and earlier results of Garfinkle. Gan and Savin description was very useful in the recent work [**Arakawa-Moreau15**] as it will explained in §4.6.2.

Let us briefly explain their description.

Suppose that $\mathfrak{g}$ is not of type A. According to Kostant, $\mathcal{J}_0$ is generated by the $\mathfrak{g}$-submodule $L_{\mathfrak{g}}(0) \oplus W$ in $S^2(\mathfrak{g})$, where W is such that, as $\mathfrak{g}$-modules,

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus L_{\mathfrak{g}}(0) \oplus W.$$

Note that the above decomposition of $S^2(\mathfrak{g})$ still holds in type A ([**Garfinkle82**, Chapter IV, Proposition 2]).

Also, note that $L_{\mathfrak{g}}(0) = \mathbb{C}\Omega$ where $\Omega = \sum_i x_i x^i$ is the Casimir element in $S(\mathfrak{g})$, with $\{x_i\}_i$ is a basis of $\mathfrak{g}$, and $\{x^i\}_i$ its dual basis with respect to $(\ |\ )$.

LEMMA 3.70. *Suppose that $\mathfrak{g}$ is not of type A. The ideal J_W in $S(\mathfrak{g})$ generated by W contains Ω^2, and hence, $\sqrt{J_W} = J_0$, where J_0 is the prime ideal of $S(\mathfrak{g})$ corresponding to the minimal nilpotent orbit closure $\overline{\mathbb{O}_{\min}}$.*

PROOF. By the proof of [**Gan-Savin04**, Theorem 3.1] $\mathcal{J}_W$ contains $\mathfrak{g} \cdot \Omega$, and the assertion follows. □

The structure of W was determined by Garfinkle [**Garfinkle82**]. Consider the $\mathfrak{sl}_2$-triple $(e_\theta, h_\theta, f_\theta)$ of $\mathfrak{g}$ where $f_\theta = e_{-\theta}$ is a θ-root vector so that it lies in $\mathbb{O}_{min}$. Set

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h_\theta, x] = 2jx\}.$$

Then (cf. Example 3.7)

$$\begin{gathered}\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,\\ \mathfrak{g}_{-1} = \mathbb{C}f_\theta,\ \mathfrak{g}_1 = \mathbb{C}e_\theta,\ \mathfrak{g}_0 = \mathbb{C}h_\theta \oplus \mathfrak{g}^\natural,\ \mathfrak{g}^\natural = \{x \in \mathfrak{g}_0 \mid (h_\theta | x) = 0\}.\end{gathered}$$

The subalgebra $\mathfrak{g}^\natural$ is a reductive subalgebra of $\mathfrak{g}$ whose simple roots are the simple roots of $\mathfrak{g}$ perpendicular to θ. Write

$$[\mathfrak{g}^\natural, \mathfrak{g}^\natural] = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$$

as a direct sum of simple summands, and let θ_i be the highest root of $\mathfrak{g}_i$.

If $\mathfrak{g}$ is neither of type A_r nor C_r,

$$W = \bigoplus_{i \geqslant 1} L_{\mathfrak{g}}(\theta + \theta_i). \tag{72}$$

If $\mathfrak{g}$ is of type C_r, then $\mathfrak{g}^\natural$ is simple of type C_{r-1}, so that there is a unique θ_1, and we have

$$W = L_{\mathfrak{g}}(\theta + \theta_1) \oplus L_{\mathfrak{g}}(\frac{1}{2}(\theta + \theta_1)).$$

By [Garfinkle82, Gan-Savin04] $\mathcal{J}_0$ is generated by W and $\Omega - c_0$, where W is identified with a $\mathfrak{g}$-submodule of $U(\mathfrak{g})$ by the $\mathfrak{g}$-module isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$ given by the symmetrization map, and c_0 is the eigenvalue of Ω for the infinitesimal character that Joseph obtained. We have

$$\operatorname{gr} \mathcal{J}_0 = J_0 = \sqrt{J_W}$$

and this shows that $\mathcal{J}_0$ is indeed completely prime.

Let $\mathcal{J}_W$ be the two-sided ideal of $U(\mathfrak{g})$ generated by W.

PROPOSITION 3.71. *We have the algebra isomorphism*

$$U(\mathfrak{g})/\mathcal{J}_W \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

PROOF. By the proof of [Gan-Savin04, Theorem 3.1], $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$. Hence it contains $(\Omega - c_0)\Omega$. Since $c_0 \neq 0$, we have the isomorphism of algebras

$$U(\mathfrak{g})/\mathcal{J}_W \xrightarrow{\simeq} U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega\rangle \times U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega - c_0\rangle.$$

As we have explained above, $\langle \mathcal{J}_W, \Omega - c_0\rangle = \mathcal{J}_0$. Also, since $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$, $\langle \mathcal{J}_W, \Omega\rangle$ contains $\mathfrak{g}$. Therefore $U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega\rangle = \mathbb{C}$ as required. □

3.4.22. Classical Miura map. Assume that f is *even*[6] (cf. Example 3.6) or, equivalently, that the grading $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ is even, so that $\mathfrak{L} = \mathfrak{g}_{\frac{1}{2}} = 0$. Then $\mathfrak{m} = \bigoplus_{j>0} \mathfrak{g}_j$.

Let $\mathfrak{m}_- = \bigoplus_{j<0} \mathfrak{g}_j$ be the opposed Lie subalgebra to $\mathfrak{m}$. We have

$$\mathfrak{g} = \mathfrak{m}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{m}. \tag{73}$$

Note that $\mathfrak{m}^\perp = \mathfrak{g}_0 \oplus \mathfrak{m}$ is a parabolic subalgebra of $\mathfrak{g}$ (containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$). Set $\mathfrak{m}_-^\perp = \mathfrak{m}_- \oplus \mathfrak{g}_0$.

Let $\{x_i\}_{1 \leqslant i \leqslant m}$ be a basis of $\mathfrak{m}$, and extend it to a basis $\{x_i\}_{1 \leqslant i \leqslant n}$ of $\mathfrak{g}$. Let $c_{i,j}^k$ be the structure constants with respect to this basis. Consider the linear map $\theta_0 \colon \mathfrak{m} \to C(\mathfrak{m})$ of Lemma 3.47 with respect to the Lie algebra $\mathfrak{m}$ (i.e., $\chi = 0$ and $\mathfrak{g} = \mathfrak{m}$ in this lemma). Extend it to a linear map $\theta_0 \colon \mathfrak{g} \to C(\mathfrak{g}, \mathfrak{m}) := U(\mathfrak{g}) \otimes Cl(\mathfrak{m})$ by setting

$$\theta_0(x_i) = x_i \otimes 1 + 1 \otimes \sum_{1 \leqslant j,k \leqslant m} c_{i,j}^k x_k x_j^*.$$

We already know that the restriction of θ_0 to $\mathfrak{m}$ is a Lie algebra homomorphism and

$$[\theta_0(x), 1 \otimes y] = 1 \otimes [x, y] \quad \text{for } x, y \in \mathfrak{m}.$$

Although θ_0 is not a Lie algebra homomorphism, we have the following.

LEMMA 3.72. *The restriction of θ_0 to $\mathfrak{m}_-^\perp$ is a Lie algebra homomorphism. We have $[\theta_0(x), 1 \otimes y^*] = 1 \otimes \operatorname{ad}^*(x)(y)$ for $x \in \mathfrak{m}_-^\perp$, $y \in \mathfrak{m}^*$, where ad^* denote the coadjoint action and $\mathfrak{m}^*$ is identified with $(\mathfrak{g}/\mathfrak{m}_-^\perp)^*$.*

Recall that $U(\mathfrak{g}, f) = H^0(C(\mathfrak{g}, \mathfrak{m}), \operatorname{ad} Q)$. Let $C(\mathfrak{g}, \mathfrak{m})_+$ denote the subalgebra of $C(\mathfrak{g}, \mathfrak{m})$ generated by $\theta_0(\mathfrak{m})$ and $\wedge(\mathfrak{m}) \subset Cl$, and let $C(\mathfrak{g}, \mathfrak{m})_-$ denote the subalgebra generated by $\theta_0(\mathfrak{m}_-^\perp)$ and $\wedge(\mathfrak{m}^*) \subset Cl$.

[6]In the setting of good gradings, one can always find a good grading for f which is even, so the assumption is not restrictive.

LEMMA 3.73. *The multiplication map gives a linear isomorphism*

$$C(\mathfrak{g},\mathfrak{m})_- \otimes C(\mathfrak{g},\mathfrak{m})_+ \xrightarrow{\simeq} C(\mathfrak{g},\mathfrak{m}).$$

LEMMA 3.74. *The subspaces $C(\mathfrak{g},\mathfrak{m}))_-$ and $C(\mathfrak{g},\mathfrak{m})_+$ are subcomplexes of $(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q)$. Hence $C(\mathfrak{g},\mathfrak{m}) \cong C(\mathfrak{g},\mathfrak{m})_- \otimes C(\mathfrak{g},\mathfrak{m})_+$ as complexes.*

PROOF. The fact that $C(\mathfrak{g},\mathfrak{m})_-$ is subcomplex is obvious (see Lemma 3.48). The fact that $C(\mathfrak{g},\mathfrak{m})_+$ is a subcomplex follows from the following formulas.

$$[Q,\theta_0(x_i)] = \sum_{m+1\leqslant j\leqslant n, 1\leqslant k\leqslant m} c_{k,i}^{j}\theta_0(x_j)(1\otimes x_k^*) - 1\otimes \sum_{1\leqslant j,k\leqslant m} c_{i,j}^{k}\chi(x_k)x_j^*$$

$$[Q,1\otimes x_i^*] = -1\otimes \frac{1}{2}\sum_{1\leqslant j,k\leqslant m} c_{j,k}^{i} x_j^* x_k^*.$$

□

PROPOSITION 3.75. $H^\bullet(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) \cong H^\bullet(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q)$.

PROOF. By Lemma 3.74 and Kunneth's Theorem,

$$H^p(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q) \cong \bigoplus_{i+j=p} H^i(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q)\otimes H^j(C(\mathfrak{g},\mathfrak{m})_+, \operatorname{ad} Q).$$

On the other hand, we have $\operatorname{ad}(Q)(1\otimes x_i) = \theta_\chi(x_i) = \theta_0(x_i) - \chi(x_i)$ for $i = 1,\dots,m$. Hence $C(\mathfrak{g},\mathfrak{m})_-$ is isomorphic to the tensor product of complexes of the form $\mathbb{C}[\theta_\chi(x_i)]\otimes\wedge(x_i)$ with the differential $\theta_\chi(x_i)\otimes x_i^*$, where x_i^* is the contraction with x_i. Each of these complexes has one-dimensional zeroth cohomology and zero first cohomology. Therefore $H^i(C(\mathfrak{g},\mathfrak{m})_+, \operatorname{ad} Q) = \delta_{i,0}\mathbb{C}$. This completes the proof. □

Note that the cohomological gradation takes only non-negative values on $C(\mathfrak{g},\mathfrak{m})_-$. Hence by Proposition 3.75 we may identify $U(\mathfrak{g},f) = H^0(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q)$ with the subalgebra $H^0(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) = \{c\in C(\mathfrak{g},\mathfrak{m})_-^0 \mid (\operatorname{ad} Q)c = 0\}$ of $C(\mathfrak{g},\mathfrak{m})_-$.

Consider the decomposition

$$C(\mathfrak{g},\mathfrak{m})_- = \bigoplus_{j\leqslant 0} C(\mathfrak{g},\mathfrak{m})_{-,j}, \quad C(\mathfrak{g},\mathfrak{m})_{-,j} = \{c\in C(\mathfrak{g},\mathfrak{m})_-^0 \mid [\theta_0(h), c] = 2jc\}.$$

Note that $C(\mathfrak{g},\mathfrak{m})_{-,0}$ is generated by $\theta_0(\mathfrak{g}_0)$ and is isomorphic to $U(\mathfrak{g}_0)$. The projection

$$C(\mathfrak{g},\mathfrak{m})_- \to C(\mathfrak{g},\mathfrak{m})_{-,0} \cong U(\mathfrak{g}_0)$$

is an algebra homomorphism, and hence, its restriction

$$\Upsilon\colon U(\mathfrak{g},f) = H^0(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) \to U(\mathfrak{g}_0)$$

is also an algebra homomorphism.

PROPOSITION 3.76. *The map Υ is an embedding.*

Let $K_\bullet C(\mathfrak{g},\mathfrak{m})_\pm$ be the filtration of $C(\mathfrak{g},\mathfrak{m})_\pm$ induced by the Kazhdan filtration of $C(\mathfrak{g},\mathfrak{m})$. We have the isomorphism

$$\mathbb{C}[\mathfrak{g}^*]\otimes\overline{Cl}(\mathfrak{m}) = \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m}) \cong \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_- \otimes \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_+$$

as complexes. Similarly as above, we have $H^i(\operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad}\bar{Q}) = \delta_{i,0}\mathbb{C}$, and

$$H^0(\overline{C}(\mathfrak{g},\mathfrak{m}), \operatorname{ad}\bar{Q}) \cong H^0(\operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad}\bar{Q}). \tag{74}$$

Proof of Proposition 3.76. The filtration $K_\bullet U(\mathfrak{g}_0)$ of $U(\mathfrak{g}_0)\cong C(\mathfrak{g},\mathfrak{m})_{-,0}$ induced by the Kazhdan filtration coincides with the usual PBW filtration. By (74) and Theorem 3.30, the induced map

$$H^0(\operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_-,\operatorname{ad}Q)\to \operatorname{gr}_K U(\mathfrak{g}_0)$$

can be identified with the restriction map

$$\bar{\Upsilon}\colon \mathbb{C}[\mathscr{S}_f]=\mathbb{C}[f+\mathfrak{m}^\perp]^{\mathbf{M}}\to \mathbb{C}[f+\mathfrak{g}_0]. \tag{75}$$

So it is sufficient to show that $\bar{\Upsilon}$ is injective.

If $f\in\mathbb{C}[f+\mathfrak{m}^\perp]^{\mathbf{M}}$ is in the kernel, $f(g.x)=0$ for all $g\in\mathbf{M}$ and $x\in f+\mathfrak{g}_0$. Hence it is enough to show that the image of the the action map

$$\mathbf{M}\times(f+\mathfrak{g}_0)\to f+\mathfrak{m}^\perp,\quad (g,x)\mapsto g.x, \tag{76}$$

is Zariski dense in $f+\mathfrak{m}^\perp$.

The differential of this morphism at $(1,x)\in\mathbf{M}\times(f+\mathfrak{g}_0)$ is given by

$$\mathfrak{m}\times\mathfrak{g}_0\to\mathfrak{m}^\perp,\quad (y,z)\mapsto[y,x]+z.$$

This is an isomorphism if $x\in f+(\mathfrak{g}_0)_{\mathrm{ss,reg}}$, where $(\mathfrak{g}_0)_{\mathrm{ss,reg}}=\{x\in(\mathfrak{g}_0)_{\mathrm{ss}}\mid \dim\mathfrak{g}_0^x=r\}$, with $(\mathfrak{g}_0)_{\mathrm{ss}}$ the set of semisimple elements of $\mathfrak{g}_0$. Indeed, if $x\in(\mathfrak{g}_0)_{\mathrm{ss,reg}}$ then $\mathfrak{g}^x=\mathfrak{g}_0^x$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}^x\cap\mathfrak{m}=\{0\}$. Hence (76) is a dominant morphism as required, see e.g. [**Tauvel-Yu**, Theorem 16.5.7]. □

Remark 3.77. In the case where f is regular, the fact that $\bar{\Upsilon}$ is injective is well-known. Indeed, in this case $\mathfrak{g}_0$ is the Cartan subalgebra $\mathfrak{h}$ and, under the identifications $\mathbb{C}[\mathscr{S}_f]\cong\mathbb{C}[\mathfrak{g}]^{\mathbf{G}}$, $\mathbb{C}[f+\mathfrak{h}]\cong\mathbb{C}[\mathfrak{h}]$, the map $\bar{\Upsilon}$ is identified with the Chevalley restriction map $\mathbb{C}[\mathfrak{g}]^{\mathbf{G}}\xrightarrow{\simeq}\mathbb{C}[\mathfrak{h}]^{\mathbf{W}}$, where $\mathbf{W}$ is the Weyl group associated with $(\mathfrak{g},\mathfrak{h})$.

It is also possible to extend to the case f is not even, for instance, using the construction of Kac, Roan and Wakimoto [**Kac-Roan-Wakimoto03**].

The advantage of the above proof is that it applies to a general finite W-algebra ([**Lynch79**]), and also, it generalizes to the affine setting, see §**??**.

The map Υ is called the classical *Miura map*.

PART 4

Geometry of jet schemes, Poisson vertex algebras and associated varieties of vertex algebras

4.1. Jet schemes and arc spaces

In this section, we present some general facts on jet schemes and arc spaces. Our main references on the topic are [**Mustata01**, **Ein-Mustata09**, **Ishii11**].

4.1.1. Definitions. Denote by *Sch* the category of schemes of finite type over $\mathbb{C}$. Let X be an object of this category, and $m \in \mathbb{Z}_{\geqslant 0}$.

DEFINITION 4.1. An *m-jet of X* is a morphism

$$\operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \longrightarrow X.$$

The set of all m-jets of X carries the structure of a scheme $J_m(X)$, called the *m-th jet scheme of X*. It is a scheme of finite type over $\mathbb{C}$ characterized by the following functorial property: for every scheme Z over $\mathbb{C}$, we have

$$\operatorname{Hom}_{Sch}(Z, J_m(X)) = \operatorname{Hom}_{Sch}(Z \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}), X).$$

The $\mathbb{C}$-points of $J_m(X)$ are thus the $\mathbb{C}[t]/(t^{m+1})$-points of X. From Definition 4.1, we have for example that $J_0(X) \simeq X$ and that $J_1(X) \simeq \mathrm{T}X$ where $\mathrm{T}X$ denotes the total tangent bundle of X.

The canonical projection $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{n+1})$, $m \geqslant n$, induces a *truncation morphism* $\pi_{X,m,n} \colon J_m(X) \to J_n(X)$. The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{m+1})$ induces a morphism $\iota_{X,m} \colon X \to J_m(X)$, and we have $\pi_{X,m} \circ \iota_{X,m,0} = \mathrm{id}_X$. Hence $\iota_{X,m}$ is injective and $\pi_{X,m,0}$ is surjective.

Define the (formal) disc as

$$D := \operatorname{Spec} \mathbb{C}[[t]].$$

The projections $\pi_{X,m,n}$ yield a projective system $\{J_m(X), \pi_{X,m,n}\}_{m \geqslant n}$ of schemes.

DEFINITION 4.2. Denote by $J_\infty(X)$ its projective limit in the category of schemes,

$$J_\infty(X) = \varprojlim J_m(X).$$

It is called the *arc space*, or *the infinite jet scheme* of X.

Thus elements of $J_\infty(X)$ are the morphisms

$$\gamma \colon D \to \mathbb{C}[[t]],$$

and for every scheme Z over $\mathbb{C}$,

$$\operatorname{Hom}_{Sch}(Z, J_m(X)) = \operatorname{Hom}_{Sch}(Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} D, X),$$

where $Z\widehat{\times}D$ is the completion of $Z\times D$ with respect to the subscheme $Z\times\{0\}$. In other words, the contravariant functor

$$Sch \to Set, \quad Z \mapsto \mathrm{Hom}_{Sch}(Z\widehat{\times}D, X)$$

is represented by the scheme $J_\infty(X)$. The reason why we need the completion $Z\widehat{\times}D$ in the definition is that, for A an algebra, $A\otimes\mathbb{C}[[t]] \subsetneq A[[t]] = A\widehat{\otimes}\mathbb{C}[[t]]$ in general.

We denote by $\pi_{X,\infty}$ the canonical projection:

$$\pi_{X,\infty}\colon J_\infty(X) \to X.$$

4.1.2. The affine case. In the case where X is affine, we have the following explicit description of $J_\infty(X)$. (We describe similarly $J_m(X)$.)

The quite uncommon notations below will be justified next subsection.

Let $N \in \mathbb{Z}_{>0}$ and $X \subset \mathbb{C}^N$ be an affine subscheme defined by an ideal $I = \langle f_1,\dots,f_r\rangle$ of $\mathbb{C}[x^1,\dots,x^N]$. Thus

$$X = \mathrm{Spec}\ \mathbb{C}[x^1,\dots,x^N]/I.$$

For $f \in \mathbb{C}[x^1,\dots,x^N]$, we extend f as a map from $\mathbb{C}[[t]]^N$ to $\mathbb{C}[[t]]$ via base extension. Then giving a morphism $\gamma\colon D\to X$ is equivalent to giving a morphism $\gamma^*\colon \mathbb{C}[x^1,\dots,x^N]/I \to \mathbb{C}[[t]]$, or to giving

$$\gamma^*(x^i) = \sum_{j\geqslant 0}\gamma^i_{(-j-1)}t^j, \qquad i=1,\dots,N,$$

such that for any $k=1,\dots,r$,

$$f_k(\gamma^*(x^1),\dots,\gamma^*(x^N)) = 0 \ \text{ in } \ \mathbb{C}[[t]].$$

For any $f\in\mathbb{C}[x^1,\dots,x^N]$, there exist functions $f^{(j)}$, $j\geqslant 0$, which only depend on f, in the variables $\gamma = (\gamma^1_{(-j-1)},\dots,\gamma^N_{(-j-1)})_{j\geqslant 0}$ such that

$$f\big(\gamma^*(x^1),\dots,\gamma^*(x^N)\big) = \sum_{j\geqslant 0}\frac{f^{(j)}}{j!}(\gamma)\,t^j. \tag{77}$$

Regarding the coordinates x^i as functions over $\mathbb{C}^N$, we set:

$$x^i_{(-j-1)} := (x^i)^{(j)}, \quad \text{that is,} \quad x^i_{(-j-1)}(\gamma) = j!\gamma^i_{(-j-1)},$$

for $i=1,\dots,N$.

The jet scheme $J_\infty(X)$ is then the closed subscheme in $\mathrm{Spec}\,\mathbb{C}[x^i_{(-j-1)}\,;\, i=1,\dots,N,\ j\geqslant 0]$ defined by the ideal generated by the polynomials $f_k^{(j)}$, for $k=1,\dots,r$ and $j\geqslant 0$, that is,

$$J_\infty(X) \cong \mathrm{Spec}\,\mathbb{C}[x^i_{(-j-1)}\,;\, i=1,\dots,N,\ j\geqslant 0]/\langle f_k^{(j)}\,;\, k=1,\dots,r,\ j\geqslant 0\rangle.$$

In particular, if X is an N-dimensional vector space, then

$$J_\infty(X) \cong \mathrm{Spec}\,\mathbb{C}[x^i_{(-j-1)}\,;\, i=1,\dots,N,\ j\geqslant 0],$$

and for $m\in\mathbb{Z}_{\geqslant 0}$, the projection $J_\infty(X)\to J_m(X)$ corresponds to the projection onto the first $(m+1)N$ coordinates.

One can also define the functions $f_k^{(j)}$ using a derivation.

LEMMA 4.3. *Define the derivation T of $\mathbb{C}[x^i_{(-j-1)}\ ;\ i=1,\dots,N,\ j\geqslant 0]$ by*

$$Tx^i_{(-j)} = jx^i_{(-j-1)}, \qquad j \geqslant 0.$$

Then $f_k^{(j)} = T^j f_k$ for $k=1,\dots,r$ and $j\geqslant 0$. Here we identify x^i with $x^i_{(-1)}$.

With the above lemma, we conclude that for the affine scheme $X = \operatorname{Spec} R$, with $R=\mathbb{C}[x^1,x^2,\cdots,x^N]/\langle f_1,f_2,\cdots,f_r\rangle$, its arc space $J_\infty X$ is the affine scheme $\operatorname{Spec}(J_\infty(R))$, where

$$J_\infty(R) := \frac{\mathbb{C}[x^i_{(-j)}\ ;\ i=1,2,\cdots,N,\ j\geqslant 0]}{\langle T^j f_i\ ;\ i=1,\dots,r,\ j\geqslant 0\rangle}$$

and T is as defined in the lemma.

The derivation T acts on the above quotient ring $J_\infty(R)$. Hence for an affine scheme $X=\operatorname{Spec} R$, the coordinate ring $J_\infty(R)=\mathbb{C}[J_\infty(X)]$ of its arc space $J_\infty(X)$ is a differential algebra, hence is a commutative vertex algebra by Theorem 2.4.

REMARK 4.4. The differential algebra $(J_\infty(R),T)$ is universal in the following sense. We have a $\mathbb{C}$-algebra homomorphism $j\colon R\to J_\infty(R)$ such that if (A,∂) is another differential algebra, and if $f\colon R\to A$ is a $\mathbb{C}$-algebra homomorphism, then there is a unique differential algebra homomorphism $h\colon J_\infty(R)\to A$ making the following diagram commutative.

$$\begin{tikzcd}
R \arrow[rr, "j"] \arrow[dr, "f"'] & & (J_\infty(R),T) \arrow[dl, dashed, "h"] \\
& (A,\partial) &
\end{tikzcd}$$

(The map h is a *differential algebra homomorphism* means that it is a $\mathbb{C}$-algebra homomorphism such that $\partial(h(u)) = h(T(u))$ for all $u\in J_\infty(R)$.)

LEMMA 4.5 ([**Ein-Mustata09**]). *Let $m\in\mathbb{Z}_{\geqslant 0}\cup\{\infty\}$. Then for every open subset U of X, $J_m(U)=\pi^{-1}_{X,m}(U)$.*

Then for a general scheme Y of finite type with an affine open covering $\{U_i\}_{i\in I}$, its arc space $J_\infty(Y)$ is obtained by glueing $J_\infty(U_i)$ (see [**Ein-Mustata09, Ishii11**]). In particular, the structure sheaf $\mathcal{O}_{J_\infty(Y)}$ is a sheaf of commutative vertex algebras.

The natural projection $\pi_{X,\infty}\colon J_\infty(X)\to X$ corresponds to the embedding $R\hookrightarrow J_\infty(R)$, $x^i\to x^i_{(-1)}$ in the case where $X=\operatorname{Spec} R$ is affine. In terms of arcs, $\pi_{X,\infty}(\alpha)=\alpha(0)$ for $\alpha\in\operatorname{Hom}_{Sch}(D,X)$, where 0 is the unique closed point of the formal disc D.

4.1.3. Basic properties. The map from a scheme to its jet schemes and arc space is functorial. If $f\colon X\to Y$ is a morphism of schemes, then we naturally obtain a morphism $J_m f\colon J_m(X)\to J_m(Y)$ making the following diagram commutative,

$$\begin{tikzcd}
J_m(X) \arrow[r, "J_m f"] \arrow[d, "\pi_{X,m,0}"'] & J_m(Y) \arrow[d, "\pi_{Y,m,0}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In terms of arcs, it means that $J_m f(\alpha) = f \circ \alpha$ for $\alpha \in J_m(X)$. This also holds for $m = \infty$.

We have also the following for $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and for every schemes X, Y,

$$J_m(X \times Y) \cong J_m(X) \times J_m(Y). \tag{78}$$

Indeed, for any scheme Z in Sch,

$$\begin{aligned} \mathrm{Hom}(Z, J_m(X \times Y)) &= \mathrm{Hom}(Z \times_{\mathrm{Spec}_{\mathbb{C}}} \mathbb{C}[t]/(t^{m+1}), X \times Y) \\ &\cong \mathrm{Hom}(Z \times_{\mathrm{Spec}_{\mathbb{C}}} \mathbb{C}[t]/(t^{m+1}), X) \times \mathrm{Hom}(Z \times_{\mathrm{Spec}_{\mathbb{C}}} \mathbb{C}[t]/(t^{m+1}), Y) \\ &= \mathrm{Hom}(Z, J_m(X)) \times \mathrm{Hom}(Z, J_m(Y)) \\ &\cong \mathrm{Hom}(Z, J_m(X) \times J_m(Y)). \end{aligned}$$

For $m = \infty$, just replace $\mathbb{C}[t]/(t^{m+1})$ with $\mathbb{C}[[t]]$ and take the completion in the product $Z \widehat{\times} \operatorname{Spec} \mathbb{C}[[t]] = Z \widehat{\times} D$.

If $\mathbf{A}$ is a group scheme over $\mathbb{C}$, then $J_m(\mathbf{A})$ is also a group scheme over $\mathbb{C}$. Moreover, by (78), if $\mathbf{A}$ acts on X, then $J_m(\mathbf{A})$ acts on $J_m(X)$.

EXAMPLE 4.6. Consider the algebra

$$\mathfrak{g}_\infty := \mathfrak{g}[[t]] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]] \cong J_\infty(\mathfrak{g}).$$

It is naturally a Lie algebra, with Lie bracket:

$$[xt^m, yt^n)] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z}_{\geqslant 0}.$$

The arc space $J_\infty(\mathbf{G})$ of the algebraic group $\mathbf{G}$ is naturally a proalgebraic group. Regarding $J_\infty(\mathbf{G})$ as the set of $\mathbb{C}[[t]]$-points of $\mathbf{G}$, we have $J_\infty(\mathbf{G}) = \mathbf{G}[[t]]$. As Lie algebras, we have

$$\mathfrak{g}_\infty \cong \mathrm{Lie}(J_\infty(\mathbf{G})).$$

The adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ induces an action of $J_\infty(\mathbf{G})$ on $\mathfrak{g}_\infty$, and the coadjoint action of $\mathbf{G}$ on $\mathfrak{g}^*$ induces an action of $J_\infty(\mathbf{G})$ on $J_\infty(\mathfrak{g}^*)$, and so on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$.

We refer to [**Mustata01**, Appendix] for the following result.

LEMMA 4.7. *For $f \in \mathbb{C}[\mathfrak{g}]^{\mathbf{G}}$, the polynomials $T^j f = f^{(j)}$, $j \geqslant 0$, are elements of $\mathbb{C}[\mathfrak{g}_\infty]^{J_\infty(\mathbf{G})}$. In particular, the arc space $J_\infty(\mathcal{N})$ of the nilpotent cone is the subscheme of $\mathfrak{g}_\infty$ defined by the equations $T^j P_i$, $i = 1 \dots, r$ and $j \geqslant 0$, if $P_1, \dots, P_r$ are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^{\mathbf{G}}$, that is,*

$$J_\infty(\mathcal{N}) = \operatorname{Spec} \mathbb{C}[\mathfrak{g}_\infty]/(T^j P_i\ ;\ i = 1 \dots, r,\ j \geqslant 0).$$

4.1.4. Geometrical results. So far, we have stated basic properties common for both jet schemes $J_m(X)$ and the arc space $J_\infty(X)$. For the geometry, arc spaces behave rather differently. The main reason is that $\mathbb{C}[[t]]$ is a domain, contrary to $\mathbb{C}[t]/(t^{m+1})$. Thereby the geometry of are spaces is somehow simpler.

However, although $J_m(X)$ is of finite type if X is, this is not anymore true for $J_\infty(X)$, and its coordinate ring $\mathbb{C}[J_\infty(X)]$ is not noetherian in general.

LEMMA 4.8. *Denote by X_{red} the reduced scheme of X. The natural morphism $X_{\mathrm{red}} \to X$ induces an isomorphism $J_\infty X_{\mathrm{red}} \xrightarrow{\simeq} J_\infty X$ of topological spaces.*

PROOF. We may assume that $X = \operatorname{Spec} R$. An arc α of X corresponds to a ring homomorphism $\alpha^* \colon R \to \mathbb{C}[[t]]$. Since $\mathbb{C}[[t]]$ is an integral domain, it decomposes as $\alpha^* \colon R \to R/\sqrt{0} \to \mathbb{C}[[t]]$. Thus, α is an arc of X_{red}. □

Similarly, if $X = X_1 \cup \ldots \cup X_r$, where all X_i are closed in X, then

$$J_\infty(X) = J_\infty(X_1) \cup \ldots \cup J_\infty(X_r).$$

(Note that Lemma 4.8 is false for the schemes $J_m(X)$.)

If X is a point, then $J_\infty(X)$ is also a point, since $\mathrm{Hom}(D, X) = \mathrm{Hom}(\mathbb{C}, \mathbb{C}[[t]])$ consists of only one element. Thus, Lemma 4.8 implies the following.

COROLLARY 4.9. *If X is zero-dimensional then $J_\infty(X)$ is also zero-dimensional.*

THEOREM 4.10 ([**Kolchin73**]). *The scheme $J_\infty(X)$ is irreducible if X is irreducible.*

Theorem 4.10 is false for the jet schemes $J_m(X)$: see for instance [**Moreau-Yu16**] for counter-examples in the setting of nilpotent orbit closures. We refer to *loc. cit.*, and reference therein, for more about existing relations between the geometry of the jet schemes $J_m(X)$, $m \in \mathbb{Z}_{\geqslant 0}$, and the singularities of X.

The following lemma will be used in Part 5.

LEMMA 4.11. *Let Y be irreducible, and let $f : X \to Y$ be a morphism that restricts to a bijection between some open subsets $U \subset X$ and $V \subset Y$. Then $J_\infty f : J_\infty(X) \to J_\infty(Y)$ is dominant.*

PROOF. The map $J_\infty f$ restricts to the isomorphism $J_\infty(U) \xrightarrow{\simeq} J_\infty(V)$, and the open subset $J_\infty(V)$ is dense in $J_\infty(Y)$ since $J_\infty(Y)$ is irreducible. ☐

4.2. Poisson vertex algebras

Let V be a commutative vertex algebra (cf. §2.2.1), or equivalently, a differential algebra. Recall that this means: $a_{(n)} = 0$ in $\mathrm{End}(V)$ for all $n \geqslant 0$.

4.2.1. Definition. The commutative vertex algebra V is called a *Poisson vertex algebras* if it is equipped with a bilinear maps

$$V \times V \to V[\lambda], \quad (a, b) \mapsto \{a_\lambda b\} = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b, \quad a_{(n)} \in \mathrm{End}\, V,$$

also called the *λ-bracket*, satisfying the following axioms:

$$\{(Ta)_\lambda b\} = -\lambda \{a_\lambda b\}, \quad \{a_\lambda (Tb)\} = (\lambda + T)\{a_\lambda b\}, \tag{79}$$

$$\{b_\lambda a\} = -\{a_{-\lambda - T} b\}, \tag{80}$$

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}, \tag{81}$$

$$\{a_\lambda (bc)\} = \{a_\lambda b\} c + \{a_\lambda c\} b, \quad \{(ab)_\lambda c\} = \{a_{\lambda+T} c\}_\to b + \{b_{\lambda+T} c\}_\to a, \tag{82}$$

where the arrow means that $\lambda + T$ should be moved to the right, that is, $\{a_{\lambda+T} c\}_\to b = \sum_{n \geqslant 0} (a_{(n)} c) \frac{(\lambda+T)^n}{n!} b$.

Here, by abuse of notations, we have set

$$a_-(z) = \sum_{n \geqslant 0} a_{(n)} z^{-n-1}$$

so that the $a_{(n)}$, $n \geqslant 0$, are "new" operators, the "old" ones given by the field $a(z)$ being zero for $n \geqslant 0$ since V is commutative.

The first equation in (82) says that $a_{(n)}$, $n \geqslant 0$, is a derivation of the ring V. (Do not confuse $a_{(n)} \in \mathrm{Der}(V)$, $n \geqslant 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

Note that (79), (80), (81) are the same as (33), (34), (35), and (82) is the same as (36) and (37) without the third terms. In particular, by (81), we have

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \quad m,n \in \mathbb{Z}_{\geqslant 0}. \tag{83}$$

4.2.2. Poisson vertex structure on arc spaces.

THEOREM 4.12 ([**Arakawa12**, Proposition 2.3.1]). *Let X be an affine Poisson scheme, that is, $X = \operatorname{Spec} R$ for some Poisson algebra R. Then there is a unique Poisson vertex algebra structure on $J_\infty(R) = \mathbb{C}[J_\infty(X)]$ such that*

$$\{a_\lambda b\} = \{a,b\} \quad \text{for} \quad a,b \in R \subset J_\infty(R),$$

where $\{\ ,\ \}$ is the Poisson bracket on R. In other words,

$$a_{(n)}b = \begin{cases} \{a,b\} & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}$$

for $a,b \in R$.

PROOF. The uniqueness is clear by (33) since $J_\infty(R)$ is generated by R as a differential algebra. We leave it to the reader to check the well-definedness. Since $J_\infty(R)$ is generated by R, the formula $\{a_\lambda b\} = \{a,b\}$ for $a,b \in R$ is sufficient to define the λ-bracket on $J_\infty(R)$ by formulas (79), whence the existence. $\square$

REMARK 4.13. More generally, let X be a Poisson scheme which is not necessarily affine. Then the structure sheaf $\mathcal{O}_{J_\infty(X)}$ carries a unique Poisson vertex algebra structure such that

$$\{f_\lambda g\} = \{f,g\}$$

for $f,g \in \mathcal{O}_X \subset \mathcal{O}_{J_\infty(X)}$, see [**Arakawa-Kuwabara-Malikov**, Lemma 2.1.3.1].

EXAMPLE 4.14. Recall that the affine space $\mathfrak{g}^*$ is a Poisson variety by the Kirillov-Kostant Poisson structure (see Example 3.17). If $\{x^1,\dots,x^N\}$ is a basis of $\mathfrak{g}$, then

$$\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[x^1,\dots,x^N].$$

Thus

$$J_\infty(\mathfrak{g}^*) = \operatorname{Spec} \mathbb{C}[x^i_{(-n)}\ ;\ i = 1,\dots,N,\ n \geqslant 1]. \tag{84}$$

So we may identify $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ with the symmetric algebra $S(\mathfrak{g}[t^{-1}]t^{-1})$ via

$$x_{(-n)} \longmapsto xt^{-n}, \qquad x \in \mathfrak{g},\ n \geqslant 1.$$

For $x \in \mathfrak{g}$, identify x with $x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where we denote by $|0\rangle$ the unit element in $S(\mathfrak{g}[t^{-1}]t^{-1})$. Then (83) gives that

$$[x_{(m)}, y_{(n)}] = (x_{(0)}y)_{m+n} = \{x,y\}_{(m+n)} = [x,y]_{(m+n)}, \tag{85}$$

for $x,y \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*] \subset \mathbb{C}[J_\infty(\mathfrak{g}^*)]$ and $m,n \in \mathbb{Z}_{\geq 0}$. So the Lie algebra $J_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$ acts on $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$. This action coincides with that obtained by differentiating the action of $J_\infty(\mathbf{G}) = \mathbf{G}[[t]]$ on $J_\infty(\mathfrak{g}^*)$ induced by the coadjoint action of $\mathbf{G}$ (see Example 4.6). In other words, the Poisson vertex algebra structure of $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ comes from the $J_\infty(\mathbf{G})$-action on $J_\infty(\mathfrak{g}^*)$.

4.2.3. Canonical filtration and Poisson vertex structure. Our second basic example of Poisson vertex algebras comes from the graded vertex algebra associated with the canonical filtration, that is, the Li filtration (see §2.4.1).

Set

$$\mathrm{gr}^F V = \bigoplus_{p \geqslant 0} F^p V / F^{p+1} V,$$

where $\{F^p V\}_p$ is the Li filtration. Recall that $\sigma_p \colon F^p V \mapsto F^p V / F^{p+1} V$, for $p \geqslant 0$, denotes the canonical quotient map.

We have already seen that $\mathrm{gr}^F V$ is a commutative vertex algebras. We can now specify Proposition 2.15 and the relations (47) in Definition 2.16.

PROPOSITION 4.15 ([**Li05**]). *The space $\mathrm{gr}^F V$ is a Poisson vertex algebra by*

$$\begin{aligned} \sigma_p(a) \cdot \sigma_q(b) &:= \sigma_{p+q}(a_{(-1)} b), \\ T\sigma_p(a) &:= \sigma_{p+1}(Ta), \\ \sigma_p(a)_{(n)} \sigma_q(b) &:= \sigma_{p+q-n}(a_{(n)} b), \end{aligned}$$

for $a \in F^p V$, $b \in F^q V$, $n \geqslant 0$.

PROPOSITION 4.16 ([**Zhu96, Li05**]). *The restriction of the Poisson structure gives to the Zhu's C_2-algebra R_V a Poisson algebra structure, that is, R_V is a Poisson algebra by*

$$\bar{a} \cdot \bar{b} := \overline{a_{(-1)} b}, \qquad \{\bar{a}, \bar{b}\} = \overline{a_{(0)} b},$$

where $\bar{a} = \sigma_0(a)$.

PROOF. It is straightforward from Proposition 4.15. □

Remember that we always assume that a vertex algebra V is finitely strongly generated (see §2.4.1).

Note that if $\phi \colon V \to W$ is a *homomorphism of vertex algebras*[1], ϕ respects the canonical filtration, that is, $\phi(F^p V) \subset F^p W$. Hence it induces a homomorphism $\mathrm{gr}^F V \to \mathrm{gr}^F W$ of Poisson vertex algebra homomorphism which we denote by $\mathrm{gr}^F \phi$.

4.3. Associated variety of a vertex algebra

4.3.1. Associated variety and singular support.

DEFINITION 4.17. Define the *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \mathrm{Spec}\, R_V, \qquad X_V := \mathrm{Specm}\, R_V = (\tilde{X}_V)_{\mathrm{red}}.$$

It was shown in [**Li05**, Lemma 4.2] that $\mathrm{gr}^F V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $J_\infty(R_V) \to \mathrm{gr}^F V$ of differential algebras by Remark 4.4 since R_V generates $J_\infty(R_V)$ as a differential algebra either.

This is in fact a homomorphism of Poisson vertex algebras.

THEOREM 4.18 ([**Li05**, Lemma 4.2], [**Arakawa12**, Proposition 2.5.1]). *The identity map $R_V \to R_V$ induces a surjective Poisson vertex algebra homomorphism*

$$J_\infty(R_V) = \mathbb{C}[J_\infty(\tilde{X}_V)] \twoheadrightarrow \mathrm{gr}^F V.$$

[1] i.e., ϕ preserves the λ-bracket, ϕ sends the vacuum vector of V to that of W, and the translation operators with respect to V and W commute with ϕ.

DEFINITION 4.19. Define the *singular support* of a vertex algebra V as

$$SS(V) := \mathrm{Spec}(\mathrm{gr}^F V) \subset J_\infty(\tilde{X}_V).$$

THEOREM 4.20. *We have* $\dim SS(V) = 0$ *if and only if* $\dim X_V = 0$.

PROOF. The "only if" part is obvious since $\pi_{\tilde{X}_V,\infty}(SS(V)) = \tilde{X}_V$. The "if" part follows from Corollary 4.9. □

Recall that V is called lisse (or C_2-cofinite) if $R_V = V/C_2(V)$ is finite dimensional. Thus we get:

LEMMA 4.21. *The vertex algebra V is lisse if and only if* $\dim X_V = 0$*, that is, if and only if* $\dim SS(V) = 0$.

REMARK 4.22. Suppose that V is $\mathbb{Z}_{\geqslant 0}$-graded by some Hamiltonian H, i.e., $V = \bigoplus_{i\geqslant 0} V_i$ with $V_i = \{x \in V \mid Hx = ix\}$, and that $V_0 = \mathbb{C}|0\rangle$. Then $\mathrm{gr}^F V$ and R_V are equipped with the induced grading:

$$\mathrm{gr}^F V = \bigoplus_{i\geqslant 0} (\mathrm{gr}^F V)_i, \quad (\mathrm{gr}^F V)_0 = \mathbb{C},$$
$$R_V = \bigoplus_{i\geqslant 0} (R_V)_i, \quad (R_V)_0 = \mathbb{C}.$$

So the following conditions are equivalent:
(1) V is lisse,
(2) $X_V = \{\text{point}\}$,
(3) the image of any vector $a \in V_i$ for $i \geqslant 1$ in $\mathrm{gr}^F V$ is nilpotent,
(4) the image of any vector $a \in V_i$ for $i \geqslant 1$ in R_V is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

REMARK 4.23. Suppose that the Poisson structure of R_V is trivial. Then the Poisson vertex algebra structure of $J_\infty(R_V)$ is trivial, and so is that of $\mathrm{gr}^F V$ by Theorem 4.18. This happens if and only if

$$(F^p V)_{(n)}(F^q V) \subset F^{p+q-n+1} V \quad \text{for all } n \geqslant 0.$$

If this is the case, one can give $\mathrm{gr}^F V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)}b) \quad \text{for } n \geqslant 0. \tag{86}$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial).

4.3.2. Comparison with weight-depending filtration. Let V be a vertex algebra that is $\mathbb{Z}$-graded by some Hamiltonian H (see §2.2.4):

$$V = \bigoplus_{\Delta\in\mathbb{Z}} V_\Delta \quad \text{where} \quad V_\Delta := \{v \in V \mid Hv = \Delta v\}.$$

Then there is another natural filtration of V defined as follows [**Li04**].

Let $G_p V$ be the subspace of V spanned by the vectors

$$a^1_{(-n_1-1)} a^2_{(-n_2-1)} \cdots a^r_{(-n_r-1)}|0\rangle$$

with $a^i \in V$ homogeneous, $\Delta_{a^1} + \dots + \Delta_{a^r} \leqslant p$. Then $G_\bullet V$ defines an increasing filtration of V:

$$0 = G_{-1}V \subset G_0 V \subset \dots G_1 V \subset \dots, \quad V = \bigcup_p G_p V.$$

Moreover we have

$$\begin{aligned} &TG_pV \subset G_pV, \\ &(G_p)_{(n)} G_q V \subset G_{p+q}V \quad \text{for } n \in \mathbb{Z}, \\ &(G_p)_{(n)} G_q V \subset G_{p+q-1}V \quad \text{for } n \in \mathbb{Z}_{\geqslant 0}, \end{aligned}$$

It follows that $\mathrm{gr}_G V = \bigoplus G_pV/G_{p-1}V$ is naturally a Poisson vertex algebras.

It is not too difficult to see the following.

LEMMA 4.24 ([**Arakawa12**, Proposition 2.6.1]). *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta,$$

where $F^pV_\Delta = V_\Delta \cap F^pV$, $G_pV_\Delta = V_\Delta \cap G_pV$. *Therefore*

$$\mathrm{gr}^F V \cong \mathrm{gr}_G V$$

as Poisson vertex algebras.

PROPOSITION 4.25 ([**Arakawa12**, Corollary 2.6.2]). *A vertex algebra V is finitely strongly generated if and only if R_V is finitely generated as a ring.*

If the images of some vectors $a^1, \dots, a^r \in V$ in R_V generate R_V, we say that V is *strongly generated by* $a^1, \dots, a^r$.

PROOF. Suppose that $a^1, \dots, a^r$ are strong generators of V. By Lemma 4.24, $C_2(V) = F^1V$ is spanned by the vectors $a^{i_1}_{(-n_1-1)} \dots a^{i_s}_{(-n_s-1)}|0\rangle$ with $s \geqslant 1$ and $n_1 + \dots + n_s \geqslant 1$. Thus $\{\bar{a}^1, \dots, \bar{a}^r\}$ generates R_V, where $\bar{a}^i$ is the image of a^i in R_V.

Conversely, suppose that $\{\bar{a}^1, \dots, \bar{a}^r\}$ generates R_V. Then by Theorem 4.18, $\{\bar{a}^1, \dots, \bar{a}^r\}$ generates $\mathrm{gr}^F V$ as a differential algebra. Since $\mathrm{gr}^F V \cong V$ as $\mathbb{C}$-vector spaces by the assumption that $F^\bullet V$ is separated, it follows that $\{a^1, \dots, a^r\}$ strongly generates V. □

REMARK 4.26. In fact a stronger fact is known: V is spanned by the above vectors with $r \geqslant 0$, $n_1 > n_2 > n_3 > \dots \geqslant 1$, see [**Gaberdiel-Neitzke03**], [**Li05**, Theorem 4.7].

4.3.3. Universal affine vertex algebras. Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$ defined by (38) as in §2.2.2.

Recall that the space $V^\kappa(\mathfrak{a})$ is naturally graded: $V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta$, where the grading is defined by setting $\deg xt^n = -n$, $\deg |0\rangle = 0$ and $|0\rangle = 1 \otimes 1$. Thus,

$$V^\kappa(\mathfrak{a})_\Delta = \{v \in V^\kappa(\mathfrak{a}) \mid Dv = -\Delta v\}.$$

We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C}|0\rangle$. We identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

We have $F^1V^\kappa(\mathfrak{a}) = \mathfrak{a}[t^{-1}]t^{-2}V^\kappa(\mathfrak{a})$, and a Poisson algebra isomorphism

$$\begin{aligned} \mathbb{C}[\mathfrak{a}^*] &\xrightarrow{\simeq} R_{V^\kappa(\mathfrak{a})} = V^\kappa(\mathfrak{a})/\mathfrak{a}[t^{-1}]t^{-2}V^\kappa(\mathfrak{a}) \\ x_1 \dots x_r &\longmapsto \overline{x_1t^{-1} \dots x_rt^{-1}|0\rangle} \quad (x_i \in \mathfrak{a}). \end{aligned} \tag{87}$$

Thus

$$X_{V^\kappa(\mathfrak{a})} = \mathfrak{a}^*.$$

We have the isomorphism

$$\mathbb{C}[J_\infty(\mathfrak{a}^*)] \cong \operatorname{gr} V^\kappa(\mathfrak{a}). \tag{88}$$

Indeed, the graded dimensions of both sides coincide. Moreover,

$$G_p V^\kappa(\mathfrak{a}) = U_p(\mathfrak{a}[t^{-1}]t^{-1})|0\rangle,$$

where $\{U_p(\mathfrak{a}[t^{-1}]t^{-1})\}_p$ is the PBW filtration of $U(\mathfrak{a}[t^{-1}]t^{-1})$, and we have the isomorphisms

$$\operatorname{gr} U(\mathfrak{a}[t^{-1}]t^{-1}) \cong S(\mathfrak{a}[t^{-1}]t^{-1}) \cong \mathbb{C}[J_\infty(\mathfrak{a}^*)].$$

As a consequence of (88), we get

$$SS(V^\kappa(\mathfrak{a})) = J_\infty(\mathfrak{a}^*).$$

4.3.4. Simple affine vertex algebras. Let $V_k(\mathfrak{g})$ be the unique simple graded quotient of the universal affine vertex algebra $V^k(\mathfrak{g}) = V^\kappa(\mathfrak{g})$, with $\kappa = k(\,|\,)$, $k \in \mathbb{C}$, as in §2.3.2. Remind that, as a $\widehat{\mathfrak{g}}$-module, $V_k(\mathfrak{g}) \cong L(kD)$, where D is the weight of the basic representation of $\widehat{\mathfrak{g}}$.

THEOREM 4.27. *The vertex algebra $V_k(\mathfrak{g})$ is lisse if and only if $V_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}$-module, which is true if and only if $k \in \mathbb{Z}_{\geqslant 0}$.*

LEMMA 4.28. *Let (R, ∂) be a differential algebra over $\mathbb{Q}$, I a differential ideal of R, i.e., I is an ideal of R such that $\partial I \subset I$. Then $\partial\sqrt{I} \subset \sqrt{I}$.*

PROOF. Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Since I is ∂-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m = \sum_{0 \leqslant i \leqslant m} \binom{m}{i} a^{m-i}(\partial a)^i \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$. □

PROOF OF THE "IF" PART OF THEOREM 4.27. Suppose that $V_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_{\geqslant 0}$ and the maximal submodule $N_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ ([**Kac1**]). The exact sequence $0 \to N_k(\mathfrak{g}) \to V^k(\mathfrak{g}) \to V_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \to I_k \to R_{V^k(\mathfrak{g})} \to R_{V_k(\mathfrak{g})} \to 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{V_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_θ^{k+1}. Therefore, $e_\theta \in \sqrt{I_k}$. On the other hand, by Lemma 4.28, $\sqrt{I_k}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I_k}$. This proves that $X_{V_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of the "only if" part follows from [**Dong-Li-Mason06**]. We will give a different proof using W-algebras in Remark **??**.

In view of Theorem 4.27, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

4.3.5. Virasoro vertex alegbras. Let Vir^c be the universal Virasoro vertex algebra with central charge c as in §2.2.3. Any Vir-module with central c (i.e., the central element C of Vir acts as a multiplication by c) on which $L(z)$ is a field can be considered as a Vir^c-module.

EXERCISE 4.29. Show that

$$R_{\mathrm{Vir}^c} \cong \mathbb{C}[x],$$

with the trivial Poisson structure, where x is the image of $L_{-2}|0\rangle$.

For the sequel, we let N_c be the unique maximal submodule of Vir^c, and $\mathrm{Vir}_c = \mathrm{Vir}^c/N_c$ the unique quotient.

4.4. Computation of Zhu's algebras

Recall that we have a well-defined algebra homomorphism (cf. Lemma 2.20)

$$\eta_V \colon R_V \twoheadrightarrow \operatorname{gr}\operatorname{Zhu}(V)$$

between the Zhu's C_2-algebra R_V and the Zhu algebra $\operatorname{Zhu}(V)$ (cf. Section 2.5).

4.4.1. PBW basis. We say that a vertex algebra V *admits a PBW basis* if R_V is a polynomial algebra and if the map $\mathbb{C}[J_\infty(X_V)] \twoheadrightarrow \operatorname{gr}^F V$ is an isomorphism.

THEOREM 4.30. *If V admits a PBW basis, then $\eta_V \colon R_V \twoheadrightarrow \operatorname{gr}\operatorname{Zhu}V$ is an isomorphism.*

PROOF. We have $\operatorname{gr}\operatorname{Zhu}(V) = V/\operatorname{gr}(V \circ V)$, where $\operatorname{gr}(V \circ V)$ is the associated graded space of $V \circ V$ with respect to the filtration induced by the filtration $V_{\leqslant p}$. We wish to show that $\operatorname{gr}(V \circ V) = F^1 V$. Since $a \circ b \equiv a_{(-2)}b \pmod{F_{\leqslant \Delta_a + \Delta_b} V}$, it is sufficient to show that $a \circ b \neq 0$ implies that $a_{(-2)}b \neq 0$.

Suppose that $a_{(-2)}b = (Ta)_{(-1)}b = 0$. Since V admits a PBW basis, $\operatorname{gr}^F V$ has no zero divisors, whence $Ta = 0$. Also, from the PBW property we find that $Ta = 0$ implies that $a = c|0\rangle$ for some constant $c \in \mathbb{C}$. Thus, a is a constant multiple of $|0\rangle$, in which case $a \circ b = 0$. $\square$

4.4.2. Universal affine vertex algebras. The universal affine vertex algebra $V^k(\mathfrak{g})$ (see §4.3.3) admits a PBW basis. Therefore

$$\eta_{V^k(\mathfrak{g})} \colon R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\simeq} \operatorname{gr}\operatorname{Zhu}V^k(\mathfrak{g}).$$

On the other hand, from Lemma 2.19 one finds that

$$\begin{aligned} U(\mathfrak{g}) &\longrightarrow \operatorname{Zhu}(V^k(\mathfrak{g})) \\ \mathfrak{g} \ni x &\longmapsto \bar{x} = \overline{x_{(-1)}|0\rangle} \end{aligned} \tag{89}$$

gives a well-defined algebra homomorphism. This map respects the filtration on both sides, where the filtration in the left side is the PBW filtration. Hence it induces a map between their associated graded algebras, which is identical to $\eta_{V^k(\mathfrak{g})}$. Therefore (89) is an isomorphism, that is to say, $V^k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})$.

EXERCISE 4.31. Extend Theorem 4.30 to the case where $\mathfrak{g}$ is a Lie superalgebra.

Theorem 2.18 gives the following in this example. The top degree component of the irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight λ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $N_k = N_k(\mathfrak{g})$ be the maximal ideal of $V^k(\mathfrak{g})$ as in §4.3.4 so that

$$V_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k.$$

We have the exact sequence $\mathrm{Zhu}(N_k) \to U(\mathfrak{g}) \to \mathrm{Zhu}(V_k(\mathfrak{g})) \to 0$ since the functor $\mathrm{Zhu}(?)$ is right exact and thus $\mathrm{Zhu}(V_k(\mathfrak{g}))$ is the quotient of $U(\mathfrak{g})$ by the image I_k of $\mathrm{Zhu}(N_k)$ in $U(\mathfrak{g})$:

$$\mathrm{Zhu}(V_k(\mathfrak{g})) = U(\mathfrak{g})/I_k.$$

Hence when the homomorphism $\eta_{V_k(\mathfrak{g})}$ of Lemma 2.20 is an isomorphism, the associated variety $X_{V_k(\mathfrak{g})}$ can be viewed as an analog of the associated variety of primitive ideals. However, we will see in Section 4.6 that there are substantial differences.

In the case where k is a nonnegative integer, we have seen (§4.3.4) that N_k is just the submodule of $V^k(\mathfrak{g})$ generated by $(e_\theta t^{-1})^{k+1}|0\rangle$. In general, it is a hard problem to compute N_k and I_k. We will see next section some examples where it is possible.

One way to achieve this is to use *singular vectors*. Recall that a *singular vector of* $\mathfrak{g}$ of a $\mathfrak{g}$-module M is a vector $v \in M$ such that $\mathfrak{n}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$. A *singular vector of* $\widehat{\mathfrak{g}}$ of a $\widehat{\mathfrak{g}}$-module M is a vector $v \in M$ such that $\widehat{\mathfrak{n}}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$, and $(f_\theta t).v = 0$. In particular, regarding $V^k(\mathfrak{g})$ as a $\widehat{\mathfrak{g}}$-module, a vector $v \in V^k(\mathfrak{g})$ is singular if and only if $\widehat{\mathfrak{n}}_+.v = 0$.

LEMMA 4.32. *We have a $\mathfrak{g}$-module embedding*

$$\sigma_d \colon S^d(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g})_d, \quad x_1 \dots x_d \mapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} (x_{\sigma(1)} t^{-1}) \dots (x_{\sigma(d)} t^{-1})|0\rangle,$$

where $S(\mathfrak{g}) = \bigoplus_d S^d(\mathfrak{g})$ is the usual grading of $S^d(\mathfrak{g})$.

Let v be a singular vector for $\mathfrak{g}$ in $S^d(\mathfrak{g})$. Then $\sigma_d(v)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $(f_\theta t)\sigma_d(v) = 0$.

4.4.3. Free fermions. Let $\mathfrak{n}$ be a finite-dimensional vector space. The *Clifford affinization* $\widehat{Cl}$ of $\mathfrak{n}$ is the Clifford algebra (see Appendix A) associated with $\mathfrak{n}[t, t^{-1}] \oplus \mathfrak{n}^*[t, t^{-1}]$ and its symmetric bilinear form defined by

$$\langle xt^m | ft^n \rangle = \delta_{m+n,0} f(x), \quad \langle xt^m | yt^n \rangle = 0 = \langle ft^m | gt^n \rangle$$

for $x, y \in \mathfrak{n}$, $f, g \in \mathfrak{n}^*$, $m, n \in \mathbb{Z}$.

Let $\{x_i\}_{1 \leqslant i \leqslant s}$ be a basis of $\mathfrak{n}$, $\{x_i^*\}_{1 \leqslant i \leqslant s}$ its dual basis. We write $\psi_{i,m}$ for $x_i t^m \in \widehat{Cl}$ and $\psi_{i,m}^*$ for $x_i^* t^m \in \widehat{Cl}$, so that $\widehat{Cl}$ is the associative superalgebra with

- odd generators: $\psi_{i,m}, \psi_{i,m}^*, m \in \mathbb{Z}, i = \{1, \dots, s\}$,
- relations: $[\psi_{i,m}, \psi_{j,n}] = [\psi_{i,m}^*, \psi_{j,n}^*] = 0$, $[\psi_{i,m}, \psi_{j,n}] = \delta_{i,j}\delta_{m+n,0}$.

Define the *charged fermion Fock space* associated with $\mathfrak{n}$ as

$$\mathcal{F}_{\mathfrak{n}} := \widehat{Cl}/(\sum_{\substack{m \geqslant 0 \\ 1 \leqslant i \leqslant s}} \widehat{Cl}\psi_{i,m} + \sum_{\substack{k \geqslant 1 \\ 1 \leqslant j \leqslant s}} \widehat{Cl}\psi_{j,k}^*).$$

It is an irreducible $\widehat{Cl}$-module, and as $\mathbb{C}$-vector spaces we have

$$\mathcal{F}_{\mathfrak{n}} \cong \wedge(\mathfrak{n}^*[t^{-1}]) \otimes \wedge(\mathfrak{n}[t^{-1}]t^{-1}).$$

There is a unique vertex (super)algebra structure on $\mathcal{F}_{\mathfrak{n}}$ such that the image of 1 is the vacuum $|0\rangle$ and

$$\begin{aligned} Y(\psi_{i,-1}|0\rangle, z) &= \psi_i(z) := \sum_{n\in\mathbb{Z}} \psi_{i,n} z^{-n-1}, \\ Y(\psi^*_{i,0}|0\rangle, z) &= \psi^*_i(z) := \sum_{n\in\mathbb{Z}} \psi^*_{i,n} z^{-n}. \end{aligned}$$

We have $F^1\mathcal{F}_{\mathfrak{n}} = \mathfrak{n}^*[t^{-1}]t^{-1}\mathcal{F}_{\mathfrak{n}} + \mathfrak{n}[t^{-1}]t^{-2}\mathcal{F}_{\mathfrak{n}}$, and it follows that there is an isomorphism

$$\begin{array}{ccc} \overline{Cl} & \stackrel{\simeq}{\longrightarrow} & R_{\mathcal{F}_{\mathfrak{n}}}, \\ x_i & \longmapsto & \overline{\psi_{i,-1}|0\rangle}, \\ x_i^* & \longmapsto & \overline{\psi^*_{i,0}|0\rangle} \end{array}$$

as Poisson superalgebras. Thus,

$$X_{\mathcal{F}_{\mathfrak{n}}} = T^*(\Pi\mathfrak{n}),$$

where $\Pi\mathfrak{n}$ is the space $\mathfrak{n}$ considered as a purely odd affine space. Its arc space $J_\infty(T^*(\Pi\mathfrak{n}))$ is also regarded as a purely odd affine space, such that $\mathbb{C}[J_\infty(T^*(\Pi\mathfrak{n}))] = \wedge(\mathfrak{n}^*[t^{-1}]) \otimes \wedge(\mathfrak{n}[t^{-1}]t^{-1})$. The map $\mathbb{C}[J_\infty(X_{\mathcal{F}_{\mathfrak{n}}})] \to \operatorname{gr}\mathcal{F}_{\mathfrak{n}}$ is an isomorphism and $\mathcal{F}_{\mathfrak{n}}$ admits a PBW basis. Therefore we have the isomorphism

$$\eta_{\mathcal{F}_{\mathfrak{n}}} : R_{\mathcal{F}_{\mathfrak{n}}} = \overline{Cl} \stackrel{\simeq}{\longrightarrow} \mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}})$$

by Exercise 4.31. On the other hand the map

$$\begin{array}{ccc} Cl & \to & \mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}}) \\ x_i & \mapsto & \overline{\psi_{i,-1}|0\rangle}, \\ x_i^* & \mapsto & \overline{\psi^*_{i,0}|0\rangle} \end{array}$$

gives an algebra homomorphism that respects the filtration. Hence we have

$$\mathrm{Zhu}(\mathcal{F}_{\mathfrak{n}}) \cong Cl.$$

That is, $\mathcal{F}_{\mathfrak{n}}$ is a chiralization of Cl.

4.5. Poisson vertex modules and their associated variety

4.5.1. Poisson vertex modules.

DEFINITION 4.33. A *Poisson vertex module* over a Poisson vertex algebra V is a V-module M in the usual sense of vertex V-module, equipped with a linear map

$$V \mapsto (\operatorname{End} M)[[z^{-1}]]z^{-1}, \quad a \mapsto Y^M_-(a,z) = \sum_{n\geqslant 0} a^M_{(n)} z^{-n-1},$$

satisfying

$$a_{(n)}^{M}m = 0 \quad \text{for} \quad n \gg 0, \tag{90}$$

$$(Ta)_{(n)}^{M} = -na_{(n-1)}^{M}, \tag{91}$$

$$a_{(n)}^{M}(bv) = (a_{(n)}^{M}b)v + b(a_{(n)}^{M}v), \tag{92}$$

$$[a_{(m)}^{M}, b_{(n)}^{M}] = \sum_{i\geqslant 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}^{M}, \tag{93}$$

$$(ab)_{(n)}^{M} = \sum_{i=0}^{\infty}(a_{(-i-1)}b_{(n+i)}^{M} + b_{(-i-1)}a_{(n+i)}^{M}) \tag{94}$$

for all $a, b \in V$, $m, n \geqslant 0$, $v \in M$.

A Poisson vertex algebra V is naturally a Poisson vertex module over itself.

EXAMPLE 4.34. Let M be a Poisson vertex module over $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$. Then by (93), the assignment

$$xt^n \mapsto x_{(n)}^{M}, \qquad x \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*] \subset \mathbb{C}[J_\infty(\mathfrak{g}^*)],\ n \geqslant 0,$$

defines a $J_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$-module structure on M. In fact, a Poisson vertex module over $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$ is the same as a $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$-module M in the usual associative sense equipped with an action of the Lie algebra $J_\infty(\mathfrak{g})$ such that $(xt^n)m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$, and

$$(xt^n)\cdot(am) = (x_{(n)}a)\cdot m + a(xt^n)\cdot m$$

for $x \in \mathfrak{g}$, $n \geqslant 0$, $a \in \mathbb{C}[J_\infty(\mathfrak{g}^*)]$, $m \in M$.

Below we often write $a_{(n)}$ for $a_{(n)}^{M}$.

The proofs of the following assertions are straightforward. (We refer to §3.4.6 for the definition of *Poisson modules*.)

LEMMA 4.35. *Let R be a Poisson algebra, E a Poisson module over R. There is a unique Poisson vertex $J_\infty(R)$-module structure on $J_\infty(R) \otimes_R E$ such that*

$$a_{(n)}(b \otimes m) = (a_{(n)}b) \otimes m + \delta_{n,0} b \otimes \{a, m\}$$

for $n \geqslant 0$, $a \in R \subset J_\infty(R)$, $b \in J_\infty(R)$, $m \in E$ (Recall that $J_\infty(R) = \mathbb{C}[J_\infty(\operatorname{Spec} R)]$.)

LEMMA 4.36. *Let R be a Poisson algebra, M a Poisson vertex module over $J_\infty(R)$. Suppose that there exists a R-submodule E of M (in the usual commutative sense) such that $a_{(n)}E = 0$ for $n > 0$, $a \in R$, and that M is generated by E (in the usual commutative sense). Then there exists a surjective homomorphism*

$$J_\infty(R) \otimes_R E \twoheadrightarrow M$$

of Poisson vertex modules.

4.5.2. Canonical filtration of modules over vertex algebras. Let V be a vertex algebra graded by a Hamiltonian H. A *compatible filtration* of a V-module M is a decreasing filtration

$$M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots$$

such that

$$\begin{aligned}
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1}M \quad \text{for } a \in F^pV,\ \forall n \in \mathbb{Z},\\
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n}M \quad \text{for } a \in F^pV,\ n \geqslant 0,\\
&H.\Gamma^p M \subset \Gamma^p M \quad \text{for all } p \geqslant 0,\\
&\bigcap_p \Gamma^p M = 0.
\end{aligned}$$

For a compatible filtration $\Gamma^\bullet M$, the associated graded space

$$\mathrm{gr}^\Gamma M = \bigoplus_{p \geqslant 0} \Gamma^p M/\Gamma^{p+1}M$$

is naturally a graded vertex Poison module over the graded vertex Poisson algebra $\mathrm{gr}^F V$, and hence, it is a graded vertex Poison module over $J_\infty(R_V) = \mathbb{C}[\tilde{X}_V]$ by Theorem 4.18.

The vertex Poisson $J_\infty(R_V)$-module structure of $\mathrm{gr}^\Gamma M$ restricts to a Poisson R_V-module structure of $M/\Gamma^1 M = \Gamma^0 M/\Gamma^1 M$, and $a_{(n)}(M/\Gamma^1 M) = 0$ for $a \in R_V \subset J_\infty(R_V)$, $n > 0$. It follows that there is a homomorphism

$$J_\infty(R_V) \otimes_{R_V} (M/\Gamma^1 M) \to \mathrm{gr}^\Gamma M, \quad a \otimes \bar{m} \mapsto a\bar{m},$$

of vertex Poisson modules by Lemma 4.36.

Suppose that V is positively graded and so is a V-module M. We denote by $F^\bullet M$ the Li filtration [**Li05**] of M, which is defined by

$$F^p M = \mathrm{span}_{\mathbb{C}}\{a^1_{(-n_1-1)} \dots a^r_{(-n_r-1)} m \mid a^i \in V,\ m \in M,\ n_1 + \dots + n_r \geqslant p\}.$$

It is a compatible filtration of M, and in fact, it is the finest compatible filtration of M, that is, $F^pM \subset \Gamma^p M$ for all p for any compatible filtration $\Gamma^\bullet M$ of M. The subspace $F^1 M$ is spanned by the vectors $a_{(-2)}m$ with $a \in V$, $m \in M$, which is often denoted by $C_2(M)$ in the literature. Set

$$\overline{M} = M/F^1M (= M/C_2(M)), \tag{95}$$

which is a Poisson module over $R_V = \overline{V}$. By [**Li05**, Proposition 4.12], the vertex Poisson module homomorphism

$$J_\infty(R_V) \otimes_{R_V} \overline{M} \to \mathrm{gr}^F M$$

is surjective.

Let $\{a^i\,;\, i \in I\}$ be elements of V such that their images generate R_V in the usual commutative sense, and let U be a subspace of M such that $M = U + F^1 M$. The surjectivity of the above map is equivalent to that

$$\begin{aligned}
&F^p M \\
&= \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(-n_1-1)} \dots a^{i_r}_{(-n_r-1)} m \mid m \in U,\ n_i \geqslant 0, n_1 + \dots + n_r \geqslant p,\ i_1, \dots, i_r \in I\}.
\end{aligned} \tag{96}$$

LEMMA 4.37. *Let V be a vertex algebra, M a V-module. The Poisson vertex algebra module structure of $\mathrm{gr}^F M$ restricts to the Poisson module structure of $\overline{M} := M/F^1M$ over R_V, that is, $\overline{M}$ is a Poisson R_V-module by*

$$\bar{a} \cdot \bar{m} = \overline{a_{(-1)}m}, \quad \mathrm{ad}(\bar{a})(\bar{m}) = \overline{a_{(0)}m}, \qquad \bar{a} \in R_V,\ m \in M.$$

A V-module M is called *finitely strongly generated* if $\overline{M}$ is finitely generated as a R_V-module in the usual associative sense.

4.5.3. Associated varieties of modules over affine vertex algebras. Recall that a $\widehat{\mathfrak{g}}$-module M of level k is called smooth if $x(z)$ is a field on M for $x \in \mathfrak{g}$, that is, $(xt^n)m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$, and that a $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k (cf. §2.3.2).

For a $V = V^k(\mathfrak{g})$-module M, or equivalently, a smooth $\widehat{\mathfrak{g}}$-module of level k, we have

$$\overline{M} = M/\mathfrak{g}[t^{-1}]t^{-2}M,$$

and the Poisson $\mathbb{C}[\mathfrak{g}^*]$-module structure is given by

$$x \cdot \bar{m} = \overline{(xt^{-1})m}, \quad \mathrm{ad}(x)\overline{m} = \overline{xm}, \qquad x \in \mathfrak{g},\, m \in M.$$

For a $\mathfrak{g}$-module E, let

$$V_E^k := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus D)} E,$$

where E is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$-module on which $\mathfrak{g}[t] \oplus \mathbb{C}D$ acts trivially and K acts as multiplication by k. Then

$$\overline{V_E^k} \cong \mathbb{C}[\mathfrak{g}^*] \otimes E,$$

where the Poisson $\mathbb{C}[\mathfrak{g}^*]$-module structure is given by

$$f \cdot g \otimes v = (fg) \otimes v, \quad \mathrm{ad}\, x(f \otimes v) = \{x, f\} \otimes v + f \otimes xv,$$

for $f, g \in \mathbb{C}[\mathfrak{g}^*]$, $v \in V$.

Recall that $\mathcal{O}_k$ denotes the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ of level k (cf. §1.5.2). Let $\mathbf{KL}_k$ be the full subcateogory of $\mathcal{O}_k$ consisting of modules M such that M_d is finite-dimensional for all $d \in \mathbb{C}$, where

$$M_d = \{m \in M \mid Dm = dm\}.$$

Note that V_E^k is an object of $\mathbf{KL}_k$ for a finite dimensional representation E of $\mathfrak{g}$. Thus, $V^k(\mathfrak{g}) = V_{\mathbb{C}}^k$ and its simple quotient $V_k(\mathfrak{g})$ are also objects of $\mathbf{KL}_k$.

Both $\mathcal{O}_k$ and $\mathbf{KL}_k$ can be regarded as full subcategories of the category of $V^k(\mathfrak{g})$-modules.

Lemma 4.38. *For $M \in \mathbf{KL}_k$ the following conditions are equivalent:*

1. *M is finitely strongly generated as a $V^k(\mathfrak{g})$-module,*
2. *M is finitely generated as a $\mathfrak{g}[t^{-1}]t^{-1}$-module,*
3. *M is finitely generated as a $\widehat{\mathfrak{g}}$-module.*

Definition 4.39. For a finitely strongly generated $V^k(\mathfrak{g})$-module M, define its *associated variety* X_M by

$$\begin{aligned} X_M &= \mathrm{supp}_{R_V}(\overline{M}) \\ &= \{\mathfrak{p} \in \mathrm{Spec}\, R_V \,;\, \mathfrak{p} \supset \mathrm{Ann}_{R_V}(\overline{M})\} \subset X_V, \end{aligned}$$

equipped with the reduced scheme structure.

Example 4.40. We have $X_{V_E^k} = \mathfrak{g}^*$ for a finite dimensional representation E of $\mathfrak{g}$.

4.5.4. Frenkel-Zhu's bimodules. Recall that for a graded vertex algebra V, its Zhu's algebra is defined by $\mathrm{Zhu}(V) = V/V \circ V$. There is a similar construction for modules due to Frenkel and Zhu [**Frenkel-Zhu92**]. For a V-module M, set

$$\mathrm{Zhu}(M) = M/V \circ M,$$

where $V \circ M$ is the subspace of M spanned by the vectors

$$a \circ m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, and $m \in M$.

Proposition 4.41 ([**Frenkel-Zhu92**]). $\mathrm{Zhu}(M)$ *is a bimodule over* $\mathrm{Zhu}(V)$ *by the multiplications*

$$a * m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} m, \quad m * a = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)} m$$

for $a \in V_{\Delta_a}$*,* $\Delta_a \in \mathbb{Z}$*, and* $m \in M$*.*

Thus, we have a right exact functor

$$V\text{-Mod} \to \mathrm{Zhu}(V)\text{-biMod}, \quad M \mapsto \mathrm{Zhu}(M),$$

where $\mathrm{Zhu}(V)$-biMod is the category of bimodules over $\mathrm{Zhu}(V)$.

Lemma 4.42. *Let* $M = \bigoplus_{d \in h + \mathbb{Z}_{\geqslant 0}} M_d$ *be a positive energy representation of a* $\mathbb{Z}_{\geqslant 0}$*-graded vertex algebra* V*. Define an increasing filtration* $\{\mathrm{Zhu}_p(M)\}_p$ *on* $\mathrm{Zhu}(V)$ *by*

$$\mathrm{Zhu}_p(M) = \mathrm{im}(\bigoplus_{d=h}^{h+p} M_p \to \mathrm{Zhu}(M)).$$

(1) *We have*

$$\mathrm{Zhu}_p(V) \cdot \mathrm{Zhu}_q(M) \cdot \mathrm{Zhu}_r(V) \subset \mathrm{Zhu}_{p+q+r}(M),$$
$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(M)] \subset \mathrm{Zhu}_{p+q-1}(M).$$

Therefore $\mathrm{gr}\,\mathrm{Zhu}(M) = \bigoplus_p \mathrm{Zhu}_p(M)/\mathrm{Zhu}_{p-1}(M)$ *is a Poisson* $\mathrm{gr}\,\mathrm{Zhu}(V)$*-module, and hence is a Poisson* R_V*-module through the homomorphism* $\eta_V \colon R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(V)$*.*

(2) *There is a natural surjective homomorphism*

$$\eta_M \colon \bar{M}(= M/F^1 M) \to \mathrm{gr}\,\mathrm{Zhu}(M)$$

of Poisson R_V*-modules. This is an isomorphism if* $\mathrm{gr}\, M$ *is free over* $\mathrm{gr}\, V$*.*

Example 4.43. Let $M = V_E^k$. Since $\mathrm{gr}\, V_E^k$ is free over $\mathbb{C}[J_\infty(\mathfrak{g}^*)]$, we have the isomorphism

$$\eta_{V_E^k} \colon \overline{V_E^k} = E \otimes \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\simeq} \mathrm{gr}\,\mathrm{Zhu}(V_E^k).$$

On the other hand, there is a $U(\mathfrak{g})$-bimodule homomorphism

$$\begin{aligned} & E \otimes U(\mathfrak{g}) \to \mathrm{Zhu}(V_E^k), \\ & v \otimes x_1 \dots x_r \mapsto (1 \otimes v) * (x_1 t^{-1}) * \dots * (x_r t^{-1}) + V^k(\mathfrak{g}) \circ V_E^k \end{aligned} \tag{97}$$

which respects the filtration. Here the $U(\mathfrak{g})$-bimodule structure of $U(\mathfrak{g})\otimes E$ is given by

$$x(v\otimes u)=(xv)\otimes u+v\otimes xu,\quad (v\otimes u)x=v\otimes(ux),$$

and the filtration of $U(\mathfrak{g})\otimes E$ is given by $\{U_i(\mathfrak{g})\otimes E\}$. Since the induced homomorphism between associated graded spaces (97) coincides with $\eta_{V_E^k}$, (97) is an isomorphism.

Recall that $\mathcal{HC}$ is the category of Harish-Chandra bimodules (cf. §3.4.16).

LEMMA 4.44. *For $M\in \mathbf{KL}_k$, we have $\mathrm{Zhu}(M)\in\mathcal{HC}$. If M is finitely generated, then so is $\mathrm{Zhu}(M)$.*

4.6. Advanced results, open problems

In previous sections, we were mostly interested in algebraic properties of the Poisson algebra R_V. We study in this section more geometrical aspects. Recall that the associated variety of a vertex algebra V is the Poisson variety

$$X_V:=\mathrm{Specm}(R_V).$$

As a Poisson variety, X_V is a finite disjoint union of smooth analytic Poisson manifolds, and it is stratified by its symplectic leaves (see §3.3.3). The case where the associated variety X_V has finitely many symplectic leaves is particularly interesting.

We present below various examples, mostly coming from simple affine vertex algebras, and also non-trivial counter-examples. Other examples coming from W-algebras will be added in Part 5.

Recall that we have $X_{V^k(\mathfrak{g})}=\mathfrak{g}^*$, and therefore the associated variety $X_{V_k(\mathfrak{g})}$ is a Poisson subscheme of $\mathfrak{g}^*$ which is $\mathbf{G}$-invariant and conic. Thus, identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\ |\ )$, the symplectic leaves of $X_{V_k(\mathfrak{g})}$ are exactly the adjoint orbits of $\mathbf{G}$ in $X_{V_k(\mathfrak{g})}$, and $X_{V_k(\mathfrak{g})}$ has finitely many symplectic leaves if and only if $X_{V_k(\mathfrak{g})}$ is contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.

4.6.1. Associated variety of admissible representations. Assume that k is an admissible level (see Definition 1.20), that is, $V_k(\mathfrak{g})=L(kD)$ is an admissible representation of $\widehat{\mathfrak{g}}$.

Recall that, according to the Irreducibility Theorem (cf. §3.4.17), the associated variety of a primitive ideal is contained in the nilpotent cone, and more specifically, it is the closure of some nilpotent orbit. Theorem 4.46 below says that we have a similar result for the associated variety of the admissible representation $L(kD)$.

The following fact was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g}=\mathfrak{sl}_2$ by Feigin and Malikov [**Feigin-Malikov97**].

THEOREM 4.45 ([**Arakawa15a**]). *If k is admissible then $SS(V_k(\mathfrak{g}))\subset J_\infty(\mathcal{N})$ or, equivalently, the associated variety $X_{V_k(\mathfrak{g})}$ is contained in $\mathcal{N}$.*

In fact, the following stronger result holds.

THEOREM 4.46 ([**Arakawa15a**]). *Assume that k is admissible. Then*

$$X_{V_k(\mathfrak{g})}=\overline{\mathbb{O}_q},$$

*where $\mathbb{O}_q$ is a nilpotent orbit which only depends on q, with q as in Proposition 1.19 (see Tables 2–10 of [**Arakawa15a**]).*

REMARK 4.47. For $\mathfrak{g} = \mathfrak{sl}_n$, Theorem 4.46 gives the following. Let k be admissible, and let $q \in \mathbb{Z}_{>0}$ be the denominator of k, that is, $k + h^\vee = p/q$, with $p \in \mathbb{Z}_{>0}$ and $(p, q) = 1$. Then

$$X_{V_k(\mathfrak{g})} = \{x \in \mathfrak{g} \mid (\operatorname{ad} x)^{2q} = 0\} = \overline{\mathbb{O}_q},$$

where $\mathbb{O}_q$ is the nilpotent orbit corresponding to the partition

$$\begin{cases} (n) & \text{if } q \geqslant n, \\ (q, q, \ldots, q, s) \quad (0 \leqslant s \leqslant n-1) & \text{if } q < n. \end{cases}$$

Remind that $h^\vee = n$ for $\mathfrak{g} = \mathfrak{sl}_n$.

REMARK 4.48. In the classical cases, one can verify that all nilpotent orbit closures $\overline{\mathbb{O}_q}$ appearing in Theorem 4.46 are all normal (see [**Arakawa-Moreau16b**, §5.1]). Hence their intersection with Slodowy slices are always irreducible by Lemma B.5. To verify this, we use the combinatorial method developed by Kraft and Procesi to determine whether a given nilpotent orbit closure is normal (cf. B.0.6).

In the exceptional cases, the nilpotent orbit closure of dimension 8 in G_2 is not normal and appears as associated variety of some simple affine vertex algebra of admissible level. Nevertheless, all nilpotent orbit closures $\overline{\mathbb{O}_q}$ appearing in Theorem 4.46 in the exceptional cases are *unibranch* (cf. §B.0.6). Hence their intersection with Slodowy slices are always irreducible too (see the comments before Lemma B.5). Moreover, except for the nilpotent orbit of dimension 8 in G_2, all nilpotent orbit closures are (conjecturally for the types E_7 and E_8) normal. To see this, just compare Tables 2–10 of [**Arakawa15a**] and Table 3 of [**Arakawa-Moreau16b**].

4.6.2. Deligne series and the Joseph ideal. There was actually a "strong Feigin-Frenkel conjecture" stating that k is admissible if and only if $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ (provided that k is not critical, that is, $k \neq -h^\vee$ in which case it is known that $X_{V_k(\mathfrak{g})} = \mathcal{N}$). Such a statement would be interesting because it would give a geometrical description of the admissible representations $L(kD)$.

This stronger conjecture is actually wrong, as shown the following.

THEOREM 4.49 ([**Arakawa-Moreau15**]). *Assume that* $\mathfrak{g}$ *belongs to the Deligne exceptional series* [**Deligne96**],

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and that $k = -\dfrac{h^\vee}{6} - 1$*. Then*

$$X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}.$$

Note that the level $k = -h^\vee/6 - 1$ is not admissible for the types D_4, E_6, E_7, E_8. Theorem 4.49 provides the first known examples of associated varieties contained in the nilpotent cone corresponding to non-admissible levels.

The proof of this result is closely related to the Joseph ideal and its description by Gan and Savin (§3.4.21).

SKETCH OF PROOF OF THEOREM 4.49. Let $W = \bigoplus_i W_i$ be the decomposition of W into irreducible submodules, and let w_i be a highest weight vector of W_i.

LEMMA 4.50. *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the types A and G_2. For any i, $\sigma_2(w_i)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = -\frac{h^\vee}{6} - 1.$$

PROOF (FOR THE TYPES E_6 AND E_7). Assume that $\mathfrak{g}$ has type E_6 or E_7. The proof is very similar for types D_4 and E_8 (only some technical changes) and we refer to [**Perše07**] for the type F_4.

By (72), $W = W_1 = L_\mathfrak{g}(\theta+\theta_1)$. Moreover, according to [**Garfinkle82**, Chapter IV, Proposition 11],

$$w_1 = e_\theta e_{\theta_1} - \sum_{j=1}^{\frac{h^\vee}{6}+1} e_{\beta_j+\theta_1} e_{\delta_j+\theta_1},$$

where (β_j, δ_j) runs through the pairs of positive roots such that

$$\beta_j + \delta_j = \theta - \theta_1.$$

The number of such pairs turns out to be equal to $h^\vee/6 + 1$.

Choose a Chevalley basis $\{h_i\}_i \cup \{e_\alpha,\, f_\alpha\}_\alpha$ of $\mathfrak{g}$ so that the conditions of [**Garfinkle82**, Chapter IV, Definition 6] are fulfilled, that is

$$\forall\, j, \quad [e_{\delta_j}, [e_{\beta_j}, e_{\theta_1}]] = e_\theta, \quad [e_{\beta_j}, e_{\theta_1}] = e_{\beta_j+\theta_1}, \quad [e_{\delta_j}, e_{\theta_1}] = e_{\delta_j+\theta_1}. \tag{98}$$

We conclude thanks to the following exercise.

EXERCISE 4.51. Verify that $\sigma_2(w_1)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $k = -h^\vee/6 - 1$.

□

For $\mathfrak{g}$ of type A_1, A_2, G_2, F_4, the number $-h^\vee/6 - 1$ is admissible, and the theorem is a special case of Theorem 4.46.

Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8 and set $k = -h^\vee/6 - 1$.

Let N be the submodule of $V^k(\mathfrak{g})$ generated by $\sigma_2(w_i)$ for all i and let

$$\tilde{V}_k(\mathfrak{g}) := V^k(\mathfrak{g})/N.$$

CLAIM 4.1 ([**Arakawa-Moreau15**, Proof of Theorem 3.1]). $\tilde{V}_k(\mathfrak{g}) = V_k(\mathfrak{g})$, that is, $\tilde{V}_k(\mathfrak{g})$ is simple.

The exact sequence $0 \to N \to V^k(\mathfrak{g}) \to \tilde{V}_k(\mathfrak{g}) \to 0$ induces an exact sequence

$$N/\mathfrak{g}[t^{-1}]t^{-2}N \to V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \to \tilde{V}_k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}\tilde{V}_k(\mathfrak{g}) \to 0.$$

Under the isomorphism $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \cong S(\mathfrak{g})$, the image of $N/\mathfrak{g}[t^{-1}]t^{-2}N$ in $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$ is identified with the ideal J of $S(\mathfrak{g})$ generated by w_i for all i. Hence $J \subset J_W \subset \sqrt{J}$. Therefore by Lemma 3.70,

$$\sqrt{J} = \sqrt{J_W} = J_0$$

as required, by Claim 4.1. Remember that J_0 is the defining ideal of $\overline{\mathbb{O}_{min}}$. □

As a consequence of the above results (specifically, Lemma 3.70, Lemma 4.50 and Claim 4.1), we obtain the following result. Recall that $\mathcal{J}_W$ is the two-sided ideal of $U(\mathfrak{g})$ generated by W.

THEOREM 4.52. *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and that $k = -\frac{h^\vee}{6} - 1$. Then $V_k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})/\mathcal{J}_W$, that is,*

$$\mathrm{Zhu}(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_W = \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

In particular (see §4.4.2), since $\mathcal{J}_0$ is maximal, the irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ is a $V_k(\mathfrak{g})$-module if and only if

$$\bar{\lambda} = 0 \quad or \quad \mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\bar{\lambda}) = \mathcal{J}_0,$$

and such λ are described by Joseph (see Table 1 of [**Arakawa-Moreau15**]*).*

Theorem 4.49 produces "new" (that is, not coming from admissible levels) examples of level k such that $X_{V_k(\mathfrak{g})}$ is contained in the nilpotent cone. In particular, such associated varieties have only finitely many symplectic leaves. Such vertex algebras are called *quasi-lisse* (cf. [**Arakawa-Kawasetsu16**]). It is shown in [**Arakawa-Kawasetsu16**] that the normalized character of an ordinary module over a quasi-lisse vertex operator algebra has a modular invariance property, in the sense that it satisfies a modular linear differential equation.

We will see that Theorem 4.49 also produces "new" examples of lisse simple W-algebras. There are actually other such examples in type D_r, $r \geqslant 5$, and in type B_r, $r \geqslant 3$; see [**Arakawa-Moreau15, Arakawa-Moreau16a, Arakawa-Moreau16b**]; (see Part 5 for more details about all this).

4.6.3. Sheets as associated variety. In all the above examples, $X_{V_k(\mathfrak{g})}$ is a closure of a some nilpotent orbit $\mathbb{O} \subset \mathcal{N}$, or $X_{V_k(\mathfrak{g})} = \mathfrak{g}^*$. The later happens if k is generic, that is, $k \notin \mathbb{Q}$ in which case $V_k(\mathfrak{g}) = V^k(\mathfrak{g})$. Therefore it is natural to ask whether there are cases when $X_{V_k(\mathfrak{g})} \not\subset \mathcal{N}$ and $X_{V_k(\mathfrak{g})}$ is a proper subvariety of $\mathfrak{g}^*$.

It is known that sheets (see §B.0.5) appear in the representation theory of finite dimensional Lie algebras (cf. e.g., [**Borho-Brylinski82, Borho-Brylinski85, Borho-Brylinski89**], and more recently of finite W-algebras [**Premet-Topley14, Premet14**]. Next result is that sheets also appear as associated varieties of some affine vertex algebras.

THEOREM 4.53 ([**Arakawa-Moreau16a**]). (1) *For $n \geqslant 4$,*

$$\tilde{X}_{V_{-1}(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}_{min}}$$

as schemes, where $\mathbb{S}_{min}$ is the unique sheet containing $\mathbb{O}_{min}$. Moreover, as schemes,

$$SS(V_{-1}(\mathfrak{sl}_n) = J_\infty(\overline{\mathbb{S}_{min}}).$$

(2) *For $m \geqslant 2$,*

$$\tilde{X}_{V_{-m}(\mathfrak{sl}_{2m})} \cong \overline{\mathbb{S}_0}$$

as schemes, where $\mathbb{S}_0$ is the unique sheet containing the nilpotent orbit $\mathbb{O}_{(2^m)}$. Moreover, as schemes,

$$\mathbb{S}(V_{-m}(\mathfrak{sl}_{2m}) = J_\infty(\overline{\mathbb{S}_0}).$$

(3) *Let r be an odd integer. Then*

$$X_{V_{2-r}(\mathfrak{so}_{2r})} = \overline{\mathbb{S}_I} \cup \overline{\mathbb{S}_{II}},$$

where $\mathbb{S}_I$ and $\mathbb{S}_{II}$ are the unique sheets containing the nilpotent orbits $\mathbb{O}^I_{(2^m)}$ and $\mathbb{O}^I_{(2^m)}$ respectively. Note they are distinct and of the same dimension. In particular the associated variety $X_{V_{2-r}(\mathfrak{so}_{2r})}$ is reducible.

By the Irreducibility Theorem, associated varieties of primitive ideals are irreducible and contained in the nilpotent cone. Theorem 4.53 shows that this is not anymore the case for affine vertex algebras.

4.6.4. Conjectures, open problems. In view of the above results (and other results, particularly, on associated varieties of simple affine W-algebras: see Part 5), we formulate a conjecture (cf. [**Arakawa-Moreau16a**, Conjecture 1]).

CONJECTURE 2. *Let $V = \oplus_{d\geqslant 0} V_d$ be a simple, finitely strongly generated, positively graded conformal vertex operator algebra such that $V_0 = \mathbb{C}$.*

(1) *X_V is equidimensional.*
(2) *Assume that X_V has finitely many symplectic leaves, that is, V is quasi-lisse. Then X_V is irreducible. In particular, $X_{V_k(\mathfrak{g})}$ is irreducible if $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$.*

We also have the following conjecture.

CONJECTURE 3. *For any vertex algebra V as in Conjecture 2, we have, as schemes, $SS(V) = J_\infty(\tilde{X}_V)$.*

Conjecture 3 is also non trivial stated for the corresponding reduced schemes. In this setting, the following result is useful (and was applied for instance in Theorem 4.53).

PROPOSITION 4.54 ([**Arakawa-Moreau16a**]). *Let V be a quotient of the vertex algebra $V^k(\mathfrak{g})$. Suppose that $X_V = \overline{G.\mathbb{C}^* x}$ for some $x \in \mathfrak{g}$. Then*

$$SS(V)_{\mathrm{red}} = J_\infty(X_V) = J_\infty(\overline{G.\mathbb{C}^* x}) = \overline{J_\infty(G.\mathbb{C}^* x)}.$$

For example, the closures of nilpotent orbits satisfy the conditions of Proposition 4.54, and also *Dixmier sheets* of rank one (cf. §B.0.5).

To conclude this section, note that there are other known examples of associated varieties with finitely many symplectic leaves: apart from the above examples, it is the case when V is the (generalized) Drinfeld-Sokolov reduction (see Part 5) of the above affine vertex algebras provided that it is nonzero ([**Arakawa15a**]). This is also expected to happen for the vertex algebras obtained from four dimensional $N = 2$ superconformal field theories ([**BLL$^+$**]), where the associated variety is expected to coincide with the spectrum of the chiral ring of the Higgs branch of the four dimensional theory. Of course, it also happens when the associated variety of V is a point, that is, when V is lisse.

PART 5

Affine W-algebras, rationality of W-algebras, and chiral differential operators on groups

APPENDIX A

Superalgebras and Clifford algebras

A *superspace* is a $\mathbb{C}$-vector space E equipped with a $\mathbb{Z}_2$-grading, $E = E^{\bar{0}} \oplus E^{\bar{1}}$. Elements in $E^{\bar{0}}$ are called *even*, elements of $E^{\bar{1}}$ are called *odd*. We denote by $|v| \in \{\bar{0}, \bar{1}\}$ the parity of homogeneous elements $v \in E$. A *morphism of superspaces* is a linear map preserving $\mathbb{Z}_2$-gradings. It is itself a superspace by:

$$\mathrm{Hom}(E,F)^{\bar{0}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{0}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{1}}),$$
$$\mathrm{Hom}(E,F)^{\bar{1}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{1}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{0}}).$$

The category of superspaces is a tensor category. Then one may define superalgebras, Lie superalgebras, Poisson superalgebras, etc. as the algebra objects, Lie algebra objects, Poisson algebra objects etc. in this tensor category.

For example, a *Lie superalgebra* is a superspace A together with a bracket $[\ ,\ ]\colon A \times A \to A$ such that for all homogeneous elements $a, b \in A$,

$$[a,b] = -(-1)^{|a||b|}[b,a],$$
$$[a,[b,c]] = \{\{a,b\},c\} + (-1)^{|a||b|}[a,[b,c]].$$

Note that any superalgebra A is naturally a Lie superalgebra by setting for all homogeneous elements $a, b \in A$,

$$[a,b] = ab - (-1)^{|a||b|}ba.$$

It is *supercommutative* if $[A,A] = 0$.

A superspace A is a *Poisson superalgebra* if it is equipped with a bracket $\{\ ,\ \}\colon A \times A \to A$ such that $(A, \{\ ,\ \})$ is a Lie superalgebra and for any $a \in A$, the operator $\{a, \cdot\}\colon A \to A$ is a *superderivation*: for all homogeneous elements $a, b \in A$,

$$\{a, bc\} = \{a,b\}a + (-1)^{|a||b|}b\{a,c\},$$

Let E be a $\mathbb{C}$-vector space. The *exterior algebra* $\wedge E$ is the quotient of the tensor algebra $T(E) = \bigoplus_{k \in \mathbb{Z}} T(E)^k$, with $T^k(E) = E \otimes \cdots \otimes E$ the k-fold tensor product, by the two-sided ideal $I(E)$ generated by elements of the form $v \otimes w + w \otimes v$ with $v, w \in E$. The product in $\wedge E$ is usually denoted by $v \wedge w$. Since $I(E)$ is graded, the exterior algebra inherits a grading

$$\wedge(E) = \bigoplus_{k \in \mathbb{Z}} \wedge^k(E).$$

Clearly, $\wedge^0(E) = \mathbb{C}$ and $\wedge^1(E) = E$. We may thus think of $\wedge E$ as the associative algebra linearly generated by E, subject to the relations $v \wedge w + w \wedge v = 0$. We will regard $\wedge(E)$ as a graded superalgebra, where the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading. Since

$$[u_1, u_2] = u_2 \wedge u_2 + (-1)^{k_1 k_2} u_2 \wedge u_2 = 0$$

for $u_1 \in \wedge^{k_1}(E)$ and $u_2 \in \wedge^{k_2}(E)$, we see that $\wedge E$ is supercommutative.

Assume that E is endowed with a symmetric bilinear form $B\colon E \times E \to E$ (possibly degenerate).

Definition A.1. The *Clifford algebra*[1] $Cl(E,B)$ is the quotient of $T(E)$ by the two-sided ideal $\mathscr{I}(E,B)$ generated by all elements of the form

$$v \otimes w + w \otimes v - B(v,w)1, \qquad v, w \in E.$$

Clearly, $Cl(E,0) = \wedge V$.

The inclusions $\mathbb{C} \to T(E)$ and $E \to T(E)$ descend to inclusions $\mathbb{C} \to Cl(E,B)$ and $E \to Cl(E,B)$ respectively. We will always view E as a subspace of $Cl(E,B)$.

Let us view $T(E) = \bigoplus_{k\in\mathbb{Z}} T^k(E)$ as a filtered superalgebra, with the $\mathbb{Z}_2$-grading and filtration inherited from the $\mathbb{Z}$-grading. Since the elements $v \otimes w + w \otimes v - B(v,w)1$ are even, of filtration degree 2, the ideal $\mathscr{I}(E,B)$ is a filtered super subspace of $T(E)$, and hence $Cl(E,B)$ inherits the structure of a filtered superalgebra. The $\mathbb{Z}_2$-grading and filtration on $Cl(E,B)$ are defined by the condition that the generators $v \in E$ are odd, of filtration degree 1. In the decomposition

$$Cl(E,B) = Cl(E,B)^{\bar{0}} \oplus Cl(E,B)^{\bar{1}}$$

the two summands are spanned by products $v_1 \dots v_k$ with k even, respectively odd. We will always regard $Cl(E,B)$ as a filtered superalgebra. Then the defining relations for the Clifford algebra become

$$[v,w] = vw + wv = B(v,w), \qquad v, w \in E.$$

The *quantization map*, given by the anti-symmetrization:

$$q\colon \wedge(E) \to Cl(E,B), \quad v_1 \wedge \ldots \wedge v_k \mapsto \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma 1} \dots v_{\sigma k},$$

with $\mathfrak{S}_k$ the permutation group of order k, is an isomorphism of superspaces. Its inverse is called the *symbol map*.

Proposition A.2. *The symbol map $\sigma\colon Cl(E,B) \to \wedge(E)$ induces an isomorphism of graded superalgebras,*

$$\operatorname{gr} Cl(E,B) \xrightarrow{\simeq} \wedge(E).$$

Since $\wedge(E)$ is supercommutative, $\operatorname{gr} Cl(E,B)$ inherits a Poisson superalgebra structure[2], and the symbol map is an isomorphism of graded Poisson superalgebras. The Poisson bracket on $\wedge(E)$ can be described by:

$$\{v,w\} = B(v,w), \qquad v, w \in E = \wedge^1(E).$$

For more about Clifford algebras, we refer to the recent book of Eckhard Meinrenken (it also adsresses *Weil algebras* and *quantized Weil algebras*) [**Meinrenken**].

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[1] In [**Meinrenken**], there is a factor 2. For some reasons, we prefer here a different normalization.

[2] The arguments are similar to the case of almost commutative algebras; see §3.3.1.

APPENDIX B

Some advanced results on the geometry of nilpotent orbit closures

In this appendix, we present some deeper aspects of the geometry of nilpotent orbit closures.

Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ be a simple Lie algebra as in Part 3.

B.0.5. Jordan classes, sheets and induced nilpotent orbits. Most of results presented in this section come from [**Borho-Kraft79, Borho81**]. Our main reference for basics about Jordan classes and sheets is [**Tauvel-Yu**, §39].

For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$, denote by $\mathfrak{z}(\mathfrak{a})$ its center. For Y a subset of $\mathfrak{g}$, denote by Y^{reg} the set of $y \in Y$ for which $\mathfrak{g}^y$ has the minimal dimension. In particular, if $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$, then

$$\mathfrak{z}(\mathfrak{l})^{\mathrm{reg}} := \{y \in \mathfrak{g} \mid \mathfrak{z}(\mathfrak{g}^y) = \mathfrak{z}(\mathfrak{l})\},$$

and $\mathfrak{z}(\mathfrak{l})^{\mathrm{reg}}$ is a dense open subset of $\mathfrak{z}(\mathfrak{l})$. For $x \in \mathfrak{g}$, denote by x_s and x_n the semisimple and the nilpotent components of x respectively.

The *Jordan class* of x is

$$J(x) := \mathbf{G}.(\mathfrak{z}(\mathfrak{g}^{x_s})^{\mathrm{reg}} + x_n).$$

It is a $\mathbf{G}$-invariant, irreducible, and locally closed subset of $\mathfrak{g}$. To a Jordan class J, we associate its *datum* which is the pair $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ defined as follows. Pick $x \in J$. Then $\mathfrak{l}$ is the Levi subalgebra $\mathfrak{g}^{x_s}$ and $\mathbb{O}_{\mathfrak{l}}$ is the nilpotent orbit in $\mathfrak{l}$ of x_n. The pair $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ does not depend on $x \in J$ up to $\mathbf{G}$-conjugacy, and there is a one-to-one correspondence between the set of pairs $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ as above, up to $\mathbf{G}$-conjugacy, and the set of Jordan classes.

A *sheet* is an irreducible component of the subsets

$$\mathfrak{g}^{(m)} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = m\}, \qquad m \in \mathbb{Z}_{\geqslant 0}.$$

It is a finite disjoint union of Jordan classes. So a sheet $\mathbb{S}$ contains a unique dense open Jordan class J and we can define the *datum* of $\mathbb{S}$ as the datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ of the Jordan class J. We have

$$\overline{\mathbb{S}} = \overline{J} \quad \text{and} \quad \mathbb{S} = (\overline{J})^{\mathrm{reg}}.$$

A sheet is called *Dixmier* if it contains a semisimple element of $\mathfrak{g}$. A sheet $\mathbb{S}$ with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is Dixmier if and only if $\mathbb{O}_{\mathfrak{l}} = \{0\}$. We shall simply denote by $\mathbb{S}_{\mathfrak{l}}$ the Dixmier sheet with datum $(\mathfrak{l}, \{0\})$.

Let $\mathfrak{l}$ be a proper Levi subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$ so that $\mathfrak{m}$ is the nilpotent radical of $\mathfrak{p}$. Let $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{M}$ be the connected closed subgroups of $\mathbf{G}$ whose Lie algebra are $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{m}$ respectively. Then $\mathbf{P} = \mathbf{LM}$.

The following definitions and results on induced nilpotent orbits are mostly extracted from [**R74**] and [**Lusztig-Spaltenstein79**]. We refer to [**Collingwood-McGovern**, Chapter 7] for a recent survey.

THEOREM B.1. *Let $\mathbb{O}_{\mathfrak{l}}$ be a nilpotent orbit of $\mathfrak{l}$. There exists a unique nilpotent orbit $\mathbb{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ whose intersection with $\mathbb{O}_{\mathfrak{l}} + \mathfrak{m}$ is a dense open subset of $\mathbb{O}_{\mathfrak{l}} + \mathfrak{m}$. Moreover, the intersection of $\mathbb{O}_{\mathfrak{g}}$ with $\mathbb{O}_{\mathfrak{l}} + \mathfrak{m}$ consists of a single $\mathbf{P}$-orbit and* $\operatorname{codim}_{\mathfrak{g}}(\mathbb{O}_{\mathfrak{g}}) = \operatorname{codim}_{\mathfrak{l}}(\mathbb{O}_{\mathfrak{l}})$.

The nilpotent orbit $\mathbb{O}_{\mathfrak{g}}$ only depends on $\mathfrak{l}$, a nd not on the choice of a parabolic subalgebra $\mathfrak{p}$ containing it. The nilpotent orbit $\mathbb{O}_{\mathfrak{g}}$ is called the *induced nilpotent orbit o f* $\mathfrak{g}$ *from* $\mathbb{O}_{\mathfrak{l}}$, and it is denoted by $\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}})$. A nilpotent orbit which is not induced in a proper way from another one is called *rigid*. In $\mathfrak{sl}_n$, only the zero orbit is rigid, and all nilpotent orbits are *Richardson*, that is, induced from 0 in some Levi subalgebra.

REMARK B.2. The induction property is transitive in the following sense: if $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Levi subalgebras of $\mathfrak{g}$ with $\mathfrak{l}_1 \subset \mathfrak{l}_2$, then

$$\operatorname{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathbb{O}_{\mathfrak{l}_1})) = \operatorname{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}_1}).$$

A Jordan class with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is a sheet if and only if $\mathbb{O}_{\mathfrak{l}}$ is rigid in $\mathfrak{l}$. So we get a one-to-one correspondence between the set of pairs $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$, up to G-conjugacy, with $\mathfrak{l}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathbb{O}_{\mathfrak{l}}$ a rigid nilpotent orbit of $\mathfrak{l}$, and the set of sheets.

Each sheet contains a unique nilpotent orbit. Namely, if $\mathbb{S}$ is a sheet with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ then the induced nilpotent orbit $\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}})$ of $\mathfrak{g}$ from $\mathbb{O}_{\mathfrak{l}}$ in $\mathfrak{l}$ is the unique nilpotent orbit contained in $\mathbb{S}$. Note that a nilpotent orbit $\mathbb{O}$ is itself a sheet if and only if $\mathbb{O}$ is rigid. For instance, outside the type A, the minimal nilpotent orbit $\mathbb{O}_{min}$ is always a sheet.

EXAMPLE B.3. The Levi subalgebras of $\mathfrak{sl}_n$, and so the (Diximer) sheets, are parametrized by compositions of n. More precisely, if $\boldsymbol{\lambda} \in \mathscr{P}(n)$, then the (Dixmier) sheet associated with $\boldsymbol{\lambda}$ is the unique sheet containing $\mathbb{O}_{{}^t\boldsymbol{\lambda}}$ where ${}^t\boldsymbol{\lambda}$ is the dual partition of $\boldsymbol{\lambda}$.

Contrary to the $\mathfrak{sl}_n$ case, it may happen in the $\mathfrak{so}_n$ case that a given nilpotent orbit belongs to different sheets, and not all sheets are Dixmier.

The *rank* of a sheet $\mathbb{S}$ with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is by definition

$$\operatorname{rank}(\mathbb{S}) := \dim \mathbb{S} - \dim \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}}) = \dim \mathfrak{z}(\mathfrak{l}).$$

If $\mathbb{S} = \mathbb{S}_{\mathfrak{l}}$ is Dixmier, then $\mathbb{O}_{\mathfrak{l}} = 0$ and we have

$$\overline{\mathbb{S}} = G.[\mathfrak{p}, \mathfrak{p}]^{\perp} = G.(\mathfrak{z}(\mathfrak{l}) + \mathfrak{m}) \quad \text{ and } \quad \mathbb{S} = (G.[\mathfrak{p}, \mathfrak{p}]^{\perp})^{\text{reg}}.$$

where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{m}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mathbb{S}$ a sheet with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$, one can assume without loss of generality that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}$. In particular, $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h}$.

The following lemma shows that Dixmier sheets of rank one are are easy to understand.

LEMMA B.4. *Let $\mathbb{S}_{\mathfrak{l}}$ be a Dixmier sheet of rank one, that is, $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}\lambda$ with $\lambda \in \mathfrak{h} \setminus \{0\}$. Then*

$$\overline{\mathbb{S}_{\mathfrak{l}}} = \overline{G.\mathbb{C}^*\lambda} = G.(\mathbb{C}\lambda + \mathfrak{m}) = G.\mathbb{C}^*\lambda \cup \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)},$$

and

$$\mathbb{S}_{\mathfrak{l}} = G.\mathbb{C}^*\lambda \cup \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0).$$

Let P be the connected parabolic subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$. The G-action on $Y := G/(P, P)$, where (P, P) is the commutator-subgroup of P, induces an algebra homomorphism

$$\psi_Y : U(\mathfrak{g}) \to \mathcal{D}_Y$$

from $U(\mathfrak{g})$ to the algebra $\mathcal{D}_Y$ of global differential operators on Y. Let

$$\mathscr{J}_Y := \ker \psi_Y$$

be the kernel of this homomorphism. It is a two-sided ideal of $U(\mathfrak{g})$. It has been shown by Borho and Brylinski [**Borho-Brylinski82**, Corollary 3.11 and Theorem4.6] that $\sqrt{\mathrm{gr}\, \mathscr{J}_Y}$ is the defining ideal of the Dixmier sheet closure determined by P, that is, $\overline{\mathbb{S}_{\mathfrak{l}}}$. Furthermore,

$$\mathscr{J}_Y = \bigcap_{\lambda \in \mathfrak{z}(\mathfrak{l})^*} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{\lambda}.$$

Here, for $\lambda \in \mathfrak{p}^*$, $\mathbb{C}_\lambda$ stands for the one-dimensional representation of $\mathfrak{p}$ corresponding to λ, and we extend a linear form $\lambda \in \mathfrak{z}(\mathfrak{l})^*$ to $\mathfrak{p}^*$ by setting $\lambda(x) = 0$ for $x \in [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{m}$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\ |\ )$, $\mathfrak{z}(\mathfrak{l})^*$ identifies with $\mathfrak{z}(\mathfrak{l})$. In particular, if $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}\lambda$ for some nonzero semisimple element $\lambda \in \mathfrak{g}$, we get

$$\mathscr{J}_Y = \bigcap_{t \in \mathbb{C}} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\lambda}.$$

In fact

$$\mathscr{J}_Y = \bigcap_{t \in Z} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\lambda}. \tag{99}$$

for any Zariski dense subset Z of $\mathbb{C}$ ([**Borho-Jantzen77**]).

B.0.6. Branching and nilpotent Slodowy slices. We collect in this paragraph some results about branchings and nilpotent Slodowy slices. We refer to [**EGA61**, Chap. III, §4.3] for the definition of *unibranchness*, and to [**Kraft-Procesi82**] or [**Fu-et-al15**] for further details on branchings and nilpotent Slodowy slices.

Consider two varieties X, Y and two points $x \in X$, $y \in Y$. The singularity of X at x is called *smoothly equivalent* to the singularity of Y at y if there is a variety Z, a point $z \in Z$ and two maps

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ \Big\downarrow{\scriptstyle \psi} & & \\ Y & & \end{array}$$

such that $\varphi(z) = x$, $\psi(z) = y$, and φ and ψ are smooth in z ([**Hesselink76**]). This clearly defines an equivalence relation between pointed varieties (X, x). We denote the equivalence class of (X, x) by $\mathrm{Sing}(X, x)$.

Various geometric properties of X at x only depends on the equivalence class $\mathrm{Sing}(X,x)$, for example: *smoothness, normality, seminormality (cf.* [**Kraft-Procesi82**, §16.1]*), unibranchness, Cohen-Macaulay, rational singularities.*

Assume that the algebraic group $\mathbf{G}$ acts regularly on the variety X. Then $\mathrm{Sing}(X,x) = \mathrm{Sing}(X,x')$ if x and x' belongs to the same $\mathbf{G}$-orbit $\mathbb{O}$. In this case, we denote the equivalence class also by $\mathrm{Sing}(X,\mathbb{O})$.

A *cross section* (or *transverse slice*) at the point $x \in X$ is defined to be a locally closed subvariety $S \subset X$ such that $x \in S$ and the map

$$G \times S \longrightarrow X, \quad (g,s) \longmapsto g.s,$$

is smooth at the point $(1,x)$. We have $\mathrm{Sing}(S,x) = \mathrm{Sing}(X,x)$.

In the case where X is the closure of some nilpotent G-orbit of $\mathfrak{g}$, there is a natural choice of a cross section. Let $\mathbb{O},\mathbb{O}'$ be two nonzero nilpotent orbits of $\mathfrak{g}$ and pick $f \in \mathbb{O}'$ that we embed f into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. The Slodowy slice $\mathscr{S}_f \cong f + \mathfrak{g}^e$ is a transverse slice of $\mathfrak{g}$ at f (cf. Theorem 3.12). The variety

$$\mathscr{S}_{\mathbb{O},f} := \overline{\mathbb{O}} \cap \mathscr{S}_f$$

is then a transverse slice of $\overline{\mathbb{O}}$ at f, which we call, following the terminology of [**Fu-et-al15**], a *nilpotent Slodowy slice*.

Note that $\mathscr{S}_{\mathbb{O},f} = \{f\}$ if and only if $\mathbb{O} = G.f$. Moreover, since the $\mathbb{C}^*$-action of $\tilde{\rho}$ on $\mathscr{S}_f$ is contracting to f and stabilizes $\mathscr{S}_{f,\mathbb{O}}$, $\mathscr{S}_{\mathbb{O},f} = \varnothing$ if and only if $G.f \not\subseteq \overline{\mathbb{O}}$. Hence we can assume that $\mathbb{O}' \subseteq \overline{\mathbb{O}}$, that is, $\mathbb{O}' \leqslant \mathbb{O}$ for the Chevalley order on nilpotent orbits. The variety $\mathscr{S}_{\mathbb{O},f}$ is equidimensional, and

$$\dim \mathscr{S}_{\mathbb{O},f} = \mathrm{codim}(\mathbb{O}',\overline{\mathbb{O}}).$$

Since any two $\mathfrak{sl}_2$-triples containing f are conjugate by an element of the isotropy group of f in $\mathbf{G}$, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of such $\mathfrak{sl}_2$-triples. Moreover, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of $f \in \mathbb{O}'$. By focussing on $\mathscr{S}_{\mathbb{O},f}$, we reduce the study of $\mathrm{Sing}(\overline{\mathbb{O}},\mathbb{O}')$ to the study of the singularity of $\mathscr{S}_{\mathbb{O},f}$ at f.

The variety $\mathscr{S}_{\mathbb{O},f}$ is not always irreducible. We are now interested in sufficient conditions for that $\mathscr{S}_{\mathbb{O},f}$ is irreducible.

Let X be an irreducible algebraic variety, and $x \in X$. We say that X is *unibranch at x* if the normalization $\pi\colon (\tilde{X},x) \to (X,x)$ of (X,x) is locally a homeomorphism at x [**Fu-et-al15**, §2.4]. Otherwise, we say that X *has branches at x* and the number of branches of X at x is the number of connected components of $\pi^{-1}(x)$ [**Beynon-Spaltenstein84**, §5,(E)].

As it is explained in [**Fu-et-al15**, Section 2.4], the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$ is equal to the number of branches of $\overline{\mathbb{O}}$ at f.

If an irreducible algebraic variety X is normal, then it is obviously unibranch at any point $x \in X$. Hence we obtain the following result.

Lemma B.5. *Let $\mathbb{O},\mathbb{O}'$ be nilpotent orbits of $\mathfrak{g}$, with $\mathbb{O}' \leqslant \mathbb{O}$ and $f \in \mathbb{O}'$. If $\overline{\mathbb{O}}$ is normal, then $\mathscr{S}_{\mathbb{O},f}$ is irreducible.*

The converse is not true. For instance, there is no branching in type G_2 but one knowns that the nilpotent orbit $\tilde{A}_1$ of G_2 of dimension 8 is not normal [**Levasseur-Smith88**].

The number of branches of $\overline{\mathbb{O}}$ at f, and so the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$, can be determined from the tables of Green functions in [**Shoji80**,

Beynon-Spaltenstein84], as discussed in [**Beynon-Spaltenstein84**, Section 5,(E)-(F)]. See Table 2 of [**Arakawa-Moreau16b**] for the complete list of the nilpotent orbits $\mathbb{O}$ which have branchings in types F_4, E_6, E_7 and E_8 (there is no branching in type G_2), and Table 3 of [**Arakawa-Moreau16b**] for the (conjectural) list a non-normal nilpotent orbit closures in the exceptional types. These results are extracted from [**Levasseur-Smith88, Kraft89, Broer98a, Broer98b, Sommers03**]. The list is known to be exhaustive for the types G_2, F_4 and E_6. It is only conjecturally exhaustive for the types E_7 and E_8.

The normality question of nilpotent orbit closures in the classical types is now completely answered ([**Kraft-Procesi79, Kraft-Procesi79, Sommers05**]). Note that, by [**Kraft-Procesi79**], if $\mathfrak{g} = \mathfrak{sl}_n$, then all nilpotent orbit closures are normal. In all the other types, there is at least one non-normal nilpotent closure.

Let $\mathbb{O}$ be a nilpotent orbit of $\mathfrak{g}$. Recall that the singular locus of $\overline{\mathbb{O}}$ is $\overline{\mathbb{O}} \setminus \mathbb{O}$. This was shown by Namikawa [**Namikawa04**] using results of Kaledin and Panyushev [**Kaledin06, Panyushev91**]; see [**Henderson15**, Section 2] for a recent review. This result also follows from Kraft and Procesi's work in the classical types [**Kraft-Procesi81, Kraft-Procesi82**], and from the main theorem of [**Fu-et-al15**] in the exceptional types.

THEOREM B.6 ([**Kraft-Procesi82**, Theorem 1]). *Let $\mathbb{O}$ be a nilpotent orbit in $\mathfrak{o}_n$ or $\mathfrak{sp}_n$.*

(1) *$\overline{\mathbb{O}}$ is normal if and only if it is unibranch.*
(2) *$\overline{\mathbb{O}}$ is normal if and only if it is normal in codimension 2.*

In particular, $\overline{\mathbb{O}}$ is normal if it does not contain a nilpotent orbit $\mathbb{O}' \leqslant \mathbb{O}$ of codimension 2. Theorem B.6 does not hold if $\mathfrak{g} = \mathfrak{so}_{2n}$ and if $\mathbb{O} = \mathbb{O}_{1,\boldsymbol{\lambda}}$, with $\boldsymbol{\lambda}$ very even. To determine the equivalence class $\mathrm{Sing}(\overline{\mathbb{O}_{\varepsilon,\boldsymbol{\lambda}}}, \mathbb{O}_{\varepsilon,\boldsymbol{\eta}})$, for $\varepsilon \in \{-1, 1\}$ and $\boldsymbol{\eta} < \boldsymbol{\lambda}$, there is a combinatorial method developed in [**Kraft-Procesi82**]. We refer to [**Arakawa-Moreau16b**, Section 4] for more details about this.

Kraft and Procesi method together with Theorem B.6 allow to deal with almost all nilpotent orbits, with exceptions for the very even nilpotent orbits in type $\mathfrak{so}_n$. For these orbits, the normality question was partially answered in [**Kraft-Procesi82**, Theorem 17.3], the remaining cases were dealt with in [**Sommers05**].

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