JET SCHEMES OF THE CLOSURE OF NILPOTENT ORBITS

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Abstract. We study in this paper the jet schemes of the closure of nilpotent orbits in a finite-dimensional complex reductive Lie algebra. For the nilpotent cone, which is the closure of the regular nilpotent orbit, all the jet schemes are irreducible. This was first observed by Eisenbud and Frenkel, and follows from a strong result of Mustaţă (2001). Using induction and restriction of "little" nilpotent orbits in reductive Lie algebras, we show that for a large number of nilpotent orbits, the jet schemes of their closure are reducible. As a consequence, we obtain certain geometrical properties of these nilpotent orbit closures.

1. Introduction

Throughout this paper, the ground field will be the field $\mathbb{C}$ of complex numbers. We shall work with the Zariski topology, and by *variety* we mean a reduced, irreducible and separated scheme of finite type over $\mathbb{C}$.

For X a scheme of finite type over $\mathbb{C}$ and $m \in \mathbb{N}$, we denote by $\mathscr{J}_m(X)$ the *m-th jet scheme of X*. It is a scheme of finite type over $\mathbb{C}$ whose $\mathbb{C}$-valued points are naturally in bijection with the $\mathbb{C}[t]/(t^{m+1})$-valued points of X, cf. e.g., [M01, EM09, Is11]. We have $\mathscr{J}_0(X) \simeq X$ and $\mathscr{J}_1(X) \simeq \mathrm{T}X$, where $\mathrm{T}X$ is the total tangent bundle of X; see Section 2 for more details about generalities on jet schemes. From Nash [Nas96], it is known that the geometry of the jet schemes is deeply related to the singularities of X. As an illustration of that phenomenon, we have the following result, first conjectured by Eisenbud and Frenkel [M01, Introduction], which will be important for us.

Theorem 1 ([M01, Thm. 1]). *Let X be an irreducible scheme of finite type over $\mathbb{C}$. If X is locally a complete intersection, then $\mathscr{J}_m(X)$ is irreducible for every $m \in \mathbb{N}$ if and only if X has rational singularities.*

According to Kolchin [Kol73], in contrast to the above theorem, the arc space $\mathscr{J}_\infty(X) = \varprojlim_m \mathscr{J}_m(X)$ of X is always irreducible when X is irreducible. In this paper, we shall be interested in the irreducibility of the jet schemes for the closure of nilpotent orbits in a complex reductive Lie algebra.

Let G be a complex connected reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $\mathcal{N}(\mathfrak{g})$ the nilpotent cone of $\mathfrak{g}$. It is the subscheme of $\mathfrak{g}$ associated to the augmentation ideal of $\mathbb{C}[\mathfrak{g}]^G$. It is a finite union of nilpotent G-orbits, and there is a unique nilpotent orbit of $\mathfrak{g}$, called the regular nilpotent orbit and denoted by $\mathcal{O}_{\mathrm{reg}}$, such that $\mathcal{N}(\mathfrak{g}) = \overline{\mathcal{O}_{\mathrm{reg}}}$.

According to Kostant [Kos63], the nilpotent cone is a complete intersection which is irreducible, reduced and normal. Furthermore, by [Hes76], it has rational singularities. Hence by Theorem 1,

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the jet scheme $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is irreducible for every $m \geqslant 1$. In fact, by [M01, Prop. 1.4 and 1.5], $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is also a complete intersection which is reduced for every $m \geqslant 1$.

In [M01, Appendix], Eisenbud and Frenkel used these results to extend certain results of Kostant [Kos63] in the setting of jet schemes. In particular, they proved that $\mathbb{C}[\mathscr{J}_m(\mathfrak{g})]$ is free over the ring $\mathbb{C}[\mathscr{J}_m(\mathfrak{g})]^{\mathscr{J}_m(G)}$ of $\mathscr{J}_m(G)$-invariants of $\mathbb{C}[\mathscr{J}_m(\mathfrak{g})]$.

Other nilpotent orbit closures do not share these geometrical properties in general. Indeed, according to a recent result of Namikawa [Nam13], for a nonzero and nonregular nilpotent orbit $\mathcal{O}$, $\overline{\mathcal{O}}$ is not a complete intersection. In addition, $\overline{\mathcal{O}}$ has not always rational singularities since it is not always normal, cf. e.g., [LeS8, KP82, K89, B98, So03].

Thus, it is quite natural to ask the following question.

Question 1. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$, and $m \in \mathbb{N}^*$. Is $\mathscr{J}_m(\overline{\mathcal{O}})$ irreducible?*

Answer Question 1 is the main purpose of this paper. For the zero orbit and the regular orbit, the answer is positive for every $m \in \mathbb{N}$. Outside these extreme cases, we will see that these jet schemes are rarely irreducible.

Motivations. Since $\overline{\mathcal{O}}$ is not a complete intersection for $\mathcal{O}$ nonzero and nonregular, Theorem 1 cannot be applied directly to answer Question 1. Very recently, Brion and Fu gave another proof of Namikawa's result, which is more uniform and slightly shorter, [BF13]. An interesting question, raised by Michel Brion to the first author, is whether jet schemes can be used to provide another proof of Namikawa's result.

Let us explain how we can tackle this problem using jet schemes. Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$. The singular locus of $\overline{\mathcal{O}}$ is exactly $\overline{\mathcal{O}} \setminus \mathcal{O}$. This follows from [Ka06, Lem. 1.4] and [P91]; see also [Hen14, Sec. 2] for a recent review. Moreover, we have

$$\operatorname{codim}_{\overline{\mathcal{O}}}(\overline{\mathcal{O}} \setminus \mathcal{O}) \geqslant 2.$$

For the nilpotent cone, we have precisely $\operatorname{codim}_{\mathcal{N}(\mathfrak{g})}(\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}_{\mathrm{reg}}) = 2$, and the equality $\mathcal{N}(\mathfrak{g})_{\mathrm{reg}} = \mathcal{O}_{\mathrm{reg}}$ is a consequence of [Kos63, Thm. 9] (thus the notation $\mathcal{O}_{\mathrm{reg}}$ does not bear any confusion).

So, if we assume that $\overline{\mathcal{O}}$ is a complete intersection, then $\overline{\mathcal{O}}$ is normal and so it has rational singularities by [Hi91] or [P91]. Hence, in that event, Mustaţă's Theorem implies that $\mathscr{J}_m(\overline{\mathcal{O}})$ is irreducible for every $m \geqslant 1$. So if we can show that $\mathscr{J}_m(\overline{\mathcal{O}})$ is reducible for some $m \geqslant 1$, then we would obtain a contradiction[1]. The above was our original motivation to look into Question 1.

It may happen that a variety X is not a complete intersection, that X has rational singularities and that nonetheless $\mathscr{J}_m(X)$ is irreducible for every $m \geqslant 1$. The cone over the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{2n-1}, \qquad n \geqslant 2,$$

shows that this situation is possible, cf. [M01, Ex. 4.7]. We do not know so far whether this situation may happen in the context of nilpotent orbit closures.

More generally, following Nash's philosophy, it would be interesting to understand what kind of properties on the singularities of $\overline{\mathcal{O}}$ we can deduce from the study of $\mathscr{J}_m(\overline{\mathcal{O}})$, $m \geqslant 1$. The fact that $\overline{\mathcal{O}}$ is not a complete intersection (with $\mathcal{O}$ nonzero and nonregular) whenever $\mathscr{J}_m(\overline{\mathcal{O}})$ is reducible for some $m \geq 1$ is one illustration of such a phenomenon.

Nilpotent orbit closures also form an interesting family of varieties providing examples and counter-examples in the context of jet schemes. For example, Examples 7.6 and 7.7 illustrate

[1]There are other approaches to use jet schemes to show that $\overline{\mathcal{O}}$ is not a complete intersection; see Example 7.2.

that the locally complete intersection hypothesis cannot be removed from Lemma 2.7,(3), and Theorem 2.8,(3). Another example is that the normality is not conserved when we pass to jet schemes. By Kostant, the nilpotent cone $\mathcal{N}(\mathfrak{g})$ is normal, and we show in Proposition 7.3 that $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$, $m \geqslant 1$, is not normal for a simple Lie algebra $\mathfrak{g}$.

Main results. Let us describe the main techniques used to study Question 1, and summarize the main results of the paper. To avoid technical details, we shall assume here that $\mathfrak{g}$ is simple.

Let X be an irreducible variety, and $m \in \mathbb{N}$. Then $\mathscr{J}_m(X)$ is irreducible if and only if

$$\pi_{X,m}^{-1}(X_{\mathrm{sing}}) \subset \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})},$$

where $\pi_{X,m} : \mathscr{J}_m(X) \to X$ is the canonical projection from $\mathscr{J}_m(X)$ onto X (cf. Section 2), X_{reg} is the smooth part of X, and X_{sing} its complement (cf. Lemma 2.7). This is our starting point.

For $\mathcal{O}$ a nilpotent orbit of $\mathfrak{g}$, the singular locus of $\overline{\mathcal{O}}$ is $\overline{\mathcal{O}} \setminus \mathcal{O}$ (cf. Section 3). The above criterion leads us to the following two conditions which will be central in our paper (cf. Definition 3.3).

Definition 1. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$.*

1) *We say that $\mathcal{O}$* verifies RC_1 *if $\pi_{\overline{\mathcal{O}},1}^{-1}(0)$ is not contained in the closure of $\pi_{\overline{\mathcal{O}},1}^{-1}(\mathcal{O})$.*
2) *Let $m \in \mathbb{N}^*$. We say that $\mathcal{O}$* verifies $\mathrm{RC}_2(m)$ *if for some nilpotent orbit $\mathcal{O}'$ contained in $\overline{\mathcal{O}} \setminus \mathcal{O}$, we have $\dim \pi_{\overline{\mathcal{O}},m}^{-1}(\mathcal{O}') \geqslant \dim \pi_{\overline{\mathcal{O}},m}^{-1}(\mathcal{O})$.*

Here the letters RC *stand for "reducibility condition".*

It follows readily (cf. Lemma 3.4) that if a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ verifies RC_1 (resp. $\mathrm{RC}_2(m)$ for some $m \in \mathbb{N}^*$), then $\mathscr{J}_1(\overline{\mathcal{O}})$ (resp. $\mathscr{J}_m(\overline{\mathcal{O}})$) is reducible.

We have a characterization for the condition RC_1 (cf. Proposition 3.6) which allows us for example to show that the nilpotent orbits of $\mathfrak{sl}_{2p}(\mathbb{C})$, with $p \geqslant 2$, associated with partitions of the form (2^p) verify RC_1 (cf. Example 3.7). Note that these orbits do not verify $\mathrm{RC}_2(1)$ (see again Example 3.7).

A nilpotent orbit $\mathcal{O}$ is called *little* if $0 < 2\dim \mathcal{O} \leqslant \dim \mathfrak{g}$ (cf. Definition 4.1). For example, the minimal nilpotent orbit of $\mathfrak{g}$ is little (cf. Corollary 4.3), and the nilpotent orbits of $\mathfrak{sl}_n(\mathbb{C})$ associated with partitions of the form $(2^p, 1^q)$, with $p, q \in \mathbb{N}^*$, are little (cf. Example 4.4). There are many other examples (see Section 4). Little nilpotent orbits verify both RC_1 and $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$ (cf. Proposition 4.2), and they turn out to be useful to study the reducibility of jet schemes of many other orbits via "restriction" or "induction" of orbits.

Firstly, by "restriction" to some Levi subalgebras of $\mathfrak{g}$ (cf. Proposition 4.6), we can obtain from nilpotent orbits $\mathcal{O}$ which verify $0 < 2\dim \mathcal{O} < \dim \mathfrak{g}$ examples of nilpotent orbits which verify RC_1 (and that are not necessarily little); see Table 1. More precisely, we have the following statement (cf. Proposition 4.6 [2]).

Proposition 1. *Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ with a center of dimension one, and such that $\mathfrak{a} := [\mathfrak{l}, \mathfrak{l}]$ is simple. Denote by A the connected subgroup of G whose Lie algebra is $\mathfrak{a}$. Let e be a nilpotent element of $\mathfrak{a}$ and suppose that the following conditions are satisfied:*

(i) *$\mathfrak{a}$ contains a regular semisimple element of $\mathfrak{g}$,*
(ii) $2\dim G.e < \dim \mathfrak{g}$.

Then $A.e$ verifies RC_1.

[2] Proposition 4.6 is stated in a slightly more general context.

Secondly, by "induction", we can reach from nilpotent orbits of reductive Lie subalgebras of $\mathfrak{g}$ many nilpotent orbits of $\mathfrak{g}$. Here, we consider induction in the sense of Lusztig-Spaltenstein [LuS79]. We refer the reader to Section 5 for the precise definition of a nilpotent orbit of $\mathfrak{g}$ induced from another one in some proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Our next statement says that condition $\mathrm{RC}_2(m)$, for $m \in \mathbb{N}^*$, passes through induction.

Theorem 2. *Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, $\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit of $\mathfrak{l}$ and $\mathcal{O}_{\mathfrak{g}}$ the induced nilpotent orbit of $\mathfrak{g}$ from $\mathcal{O}_{\mathfrak{l}}$. If $\mathcal{O}_{\mathfrak{l}}$ verifies $\mathrm{RC}_2(m)$ for some $m \in \mathbb{N}^*$, then $\mathcal{O}_{\mathfrak{g}}$ also verifies $\mathrm{RC}_2(m)$.*

From this result, we are able to deal with a large number of nilpotent orbits. First of all, any nilpotent orbit induced from a nilpotent orbit that has a little factor verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$ (cf. Theorem 6.1). In particular, if $\mathfrak{g}$ is not of type $\mathbf{A_1}$, $\mathbf{B_2} = \mathbf{C_2}$ or $\mathbf{G_2}$, then the subregular nilpotent orbit $\mathcal{O}_{\mathrm{subreg}}$ of $\mathfrak{g}$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$ (cf. Corollary 6.2), and so $\mathscr{J}_m(\overline{\mathcal{O}_{\mathrm{subreg}}})$ is reducible for every $m \in \mathbb{N}^*$.

It turns out that many nilpotent orbits can be induced from a nilpotent orbit that has a little factor. This allows us to obtain the following result when $\mathfrak{g}$ is of type $\mathbf{A}$ (cf. Theorem 6.5).

Theorem 3. *Any nilpotent orbit of $\mathfrak{sl}_n(\mathbb{C})$ associated with a non rectangular partition of n verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*

For the other simple Lie algebras of classical types, we have the following (cf. Theorem 6.7).

Theorem 4. *Let $n \in \mathbb{N}^*$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t)$ be a partition of n and set $\lambda_{t+1} = 0$. Suppose that there exist $1 \leqslant k < \ell \leqslant t$ such that $\lambda_k \geqslant \lambda_{k+1} + 2$ and $\lambda_\ell \geqslant \lambda_{\ell+1} + 2$.*

1) *If $\mathcal{O}$ is a nilpotent orbit of $\mathfrak{so}_n(\mathbb{C})$ whose associated partition is $\boldsymbol{\lambda}$, then $\mathcal{O}$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*
2) *If n is even and $\mathcal{O}$ is a nilpotent orbit of $\mathfrak{sp}_n(\mathbb{C})$ whose associated partition is $\boldsymbol{\lambda}$, then $\mathcal{O}$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*

While our result in the special linear case is exhaustive relative to induction, in the orthogonal and symplectic cases, other nilpotent orbits can be obtained by induction from a little orbit (cf. Theorem 6.7 and Remark 6.8). For a simple Lie algebra of exceptional type, we have a list of nilpotent orbits which can be induced from a little one (cf. Appendix C).

Organization of the paper. In Section 2, we state some basic properties on jet schemes with some proofs for the convenience of the reader.

In Section 3, we recall some standard properties of nilpotent orbit closures, and of their jet schemes. We introduce here the two sufficient conditions RC_1 and $\mathrm{RC}_2(m)$, $m \geqslant 1$, to study the reducibility of these jet schemes, and we state some first properties of these conditions.

Section 4 is devoted to little nilpotent orbits. We show that little nilpotent orbits verify both RC_1 and $\mathrm{RC}_2(m)$ for every $m \geqslant 1$, and we show how they can be used to prove condition RC_1 via the "restriction" of orbits (cf. Proposition 4.6).

In Section 5, we study the induction of nilpotent orbits the sense of Luzstig-Spaltenstein, [LuS79]. The main result is that condition $\mathrm{RC}_2(m)$, for $m \geqslant 1$, passes through induction (cf. Theorem 5.6). We describe in Section 6 how to use Theorem 5.6 to obtain the reducibility of nilpotent orbit closures in simple Lie algebras according to their Dynkin type. The details of some of the conclusions are presented in Appendices B and C.

We present in Section 7 some applications of our results to geometrical properties of nilpotent orbit closures. We also discuss in this section some open problems.

The standard notations relative to nilpotent orbits in classical simple Lie algebras are gathered together in Appendix A. Appendix B contains some numerical data for classical simple Lie algebras, and Appendix C summarizes our conclusions for simple Lie algebras of exceptional type.

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Contents

2. Generalities on jet schemes

In this section, we present some general facts on jet schemes. Our main references on the topic are [M01, EM09, Is11], and [DEM, Chap. 8].

Let X be a scheme of finite type over $\mathbb{C}$, and $m \in \mathbb{N}$.

Definition 2.1. *An m-*jet of X *is a morphism*

$$\operatorname{Spec}\mathbb{C}[t]/(t^{m+1}) \longrightarrow X.$$

*The set of all m-jets of X carries the structure of a scheme $\mathscr{J}_m(X)$, called the m-*th jet scheme of X*. It is a scheme of finite type over $\mathbb{C}$ characterized by the following functorial property: for every scheme Z over $\mathbb{C}$, we have*

$$\operatorname{Hom}(Z, \mathscr{J}_m(X)) = \operatorname{Hom}(Z \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[t]/(t^{m+1}), X).$$

The $\mathbb{C}$-points of $\mathscr{J}_m(X)$ are thus the $\mathbb{C}[t]/(t^{m+1})$-points of X. From Definition 2.1, we have for example that $\mathscr{J}_0(X) \simeq X$ and that $\mathscr{J}_1(X) \simeq \mathrm{T}X$ where $\mathrm{T}X$ denotes the total tangent bundle of X.

For $p \in \{0, \ldots, m\}$, the canonical projection $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{p+1})$ induces a *truncation morphism*,

$$\pi_{X,m,p} : \mathscr{J}_m(X) \to \mathscr{J}_p(X).$$

We shall simply denote by $\pi_{X,m}$ the morphism $\pi_{X,m,0}$,

$$\pi_{X,m} : \mathscr{J}_m(X) \to \mathscr{J}_0(X) \simeq X.$$

Also, the canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{m+1})$ induces a morphism $\iota_{X,m} : X \to \mathscr{J}_m(X)$, and we have $\pi_{X,m} \circ \iota_{X,m} = \mathrm{Id}_X$. Hence $\iota_{X,m}$ is injective and $\pi_{X,m}$ is surjective. We shall always view X as a subscheme of $\mathscr{J}_m(X)$.

If $f : X \to Y$ is a morphism of schemes, then we naturally obtain a morphism $f_m : \mathscr{J}_m(X) \to \mathscr{J}_m(Y)$ making the following diagram commutative,

$$\begin{tikzcd}
\mathscr{J}_m(X) \arrow[r, "f_m"] \arrow[d, "\pi_{X,m}"'] & \mathscr{J}_m(Y) \arrow[d, "\pi_{Y,m}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Remark 2.2. *In the case where X is affine, we have the following explicit description of $\mathscr{J}_m(X)$.*

Let $n \in \mathbb{N}^*$ and $X \subset \mathbb{C}^n$ be the affine subscheme defined by an ideal $I = (f_1, \ldots, f_r)$ of $\mathbb{C}[x_1, \ldots, x_n]$. Thus

$$X = \mathrm{Spec}\ \mathbb{C}[x_1, \ldots, x_n]/I.$$

For $k \in \{1, \ldots, r\}$, we extend f_k as a map from $(\mathbb{C}[t]/(t^{m+1}))^n$ to $\mathbb{C}[t]/(t^{m+1})$ via base extension. Then giving a morphism $\gamma : \mathrm{Spec}\,\mathbb{C}[t]/(t^{m+1}) \to X$ is equivalent to giving a morphism $\gamma^* : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}[t]/(t^{m+1})$, or to giving

$$\gamma^*(x_i) = \sum_{j=0}^{m} \gamma_i^{(j)} t^j \qquad (1 \leqslant i \leqslant n)$$

such that for any $k \in \{1, \ldots, r\}$,

$$f_k(\gamma^*(x_1), \ldots, \gamma^*(x_n)) = 0 \ \text{ in } \ \mathbb{C}[t]/(t^{m+1}).$$

For $k \in \{1, \ldots, r\}$, there exist functions $f_k^{(0)}, \ldots, f_k^{(m)}$, which depend only on f, in the variables $\boldsymbol{\gamma} = (\gamma_i^{(j)})_{\substack{1 \leqslant i \leqslant n, \\ 0 \leqslant j \leqslant m}}$ such that

$$f_k\big(\gamma^*(x_1), \ldots, \gamma^*(x_n)\big) = \sum_{j=0}^{m} f_k^{(j)}(\boldsymbol{\gamma})\, t^j. \tag{1}$$

The jet scheme $\mathscr{J}_m(X)$ is then the closed subscheme in $\mathbb{C}^{(m+1)n}$ defined by the ideal generated by the polynomials $f_k^{(j)}$, where $k \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, m\}$. More precisely,

$$\mathscr{J}_m(X) \simeq \mathrm{Spec}\,\mathbb{C}[x_1^{(j)}, \ldots, x_n^{(j)}; j = 0, \ldots, m]/(f_k^{(j)}\,;\, k = 1, \ldots, r,\, j = 0, \ldots, m).$$

In particular, if X is an n-dimensional vector space, then $\mathscr{J}_m(X) \simeq \mathbb{C}^{(m+1)n}$ and for $p \in \{0, \ldots, m\}$, the projection $\mathscr{J}_m(X) \to \mathscr{J}_p(X)$ corresponds to the projection onto the first $(p+1)n$ coordinates.

Example 2.3. *Let us consider a concrete example. Let* $X = \operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+yz) \subset \mathbb{C}^3$ *and let us compute* $\mathscr{J}_1(X)$ *and* $\mathscr{J}_2(X)$*. We have*

$$\begin{aligned}&(x_0+x_1t+x_2t^2)^2+(y_0+y_1t+y_2t^2)(z_0+z_1t+z_2t^2)\\ &= x_0^2+y_0z_0+(2x_0x_1+y_0z_1+y_1z_0)t+(2x_0x_2+x_1^2+y_0z_2+y_2z_0+y_1z_1)t^2 \mod t^3.\end{aligned}$$

Hence $\mathscr{J}_1(X)$ *is the subscheme of*

$$\mathscr{J}_1(\mathbb{C}^3) \simeq \mathbb{C}[x_0,y_0,z_0,x_1,y_1,z_1]$$

defined by the ideal

$$(x_0^2+y_0z_0, 2x_0x_1+y_0z_1+y_1z_0),$$

and $\mathscr{J}_2(X)$ *is the subscheme of*

$$\mathscr{J}_2(\mathbb{C}^3) \simeq \mathbb{C}[x_0,y_0,z_0,x_1,y_1,z_1,x_2,y_2,z_3]$$

defined by the ideal

$$(x_0^2+y_0z_0, 2x_0x_1+y_0z_1+y_1z_0, 2x_0x_2+x_1^2+y_0z_2+y_1z_1+y_2z_0).$$

We now list some basic properties that we need in the sequel. Their proofs can found in [EM09, Lem. 2.3, Rem. 2.8, Rem. 2.10].

Lemma 2.4. 1) *For every open subset* U *of* X*, we have* $\mathscr{J}_m(U) = \pi_{X,m}^{-1}(U)$*.*
2) *For every scheme* Y*, we have a canonical isomorphism* $\mathscr{J}_m(X \times Y) \simeq \mathscr{J}_m(X) \times \mathscr{J}_m(Y)$*.*
3) *If* G *is a group scheme over* $\mathbb{C}$*, then* $\mathscr{J}_m(G)$ *is also a group scheme over* $\mathbb{C}$*. Moreover, if* G *acts on* X*, then* $\mathscr{J}_m(G)$ *acts on* $\mathscr{J}_m(X)$*.*
4) *If* $f : X \to Y$ *is a smooth surjective morphism between schemes, then* f_m *is also smooth and surjective for every* $m \in \mathbb{N}^*$*.*

Geometrical properties. It is known that the geometry of the jet schemes $\mathscr{J}_m(X)$, $m \geqslant 1$, is closely linked to that of X. More precisely, we can transport some geometrical properties from $\mathscr{J}_m(X)$ to X.

The following proposition gives examples of such phenomena ([MFK94] and [Is11, Thm. 3.5]).

Proposition 2.5. *Let* $m \in \mathbb{N}^*$*. If* $\mathscr{J}_m(X)$ *is smooth (resp., irreducible, reduced, normal, locally a complete intersection) for some* m*, then so is* X*.*

For smoothness, the converse is true, even with "every m" instead of "for some m". In fact, for smooth varieties, we have the following more precise statement, [EM09, Cor. 2.12].

Proposition 2.6. *If* X *is a smooth variety of dimension* n*, then the truncation morphism* $\pi_{m,p}$*, for* $p \in \{0,\dots,m\}$*, is a locally trivial projection with fiber isomorphic to* $\mathbb{C}^{(m-p)n}$*. In particular,* $\mathscr{J}_m(X)$ *is a smooth variety of dimension* $(m+1)n$*.*

For the other properties stated in Proposition 2.5, the converse is not true in general. We refer to [Is11, §3] for counter-examples. We shall encounter other counter-examples in this paper in the setting of nilpotent orbit closures. In this setting, our main purpose is to study the irreducibility of jet schemes. The following lemma gives a necessary and sufficient condition for the converse of Proposition 2.5 to hold for irreducibility.

We denote by X_{reg} the smooth part of X, and by X_{sing} its complement.

Lemma 2.7. *Assume that* X *is an irreducible reduced scheme of finite type over* $\mathbb{C}$*, and let* $m \in \mathbb{N}^*$*.*

1) $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$ *is an irreducible component of* $\mathscr{J}_m(X)$.
2) $\mathscr{J}_m(X)$ *is irreducible if and only if* $\pi_{X,m}^{-1}(X_{\mathrm{sing}})$ *is contained in* $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$.
3) *If* X *is a complete intersection, then* $\mathscr{J}_m(X)$ *is irreducible if and only if* $\dim \pi_{X,m}^{-1}(X_{\mathrm{sing}}) < \dim \pi_{X,m}^{-1}(X_{\mathrm{reg}})$.

In particular, if $\dim \pi_{X,m}^{-1}(X_{\mathrm{sing}}) \geqslant \dim \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$, then $\mathscr{J}_m(X)$ is reducible.

Proof. Part (3) is proved in [M01, Prop .1.4], and its proof implies parts (1) and (2). More precisely, since X_{reg} is smooth and irreducible, $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$ is an irreducible closed subset of $\mathscr{J}_m(X)$ of dimension $(m+1)\dim X$, cf. Proposition 2.6. Then parts (1) and (2) follow easily from the fact that we have the decomposition

$$\mathscr{J}_m(X) = \pi_{X,m}^{-1}(X_{\mathrm{sing}}) \cup \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$$

of closed subsets, and that $\pi_{X,m}^{-1}(X_{\mathrm{sing}}) \not\supset \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$. ☐

There are also subtle connections between the geometry of $\mathscr{J}_m(X)$, $m \geqslant 1$, and the singularities of X which are important for us. In particular, according to [M01, Thm. 0.1, Prop. 1.5 and 4.12], we have:

Theorem 2.8 (Mustaţă)**.** *Let* X *be an irreducible variety over* $\mathbb{C}$.

1) *If* X *is locally a complete intersection, then* $\mathscr{J}_m(X)$ *is irreducible for every* $m \geqslant 1$ *if and only if* X *has rational singularities.*
2) *If* X *is locally a complete intersection and if* $\mathscr{J}_m(X)$ *is irreducible for some* $m \geqslant 1$*, then* $\mathscr{J}_m(X)$ *is also reduced.*
3) *If* X *is locally a complete intersection, then* $(\mathscr{J}_1(X))_{\mathrm{reg}} = \pi_{X,1}^{-1}(X_{\mathrm{reg}})$.

Let us give an easy counter-example to the converse implication of Proposition 2.5 for normality. This example turns out to be a particular case of a more general situation that will be studied in Proposition 7.3.

Example 2.9. *Let* X *be as in Example 2.3. Then* X *is a complete intersection and it is normal since the singular locus is reduced to* $\{0\}$ *which has codimension 2 in* X*. Next, it is not difficult to verify that* $\mathscr{J}_1(X)$ *is irreducible, reduced and that it is a complete intersection. But* $\mathscr{J}_1(X)$ *is not normal. Indeed, by Theorem 2.8,(3),*

$$(\mathscr{J}_1(X))_{\mathrm{sing}} = \pi_{X,1}^{-1}(\{0\}) \simeq \{0\} \times \mathbb{C}^3.$$

Hence, the singular locus of $\mathscr{J}_1(X)$ *has codimension 1 in* $\mathscr{J}_1(X)$ *since* $\dim \mathscr{J}_1(X) = 2\dim X = 4$.

Group actions. Let G be a connected algebraic group, acting on a variety X, and $m \in \mathbb{N}$. Denote by

$$\rho\colon G \times X \to X, \quad (g,x) \mapsto g.x$$

the corresponding action. As stated in Lemma 2.4, the morphism

$$\rho_m : \mathscr{J}_m(G \times X) \simeq \mathscr{J}_m(G) \times \mathscr{J}_m(X) \to \mathscr{J}_m(X)$$

defines an action of $\mathscr{J}_m(G)$ on $\mathscr{J}_m(X)$.

Recall that we embed X into $\mathscr{J}_m(X)$ through $\iota_{X,m}$. For $x \in X$, let us denote by G^x the stabilizer of x in G, and for $m \in \mathbb{N}$, we denote by $\mathscr{J}_m(G)^x$ its stabilizer in $\mathscr{J}_m(G)$. The following results are probably standard. Since we have not found any reference, we shall include their proofs.

Lemma 2.10. *Let $x \in X$. Then,*

$$\mathscr{J}_m(G).x = \mathscr{J}_m(G.x), \quad \mathscr{J}_m(G^x) = \mathscr{J}_m(G)^x \quad \text{and} \quad \pi^{-1}_{\overline{G.x},m}(G.x) = \mathscr{J}_m(G.x).$$

Proof. The morphism $G \times \{x\} \to G.x$, $(g, x) \mapsto g.x$ is a submersion at all points of $G \times \{x\}$. Hence, according to [Ha76, Ch. III, Prop. 10.4], it is a smooth morphism onto $G.x$. So, by Lemma 2.4,(4), the induced morphism $\mathscr{J}_m(G) \times \{x\} \to \mathscr{J}_m(G.x)$ is also smooth and surjective. Consequently, we have the first equality $\mathscr{J}_m(G).x = \mathscr{J}_m(G.x)$.

By applying the first equality to the algebraic group G^x, we get $\mathscr{J}_m(G^x).x = \mathscr{J}_m(G^x.x)$, and whence the inclusion $\mathscr{J}_m(G^x) \subset \mathscr{J}_m(G)^x$.

Conversely, let $\gamma : \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to G$ be an element of $\mathscr{J}_m(G)^x$. Then $\rho_m(\gamma, x) = x$, and hence viewing x as a morphism $x : \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to X$, we have

$$\rho(\gamma(\tau), x(\tau)) = x(\tau)$$

where τ is the unique element of $\operatorname{Spec} \mathbb{C}[t]/(t^{m+1})$. Thus $\gamma(\tau) \in G^x$ and $x(\tau) = x$. So we have $\gamma \in \mathscr{J}_m(G^x)$, and the second equality follows.

The third equality is a direct consequence of Lemma 2.4,(1) since $G.x$ is open in its closure. ∎

Let $\mathfrak{g}$ be the Lie algebra of G. We consider now the adjoint action of G on $\mathfrak{g}$. For the results we present here, we refer the reader to [M01, Appendix]. Denote by

$$\mathfrak{g}_m := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]/(t^{m+1})$$

the *generalized Takiff Lie algebra* whose Lie bracket is given by

$$[u \otimes x(t), v \otimes y(t)] = [u, v] \otimes x(t)y(t) \qquad (u, v \in \mathfrak{g},\ x(t), y(t) \in \mathbb{C}[t]/(t^{m+1})).$$

As Lie algebras, we have

$$\mathscr{J}_m(\mathfrak{g}) \simeq \mathfrak{g}_m \simeq \operatorname{Lie}(\mathscr{J}_m(G)).$$

In the sequel, when there is no confusion, we shall use the notations $\mathfrak{g}_m$ and G_m for $\mathscr{J}_m(\mathfrak{g})$ and $\mathscr{J}_m(G)$ respectively. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, then $\mathscr{J}_m(\mathfrak{a}) \simeq \mathfrak{a}_m$ is a Lie subalgebra of $\mathfrak{g}_m$. In particular, for $x \in \mathfrak{g}$, we have $(\mathfrak{g}_m)^x = (\mathfrak{g}^x)_m$, where for any subalgebra $\mathfrak{m}$ of $\mathfrak{g}_k$, with $k \geqslant 0$, $\mathfrak{m}^x$ stands for the centralizer of x in $\mathfrak{m}$.

We can identify $\mathfrak{g}_m$ with $\mathfrak{g}^{m+1} \simeq \mathscr{J}_m(\mathfrak{g})$ as a variety through the map

$$\mathfrak{g}^{m+1} \to \mathfrak{g}_m, \quad (x_0, x_1, \ldots, x_m) \mapsto x_0 + x_1 \otimes t + \cdots + x_m \otimes t^m.$$

Let G_m be a connected algebraic group whose Lie algebra is $\mathfrak{g}_m$. Let $\mathbb{C}[\mathfrak{g}_m]$ be the coordinate ring of $\mathfrak{g}_m$, and let $\mathbb{C}[\mathfrak{g}_m]^{G_m}$ be the subring of G_m-invariants. We conclude in this section with the following result.

Lemma 2.11. *For $f \in \mathbb{C}[\mathfrak{g}]^G$, the polynomials $f^{(0)}, \ldots, f^{(m)}$, as defined in Remark 2.2, are elements of $\mathbb{C}[\mathfrak{g}_m]^{G_m}$.*

Proof. This is straightforward from the explicit description of the polynomials $f^{(0)}, \ldots, f^{(m)}$ given in Remark 2.2. ∎

3. Nilpotent orbit closures

From now on, we let G to be a connected reductive algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra and $\mathcal{N}(\mathfrak{g})$ the nilpotent cone of $\mathfrak{g}$. Recall that $\mathcal{N}(\mathfrak{g})$ is the subscheme of $\mathfrak{g}$ defined by the augmentation ideal of $\mathbb{C}[\mathfrak{g}]^G$, and that $\mathcal{N}(\mathfrak{g}) = \overline{\mathcal{O}_{\mathrm{reg}}}$ where $\mathcal{O}_{\mathrm{reg}}$ is the regular nilpotent orbit of $\mathfrak{g}$ (cf. Introduction). As mentioned in the Introduction, we are interested in this paper in the irreducibility of jet schemes of the closure of nilpotent orbits.

Recall that for an arbitrary nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$, the singular locus of $\overline{\mathcal{O}}$ is $\overline{\mathcal{O}} \setminus \mathcal{O}$ and that $\mathrm{codim}_{\overline{\mathcal{O}}}(\overline{\mathcal{O}} \setminus \mathcal{O}) \geqslant 2$ (cf. Introduction).

Definition 3.1. *Let $\mathcal{O}$ be a nonzero nilpotent orbit of $\mathfrak{g}$. Define $\mathfrak{g}_{\mathcal{O}}$ to be the smallest semisimple ideal of $\mathfrak{g}$ containing $\mathcal{O}$.*

More precisely, if $\mathfrak{g} \simeq \mathfrak{z}(\mathfrak{g}) \times \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_m$, with $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$ and $\mathfrak{s}_1, \ldots, \mathfrak{s}_m$ the simple factors of $\mathfrak{g}$, then $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$, with $\mathcal{O}_i$ a nilpotent orbit of $\mathfrak{s}_i$ for $i = 1, \ldots, m$, and

$$\mathfrak{g}_{\mathcal{O}} = \mathfrak{s}_{i_1} \times \cdots \times \mathfrak{s}_{i_k}$$

where $\{i_1, \ldots, i_k\}$ is the set of integers $j \in \{1, \ldots, m\}$ such that $\mathcal{O}_j$ is nonzero. In particular, if $\mathcal{O}$ is zero, then $\mathfrak{g}_{\mathcal{O}} = 0$, and if $\mathcal{O}$ is nonzero and $\mathfrak{g}$ is simple, then $\mathfrak{g}_{\mathcal{O}} = \mathfrak{g}$.

For $\mathcal{O}$ a nilpotent orbit of $\mathfrak{g}$, we denote by $\mathcal{I}_{\overline{\mathcal{O}}}$ the defining ideal of $\overline{\mathcal{O}}$ in $\mathfrak{g}_{\mathcal{O}}$. Thus,

$$\overline{\mathcal{O}} = \operatorname{Spec} \mathbb{C}[\mathfrak{g}_{\mathcal{O}}]/\mathcal{I}_{\overline{\mathcal{O}}}.$$

Recall that $\overline{\mathcal{O}}$ is conical, so $\mathcal{I}_{\overline{\mathcal{O}}}$ is a homogeneous ideal.

Lemma 3.2. *Let $\mathcal{O}$ be a nonzero nilpotent orbit of $\mathfrak{g}$. If $f_1, \ldots, f_s$ are homogeneous generators of $\mathcal{I}_{\overline{\mathcal{O}}}$, then the minimum degree of the f_i's is exactly 2.*

Proof. By the above discussion, $\mathcal{O}$ is a product of nilpotent orbits. We may therefore assume that $\mathfrak{g} = \mathfrak{g}_{\mathcal{O}}$ is simple.

Assume that for some $i \in \{1, \ldots, s\}$, $\deg f_i = 1$. A contradiction is expected. Let $\mathcal{V}$ be the intersection of all the hyperplanes $\mathcal{H}_g$, $g \in G$, defined by the linear form

$$g.f_i \colon \mathfrak{g} \to \mathbb{C}, \quad x \longmapsto f_i(g^{-1}(x)).$$

Since $\overline{\mathcal{O}}$ is G-invariant and is contained in the zero locus of f_i, $\overline{\mathcal{O}}$ is contained in $\mathcal{V}$. Thus $\mathcal{V}$ is a nonzero G-invariant subspace of $\mathfrak{g}$ which is different from $\mathfrak{g}$ (because $\mathcal{V}$ is contained in the hyperplane $\mathcal{H}_{1_G}$), whence the contradiction since $\mathfrak{g}$ is simple.

The Casimir element, $x \mapsto \langle x, x \rangle$ with $\langle\, , \rangle$ the Killing form of $\mathfrak{g}$, vanishes on the nilpotent cone of $\mathfrak{g}$. Hence it is contained in $\mathcal{I}_{\overline{\mathcal{O}}}$. Since it has degree 2, the minimal degree of the f_i's is exactly 2. □

To determine the reducibility of $\mathscr{J}_m(\overline{\mathcal{O}})$ for $\mathcal{O}$ a (nonzero) nilpotent orbit of $\mathfrak{g}$, we introduce the two sufficient conditions below.

Definition 3.3. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$.*

1) *We say that $\mathcal{O}$* verifies RC_1 *if $\pi_{\overline{\mathcal{O}},1}^{-1}(0)$ is not contained in the closure of $\pi_{\overline{\mathcal{O}},1}^{-1}(\mathcal{O})$.*
2) *Let $m \in \mathbb{N}^*$. We say that $\mathcal{O}$* verifies $\mathrm{RC}_2(m)$ *if for some nilpotent orbit $\mathcal{O}'$ contained in $\overline{\mathcal{O}} \setminus \mathcal{O}$, we have $\dim \pi_{\overline{\mathcal{O}},m}^{-1}(\mathcal{O}') \geqslant \dim \pi_{\overline{\mathcal{O}},m}^{-1}(\mathcal{O}) = (m+1)\dim \mathcal{O}$.*

The following Lemma directly results from Lemma 2.7,(2).

Lemma 3.4. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$.*

1) *If $\mathcal{O}$ verifies RC_1, then $\mathscr{J}_1(\overline{\mathcal{O}})$ is reducible.*
2) *If $\mathcal{O}$ verifies $\mathrm{RC}_2(m)$ for some $m \in \mathbb{N}^*$, then $\mathscr{J}_m(\overline{\mathcal{O}})$ is reducible.*

The zero nilpotent orbit verifies neither RC_1 nor $\mathrm{RC}_2(m)$ for $m \in \mathbb{N}^*$. Since $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is irreducible for every $m \in \mathbb{N}^*$ (cf. Introduction), the same goes for the regular nilpotent orbit according to Lemma 3.4.

In view of the conditions above, let us study the zero fiber of $\pi_{\overline{\mathcal{O}},1} \colon \mathscr{J}_1(\overline{\mathcal{O}}) \to \overline{\mathcal{O}}$. As in Section 2, we identify $(\mathfrak{g}_{\mathcal{O}})_m$ with $(\mathfrak{g}_{\mathcal{O}})^{m+1} = \underbrace{\mathfrak{g}_{\mathcal{O}} \times \cdots \times \mathfrak{g}_{\mathcal{O}}}_{(m+1) \text{ times}}$.

Lemma 3.5. *Let $\mathcal{O}$ be a nonzero nilpotent orbit of $\mathfrak{g}$, and $m \in \mathbb{N}^*$.*

1) *We have $\pi^{-1}_{\overline{\mathcal{O}},1}(0) \simeq \{0\} \times \mathfrak{g}_{\mathcal{O}}$. In particular, $\dim \pi^{-1}_{\overline{\mathcal{O}},1}(0) = \dim \mathfrak{g}_{\mathcal{O}}$.*
2) *If $m \geqslant 2$, then $\dim \pi^{-1}_{\overline{\mathcal{O}},m}(0) \geqslant \dim \mathscr{J}_{m-2}(\overline{\mathcal{O}}) + \dim \mathfrak{g}_{\mathcal{O}} \geqslant m \dim \mathcal{O} + \operatorname{codim}_{\mathfrak{g}_{\mathcal{O}}}(\mathcal{O})$.*

Part (2) of Lemma 3.5 remains valid for an affine variety in $\mathbb{C}^n$ defined by homogeneous polynomials of degree at least 2. The special case where all the generators have the same degree is treated in [Y07, Prop. 5.2].

Proof. Clearly we may assume that $\mathfrak{g}_{\mathcal{O}} = \mathfrak{g}$. Let $f_1, \dots, f_r$ be homogeneous generators of $\mathcal{I}_{\overline{\mathcal{O}}}$ that we order so that $2 = d_1 \leqslant \cdots \leqslant d_r$, with $d_i = \deg f_i$ for any $i = 1, \dots, r$ (cf. Lemma 3.2).

1) Through our identification, we can write

$$\pi^{-1}_{\overline{\mathcal{O}},1}(0) \simeq \{0\} \times \{x \in \mathfrak{g} \mid f_i(tx) = 0 \bmod t^2 \text{ for any } i = 1, \dots, r\},$$

whence the statement since for any $x \in \mathfrak{g}$ and $i \in \{1, \dots, r\}$, we have $f_i(tx) = t^{d_i} f_i(x)$ and $d_i \geqslant 2$.

2) Assume that $m \geqslant 2$. Let $(x_1, x_2, \dots, x_{m-1})$ be an element of $\mathscr{J}_{m-2}(\overline{\mathcal{O}})$, and let $x_m \in \mathfrak{g}$. Then for any $i \in \{1, \dots, r\}$, we get

$$f_i(tx_1 + t^2 x_2 + \cdots + t^m x_m) = f_i(tx_1 + t^2 x_2 + \cdots + t^{m-1} x_{m-1}) \bmod t^{m+1}$$

since f_i is homogeneous of degree at least 2. Hence,

$$f_i(tx_1 + t^2 x_2 + \cdots + t^m x_m) = t^{d_i} f_i(x_1 + tx_2 + \cdots + t^{m-2} x_{m-1}) \bmod t^{m+1}.$$

But $f_i(x_1 + tx_2 + \cdots + t^{m-2} x_{m-1}) = 0 \bmod t^{m-1}$ because $(x_1, x_2, \dots, x_{m-1}) \in \mathscr{J}_{m-2}(\overline{\mathcal{O}})$. So,

$$t^{d_i} f_i(x_1 + tx_2 + \cdots + t^{m-2} x_{m-1}) = 0 \bmod t^{m+1}$$

since $d_i \geqslant 2$. In other words, $(0, x_1, x_2, \dots, x_m)$ is an element of $\pi^{-1}_{\overline{\mathcal{O}},m}(0)$.

Thus we obtain an embedding from $\mathscr{J}_{m-2}(\overline{\mathcal{O}}) \times \mathfrak{g}$ into $\pi^{-1}_{\overline{\mathcal{O}},m}(0)$ given by

$$\mathscr{J}_{m-2}(\overline{\mathcal{O}}) \times \mathfrak{g} \longrightarrow \pi^{-1}_{\overline{\mathcal{O}},m}(0), \quad ((x_1, x_2, \dots, x_{m-1}), x_m) \longmapsto (0, x_1, x_2, \dots, x_{m-1}, x_m).$$

The assertions follows. ∎

Let $\mathcal{O}$ be a nonzero nilpotent orbit of $\mathfrak{g}$, and fix $e \in \mathcal{O}$. The tangent space at e to $\overline{\mathcal{O}}$ is the space $[e, \mathfrak{g}]$. Consider the morphism

$$\eta_{\mathfrak{g},e} \colon G \times [e, \mathfrak{g}] \longrightarrow \mathfrak{g}, \quad (g, x) \longmapsto g(x).$$

Proposition 3.6. *The nonzero nilpotent orbit $\mathcal{O}$ verifies RC_1 if and only if the closure of the image of $\eta_{\mathfrak{g},e}$ is strictly contained in $\mathfrak{g}_{\mathcal{O}}$.*

Proof. Since $[e,\mathfrak{g}] = [e,\mathfrak{g}_{\mathcal{O}}]$, we may assume that $\mathfrak{g} = \mathfrak{g}_{\mathcal{O}}$. Thus, by the definition of condition RC_1, we have to show that $\pi^{-1}_{\overline{\mathcal{O}},1}(0)$ is contained in $\overline{\pi^{-1}_{\overline{\mathcal{O}},1}(\mathcal{O})}$ if and only if $\eta_{\mathfrak{g},e}$ is dominant, i.e., $\overline{G.[e,\mathfrak{g}]} = \mathfrak{g}$.

By Lemma 3.5,(1), we have $\pi^{-1}_{\overline{\mathcal{O}},1}(0) \simeq \{0\} \times \mathfrak{g}$. On the other hand,

$$\pi^{-1}_{\overline{\mathcal{O}},1}(\mathcal{O}) = G.(\{e\} \times [e,\mathfrak{g}]).$$

So, if $\pi^{-1}_{\overline{\mathcal{O}},1}(0) \subset \overline{\pi^{-1}_{\overline{\mathcal{O}},1}(\mathcal{O})}$, then

$$\{0\} \times \mathfrak{g} \subset \overline{G.(\{e\} \times [e,\mathfrak{g}])} \subset \overline{G.e} \times \overline{G.[e,\mathfrak{g}]},$$

whence the inclusion $\mathfrak{g} \subset \overline{G.[e,\mathfrak{g}]}$, and $\eta_{\mathfrak{g},e}$ is dominant.

For the other direction, observe that $\overline{\pi^{-1}_{\overline{\mathcal{O}},1}(\mathcal{O})}$ is a closed bicone of $\mathfrak{g} \times \mathfrak{g}$ since $\mathcal{O}$ and $\overline{\mathcal{O}}$ are both subcones of $\mathfrak{g}$. Here, by *bicone*, we mean a subset of $\mathfrak{g} \times \mathfrak{g}$ stable under the natural $(\mathbb{C}^* \times \mathbb{C}^*)$-action on $\mathfrak{g} \times \mathfrak{g}$. Therefore, if $\overline{G.[e,\mathfrak{g}]} = \mathfrak{g}$, then

$$\overline{G.(\{e\} \times [e,\mathfrak{g}])} = \overline{G.(\mathbb{C}^* e \times [e,\mathfrak{g}])} \supset \{0\} \times \overline{G.[e,\mathfrak{g}]} = \{0\} \times \mathfrak{g},$$

whence $\pi^{-1}_{\overline{\mathcal{O}},1}(0) \subset \overline{\pi^{-1}_{\overline{\mathcal{O}},1}(\mathcal{O})}$. □

Example 3.7. *Let $p \in \mathbb{N}^*$ with $p \geqslant 2$, and $\mathfrak{g} = \mathfrak{sl}_{2p}(\mathbb{C})$. In the notations of Appendix A, we claim that the nilpotent orbit $\mathcal{O}_{(2^p)}$ of $\mathfrak{g}$ associated with the partition (2^p) verifies* RC_1. *According to Proposition 3.6, it suffices to prove that for the element*

$$e := \begin{pmatrix} 0 & I_p \\ 0 & 0 \end{pmatrix} \in \mathcal{O}_{(2^p)},$$

the morphism $\eta_{\mathfrak{g},e}$ is not dominant. We readily verify that $[e,\mathfrak{g}]$ consists of matrices of the form

$$\begin{pmatrix} A & C \\ 0 & -A \end{pmatrix}$$

with A and C of size p. In particular, $[e,\mathfrak{g}]$ is contained in the closed subset $\mathcal{Z}$ of $\mathfrak{g}$ consisting of the matrices whose characteristic polynomial is even. Since $\overline{G([e,\mathfrak{g}])}$ and $\mathcal{Z}$ are both closed G-stable subsets of $\mathfrak{g}$, we get

$$\overline{G([e,\mathfrak{g}])} \subset \mathcal{Z}.$$

The diagonal matrix $\mathrm{diag}(1,\ldots,1,-2p+1)$ *is in $\mathfrak{g}$ but does not lie in $\mathcal{Z}$, for $p \geqslant 2$. Hence, $\mathcal{Z}$ is strictly contained in $\mathfrak{g}$, and $\eta_{\mathfrak{g},e}$ is not dominant. Thus $\mathcal{O}_{(2^p)}$ verifies* RC_1.

According to Lemma 3.4,(1), $\mathscr{J}_1(\overline{\mathcal{O}_{(2^p)}})$ is reducible. In fact, we can be more precise. By [W02, Thm. 1] *(see also* [W89] *or* [W03, Prop. 8.2.15]*), the defining ideal of $\overline{\mathcal{O}_{(2^p)}}$ is generated by the entries of the matrix X^2 as functions of $X \in \mathfrak{sl}_{2p}(\mathbb{C})$. It follows that $\mathscr{J}_1(\overline{\mathcal{O}_{(2^p)}})$ can be identified with the scheme of pairs $(X_0, X_1) \in \mathfrak{sl}_{2p}(\mathbb{C}) \times \mathfrak{sl}_{2p}(\mathbb{C})$ defined by the equations $X_0^2 = 0$ and $X_0X_1 + X_1X_0 = 0$. Using this identification, we obtain from direct computations that*

- *$\mathscr{J}_1(\overline{\mathcal{O}_{(2^p)}})$ has exactly one irreducible component of dimension $4p^2 = 2\dim \mathcal{O}_{(2^p)}$,*
- *all the other irreducible components have dimension $4p^2 - 1$, and $\pi^{-1}_{\overline{\mathcal{O}_{(2^p)}},1}(0)$ is one of them.*

Remark 3.8. *Assume that $\mathfrak{g} = \mathfrak{g}_{\mathcal{O}}$. A nilpotent element e is distinguished if its centralizer is contained in the nilpotent cone. In particular, if e is distinguished, then the centralizer of an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$ is zero, and the theory of representations of $\mathfrak{sl}_2$ shows that $[e,\mathfrak{g}]$ contains $\mathfrak{g}^h$, and hence contains a Cartan subalgebra of $\mathfrak{g}$. Consequently, $G.e$ does not verify* RC_1.

Remark 3.9. *Assume that $\mathfrak{g} = \mathfrak{g}_{\mathcal{O}}$. Since $G \times [e, \mathfrak{g}]$ and $\mathfrak{g}$ are irreducible varieties, $\eta_{\mathfrak{g},e}$ is dominant if and only if there is a nonempty open set U consisting of points $a \in G \times [e, \mathfrak{g}]$ such that $(\mathrm{d}\eta_{\mathfrak{g},e})_a$ is surjective. The differential of $\eta_{\mathfrak{g},e}$ at $a = (g, [e, x])$, with $(g, x) \in G \times \mathfrak{g}$ is given by*

$$\mathfrak{g} \times [e, \mathfrak{g}] \longrightarrow \mathfrak{g}, \quad (v, [e, w]) \longmapsto [v, [e, x]] + g([e, w]).$$

Let us endow $G \times [e, \mathfrak{g}]$ with the action of G by left multiplication on the first factor. Since $\eta_{\mathfrak{g},e}$ is G-equivariant, we may assume that a is of the form $a = (1_G, [e, x])$ with $x \in \mathfrak{g}$. Then $(\mathrm{d}\eta_{\mathfrak{g},e})_a$ is surjective if and only if $[\mathfrak{g}, [e, x]] + [e, \mathfrak{g}] = \mathfrak{g}$.

Consequently, $\eta_{\mathfrak{g},e}$ is dominant if and only if there exists $x \in \mathfrak{g}$ such that $[\mathfrak{g}, [e, x]] + [e, \mathfrak{g}] = \mathfrak{g}$. This allows us to affirm in some cases that $\eta_{\mathfrak{g},e}$ is dominant. For example, for e in the non-distinguished nilpotent orbit $\mathcal{O}_{(3^2)}$ of $\mathfrak{sl}_6(\mathbb{C})$, the map $\eta_{\mathfrak{g},e}$ is dominant.

4. Little nilpotent orbits

We introduce in this section a family of nonzero nilpotent orbits which verify both RC_1 and $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$. This family turns out to be useful to study the reducibility of jet schemes of many other orbits.

Lemma 3.5 leads us to the following definition.

Definition 4.1. *Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$ and let $\mathfrak{g}_{\mathcal{O}}$ be as defined in Definition 3.1. We say that $\mathcal{O}$ is* little *if $0 < 2 \dim \mathcal{O} \leqslant \dim \mathfrak{g}_{\mathcal{O}}$.*

In particular, neither the zero orbit nor the regular nilpotent orbit is little.

Proposition 4.2. *If $\mathcal{O}$ is a little nilpotent orbit of $\mathfrak{g}$, then $\mathcal{O}$ verifies RC_1 and $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*

Proof. Let $\mathcal{O}$ be a little nilpotent orbit of $\mathfrak{g}$. As in the preceding proofs, we may assume that $\mathfrak{g} = \mathfrak{g}_{\mathcal{O}}$. According to Lemma 3.5,(1), we have $\dim \pi_{\overline{\mathcal{O}},1}^{-1}(0) = \dim \mathfrak{g}$. Since $\pi_{\overline{\mathcal{O}},1}^{-1}(\mathcal{O})$ has dimension $2 \dim \mathcal{O} \leqslant \dim \mathfrak{g}$, $\mathcal{O}$ verifies $\mathrm{RC}_2(1)$ and RC_1. Now let $m \geqslant 2$. According to Lemma 3.5,(2), we have

$$\dim \pi_{\overline{\mathcal{O}},m}^{-1}(0) \geqslant m \dim \mathcal{O} + \mathrm{codim}_{\mathfrak{g}}(\mathcal{O}) \geqslant (m+1) \dim \mathcal{O},$$

since $\mathrm{codim}_{\mathfrak{g}}(\mathcal{O}) \geqslant \dim \mathcal{O}$ because $\mathcal{O}$ is little. Hence $\mathcal{O}$ verifies $\mathrm{RC}_2(m)$. □

When $\mathfrak{g}$ is simple, there is a unique nonzero nilpotent orbit $\mathcal{O}_{\min}$, called the *minimal* nilpotent orbit of $\mathfrak{g}$, of minimal dimension and it is contained in the closure of all nonzero nilpotent orbits.

Corollary 4.3. *Assume that $\mathfrak{g}$ is simple and not of type $\mathbf{A_1}$. Then $\mathcal{O}_{\min}$ is little. In particular, $\mathscr{J}_m(\overline{\mathcal{O}_{\min}})$ is reducible for every $m \in \mathbb{N}^*$.*

Proof. Let $e \in \mathcal{O}_{\min}$ that we embed into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$, and consider the corresponding Dynkin grading,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \quad \text{with} \quad \mathfrak{g}(i) := \{x \in \mathfrak{g} \; ; \; [h, x] = ix\}.$$

By [CM93, Lem. 4.1.3], $\dim \mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1)$. In addition, since $e \in \mathcal{O}_{\min}$, we have $\dim \mathfrak{g}(2) = 1$ and $\mathfrak{g} = \sum_{-2 \leqslant i \leqslant 2} \mathfrak{g}(i)$, [TY05, Prop. 34.4.1]. As a result, we obtain that

$$\dim \mathfrak{g} - 2 \dim \mathcal{O} = \dim \mathfrak{g}(0) - 2.$$

The Levi subalgebra $\mathfrak{g}(0)$ contains a Cartan subalgebra which has dimension at least two by our hypothesis. Hence, $\dim \mathfrak{g} - 2 \dim \mathcal{O} \geqslant 0$, and so $\mathcal{O}_{\min}$ is little. □

For classical simple Lie algebras, there are explicit formulas (see Appendix A) for the dimension of nilpotent orbits. This allows to obtain readily examples of little nilpotent orbits.

Example 4.4. *Let $n \in \mathbb{N}^*$ and $p, q \in \mathbb{N}$.*

(i) *A nilpotent orbit of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to a rectangular partition is never little.*
(ii) *The nilpotent orbit $\mathcal{O}_{(2^p,1^q)}$ of $\mathfrak{sl}_{2p+q}(\mathbb{C})$ is little if and only if $p, q \in \mathbb{N}^*$.*
(iii) *The nilpotent orbit $\mathcal{O}_{(p,1^q)}$ of $\mathfrak{sl}_{p+q}(\mathbb{C})$ is little for $q \gg p$.*

Explicit computations suggest that it is unlikely that there is a nice description of little nilpotent orbits in terms of partitions.

We refer to Appendix A for the notations $\mathscr{P}_\varepsilon(n)$, $\varepsilon \in \{0, 1\}$, and $\mathcal{O}_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \mathscr{P}_\varepsilon(n)$, $n \in \mathbb{N}^*$.

Example 4.5. *Let $\boldsymbol{\lambda} = (2^p, 1^q)$, with $p \in \mathbb{N}^*$ and $q \in \mathbb{N}$.*

(i) *If p is even, then $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, and the nilpotent orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathfrak{so}_{2p+q}(\mathbb{C})$ is little.*
(ii) *If q is even, then $\boldsymbol{\lambda} \in \mathscr{P}_{-1}(n)$, and the nilpotent orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathfrak{sp}_{2p+q}(\mathbb{C})$ is little if and only if $p \leqslant q(q+1)/2$.*

The next proposition will allow us to produce new examples of nilpotent orbits which verify RC_1 by the "restriction" of certain little nilpotent orbits to Levi subalgebras.

Recall that for $\mathcal{O}$ a nilpotent orbit of some reductive Lie algebra $\mathfrak{a}$, the semisimple Lie algebra $\mathfrak{a}_{\mathcal{O}}$ was defined in Definition 3.1.

Proposition 4.6. *Assume that $\mathfrak{g}$ is simple. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ with center $\mathfrak{z}(\mathfrak{l})$, and denote by A the connected subgroup of G whose Lie algebra is $\mathfrak{a} := [\mathfrak{l}, \mathfrak{l}]$. Let e be a nilpotent element of $\mathfrak{a}$ and suppose that the following conditions are satisfied:*

(i) *$\mathfrak{a}$ contains a regular semisimple element of $\mathfrak{g}$,*
(ii) *$\mathfrak{a}_{A.e} = \mathfrak{a}$,*
(iii) *$2 \dim G.e \leqslant \dim \mathfrak{g} - \dim \mathfrak{z}(\mathfrak{l})$.*

Then $A.e$ verifies RC_1.

Proof. Define the following maps

$$\theta \colon G \times \mathfrak{a} \to \mathfrak{g},\ (g, x) \mapsto g(x), \qquad \eta = \eta_{\mathfrak{g},e} \colon G \times [e, \mathfrak{g}] \to \mathfrak{g},\ (g, x) \mapsto g(x).$$

Observe that the image of each of the above maps is irreducible. Moreover, for any $x \in \mathfrak{g}$, the map $g \mapsto (g^{-1}, g(x))$ defines a bijection between $G_\theta(x) := \{g \in G\ ;\ g(x) \in \mathfrak{a}\}$ and $\theta^{-1}(\{x\})$. Similarly, we have a bijection between $G_\eta(x) := \{g \in G\ ;\ g(x) \in [e, \mathfrak{g}]\}$ and $\eta^{-1}(\{x\})$. These bijections are isomorphisms of varieties.

Step 1. We shall first compute the dimension of the image of θ.

Let L be the connected subgroup of G whose Lie algebra is $\mathfrak{l}$. By condition (i), $\mathfrak{a}$ contains regular semisimple elements of $\mathfrak{g}$. If s is such an element, then $\mathfrak{g}^s$ is a Cartan subalgebra of $\mathfrak{l}$. Let $g \in G_\theta(s)$. Then $g(s) \in \mathfrak{a}$ and $\mathfrak{g}^{g(s)} = g(\mathfrak{g}^s)$ is another Cartan subalgebra of $\mathfrak{l}$. It follows that there exists $\tau \in L$ such that $\tau g \in N_G(\mathfrak{g}^s)$, with $N_G(\mathfrak{g}^s)$ the normalizer of $\mathfrak{g}^s$ in G. Hence, $g \in LN_G(\mathfrak{g}^s)$. Thus, we have obtained the inclusion $G_\theta(s) \subset LN_G(\mathfrak{g}^s)$. On the other hand, since L normalizes $\mathfrak{a}$, we get $L \subset G_\theta(s)$ and therefore $\dim L \leqslant \dim G_\theta(s)$.

Let $C_G(\mathfrak{g}^s)$ and $C_L(\mathfrak{g}^s)$ be the centralizers of $\mathfrak{g}^s$ in G and L respectively. Since $\mathfrak{g}^s$ is a Cartan subalgebra, $C_G(\mathfrak{g}^s)$ is connected and so, $C_G(\mathfrak{g}^s) = C_L(\mathfrak{g}^s)$ is contained in L. It follows that $LN_G(\mathfrak{g}^s)$

is a finite union of right L-cosets. We deduce that

$$\dim \theta^{-1}(\{s\}) = \dim G_\theta(s) = \dim L = \dim \mathfrak{a} + \mathfrak{z}(\mathfrak{l}).$$

Since the set of regular semisimple elements in $\mathfrak{g}$ is open and dense, we obtain that for s as above,

$$\dim \overline{\operatorname{im}\theta} = \dim \mathfrak{g} + \dim \mathfrak{a} - \dim \theta^{-1}(\{s\}) = \dim \mathfrak{g} - \dim \mathfrak{z}(\mathfrak{l}).$$

Step 2. We now consider the image of η.

Let (e,h,f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. We easily check that $\mathfrak{c} := \mathbb{C}h \oplus \mathfrak{g}^e$ is a Lie subalgebra, and that $\mathfrak{c}$ stabilizes $[e,\mathfrak{g}]$. Let C be the connected subgroup of G whose Lie algebra is $\mathfrak{c}$. Then C is contained in $G_\eta(x)$ for any $x \in [e,\mathfrak{g}]$. In particular, $\dim G_\eta(x) \geqslant \dim C = 1 + \dim \mathfrak{g}^e$ for $x \in [e,\mathfrak{g}]$, and so

$$\dim \overline{\operatorname{im}\eta} \leqslant \dim \mathfrak{g} + \dim\,[e,\mathfrak{g}] - 1 - \dim \mathfrak{g}^e = 2\dim G.e - 1.$$

Step 3. By condition (iii) and Steps 1 and 2, we deduce that $\dim \overline{\operatorname{im}\theta} > \dim \overline{\operatorname{im}\eta}$. Thus $\overline{\operatorname{im}\theta} \not\subset \overline{\operatorname{im}\eta}$. We claim that this implies that $A.e$ is RC_1. Let us suppose on the contrary that $A.e$ is not RC_1. By condition (ii) and Lemma 3.5,(1), $\pi_{A.e,1}^{-1}(0) = \{0\} \times \mathfrak{a}$. So, $\pi_{A.e,1}^{-1}(0)$ is contained in $\overline{\pi_{A.e,1}^{-1}(A.e)}$. Recall from the end of Section 2 the notations G_1 and A_1 for $\mathscr{J}_1(G)$ and $\mathscr{J}_1(A)$ respectively. It follows that

$$\{0\} \times G.\mathfrak{a} \subset G_1.(\{0\} \times \mathfrak{a}) \subset G_1\overline{A_1.e} \subset \overline{G_1.e},$$

whence

$$\{0\} \times \overline{G.\mathfrak{a}} \subset \overline{G_1.e}.$$

Since $\overline{\pi_{G.e,1}^{-1}(G.e)} = \overline{G_1.e}$ (cf. Lemma 2.10), it follows from the proof of Proposition 3.6 that

$$\overline{G_1.e} \cap (\{0\} \times \mathfrak{g}) = \overline{\pi_{G.e,1}^{-1}(G.e)} \cap (\{0\} \times \mathfrak{g}) = \{0\} \times \overline{G.[e,\mathfrak{g}]}.$$

Hence we get $\overline{\operatorname{im}\theta} \subset \overline{\operatorname{im}\eta}$ and the contradiction. $\square$

Suppose that $\mathfrak{g}$ is simple. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by Δ the root system relative to $(\mathfrak{g},\mathfrak{h})$ and let us fix a system of simple roots Π. Given $S \subset \Pi$, we denote $\Delta_S = \mathbb{Z}S \cap \Delta$ the subroot system generated by S, and

$$\mathfrak{l}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_S} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ denotes the root subspace relative to α. Then $\mathfrak{l}_S$ is a Levi subalgebra of $\mathfrak{g}$ and any Levi subalgebra of $\mathfrak{g}$ is conjugate to one in this form.

Given $S \subset \Pi$, denote $\mathfrak{t} = [\mathfrak{l}_S, \mathfrak{l}_S] \cap \mathfrak{h}$. Then, $\mathfrak{l}_S$ verifies condition (i) if and only if $\mathfrak{t} \not\subset \cup_{\alpha \in \Delta} \ker \alpha$. To check the latter condition, it is enough to verify that for every $\alpha \in \Delta$, there is $\beta \in S$ such that $\langle \beta^\vee, \alpha \rangle \neq 0$.

Thus not all Levi subalgebras of $\mathfrak{g}$ verify condition (i) of Proposition 4.6. For example, if $\mathfrak{g}$ is simple of type $\mathbf{B}_{\boldsymbol{\ell}}$, then a (maximal) Levi subalgebra whose semisimple part is simple of type $\mathbf{B}_{\boldsymbol{\ell-1}}$ does not verify the condition. The same goes for a Levi subalgebra in type $\mathbf{C}_{\boldsymbol{\ell}}$ whose semisimple part is simple of type $\mathbf{C}_{\boldsymbol{\ell-1}}$.

However, if $\mathfrak{g}$ is simple of type $\mathbf{D}_{\boldsymbol{\ell}}$ and if $\mathfrak{l}$ is a Levi subalgebra whose semisimple part is simple of type $\mathbf{D}_{\boldsymbol{\ell-1}}$, then $\mathfrak{l}$ verifies the condition (i). Likewise, if $\mathfrak{g}$ is simple of type $\mathbf{E_7}$ and if $\mathfrak{l}$ is a Levi subalgebra whose semisimple part is simple of type $\mathbf{E_6}$, then $\mathfrak{l}$ verifies the condition (i). Applying Proposition 4.6, we obtain examples of nilpotent orbits in types $\mathbf{D}$ or $\mathbf{E_6}$ which verify RC_1 that are not little.

We list in Table 1 some nilpotent orbits that we obtain in this way. In all the examples presented in the table, the center of the Levi subalgebra is 1-dimensional, and $\mathfrak{a}$ is simple. The first and second columns give the type of the simple Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$. Condition (ii) is verified in view of the discussion above. We describe the nilpotent orbits $G.e$ and $A.e$ in the third and fourth columns respectively. The description for an orbit in $\mathfrak{g}$ of type $\mathbf{D}$ is given in terms of partitions (cf. Appendix A), while for an orbit in $\mathfrak{g}$ of type $\mathbf{E_6}$ or $\mathbf{E_7}$, it is given by its Bala-Carter label.

$\mathfrak{g}$	$\mathfrak{a}$	$G.e$	$A.e$
$\mathbf{D_6}$	$\mathbf{D_5}$	$(3,2^2,1^5)$	$(3,2^2,1^3)$
$\mathbf{D_7}$	$\mathbf{D_6}$	$(3^2,1^8)$	$(3^2,1^6)$
$\mathbf{D_9}$	$\mathbf{D_8}$	$(3^2,2^2,1^8)$	$(3^2,2^2,1^6)$
$\mathbf{D_{10}}$	$\mathbf{D_9}$	$(3^3,1^{11})$	$(3^3,1^9)$
$\mathbf{D_{10}}$	$\mathbf{D_9}$	$(4^2,1^{12})$	$(4^2,1^{10})$
$\mathbf{D_{10}}$	$\mathbf{D_9}$	$(5,2^2,1^{11})$	$(5,2^2,1^9)$
$\mathbf{D_{10}}$	$\mathbf{D_9}$	$(5,3,1^{12})$	$(5,3,1^{10})$
$\mathbf{E_7}$	$\mathbf{E_6}$	$(3A_1)'$	$3A_1$
$\mathbf{E_7}$	$\mathbf{E_6}$	A_2	A_2

TABLE 1. Examples of non-little nilpotent orbits satisfying RC_1 obtained by restriction.

Remark 4.7.

1) *The first (and also the last) line of Table 1 provides an example of a* rigid[3] *nilpotent orbit which verifies* RC_1 *and which is not little.*
2) *Propositions 3.6, 4.2 and 4.6, together with Remark 3.9, allow us to classify all nilpotent orbits verifying* RC_1 *in simple Lie algebras of exceptional type. They are listed in Appendix C.*

5. Induced nilpotent orbits

Let $\mathfrak{l}$ be a proper Levi subalgebra of $\mathfrak{g}$, and let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ so that $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{p}$. Let P, L and U be the connected closed subgroups of G whose Lie algebra are $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{u}$ respectively. Then $P = LU$.

The following definitions and results on induced nilpotent orbits are mostly extracted from [R74] and [LuS79]. We refer to [CM93, Chap. 7] for a recent survey.

Theorem 5.1. *Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit of $\mathfrak{l}$. There exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ whose intersection with $\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$ is a dense open subset of $\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$. Moreover, the intersection of $\mathcal{O}_{\mathfrak{g}}$ with $\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$ consists of a single P-orbit and* $\operatorname{codim}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \operatorname{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}})$.

The nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ only depends on $\mathfrak{l}$, and not on the choice of a parabolic subalgebra $\mathfrak{p}$ containing it. The nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ is called the *induced nilpotent orbit of* $\mathfrak{g}$ *from* $\mathcal{O}_{\mathfrak{l}}$, and it is denoted by $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$. A nilpotent orbit which is not induced in a proper way from another one is called *rigid*. In type $\mathbf{A}$, only the zero orbit is rigid.

[3] See Section 5 for the notion of rigid nilpotent orbit, and Appendices A and C for the description of rigid nilpotent orbits in simple Lie algebras.

Remark 5.2.

1) *Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ be the simple factors of $[\mathfrak{g}, \mathfrak{g}]$ and denote by $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. Then there are Levi subalgebras $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ of $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ respectively such that*
$$\mathfrak{l} = \mathfrak{z}(\mathfrak{g}) \times \mathfrak{r}_1 \times \cdots \times \mathfrak{r}_n.$$
If $\mathcal{O}_{\mathfrak{l}}$ is a nilpotent orbit of $\mathfrak{l}$, then $\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_{\mathfrak{r}_1} \times \cdots \times \mathcal{O}_{\mathfrak{r}_n}$, where $\mathcal{O}_{\mathfrak{r}_1}, \ldots, \mathcal{O}_{\mathfrak{r}_n}$ are nilpotent orbits in the semisimple parts of $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ respectively. Then
$$\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \mathrm{Ind}_{\mathfrak{r}_1}^{\mathfrak{s}_1}(\mathcal{O}_{\mathfrak{r}_1}) \times \cdots \times \mathrm{Ind}_{\mathfrak{r}_n}^{\mathfrak{s}_n}(\mathcal{O}_{\mathfrak{r}_n}) = \mathrm{Ind}_{[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{l}}^{[\mathfrak{g},\mathfrak{g}]}(\mathcal{O}_{\mathfrak{l}}).$$
2) *The induction property is transitive in the following sense,* [CM93, Prop. 7.1.4]*: if $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Levi subalgebras of $\mathfrak{g}$ with $\mathfrak{l}_1 \subset \mathfrak{l}_2$, then*
$$\mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = \mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1}).$$
3) *If $\Omega_{\mathfrak{l}}$ is an L-orbit in $\overline{\mathcal{O}_{\mathfrak{l}}} \setminus \mathcal{O}_{\mathfrak{l}}$, then the induced nilpotent orbit of $\mathfrak{g}$ from $\Omega_{\mathfrak{l}}$ is contained in $\overline{\mathcal{O}_{\mathfrak{g}}} \setminus \mathcal{O}_{\mathfrak{g}}$.*

Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit of $\mathfrak{l}$ and denote by $\mathcal{O}_{\mathfrak{g}}$ the induced nilpotent orbit of $\mathfrak{g}$ from $\mathcal{O}_{\mathfrak{l}}$. According to Theorem 5.1, $\mathcal{O}_{\mathfrak{g}} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{u})$ is a single P-orbit that we shall denote by $\mathcal{O}_{\mathfrak{p}}$, that is
$$\mathcal{O}_{\mathfrak{p}} := \mathcal{O}_{\mathfrak{g}} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}).$$

Lemma 5.3. *We have:*
$$\overline{\mathcal{O}_{\mathfrak{p}}} = \overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}, \qquad \overline{\mathcal{O}_{\mathfrak{p}}} \cap \mathcal{O}_{\mathfrak{g}} = \mathcal{O}_{\mathfrak{p}} \qquad \textit{and} \qquad \overline{\mathcal{O}_{\mathfrak{g}}} = G.(\overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}).$$

Proof. The first equality is obvious since $\mathcal{O}_{\mathfrak{p}}$ is dense in $\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$ by definition.

Next, the inclusion $\mathcal{O}_{\mathfrak{p}} \subset \overline{\mathcal{O}_{\mathfrak{p}}} \cap \mathcal{O}_{\mathfrak{g}}$ is clear. To show the other inclusion, assume that there is $x \in \overline{\mathcal{O}_{\mathfrak{p}}} \cap \mathcal{O}_{\mathfrak{g}}$, with $x \notin \mathcal{O}_{\mathfrak{p}}$. A contradiction is expected. Since $x \in \overline{\mathcal{O}_{\mathfrak{p}}} \setminus \mathcal{O}_{\mathfrak{p}}$, $\dim P.x < \dim P.e$. Hence,
$$\dim \mathfrak{g}^x \geqslant \dim \mathfrak{p}^x > \dim \mathfrak{p}^e = \dim \mathfrak{g}^e.$$
As a consequence, x is not in $\mathcal{O}_{\mathfrak{g}}$, whence the contradiction.

A proof of the last equality can be found in [CM93, Thm. 7.1.3]. □

For jet schemes, we have the following generalization.

Lemma 5.4. *We have*

1) $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})} = \overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m$,
2) $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})} \cap \mathscr{J}_m(\mathcal{O}_{\mathfrak{g}}) = \mathscr{J}_m(\mathcal{O}_{\mathfrak{p}}) = (\mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{l}}}) + \mathfrak{u}_m) \cap \mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})$,
3) $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})}$ *is the closure of* $G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}$.

Proof. 1) Since $\mathcal{O}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$, we get $\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}}) \subset \overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m$ because $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m$ is closed. Let $e' \in \mathcal{O}_{\mathfrak{l}}$ and $x \in \mathfrak{u}$ be such that $e := e' + x$ is in $\mathcal{O}_{\mathfrak{p}}$. From the above inclusion, we deduce that
$$\dim \mathfrak{p} - \dim \mathfrak{p}^e \leqslant \dim \mathfrak{l} - \dim \mathfrak{l}^{e'} + \dim \mathfrak{u} = \dim \mathfrak{p} - \dim \mathfrak{g}^e,$$
because $\dim \mathfrak{l}^{e'} = \dim \mathfrak{g}^e$ by Theorem 5.1. Since $\dim \mathfrak{p}^e \leqslant \dim \mathfrak{g}^e$, we get $\mathfrak{p}^e = \mathfrak{g}^e$, whence $\dim \overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})} = \dim(\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}}) + \mathfrak{u}_m)$ by Lemma 2.10 and Proposition 2.6. So $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}$ and $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m$ are irreducible varieties of the same dimension, and the equality follows.

2) Taking into account Lemma 2.10 and Proposition 2.6, the result follows from the same arguments as in the proof of Lemma 5.3, second equality.

3) By Lemma 2.10, we have

$$\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}}) = G_m.\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}}) \subset G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}.$$

As a result, $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})}$ is contained in the closure of $G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}$. On the other hand, since $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})}$ is G_m-stable, we get

$$G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})} \subset \overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})}.$$

So the closure of $G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}$ is contained in $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})}$, whence the expected equality. $\square$

Question 5.5. *For $m = 0$, $G_m.\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{p}})}$ is closed (cf. Lemma 5.3) essentially because G/P is compact. For $m \geqslant 1$, G_m/P_m is a trivial fibration over G/P with m-dimensional affine fiber. Can we show nevertheless that $G_m.(\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m)$ is closed, in other words that $\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{g}})} = G_m.(\overline{\mathscr{J}_m(\mathcal{O}_{\mathfrak{l}})} + \mathfrak{u}_m)$?*

Theorem 5.6. *Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, $\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit of $\mathfrak{l}$ and $\mathcal{O}_{\mathfrak{g}}$ the induced nilpotent orbit of $\mathfrak{g}$ from $\mathcal{O}_{\mathfrak{l}}$. If $\mathcal{O}_{\mathfrak{l}}$ verifies $\mathrm{RC}_2(m)$ for some $m \in \mathbb{N}^*$, then $\mathcal{O}_{\mathfrak{g}}$ also verifies $\mathrm{RC}_2(m)$.*

The rest of the section will be devoted to the proof of Theorem 5.6.

Definition 5.7. *Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. We say that $\mathfrak{l}$ is a* maximal Levi subalgebra of $\mathfrak{g}$ *if the center of $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{l}$ has dimension one.*

Let us first assume that $\mathfrak{g}$ is simple and that $\mathfrak{l}$ is a maximal Levi subalgebra of $\mathfrak{g}$. Thus, the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ has dimension one. Let us fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{l}$ and Δ the root system relative to $(\mathfrak{g}, \mathfrak{h})$. There exists a simple root system Π and a subset $\Pi' \subseteq \Pi$ verifying $\mathrm{card}(\Pi \setminus \Pi') = 1$ such that $\mathfrak{l}$ is the sum of $\mathfrak{h}$ and all the α-root spaces for α in the root subsystem generated by Π'. Define z to be the element in $\mathfrak{h}$ such that

$$\alpha(z) = 0 \;\text{ if }\; \alpha \in \Pi' \quad \text{ and } \quad \alpha(z) = 1 \;\text{ if }\; \alpha \in \Pi \setminus \Pi'.$$

Then z is a generator of $\mathfrak{z}(\mathfrak{l})$ and all the eigenvalues of $\mathrm{ad}\, z$ are integers.

Let $m \in \mathbb{N}$. Then $\mathrm{ad}\, z$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}_m$,

$$\mathfrak{g}_m = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_m(k) \qquad \text{with} \qquad \mathfrak{g}_m(k) := \{y \in \mathfrak{g}_m \mid [z, y] = ky\}.$$

Set

$$\mathfrak{p} = \bigoplus_{k \geqslant 0} \mathfrak{g}_0(k) \qquad \text{and} \qquad \mathfrak{u} = \bigoplus_{k > 0} \mathfrak{g}_0(k).$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ where $\mathfrak{l} = \mathfrak{g}_0(0)$ is a Levi factor, and whose nilpotent radical is $\mathfrak{u}$. Denote by P, L and U the connected closed subgroups of G whose Lie algebra is $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{u}$ respectively.

Observe that

$$\mathfrak{l}_m = \mathfrak{z}(\mathfrak{l})_m \oplus [\mathfrak{l}_m, \mathfrak{l}_m] = \mathfrak{g}_m(0), \qquad \mathfrak{p}_m = \bigoplus_{k \geqslant 0} \mathfrak{g}_m(k), \qquad \mathfrak{u}_m = \bigoplus_{k > 0} \mathfrak{g}_m(k).$$

Remark 5.8. *Clearly, for any nonzero integer k, we have $[z, \mathfrak{g}_m(k)] = \mathfrak{g}_m(k)$. In particular, $\mathfrak{g}_m(0) = (\mathfrak{g}_m)^z = \mathfrak{n}_{\mathfrak{g}_m}(\mathbb{C}z)$ where $\mathfrak{n}_{\mathfrak{g}_m}(\mathbb{C}z)$ is the normalizer of z in $\mathfrak{g}_m$. Also, if $x \in \mathfrak{g}_m(k)$, with $k \in \mathbb{N}^*$, then x is* ad*-nilpotent, and $e^{\mathrm{ad}\, x} z = z + [x, z] = z - kx$.*

Lemma 5.9. *Let $\lambda \in \mathbb{C}^*$, $x \in \mathfrak{g}_m(0)$ and $y \in \mathfrak{u}_m$. If x is* ad*-nilpotent in $\mathfrak{g}_m$ then there exists $\tau \in U_m$ such that $\tau(\lambda z + x + y) = \lambda z + x$.*

Proof. For some $p > 0$, $y = y_p + t$ with $y_p \in \mathfrak{g}_m(p)$ and $t \in \sum_{k \geqslant p+1} \mathfrak{g}_m(k)$. Since x is ad-nilpotent, the sequence $((\operatorname{ad} x)^n \mathfrak{g}_m(p))_{n \in \mathbb{N}}$ is decreasing and $(\operatorname{ad} x)^n \mathfrak{g}_m(p) = \{0\}$ for $n \geqslant \dim \mathfrak{g}_m(p)$. Let $q \in \mathbb{N}$ be such that $y_p \in (\operatorname{ad} x)^q \mathfrak{g}_m(p)$. Then

$$\begin{aligned} e^{(1/p\lambda)\operatorname{ad} y_p}(\lambda z + x + y) &= \lambda z + e^{(1/p\lambda)\operatorname{ad} y_p} x + e^{(1/p\lambda)\operatorname{ad} y_p} t \\ &= \lambda z + x + (1/p\lambda)[y_p, x] + t' = \lambda z + x + y' \end{aligned}$$

with $t' \in \sum_{k \geqslant p+1} \mathfrak{g}_m(k)$, $y' := (1/p\lambda)[y_p, x] + t'$ and

$$(1/p\lambda)[y_p, x] \in (\operatorname{ad} x)^{q+1} \mathfrak{g}_m(p).$$

Therefore we may start again with y'. After a finite number of steps, we come to an element in $\sum_{k \geqslant p+1} \mathfrak{g}_m(k)$. Then we can start again with $p+1$ instead of p and, after a finite number of steps, we come to an element of the expected form $\lambda z + x$. $\square$

Lemma 5.10. *Let Ω be an L-orbit contained in $\overline{\mathcal{O}_{\mathfrak{l}}}$ and let X be an irreducible component of $\pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega})$. Then*

$$\dim G_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m) = \dim X + 2 \dim \mathfrak{u}_m + 1.$$

Proof. Set

$$C := \mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m.$$

Since Ω and $\overline{\Omega}$ are L-stable, $\pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega})$ is L_m-stable and so is X. In addition, $\mathfrak{z}(\mathfrak{l})$ is L_m-stable too. Hence, C is P_m-stable because

$$P_m.C = L_m U_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m) = L_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m) \subset C.$$

Observe also that the elements of X are all ad-nilpotent.

Consider the action of P_m on $G_m \times C$ given by $\rho.(\sigma, c) = (\sigma\rho^{-1}, \rho(c))$. Denote by $\overline{(\sigma, c)}$ the P_m-orbit of $(\sigma, c) \in G_m \times C$ with respect to this action, and denote by $G_m \times_{P_m} C$ the corresponding quotient space. The natural morphism

$$G_m \times C \to \mathfrak{g}, \quad (\sigma, c) \mapsto \sigma(c)$$

factors through the quotient and we obtain a morphism

$$\psi\colon G_m \times_{P_m} C \to \mathfrak{g}$$

whose image is $G_m.C$. Since X and $\mathfrak{u}_m$ are both closed cones, $z = 1_{G_m}(z)$ lies in the image of ψ and

$$\psi^{-1}(z) = \{\overline{(\sigma, c)} \in G_m \times_{P_m} C \,;\, \sigma(c) = z\}.$$

Let $\overline{(\sigma, c)} \in \psi^{-1}(z)$. Because z is ad-semisimple, c is also ad-semisimple. Since all elements of X are ad-nilpotent, we deduce that c does not belong to $X + \mathfrak{u}_m$. Also, since $U_m \subset P_m$, we may assume by Lemma 5.9 that c is of the form $\lambda z + x$ with $\lambda \in \mathbb{C}^*$ and $x \in X$. Since $x \in \mathfrak{g}_m(0) = (\mathfrak{g}_m)^z$, we deduce from the uniqueness of the Jordan decomposition that $c = \lambda z$. In particular, σ is in the normalizer $N_G(\mathbb{C}z)$ of z in G, and $c = \sigma^{-1}(z)$.

According to Remark 5.8, the identity component of the centralizer $C_{G_m}(z)$ of z in G_m is contained in P_m and it has finite index in $N_{G_m}(\mathbb{C}z)$. Consequently, $\psi^{-1}(z)$ is a finite set. Thus, we get that $\dim G_m.C = \dim G_m \times_{P_m} C$ because they are both irreducible subsets. To conclude, it suffices to observe that $\dim G_m - \dim P_m = \dim \mathfrak{u}_m$ and $\dim C = 1 + \dim X + \dim \mathfrak{u}_m$ since $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}z$. $\square$

Since $\mathfrak{g}$ is simple, its Killing form $\langle\, ,\rangle$ is non-degenerate. Let us denote by ϕ the element of $\mathbb{C}[\mathfrak{g}]^G$ defined by

$$\forall\, x \in \mathfrak{g}, \qquad \phi(x) = \langle x, x\rangle.$$

By our choice of z, $\phi(z)$ is a nonzero positive integer. Set

$$\mathscr{C} := \mathfrak{z}(\mathfrak{l}) + \overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}$$

Lemma 5.11. *The nullvariety in $\mathscr{C}$ of ϕ is $\overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}$.*

Proof. First of all, $\overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}$ is contained in the nullvariety in $\mathscr{C}$ of ϕ. For the other inclusion, let $u = \lambda z + x + y$ be in $\mathscr{C}$, with $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{O}_{\mathfrak{l}}}$ and $y \in \mathfrak{u}$ such that $\phi(u) = 0$. We have

$$0 = \phi(u) = \langle \lambda z + x + y, \lambda z + x + y\rangle = \lambda^2\langle z, z\rangle + \langle x, x\rangle = \lambda^2\langle z, z\rangle$$

since $\mathfrak{u}$ is orthogonal to $\mathfrak{p}$, $\mathfrak{z}(\mathfrak{l})$ is orthogonal to $[\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{u}$ and since $\langle x, x\rangle = \phi(x) = 0$. Hence $\lambda = 0$ since $\phi(z) \neq 0$. So, u lies in $\overline{\mathcal{O}_{\mathfrak{l}}} + \mathfrak{u}$, whence the other inclusion. $\square$

Let $\phi^{(0)}, \dots, \phi^{(m)} \in \mathbb{C}[\mathfrak{g}_m]$ be the polynomials as defined in Remark 2.2 relative to ϕ. According to Lemma 2.11, they are G_m-invariant. In particular, $\phi^{(0)}$ is G_m-invariant.

Lemma 5.12. *Let $\Omega_{\mathfrak{l}}$ be an L-orbit contained in $\overline{\mathcal{O}_{\mathfrak{l}}}$ and set $\Omega_{\mathfrak{g}} := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\Omega_{\mathfrak{l}})$. Then:*

1) *the nullvariety in $G_m.(\mathfrak{z}(\mathfrak{l}) + \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + \mathfrak{u}_m)$ of $\phi^{(0)}$ is contained in $\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{g}}},m}(\overline{\Omega_{\mathfrak{g}}})$,*
2) $\dim \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{g}}},m}(\overline{\Omega_{\mathfrak{g}}}) \geqslant \dim \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + 2\dim \mathfrak{u}_m$.

Proof. Let us denote by Y the nullvariety in $G_m.(\mathfrak{z}(\mathfrak{l}) + \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + \mathfrak{u}_m)$ of $\phi^{(0)}$. First of all, observe that Y contains 0 because $\mathfrak{z}(\mathfrak{l})$, $\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}})$ and $\mathfrak{u}_m$ are closed cones. In particular, Y is nonempty.

1) Let $u = g.(\lambda z + x + y)$ be in Y, with $g \in G_m$, $\lambda \in \mathbb{C}$, $x \in \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}})$ and $y \in \mathfrak{u}_m$ such that $\phi^{(0)}(u) = 0$. Since $\phi^{(0)}$ is G_m-invariant, we get, setting $x_0 := \pi_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(x)$ and $y_0 := \pi_{\mathfrak{u},m}(y)$,

$$0 = \phi^{(0)}(u) = \phi^{(0)}(\lambda z + x + y) = \phi(\lambda z + x_0 + y_0) = \lambda^2\phi(z)$$

by the computations of the proof of Lemma 5.11. Hence $\lambda = 0$ since $\phi(z) \neq 0$. So u lies in $G_m.(\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + \mathfrak{u}_m)$. But

$$G_m.(\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + \mathfrak{u}_m) \subset G_m.(\mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{l}}}) + \mathfrak{u}_m) \subset G_m.\mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{g}}}) = \mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{g}}})$$

because $\mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{g}}})$ is G_m-invariant. Thus Y is contained in $\mathscr{J}_m(\overline{\mathcal{O}_{\mathfrak{g}}})$. Then it remains to observe that for $u \in Y$,

$$\pi_{\overline{\mathcal{O}_{\mathfrak{g}}},m}(u) \in G.(\overline{\Omega_l} + \mathfrak{u}) = \overline{\Omega_{\mathfrak{g}}}$$

by Lemma 5.3. In conclusion, Y is contained in $\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{g}}},m}(\overline{\Omega_{\mathfrak{g}}})$.

2) Let X be an irreducible component of $\pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega})$ of maximal dimension, and let Y' be the nullvariety in $G_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m)$ of $\phi^{(0)}$. The function $\phi^{(0)}$ is not identically zero on $G_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m)$ since $z \in G_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m)$ and $\phi^{(0)}(z) = \phi(z) \neq 0$. Since Y' is irreducible, we deduce by Lemma 5.10 and our choice of X that

$$\dim Y' = \dim G_m.(\mathfrak{z}(\mathfrak{l}) + X + \mathfrak{u}_m) - 1 = \dim X + 2\dim \mathfrak{u}_m = \dim \pi^{-1}_{\overline{\mathcal{O}_{\mathfrak{l}}},m}(\overline{\Omega_{\mathfrak{l}}}) + 2\dim \mathfrak{u}_m,$$

whence the statement by 1). $\square$

Proposition 5.13. *If for some L-orbit $\Omega_{\mathfrak{l}}$ in $\overline{\mathcal{O}_{\mathfrak{l}}}$, we have $\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega_{\mathfrak{l}}}) \geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\mathcal{O}_{\mathfrak{l}})$, then $\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\overline{\Omega_{\mathfrak{g}}}) \geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\mathcal{O}_{\mathfrak{g}})$, where $\Omega_{\mathfrak{g}}$ is the induced nilpotent orbit of $\mathfrak{g}$ from $\Omega_{\mathfrak{l}}$.*

Proof. Assume that for some L-orbit $\Omega_{\mathfrak{l}}$ in $\overline{\mathcal{O}_{\mathfrak{l}}}$, we have $\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega_{\mathfrak{l}}}) \geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\mathcal{O}_{\mathfrak{l}})$. Then by Lemma 5.12, we have

$$\begin{aligned}\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\overline{\Omega_{\mathfrak{g}}}) \geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega_{\mathfrak{l}}}) + 2\dim \mathfrak{u}_m &\geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\mathcal{O}_{\mathfrak{l}}) + 2\dim \mathfrak{u}_m \\ &= (m+1)\dim \mathcal{O}_{\mathfrak{l}} + 2(m+1)\dim \mathfrak{u}.\end{aligned}$$

To conclude, it remains to observe that $\pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\mathcal{O}_{\mathfrak{g}})$ has dimension $(m+1)\dim \mathcal{O}_{\mathfrak{l}} + 2(m+1)\dim \mathfrak{u}$ because $\dim \mathcal{O}_{\mathfrak{g}} = 2\dim \mathfrak{u} + \dim \mathcal{O}_{\mathfrak{l}}$ from Theorem 5.1. $\square$

Remark 5.14. *The above proof actually shows that $\pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\overline{\Omega_{\mathfrak{g}}})$ has dimension at least $2(m+1)\dim \mathfrak{u} + \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{l}},m}(\overline{\Omega_{\mathfrak{l}}})$ even if $\Omega_{\mathfrak{l}}$ does not verify the hypothesis of the proposition. This can be used in practice to give an estimation of $\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{g}},m}(\overline{\mathcal{O}_{\mathfrak{g}}} \setminus \mathcal{O}_{\mathfrak{g}})$.*

We are now in a position to prove the main result of the section.

Proof of Theorem 5.6. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. Then there is a finite sequence of Levi subalgebras

$$\mathfrak{l} = \mathfrak{l}_0 \subset \mathfrak{l}_1 \subset \mathfrak{l}_1 \subset \cdots \subset \mathfrak{l}_k = \mathfrak{g}$$

such that $\mathfrak{l}_{i-1}$ is a maximal Levi subalgebra of $\mathfrak{l}_i$ for every $i \in \{1, \ldots, k\}$.

Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit of $\mathfrak{l} = \mathfrak{l}_0$ verifying $\mathrm{RC}_2(m)$ for some $m \in \mathbb{N}$, and set for $i \in \{1, \ldots, k\}$,

$$\mathcal{O}_{\mathfrak{l}_i} = \mathrm{Ind}_{\mathfrak{l}_{i-1}}^{\mathfrak{l}_i}(\mathcal{O}_{\mathfrak{l}_{i-1}}).$$

Since induction is transitive, cf. Remark 5.2,(2), we get

$$\mathcal{O}_{\mathfrak{g}} := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \mathrm{Ind}_{\mathfrak{l}_{k-1}}^{\mathfrak{l}_k}(\mathrm{Ind}_{\mathfrak{l}_{k-2}}^{\mathfrak{l}_{k-1}}(\ldots(\mathrm{Ind}_{\mathfrak{l}_0}^{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_0})))).$$

So, in order to proof Theorem 5.6, we may assume that $\mathfrak{l}$ is maximal in $\mathfrak{g}$. Let us write $\mathcal{O}_{\mathfrak{l}}$ as a product $\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_{\mathfrak{r}_1} \times \cdots \times \mathcal{O}_{\mathfrak{r}_n}$, with the $\mathfrak{r}_j$'s as in Remark 5.2,(1). Since $\mathcal{O}_{\mathfrak{l}}$ verifies $\mathrm{RC}_2(m)$, $\mathcal{O}_{\mathfrak{r}_j}$ verifies $\mathrm{RC}_2(m)$ for some $j \in \{1, \ldots, n\}$. Since $\mathfrak{l}$ is maximal in $\mathfrak{g}$, either $\mathfrak{r}_j = \mathfrak{s}_j$ and $\mathrm{Ind}_{\mathfrak{r}_j}^{\mathfrak{s}_j}(\mathcal{O}_{\mathfrak{r}_j})$ obviously verifies $\mathrm{RC}_2(m)$ too, or $\mathfrak{r}_j$ is maximal in $\mathfrak{s}_j$ and by Proposition 5.13, $\mathrm{Ind}_{\mathfrak{r}_j}^{\mathfrak{s}_j}(\mathcal{O}_{\mathfrak{r}_j})$ verifies $\mathrm{RC}_2(m)$ as well. Indeed, since $\mathcal{O}_{\mathfrak{r}_j}$ verifies $\mathrm{RC}_2(m)$, for some $\Omega_{\mathfrak{r}_j}$ in $\overline{\mathcal{O}_{\mathfrak{r}_j}} \setminus \mathcal{O}_{\mathfrak{r}_j}$, $\dim \pi^{-1}_{\mathcal{O}_{\mathfrak{r}_j},m}(\overline{\Omega_{\mathfrak{r}_j}}) \geqslant \dim \pi^{-1}_{\mathcal{O}_{\mathfrak{r}_j},m}(\mathcal{O}_{\mathfrak{r}_j})$ and Proposition 5.13 applies. In both cases, by Remark 5.2,(3), we conclude that $\mathcal{O}_{\mathfrak{g}} := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$ verifies $\mathrm{RC}_2(m)$. $\square$

6. Consequence of Theorem 5.6

Theorem 5.6 allows us to answer the reducibility problem for many nilpotent orbits.

Recall from the beginning of Section 3 that if $\mathcal{O}$ is a nilpotent orbit of a reductive Lie algebra $\mathfrak{g}$ with simple factors $\mathfrak{s}_1, \ldots, \mathfrak{s}_m$, then $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$ where $\mathcal{O}_i$ is a nilpotent orbit of $\mathfrak{s}_i$. We shall say that $\mathcal{O}$ *has a little factor* if there exists i such that $\mathcal{O}_i$ is a little nilpotent orbit of $\mathfrak{s}_i$.

The following result is a direct consequence of Theorem 5.6 and Proposition 4.2.

Theorem 6.1. *Any nilpotent orbit induced from a nilpotent orbit that has a little factor verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*

When $\mathfrak{g}$ is simple, there is a unique nilpotent orbit $\mathcal{O}_{\text{subreg}}$ of $\mathfrak{g}$, called the subregular nilpotent orbit, such that $\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}_{\text{reg}} = \overline{\mathcal{O}_{\text{subreg}}}$. It has codimension $\operatorname{rk}\mathfrak{g} + 2$ in $\mathfrak{g}$.

Corollary 6.2. *Assume that $\mathfrak{g}$ simple and not of type $\mathbf{A_1}$, $\mathbf{B_2} = \mathbf{C_2}$ or $\mathbf{G_2}$. Then the subregular nilpotent orbit $\mathcal{O}_{\text{subreg}}$ of $\mathfrak{g}$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$. In particular, $\mathscr{J}_m(\overline{\mathcal{O}_{\text{subreg}}})$ is reducible for every $m \in \mathbb{N}^*$.*

Proof. Assume first that $\mathfrak{g}$ has type $\mathbf{A_2}$. Then $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mathcal{O}_{\text{subreg}} = \mathcal{O}_{\min} = \mathcal{O}_{(2,1)}$. Hence, $\mathcal{O}_{\text{subreg}}$ is little and verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$ according to Corollary 4.3.

Assume now that $\mathfrak{g}$ is simple with rank $\geqslant 3$. Then there exists a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that $[\mathfrak{l},\mathfrak{l}]$ is simple of type $\mathbf{A_2}$, and the subregular nilpotent orbit of $\mathfrak{g}$ is induced from that of $[\mathfrak{l},\mathfrak{l}]$ for dimension reasons (cf. Theorem 5.1). Therefore, the theorem follows from the case $\mathfrak{sl}_3(\mathbb{C})$ and Theorem 6.1. $\square$

Remark 6.3. *Outside types **A** and **B**, the subregular nilpotent orbit of a simple Lie algebra is distinguished. Thus Corollary 6.2 provides examples of distinguished nilpotent orbits which verify $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$. In particular, according to Remark 3.8, these nilpotent orbits verify $\mathrm{RC}_2(1)$ but not RC_1.*

Remark 6.4. *For $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C}) \simeq \mathfrak{so}_5(\mathbb{C})$, we can show that $\mathscr{J}_1(\overline{\mathcal{O}_{\text{subreg}}})$ is irreducible.*

Let us detail this example where the computations are explicit. Let $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$. The subregular nilpotent orbit is $\mathcal{O}_{(2^2)}$. By Appendix A, it has dimension 6, and its singular locus is the union of two nilpotent orbits, $\mathcal{O}_{(2,1^2)} = \mathcal{O}_{\min}$ and the zero orbit.

Using [W02, Thm. 1] *(see also* [W89] *or* [W03, Prop. 8.2.15]*) and the realization of $\mathfrak{sp}_4(\mathbb{C})$ as the set of anti-self-adjoint matrices for the symplectic form, we can show that the defining ideal of $\overline{\mathcal{O}_{(2^2)}}$ is generated by the entries of the matrix X^2 as functions of $X \in \mathfrak{sp}_4(\mathbb{C})$*[4]*. It follows that $\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})$ can be identified with the scheme of pairs $(X_0, X_1) \in \mathfrak{sp}_4(\mathbb{C}) \times \mathfrak{sp}_4(\mathbb{C})$ defined by the equations $X_0^2 = 0$ and $X_0X_1 + X_1X_0 = 0$.*

Using this identification, we obtain from direct computations that

$$\dim \pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2,1^2)}) = 11 \qquad \textit{and} \qquad \dim \pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(0) = 10.$$

Furthermore, there is no smooth points of $\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})$ in $\pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2,1^2)}) \cup \pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(0)$. To see this, we have computed the dimension of the tangent space to $\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})$ at generic points in $\pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2,1^2)})$ and $\pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(0)$. For the points in $\pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2,1^2)})$, the smallest dimension for the tangent space is 13; for the points in $\pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(0)$, the dimension is 14.

Now, if $\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})$ were reducible, it would have an irreducible component of dimension 10 or 11 by the above equalities. This is not possible according to the computations of the tangent space dimensions. Hence, $\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})$ is irreducible.

Classical types. We now summarize our conclusions for the case where $\mathfrak{g}$ is simple of classical type. We refer to Appendix A for the notations relative to the induction of nilpotent orbits in the classical cases.

[4]Here, we have used the computer program `Macaulay2` to check that these equations indeed generate a reduced ideal.

Theorem 6.5 (Type **A**)**.** *Let $n \in \mathbb{N}^*$, $n \geqslant 2$, and let $\boldsymbol{\lambda} \in \mathscr{P}(n)$. Suppose that $\boldsymbol{\lambda}$ is non rectangular, then the nilpotent orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathfrak{sl}_n(\mathbb{C})$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$. In particular, $\mathscr{J}_m(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ is reducible for every $m \in \mathbb{N}^*$.*

Proof. Suppose that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}(n)$ is non rectangular, with $1 < r < n$. Then there exists $1 \leqslant p < r$ such that $\lambda_p > \lambda_{p+1}$. It follows that

$$\boldsymbol{\lambda} = \mathrm{Ind}^n_{(n-p-r,p+r)} \boldsymbol{\Lambda}$$

where

$$\boldsymbol{\Lambda} = \left((\lambda_1 - 2, \dots, \lambda_p - 2, \lambda_{p+1} - 1, \dots, \lambda_r - 1), (2^p, 1^{r-p})\right).$$

Thus any non rectangular partition of n can be induced from a partition of the form $(2^p, 1^q)$ with $p, q \in \mathbb{N}^*$. According to Example 4.4, $\mathcal{O}_{(2^p,1^q)}$ is little for $p, q \in \mathbb{N}^*$. Hence the theorem follows from Theorem 6.1. ☐

Remark 6.6. *It is not difficult to see that rectangular partitions can only be induced from a rectangular one. So they cannot be induced from a nilpotent orbit that has a little factor (cf. Example 4.4).*

In fact, for the rectangular case, the theorem is not true. First of all, it is obvious not true for $\boldsymbol{\lambda} = (n)$ and $\boldsymbol{\lambda} = (1^n)$. Let us look at some special cases.

1) *Let $\boldsymbol{\lambda} = (2^p)$ with $2p = n$. Then we saw in Example 3.7 that $\mathcal{O}_{\boldsymbol{\lambda}}$ is RC_1, and that all the irreducible components of $\mathscr{J}_1(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ different from $\pi^{-1}_{\overline{\mathcal{O}_{\boldsymbol{\lambda}}},1}(\mathcal{O}_{\boldsymbol{\lambda}})$ has codimension one. In particular, it is not $\mathrm{RC}_2(1)$.*
2) *Let $\boldsymbol{\lambda} = (3^2)$. By [W02] (see also [W89] or [W03, Prop. 8.2.15]), the defining ideal of $\overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ is generated by $\mathrm{tr}(X^2)$ and the entries of the matrix X^3 as functions of $X \in \mathfrak{sl}_6(\mathbb{C})$. By Appendix A, the singular locus of $\overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ is the finite union of the nilpotent orbits $\mathcal{O}_{\boldsymbol{\mu}}$ with*

$$\mu \in \{(3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)\} \subset \mathscr{P}(6),$$

and the respective dimensions of $\pi^{-1}_{\overline{\mathcal{O}_{\boldsymbol{\lambda}}},1}(\mathcal{O}_{\boldsymbol{\mu}})$ are 47, 44, 44, 47, 44, 35. Note that $\mathscr{J}_1(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ has dimension 48. Next, we obtain that the respective dimensions of the tangent space to $\mathscr{J}_1(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ at generic points in $\pi^{-1}_{\overline{\mathcal{O}_{\boldsymbol{\lambda}}},1}(\mathcal{O}_{\boldsymbol{\mu}})$, with $\boldsymbol{\mu}$ running through the above set, are 49, 51, 51, 48, 52, 69. Arguing as in Remark 6.4, we conclude that $\mathscr{J}_1(\overline{\mathcal{O}})$ is irreducible.

Thereby, from Remark 6.6,(1) and (2), we have complete answers for the reducibility of $\mathscr{J}_1(\overline{\mathcal{O}})$ for any nilpotent orbit $\mathcal{O}$ in $\mathfrak{sl}_n(\mathbb{C})$, for $n \leqslant 7$, and for any nilpotent orbit $\mathcal{O}$ in $\mathfrak{sl}_p(\mathbb{C})$, with p a prime number.

In the other classical simple Lie algebras, we have the following result.

Theorem 6.7 (Types **B**, **C**, **D**)**.** *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t) \in \mathscr{P}_\varepsilon(n)$ with $\varepsilon \in \{1, -1\}$, and set $\lambda_{t+1} = 0$.*

1) *Suppose that $\varepsilon = 1$ and there exist $1 \leqslant k < \ell \leqslant t$ such that $\lambda_k \geqslant \lambda_{k+1} + 2$ and $\lambda_\ell \geqslant \lambda_{\ell+1} + 2$, then the nilpotent orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathfrak{so}_n(\mathbb{C})$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*
2) *Suppose that $\varepsilon = -1$ and there exist $1 \leqslant k < \ell \leqslant t$ such that $\lambda_k \geqslant \lambda_{k+1} + 2$ and $\lambda_\ell \geqslant \lambda_{\ell+1} + 2$, then the nilpotent orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathfrak{sp}_n(\mathbb{C})$ verifies $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*
3) *Suppose that $\varepsilon = 1$ and that $\boldsymbol{\lambda}$ is very even. Then both $\mathcal{O}^I_{\boldsymbol{\lambda}}$ and $\mathcal{O}^{II}_{\boldsymbol{\lambda}}$ verfiy $\mathrm{RC}_2(m)$ for every $m \in \mathbb{N}^*$.*

In particular, $\mathscr{J}_m(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ is reducible for every $m \in \mathbb{N}^*$.

Proof. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t) \in \mathscr{P}_\varepsilon(n)$, set $\lambda_{t+1} = 0$, and suppose that there exist $1 \leqslant k < \ell \leqslant t$ such that $\lambda_k \geqslant \lambda_{k+1} + 2$ and $\lambda_\ell \geqslant \lambda_{\ell+1} + 2$ as in the theorem. Then

$$\boldsymbol{\lambda} = \mathrm{Ind}^{n,\varepsilon}_{(\ell+k;n-2(\ell+k))} \boldsymbol{\Gamma}$$

where

$$\boldsymbol{\Gamma} := \Big((2^k, 1^{\ell-k}); (\lambda_1 - 4, \dots, \lambda_k - 4, \lambda_{k+1} - 2, \dots, \lambda_\ell - 2, \lambda_{\ell+1}, \dots, \lambda_t) \Big).$$

So $\boldsymbol{\lambda}$ is induced from a partition in $\mathscr{P}(n)$ of the form $(2^p, 1^q)$, with $p, q \in \mathbb{N}^*$. By Example 4.4, the partition $(2^p, 1^q)$ is little. This concludes the proof of parts (1) and (2) according to Theorem 6.1.

Finally, if $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ is very even, then $\mathcal{O}_{\boldsymbol{\lambda}}$ is induced from the nilpotent orbit $\mathcal{O}_{(2^t)}$ of $\mathfrak{so}_{2t}(\mathbb{C})$ which is little by Example 4.5. Again, we conclude thanks to Theorem 6.1. $\square$

Remark 6.8. *Unlike the type* **A** *case, in types* **B**, **C**, **D**, *orbits other than the ones considered in Theorem 6.7 can be induced from little ones. For example, for $\lambda, p, q \in \mathbb{N}^*$ with p even, we have $\boldsymbol{\lambda} = \big((2\lambda)^p, (2\lambda-1)^q\big) \in \mathscr{P}_1\big(2\lambda(p+q) - q\big)$ and $\boldsymbol{\lambda}$ does not verify the conditions of Theorem 6.7. However, we have*

$$\boldsymbol{\lambda} = \big((2\lambda)^p, (2\lambda-1)^q\big) = \mathrm{Ind}^{2\lambda(p+q)-q,1}_{((\lambda-1)(p+q);2p+q)} \big((\lambda-1)^{p+q}; (2^p, 1^q)\big)$$

Since the nilpotent orbit of $\mathfrak{so}_{2p+q}(\mathbb{C})$ corresponding to the partition $(2^p, 1^q)$ is little (cf. Example 4.5), $\mathcal{O}_{\boldsymbol{\lambda}}$ verifies $\mathrm{RC}_2(m)$ for all $m \in \mathbb{N}^*$.

Unfortunately, in types **B**, **C**, **D**, *we have not found a nice exhaustive description of nilpotent orbits that can be reached by induction from a little nilpotent orbit. Computations using* `GAP4` *show that a big proportion of partitions can be induced from little ones. See Appendix B for some numerical data.*

Exceptional types. Our conclusions for the exceptional types are summarized in Appendix C. More precisely, we can find in Appendix C the list of nilpotent orbits in a simple Lie algebra of exceptional type which can be induced from a little one.

7. Applications, remarks and comments

We give in this section applications to geometrical properties of nilpotent orbit closures.

Nilpotent orbits closures and complete intersections. Let $\mathcal{O}$ be a nilpotent orbit of the reductive Lie algebra $\mathfrak{g}$.

Theorem 7.1. *If $\mathcal{O}$ verifies RC_1 or $\mathrm{RC}_2(m)$ for some $m \geqslant 1$, then $\overline{\mathcal{O}}$ is not a complete intersection.*

Proof. Since the singular locus of $\overline{\mathcal{O}}$ is $\overline{\mathcal{O}} \setminus \mathcal{O}$ (cf. Introduction), it has codimension at least two in $\overline{\mathcal{O}}$. Hence, $\overline{\mathcal{O}}$ is normal if it is a complete intersection. If so, by [Hi91] or [P91], it has rational singularities. The theorem is then of direct consequence of Theorem 2.8. $\square$

In the papers of Namikawa, [Nam13], and Brion-Fu, [BF13], the authors use symplectic resolutions of singularities of nilpotent orbit closures to prove the above corollary for arbitrary nilpotent orbits in $\mathfrak{g}$. The foregoing provides an alternative method to obtain that result through jet schemes in a large number of cases (see Section 6). There are other approaches in the jet scheme setting to show that $\overline{\mathcal{O}}$ is not a complete intersection. Let us give an example.

Example 7.2. *The computations described in Remark 6.6,(2), show that for generic $x \in \pi^{-1}_{\overline{\mathcal{O}_{(3^2)}},1}(\mathcal{O}_{(2^2,1^2)})$, the tangent space at x of $\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})$ has dimension $48 = \dim \mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})$. Hence, such an x is a smooth point of $\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})$, because $\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})$ is irreducible, which does not belong to $\pi^{-1}_{\overline{\mathcal{O}_{(3^2)}},1}(\mathcal{O}_{(3^2)})$. So, $(\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}}))_{\rm reg} \neq \pi^{-1}_{\overline{\mathcal{O}_{(3^2)}},1}(\mathcal{O}_{(3^2)})$ and by Theorem 2.8,(3), $\overline{\mathcal{O}_{(3^2)}}$ is not a complete intersection.*

Unfortunately, theses arguments cannot be used for the nilpotent orbit $\mathcal{O}_{(2^2)}$ of $\mathfrak{sp}_4(\mathbb{C})$ because, in this case, the computations of Remark 6.4 show that we exactly have $(\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}}))_{\rm reg} = \pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2^2)})$.

Examples and counter-examples. Our results provide many examples showing that the converse of Proposition 2.5 for irreducibility is not true. Since the nilpotent cone $\mathcal{N}(\mathfrak{g})$ is normal, the following result illustrates that the converse of Proposition 2.5 for normality is also not true.

Proposition 7.3. *Assume that $\mathfrak{g}$ simple, and let $m \in \mathbb{N}$. Then $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is normal if and only if $m = 0$.*

Proof. Since $\mathscr{J}_0(\mathcal{N}(\mathfrak{g})) \simeq \mathcal{N}(\mathfrak{g})$ is normal, we have to show that for any $m \in \mathbb{N}^*$, $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is not normal.

Fix $m \in \mathbb{N}^*$. Let ℓ be the rank of $\mathfrak{g}$, and let $p_1, \dots, p_\ell$ be homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ so that

$$\mathcal{N}(\mathfrak{g}) = \operatorname{Spec} \mathbb{C}[\mathfrak{g}]/(p_1, \dots, p_\ell).$$

By Remark 2.2, we get

$$\mathscr{J}_m(\mathcal{N}(\mathfrak{g})) \simeq \operatorname{Spec} \mathbb{C}[\mathfrak{g}_m]/(p_i^{(j)};\, i = 1, \dots, \ell,\, j = 0, \dots, m).$$

Since $\mathcal{N}(\mathfrak{g})$ is a complete intersection with rational singularities, $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$ is irreducible and reduced by Theorem 2.8. So, it is generically reduced and we have

$$(\mathscr{J}_m(\mathcal{N}(\mathfrak{g})))_{\rm reg} = \{x = x_0 + x_1 t + \cdots x_m t^m \in \mathscr{J}_m(\mathcal{N}(\mathfrak{g})) \mid {\rm d}p_i^{(j)}(x_0, x_1, \dots, x_m) \text{ are linearly independent for } i = 1, \dots, \ell \text{ and } j = 0, \dots, m\}. \tag{2}$$

According to [RT92, Lem. 3.3,(i)], the vectors ${\rm d}p_i^{(j)}(x_0, x_1, \dots, x_m)$, for $i \in \{1, \dots, \ell\}$, $j \in \{0, \dots, m\}$ and $x_0 + x_1 t + \cdots x_m t^m \in \mathfrak{g}_m$, are linearly independent if and only if the vectors ${\rm d}p_1(x_0), \dots, {\rm d}p_\ell(x_0)$ are linearly independent. But by [Kos63], the later condition is satisfied if and only if x_0 is a regular element of $\mathfrak{g}$. Therefore by (2), we get

$$(\mathscr{J}_m(\mathcal{N}(\mathfrak{g})))_{\rm reg} = \pi^{-1}_{\mathcal{N}(\mathfrak{g}),m}(\mathcal{O}_{\rm reg}) \quad \text{and} \quad (\mathscr{J}_m(\mathcal{N}(\mathfrak{g})))_{\rm sing} = \pi^{-1}_{\mathcal{N}(\mathfrak{g}),m}(\overline{\mathcal{O}_{\rm subreg}}) \tag{3}$$

since $\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}_{\rm reg} = \overline{\mathcal{O}_{\rm subreg}}$. Then by Serre's criterion, it is enough to show that $\pi^{-1}_{\mathcal{N}(\mathfrak{g}),m}(\overline{\mathcal{O}_{\rm subreg}})$ has codimension one in $\mathscr{J}_m(\mathcal{N}(\mathfrak{g}))$, or else that

$$\dim \pi^{-1}_{\mathcal{N}(\mathfrak{g}),m}(\overline{\mathcal{O}_{\rm subreg}}) \geqslant \dim \mathscr{J}_m(\mathcal{N}(\mathfrak{g})) - 1. \tag{4}$$

The zero orbit of $\mathfrak{sl}_2(\mathbb{C})$ has codimension 2 in $\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))$. Hence, for dimension reasons, $\mathcal{O}_{\rm subreg}$ is the induced nilpotent orbit from 0 in any Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with semisimple part $[\mathfrak{l}, \mathfrak{l}]$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. So by Remark 5.14, in order to prove (4), it suffices to show the statement for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $\overline{\mathcal{O}_{\rm subreg}} = 0$ but by Lemma 3.5,(2),

$$\dim \pi^{-1}_{\mathcal{N}(\mathfrak{sl}_2(\mathbb{C})),m}(0) \geqslant \dim \mathscr{J}_{m-2}(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))) + \dim \mathfrak{sl}_2(\mathbb{C}) = 2(m-1) + 3 = 2m + 1,$$

whence the expected result since $\dim \mathscr{J}_m(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))) = 2(m+1) = 2m + 2$. $\square$

Remark 7.4. *The equalities (3) for $m = 1$ is also a consequence of Theorem 2.8,(3).*

We now give an example illustrating that the converse of Proposition 2.5 is also not true for reducedness.

Example 7.5. *The scheme $\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C})))$ is irreducible and reduced. We readily obtain from the description of $\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C})))$ given in Example 2.3 that $\mathscr{J}_1(\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))))$ is defined by the ideal $\mathcal{J}$ of*

$$\mathbb{C}[x_0, y_0, z_0, x_1, y_1, z_1, x'_0, y'_0, z'_0, x'_1, y'_1, z'_1]$$

generated by the polynomials

$$x_0^2 + y_0z_0,\ 2x_0x_1 + y_0z_1 + y_1z_0,$$
$$2x_0x'_0 + y_0z'_0 + z_0y'_0,\ 2x_0x'_1 + 2x_1x'_0 + y_0z'_1 + y_1z'_0 + z_1y'_0 + z_0y'_1.$$

A computation made with the program `Macaulay2` *shows that $\mathcal{J}$ is not radical, and that the radical of $\mathcal{J}$ is the intersection of two prime ideals. So, $\mathscr{J}_1(\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))))$ is neither reduced nor irreducible.*

Example 7.5 gives another evidence that $\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C})))$ does not have rational singularities (cf. Proposition 7.3). Indeed, if it had so, then by Theorem 2.8, $\mathscr{J}_1(\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C}))))$ would be irreducible (and reduced) because $\mathscr{J}_1(\mathcal{N}(\mathfrak{sl}_2(\mathbb{C})))$ is a complete intersection.

We now turn to other interesting phenomena.

Example 7.6. *As it has been observed in Example 3.7, for the nilpotent orbit $\mathcal{O}_{(2^p)}$ of $\mathfrak{sl}_{2p}(\mathbb{C})$, with $p \geqslant 2$, $\mathscr{J}_1(\overline{\mathcal{O}_{(2^p)}})$ is reducible and*

$$\dim \pi^{-1}_{\overline{\mathcal{O}_{(2^p)}},1}\left((\overline{\mathcal{O}_{(2^p)}})_{\mathrm{sing}}\right) < \dim \pi^{-1}_{\overline{\mathcal{O}_{(2^p)}},1}(\mathcal{O}_{(2^p)}).$$

This shows that Lemma 2.7,(3), does not hold in general if X is not a complete intersection.

Example 7.7. *As it has been observed in Remark 6.6,(2), for the nilpotent orbit $\mathcal{O}_{(3^2)}$ of $\mathfrak{sl}_6(\mathbb{C})$, $\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})$ is irreducible and*

$$\left(\mathscr{J}_1(\overline{\mathcal{O}_{(3^2)}})\right)_{\mathrm{reg}} \neq \pi^{-1}_{\overline{\mathcal{O}_{(3^2)}},1}(\mathcal{O}_{(3^2)}).$$

This shows that Theorem 2.8,(3), is not true for non locally complete intersection varieties.

Example 7.8. *For the nilpotent orbit $\mathcal{O}_{(2^2)}$ of $\mathfrak{sp}_4(\mathbb{C})$, we have observed (cf. Remark 6.4) that*

$$\left(\mathscr{J}_1(\overline{\mathcal{O}_{(2^2)}})\right)_{\mathrm{reg}} = \pi^{-1}_{\overline{\mathcal{O}_{(2^2)}},1}(\mathcal{O}_{(2^2)}).$$

This shows that the equality of Theorem 2.8,(3) may hold even if X is not locally a complete intersection.

Questions and remarks. Although we have determined the reducibliity of the closure of many nilpotent orbits, we would like to complete the cases which our methods do not apply. Here are some open questions.

Question 7.9. *We have seen that jet schemes of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{C})$ corresponding to rectangular partitions can be irreducible or reducible. Is there an explicit characterization ?*

Question 7.10. *In all our examples of nilpotent orbits $\mathcal{O}$ with $\mathscr{J}_1(\overline{\mathcal{O}})$ reducible, the orbit $\mathcal{O}$ verifies* RC_1 *or* $\mathrm{RC}_2(1)$*. Are these conditions necessary or are there examples of $\mathcal{O}$ for which $\mathscr{J}_1(\overline{\mathcal{O}})$ is reducible and that verify neither* RC_1 *nor* $\mathrm{RC}_2(1)$*?*

We have used the reducibility of jet schemes to study the property of complete intersection for nilpotent orbit closures. It is very likely that other geometrical properties of nilpotent orbit closures can be studied using jet schemes.

Appendix A. Nilpotent orbits in classical simple Lie algebras

We fix in this appendix some notations, and basic results, relative to nilpotent orbits in simple Lie algebras of classical type. Our main references are [CM93, Ke83]. The results concerning the induction of nilpotent orbits are mostly taken from [Ke83].

Let $n \in \mathbb{N}^*$, and denote by $\mathscr{P}(n)$ the set of partitions of n. As a rule, unless otherwise specified, we write an element $\boldsymbol{\lambda}$ of $\mathscr{P}(n)$ as a decreasing sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ omitting the zeroes. Thus,

$$\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 1 \quad \text{and} \quad \lambda_1 + \cdots + \lambda_r = n.$$

We shall denote the dual partition of a partition $\boldsymbol{\lambda} \in \mathscr{P}(n)$ by ${}^t\boldsymbol{\lambda}$. The *concatenation* of two partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ will be the rearrangement of the parts in decreasing order, and shall be denoted by $\boldsymbol{\lambda} \smile \boldsymbol{\lambda}'$.

Let us denote by $\geqslant$ the partial order on $\mathscr{P}(n)$ relative to dominance. More precisely, given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_r), \boldsymbol{\mu} = (\mu_1, \dots, \mu_s) \in \mathscr{P}(n)$, we have $\boldsymbol{\lambda} \geqslant \boldsymbol{\mu}$ if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i$$

for $1 \leqslant k \leqslant \min(r, s)$.

Case $\mathfrak{sl}_n(\mathbb{C})$.

According to [CM93, Thm. 5.1.1], nilpotent orbits of $\mathfrak{sl}_n(\mathbb{C})$ are parametrized by $\mathscr{P}(n)$. For $\boldsymbol{\lambda} \in \mathscr{P}(n)$, we shall denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{sl}_n(\mathbb{C})$, and if we write ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_s)$, then

$$\dim \mathcal{O}_{\boldsymbol{\lambda}} = n^2 - \sum_{i=1}^{s} d_i^2.$$

Also, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(n)$, then $\mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\mu} \leqslant \boldsymbol{\lambda}$.

The Levi subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ are parametrized by compositions of n. Let $\mathbf{m} = (m_1, \dots, m_r)$ be a composition of n, and let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)}) \in \mathscr{P}(m_1) \times \cdots \times \mathscr{P}(m_r)$. It corresponds to a nilpotent orbit in the Levi subalgebra associated to the composition $\mathbf{m}$. Set

$$\boldsymbol{\mu} := {}^t\boldsymbol{\lambda}^{(1)} \smile \cdots \smile {}^t\boldsymbol{\lambda}^{(r)} \quad \text{and} \quad \boldsymbol{\nu} = {}^t\boldsymbol{\mu}.$$

Then the partition associated to the induced nilpotent orbit from $\mathcal{O}_{(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)})}$ is $\boldsymbol{\nu}$. Note that we have $\nu_i = \lambda_i^{(1)} + \cdots + \lambda_i^{(k)}$ which is much simpler to compute in practice. We shall denote $\boldsymbol{\nu}$ by $\mathrm{Ind}_{\mathbf{m}}^{n}(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)})$ and we shall say that $\boldsymbol{\nu}$ is *induced from* $(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)})$.

Case $\mathfrak{so}_n(\mathbb{C})$.

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_1(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) \; ; \; \text{number of parts of each even number is even}\}.$$

According to [CM93, Thm. 5.1.2 and 5.1.4], nilpotent orbits of $\mathfrak{so}_n(\mathbb{C})$ are parametrized by $\mathscr{P}(n)$, with the exception that each *very even* partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ (i.e., $\boldsymbol{\lambda}$ has only even parts) corresponds to two nilpotent orbits. For $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, not very even, we shall denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{so}_n(\mathbb{C})$. For very even $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, we shall denote by $\mathcal{O}_{\boldsymbol{\lambda}}^{I}$ and $\mathcal{O}_{\boldsymbol{\lambda}}^{II}$ the two corresponding nilpotent orbits of $\mathfrak{so}_n(\mathbb{C})$. In fact, their union form a single $\mathrm{O}_n(\mathbb{C})$-orbit.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_1(n)$ and ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_s)$, then

$$\dim \mathcal{O}^{\bullet}_{\boldsymbol{\lambda}} = \frac{n(n-1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{s} d_i^2 - \sharp\{i; \lambda_i \text{ odd}\}\right)$$

where $\mathcal{O}^{\bullet}_{\boldsymbol{\lambda}}$ is either $\mathcal{O}_{\boldsymbol{\lambda}}$, $\mathcal{O}^{I}_{\boldsymbol{\lambda}}$ or $\mathcal{O}^{II}_{\boldsymbol{\lambda}}$ according to whether $\boldsymbol{\lambda}$ is very even or not. Using the same notations, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_1(n)$, then $\overline{\mathcal{O}^{\bullet}_{\boldsymbol{\mu}}} \subsetneq \overline{\mathcal{O}^{\bullet}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\mu} < \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^+ \in \mathscr{P}_1(n)$ such that $\boldsymbol{\lambda}^+ \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\mu} \in \mathscr{P}_1(n)$ verifies $\boldsymbol{\mu} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\mu} \leqslant \boldsymbol{\lambda}^+$. More precisely, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ (adding zeroes if necessary). If $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, then $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}$. Otherwise if $\boldsymbol{\lambda} \notin \mathscr{P}_1(n)$, set

$$\boldsymbol{\lambda}' = (\lambda_1, \dots, \lambda_r, \lambda_{r+1} - 1, \lambda_{r+2}, \dots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \dots, \lambda_n)$$

where r is maximum such that $(\lambda_1, \dots, \lambda_r) \in \mathscr{P}_1(\lambda_1 + \cdots + \lambda_r)$, and s is the index of the first even part in $(\lambda_{r+2}, \dots, \lambda_n)$. Note that $r = 0$ if such a maximum does not exist, while s is always defined. If $\boldsymbol{\lambda}'$ is not in $\mathscr{P}_1(n)$, then we repeat the process until we obtain an element of $\mathscr{P}_1(n)$ which will be our $\boldsymbol{\lambda}^+$.

The Levi subalgebras in $\mathfrak{so}_n(\mathbb{C})$ are parametrized by

$$\mathcal{L}(n) := \left\{(p_1, \dots, p_k; r)\, ;\, 2\sum_{i=1}^{k} p_i + r = n\right\}.$$

Let $(p_1, \dots, p_k; r) \in \mathcal{L}(n)$, $(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}) \in \mathscr{P}(p_1) \times \cdots \times \mathscr{P}(p_k)$ and $\boldsymbol{\mu} \in \mathscr{P}_1(r)$, and set

$$\boldsymbol{\nu} := \operatorname{Ind}^{n}_{(p_1,\dots,p_k,r,p_k,\dots,p_1)}(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\mu}, \boldsymbol{\lambda}^{(k)}, \dots, \boldsymbol{\lambda}^{(1)})$$

in the notations of the $\mathfrak{sl}_n(\mathbb{C})$ case. Thus $\boldsymbol{\nu}$ is the partition associated to the nilpotent orbit in $\mathfrak{sl}_n(\mathbb{C})$ induced from the nilpotent orbit in the Levi subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ associated to the composition $(p_1, \dots, p_k, r, p_k, \dots, p_1)$ and the multi-partition $(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\mu}, \boldsymbol{\lambda}^{(k)}, \dots, \boldsymbol{\lambda}^{(1)})$. The partition associated to the nilpotent orbit induced from $(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}; \boldsymbol{\mu})$ is $\boldsymbol{\nu}^+$. We shall denote $\boldsymbol{\nu}^+$ by $\operatorname{Ind}^{n,+}_{(p_1,\dots,p_k;r)}(\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}; \boldsymbol{\mu})$. The partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ corresponds to a rigid orbit if and only if

(i) $\lambda_i - \lambda_{i+1} \leqslant 1$ for all i, so the last part of $\boldsymbol{\lambda}$ is 1.
(ii) No odd number occurs exactly twice in $\boldsymbol{\lambda}$.

Note that in the case of a very even partition $\boldsymbol{\lambda}$, $\boldsymbol{\nu}^+$ is also very even partition, and we obtain both nilpotent orbits corresponding to $\boldsymbol{\nu}^+$ via induction of the nilpotent orbits corresponding to $\boldsymbol{\lambda}$, cf. [CM93, Thm. 7.3.3,(iii)].

Case $\mathfrak{sp}_{2n}(\mathbb{C})$.

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_{-1}(2n) := \{\boldsymbol{\lambda} \in \mathscr{P}(2n)\ ;\ \text{number of parts of each odd number is even}\}.$$

According to [CM93, Thm. 5.1.3], nilpotent orbits of $\mathfrak{sp}_{2n}(\mathbb{C})$ are parametrized by $\mathscr{P}_{-1}(2n)$. For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_{-1}(2n)$, we shall denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{sp}_{2n}(\mathbb{C})$, and if we write ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_s)$, then

$$\dim \mathcal{O}_{\boldsymbol{\lambda}} = n(2n+1) - \frac{1}{2}\left(\sum_{i=1}^{s} d_i^2 + \sharp\{i; \lambda_i \text{ odd}\}\right).$$

As in the case of $\mathfrak{sl}_n(\mathbb{C})$, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{-1}(2n)$, then $\mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\mu} \leqslant \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} \in \mathscr{P}(2n)$, there exists a unique $\boldsymbol{\lambda}^- \in \mathscr{P}_{-1}(2n)$ such that $\boldsymbol{\lambda}^- \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\mu} \in \mathscr{P}_{-1}(2n)$ verifies $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$, then $\boldsymbol{\mu} \leq \boldsymbol{\lambda}^-$. The construction of $\boldsymbol{\lambda}^-$ is the same as in the orthogonal case except that s is the index of the first odd part in $(\lambda_{r+2}, \ldots, \lambda_{2n})$.

As in the orthogonal case, Levi subalgebras are parametrized by $\mathcal{L}(2n)$. Let us conserve the same notations as in the orthogonal case. The partition associated to the nilpotent orbit induced from $(\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}; \boldsymbol{\mu})$ is $\boldsymbol{\nu}^-$. We shall denote $\boldsymbol{\nu}^-$ by $\mathrm{Ind}^{2n,-}_{(p_1,\ldots,p_k;r)}(\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}; \boldsymbol{\mu})$. The partition $\boldsymbol{\lambda} \in \mathscr{P}_{-1}(2n)$ corresponds to a rigid orbit if and only if

(i) $\lambda_i - \lambda_{i+1} \leqslant 1$ for all i, so the last part of $\boldsymbol{\lambda}$ is 1.
(ii) No even number occurs exactly twice in $\boldsymbol{\lambda}$.

Appendix B. Statistics in types B, C and D

As mentioned in Remark 6.8, many nilpotent orbits in $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$ can be obtained by induction from little nilpotent orbits. In particular, these induced orbits verify $\mathrm{RC}_2(m)$ for all $m \in \mathbb{N}^*$. Computations using `GAP4` gave us the following numerical data supporting our claim.

For $\varepsilon \in \{-1, 1\}$ and $n \in \mathbb{N}^*$, we denote by $\mathscr{P}^\ell_\varepsilon(n)$ the set of partitions in $\mathscr{P}_\varepsilon(n)$ induced from little ones.

Case $\mathfrak{so}_n(\mathbb{C})$.

n	$\sharp\mathscr{P}^\ell_1(n)$	$\sharp\mathscr{P}_1(n)$	n	$\sharp\mathscr{P}^\ell_1(n)$	$\sharp\mathscr{P}_1(n)$	n	$\sharp\mathscr{P}^\ell_1(n)$	$\sharp\mathscr{P}_1(n)$	n	$\sharp\mathscr{P}^\ell_1(n)$	$\sharp\mathscr{P}_1(n)$	n	$\sharp\mathscr{P}^\ell_1(n)$	$\sharp\mathscr{P}_1(n)$
2	0	1	12	20	28	22	195	236	32	1223	1431	42	6064	6868
3	0	2	13	27	35	23	250	287	33	1474	1687	43	7086	7967
4	1	3	14	32	43	24	291	350	34	1710	1981	44	8182	9233
5	1	4	15	45	55	25	367	420	35	2039	2331	45	9536	10670
6	2	5	16	52	70	26	423	501	36	2370	2741	46	10986	12306
7	4	7	17	73	86	27	527	602	37	2821	3206	47	12748	14193
8	6	10	18	83	105	28	609	722	38	3265	3740	48	14667	16357
9	9	13	19	111	130	29	751	858	39	3852	4368	49	16974	18803
10	10	16	20	130	161	30	869	1016	40	4460	5096	50	19485	21581
11	16	21	21	170	196	31	1055	1206	41	5242	5922	51	22464	24766

Case $\mathfrak{sp}_{2n}(\mathbb{C})$.

n	$\sharp\mathscr{P}^\ell_{-1}(2n)$	$\sharp\mathscr{P}_{-1}(2n)$	n	$\sharp\mathscr{P}^\ell_{-1}(2n)$	$\sharp\mathscr{P}_{-1}(2n)$	n	$\sharp\mathscr{P}^\ell_{-1}(2n)$	$\sharp\mathscr{P}_{-1}(2n)$	n	$\sharp\mathscr{P}^\ell_{-1}(2n)$	$\sharp\mathscr{P}_{-1}(2n)$
1	0	2	7	45	64	13	594	728	19	4652	5400
2	1	4	8	77	100	14	857	1040	20	6374	7336
3	3	8	9	119	154	15	1223	1472	21	8677	9904
4	9	14	10	182	232	16	1726	2062	22	11728	13288
5	15	24	11	273	344	17	2421	2864	23	15755	17728
6	28	40	12	409	504	18	3378	3948	24	21061	23528

Appendix C. Tables for exceptional types

We list below nilpotent orbits in a simple Lie algebra of exceptional type precising when possible whether they are RC_1 or $\mathrm{RC}_2(m)$. Condition RC_1 is checked using Propositions 3.6, 4.2, 4.6 and Remark 3.9. When condition RC_1 is obtained via Proposition 4.6, we give an example of the bigger simple Lie algebra and the little nilpotent orbit satisfying condition (iii) of Proposition 4.6 from which it is obtained.

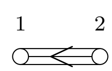
As for determining whether condition $\mathrm{RC}_2(m)$ is verified, our main method is to list the orbits induced by nilpotent orbits that have a little factor (Theorem 6.1). Thus they are $\mathrm{RC}_2(m)$ for all $m \in \mathbb{N}^*$. Since induction is transitive, we can proceed by induction on the rank of the Lie algebra, where at each step, we only need to consider induction from orbits in maximal Levi subalgebras which are themselves induced from nilpotent orbits with a little factor. For an orbit verifying condition $\mathrm{RC}_2(m)$, we give an example of a maximal Levi subalgebra $\mathfrak{l}$ and an orbit in $\mathfrak{l}$ induced from a nilpotent orbit with a little factor.

In both cases, if the orbit is little, then we just label it little. The subscript of an orbit indicates either its characteristics or the associated partition or its Bala-Carter label. If a superscript of an orbit is present, it indicates the corresponding maximal Levi subalgebra.

We have omitted the zero orbit and the regular orbit because they are neither RC_1 nor $\mathrm{RC}_2(m)$.

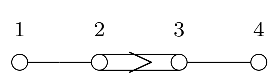
All the computations are done using the package `sla` of `GAP4`.

Type $\mathbf{G_2}$.



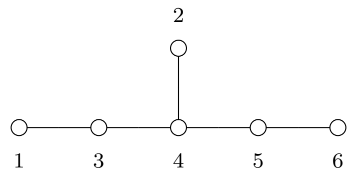
$\mathcal{O}$		$\dim \mathcal{O}$	RC_1	RC_2	rigid
A_1	$[0,1]$	6	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$\tilde{A}_1$	$[1,0]$	8	$\times$	?	$\surd$
$G_2(a_1)$	$[2,0]$	10	$\times$	?	$\times$

Type $\mathbf{F_4}$.



$\mathcal{O}$		$\dim \mathcal{O}$	RC_1	RC_2	rigid
A_1	$[1,0,0,0]$	16	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$\tilde{A}_1$	$[0,0,0,1]$	22	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$A_1+\tilde{A}_1$	$[0,1,0,0]$	28	$\times$	?	$\surd$
A_2	$[2,0,0,0]$	30	$\times$	?	$\times$
$\tilde{A}_2$	$[0,0,0,2]$	30	$\times$	?	$\times$
$A_2+\tilde{A}_1$	$[0,0,1,0]$	34	$\times$	?	$\surd$
B_2	$[2,0,0,1]$	36	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4\}}_{\mathrm{min}}$	$\times$
$\tilde{A}_2+A_1$	$[0,1,0,1]$	36	$\times$	?	$\surd$
$C_3(a_1)$	$[1,0,1,0]$	38	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3\}}_{\mathrm{min}}$	$\times$
$F_4(a_3)$	$[0,2,0,0]$	40	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4\}}_{[0,1,0]}$	$\times$
B_3	$[2,2,0,0]$	42	$\times$	?	$\times$
C_3	$[1,0,1,2]$	42	$\times$	?	$\times$
$F_4(a_2)$	$[0,2,0,2]$	44	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,4\}}_{(2,1),(1^2)}$	$\times$
$F_4(a_1)$	$[2,2,0,2]$	46	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,4\}}_{(2,1),(2)}$	$\times$

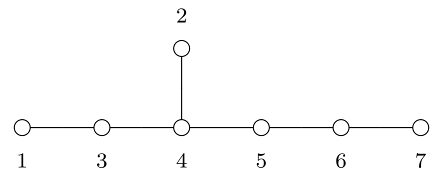
Type $\mathbf{E_6}$.



$\mathcal{O}$		$\dim\mathcal{O}$	RC_1	RC_2	rigid
A_1	$[0,1,0,0,0,0]$	22	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$2A_1$	$[1,0,0,0,0,1]$	32	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\times$
$3A_1$	$[0,0,0,1,0,0]$	40	$\surd \leftarrow \mathrm{Res}_{E_6}^{E_7}\mathcal{O}_{(3A_1)'}$	?	$\surd$
A_2	$[0,2,0,0,0,0]$	42	$\surd \leftarrow \mathrm{Res}_{E_6}^{E_7}\mathcal{O}_{A_2}$	?	$\times$
A_2+A_1	$[1,1,0,0,0,1]$	46	$\times$	$\surd \leftarrow \mathcal{O}_{min}^{\{1,2,3,4,5\}}$	$\times$
$2A_2$	$[2,0,0,0,0,2]$	48	$\times$	$\surd \leftarrow \mathcal{O}_{(3,1^7)}^{\{1,2,3,4,5\}}$	$\times$
A_2+2A_1	$[0,0,1,0,1,0]$	50	$\times$	?	$\times$
A_3	$[1,2,0,0,0,1]$	52	$\times$	$\surd \leftarrow \mathcal{O}_{(2,1^4)}^{\{1,3,4,5,6\}}$	$\times$
$2A_2+A_1$	$[1,0,0,1,0,1]$	54	$\times$	?	$\surd$
A_3+A_1	$[0,1,1,0,1,0]$	56	$\times$	?	$\times$
$D_4(a_1)$	$[0,0,0,2,0,0]$	58	$\times$	$\surd \leftarrow \mathcal{O}_{(2^2,1^2)}^{\{1,3,4,5,6\}}$	$\times$
A_4	$[2,2,0,0,0,2]$	60	$\times$	$\surd \leftarrow \mathcal{O}_{(3,1^3)}^{\{1,3,4,5,6\}}$	$\times$
D_4	$[0,2,0,2,0,0]$	60	$\times$	?	$\times$
A_4+A_1	$[1,1,1,0,1,1]$	62	$\times$	$\surd \leftarrow \mathcal{O}_{((2,1^2),(1^2))}^{\{1,2,3,4,6\}}$	$\times$
A_5	$[2,1,1,0,1,2]$	64	$\times$	?	$\times$
$D_5(a_1)$	$[1,2,1,0,1,1]$	64	$\times$	$\surd \leftarrow \mathcal{O}_{(3,2,1)}^{\{1,3,4,5,6\}}$	$\times$
$E_6(a_3)$	$[2,0,0,2,0,2]$	66	$\times$	$\surd \leftarrow \mathcal{O}_{(4,1^2)}^{\{1,3,4,5,6\}}$	$\times$
D_5	$[2,2,0,2,0,2]$	68	$\times$	$\surd \leftarrow \mathcal{O}_{(4,2)}^{\{1,3,4,5,6\}}$	$\times$
$E_6(a_1)$	$[2,2,2,0,2,2]$	70	$\times$	$\surd \leftarrow \mathcal{O}_{(5,1)}^{\{1,3,4,5,6\}}$	$\times$

The notation $\mathrm{Res}_{E_6}^{E_7}\mathcal{O}$ means that the orbit is obtained by restriction from the little nilpotent orbit $\mathcal{O}$ in E_7 as explained in Table 1.

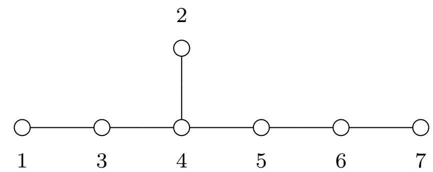
Type $\mathbf{E_7}$.



$\mathcal{O}$		$\dim\mathcal{O}$	RC_1	RC_2	rigid
A_1	$[1,0,0,0,0,0,0]$	34	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$2A_1$	$[0,0,0,0,0,1,0]$	52	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$(3A_1)''$	$[0,0,0,0,0,0,2]$	54	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\times$
$(3A_1)'$	$[0,0,1,0,0,0,0]$	64	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
A_2	$[2,0,0,0,0,0,0]$	66	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\times$
$4A_1$	$[0,1,0,0,0,0,1]$	70	$\times$	?	$\surd$
A_2+A_1	$[1,0,0,0,0,1,0]$	76	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6\}}_{[0,1,0,0,0,0]}$	$\times$
A_2+2A_1	$[0,0,0,1,0,0,0]$	82	$\times$	?	$\surd$
A_3	$[2,0,0,0,0,1,0]$	84	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(2^2,1^8)}$	$\times$
$2A_2$	$[0,0,0,0,0,2,0]$	84	$\times$	?	$\times$
A_2+3A_1	$[0,2,0,0,0,0,0]$	84	$\times$	?	$\times$
$(A_3+A_1)''$	$[2,0,0,0,0,0,2]$	86	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(3,1^9)}$	$\times$
$2A_2+A_1$	$[0,0,1,0,0,1,0]$	90	$\times$	?	$\surd$
$(A_3+A_1)'$	$[1,0,0,1,0,0,0]$	92	$\times$	?	$\surd$
$D_4(a_1)$	$[0,0,2,0,0,0,0]$	94	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(2^4,1^4)}$	$\times$
A_3+2A_1	$[1,0,0,0,1,0,1]$	94	$\times$	?	$\times$
D_4	$[2,0,2,0,0,0,0]$	96	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,0,0,0,2,0]}$	$\times$
$D_4(a_1)+A_1$	$[0,1,1,0,0,0,1]$	96	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,0,0,0,0,2]}$	$\times$
A_3+A_2	$[0,0,0,1,0,1,0]$	98	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(3,2^2,1^5)}$	$\times$
A_4	$[2,0,0,0,0,2,0]$	100	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(3^2,1^6)}$	$\times$
$A_3+A_2+A_1$	$[0,0,0,0,2,0,0]$	100	$\times$	?	$\times$
$(A_5)''$	$[2,0,0,0,0,2,2]$	102	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(5,1^7)}$	$\times$

Note that the characteristics $[0,0,0,0,2,0]$ and $[0,0,0,0,0,2]$ of nilpotent orbits in $\mathbf{D_6}$ correspond to the very even partition (2^6).

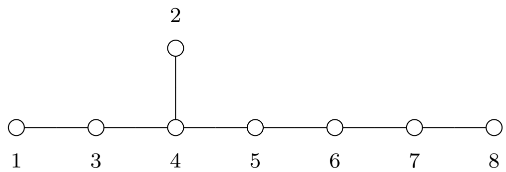
Type $\mathbf{E_7}$ (cont'd).



$\mathcal{O}$		$\dim\mathcal{O}$	RC_1	RC_2	rigid
D_4+A_1	$[2,1,1,0,0,0,1]$	102	$\times$	?	$\times$
A_4+A_1	$[1,0,0,1,0,1,0]$	104	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,3,4,5,6,7\}}_{(2^2,1^3)}$	$\times$
$D_5(a_1)$	$[2,0,0,1,0,1,0]$	106	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(3^2,2^2,1^2)}$	$\times$
A_4+A_2	$[0,0,0,2,0,0,0]$	106	$\times$	?	$\times$
$(A_5)'$	$[1,0,0,1,0,2,0]$	108	$\times$	?	$\times$
A_5+A_1	$[1,0,0,1,0,1,2]$	108	$\times$	?	$\times$
$D_5(a_1)+A_1$	$[2,0,0,0,2,0,0]$	108	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,6,7\}}_{(2,1^3),(1^3)}$	$\times$
$D_5(a_2)$	$[0,1,1,0,1,0,2]$	110	$\times$	?	$\times$
$E_6(a_3)$	$[0,0,2,0,0,2,0]$	110	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,5,6,7\}}_{(2,1),(1^2),(1^4)}$	$\times$
D_5	$[2,0,2,0,0,2,0]$	112	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,2,0,0,2,0]}$	$\times$
$E_7(a_5)$	$[0,0,0,2,0,0,2]$	112	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,2,0,0,0,2]}$	$\times$
A_6	$[0,0,0,2,0,2,0]$	114	$\times$	?	$\times$
D_5+A_1	$[2,1,1,0,1,1,0]$	114	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,6,7\}}_{(3,1^2),(1^3)}$	$\times$
$D_6(a_1)$	$[2,1,1,0,1,0,2]$	114	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,4,5,6,7\}}_{(2),(3,1^3)}$	$\times$
$E_7(a_4)$	$[2,0,0,2,0,0,2]$	116	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,3,4,5,6,7\}}_{(3^2,1)}$	$\times$
D_6	$[2,1,1,0,1,2,2]$	118	$\times$	?	$\times$
$E_6(a_1)$	$[2,0,0,2,0,2,0]$	118	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{(5^2,1^2)}$	$\times$
E_6	$[2,0,2,2,0,2,0]$	120	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,2,0,2,2,0]}$	$\times$
$E_7(a_3)$	$[2,0,0,2,0,2,2]$	120	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7\}}_{[0,2,0,2,0,2]}$	$\times$
$E_7(a_2)$	$[2,2,2,0,2,0,2]$	122	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,3,4,5,6,7\}}_{(5,2)}$	$\times$
$E_7(a_1)$	$[2,2,2,0,2,2,2]$	124	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,3,4,5,6,7\}}_{(6,1)}$	$\times$

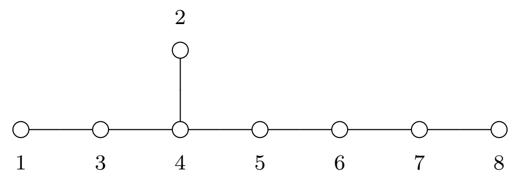
Note that the characteristics $[0,2,0,0,2,0]$ and $[0,2,0,0,0,2]$ of nilpotent orbits in $\mathbf{D_6}$ correspond to the very even partition $(4^2,2^2)$, while $[0,2,0,2,2,0]$ and $[0,2,0,2,0,2]$ correspond to (6^2).

Type $\mathbf{E_8}$.



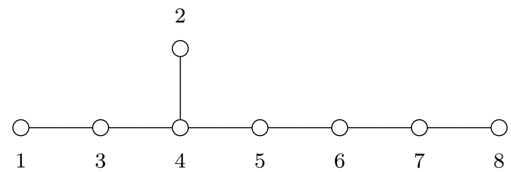
$\mathcal{O}$		$\dim \mathcal{O}$	RC_1	RC_2	rigid
A_1	$[0,0,0,0,0,0,0,1]$	58	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$2A_1$	$[1,0,0,0,0,0,0,0]$	92	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
$3A_1$	$[0,0,0,0,0,0,1,0]$	112	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\surd$
A_2	$[0,0,0,0,0,0,0,2]$	114	$\surd \leftarrow$ little	$\surd \leftarrow$ little	$\times$
$4A_1$	$[0,1,0,0,0,0,0,0]$	128	$\times$	?	$\surd$
A_2+A_1	$[1,0,0,0,0,0,0,1]$	136	$\times$	?	$\surd$
A_2+2A_1	$[0,0,0,0,0,1,0,0]$	146	$\times$	?	$\surd$
A_3	$[1,0,0,0,0,0,0,2]$	148	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6,7\}}_{[1,0,0,0,0,0,0]}$	$\times$
A_2+3A_1	$[0,0,1,0,0,0,0,0]$	154	$\times$	?	$\surd$
$2A_2$	$[2,0,0,0,0,0,0,0]$	156	$\times$	?	$\times$
$2A_2+A_1$	$[1,0,0,0,0,0,1,0]$	162	$\times$	?	$\surd$
A_3+A_1	$[0,0,0,0,0,1,0,1]$	164	$\times$	?	$\surd$
$D_4(a_1)$	$[0,0,0,0,0,0,2,0]$	166	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6,7\}}_{[0,0,0,0,0,1,0]}$	$\times$
D_4	$[0,0,0,0,0,0,2,2]$	168	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6,7\}}_{[0,0,0,0,0,0,2]}$	$\times$
$2A_2+2A_1$	$[0,0,0,0,1,0,0,0]$	168	$\times$	?	$\surd$
A_3+2A_1	$[0,0,1,0,0,0,0,1]$	172	$\times$	?	$\surd$
$D_4(a_1)+A_1$	$[0,1,0,0,0,0,1,0]$	176	$\times$	?	$\surd$
A_3+A_2	$[1,0,0,0,0,1,0,0]$	178	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(2^2,1^{10})}$	$\times$
A_4	$[2,0,0,0,0,0,0,2]$	180	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(3,1^{11})}$	$\times$
$A_3+A_2+A_1$	$[0,0,0,1,0,0,0,0]$	182	$\times$	?	$\surd$
D_4+A_1	$[0,1,0,0,0,0,1,2]$	184	$\times$	?	$\times$
$D_4(a_1)+A_2$	$[0,2,0,0,0,0,0,0]$	184	$\times$	?	$\times$
A_4+A_1	$[1,0,0,0,0,1,0,1]$	188	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6,8\}}_{[0,1,0,0,0,0],(1^2)}$	$\times$

Type $\mathbf{E_8}$ (cont'd).



$\mathcal{O}$		$\dim \mathcal{O}$	RC_1	RC_2	rigid
$2A_3$	$[1,0,0,0,1,0,0,0]$	188	$\times$	?	$\surd$
$D_5(a_1)$	$[1,0,0,0,0,1,0,2]$	190	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(2^4,1^6)}$	$\times$
A_4+2A_1	$[0,0,0,1,0,0,0,1]$	192	$\times$	?	$\times$
A_4+A_2	$[0,0,0,0,0,2,0,0]$	194	$\times$	?	$\times$
A_5	$[2,0,0,0,0,1,0,1]$	196	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(3,2^2,1^7)}$	$\times$
$D_5(a_1)+A_1$	$[0,0,0,1,0,0,0,2]$	196	$\times$	?	$\times$
$A_4+A_2+A_1$	$[0,0,1,0,0,1,0,0]$	196	$\times$	?	$\times$
D_4+A_2	$[0,2,0,0,0,0,0,2]$	198	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(2^6,1^2)}$	$\times$
$E_6(a_3)$	$[2,0,0,0,0,0,2,0]$	198	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(3^2,1^8)}$	$\times$
D_5	$[2,0,0,0,0,0,2,2]$	200	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5,1^9)}$	$\times$
A_4+A_3	$[0,0,0,1,0,0,1,0]$	200	$\times$	?	$\surd$
A_5+A_1	$[1,0,0,1,0,0,0,1]$	202	$\times$	?	$\surd$
$D_5(a_1)+A_2$	$[0,0,1,0,0,1,0,1]$	202	$\times$	?	$\surd$
$D_6(a_2)$	$[0,1,1,0,0,0,1,0]$	204	$\times$	?	$\times$
$E_6(a_3)+A_1$	$[1,0,0,0,1,0,1,0]$	204	$\times$	?	$\times$
$E_7(a_5)$	$[0,0,0,1,0,1,0,0]$	206	$\times$	?	$\times$
D_5+A_1	$[1,0,0,0,1,0,1,2]$	208	$\times$	?	$\times$
$E_8(a_7)$	$[0,0,0,0,2,0,0,0]$	208	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(3^2,2^2,1^4)}$	$\times$
A_6	$[2,0,0,0,0,2,0,0]$	210	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,7,8\}}_{(3,1^7),(1^3)}$	$\times$
$D_6(a_1)$	$[0,1,1,0,0,0,1,2]$	210	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(3^2,2^4)}$	$\times$
A_6+A_1	$[1,0,0,1,0,1,0,0]$	212	$\times$	?	$\times$
$E_7(a_4)$	$[0,0,0,1,0,1,0,2]$	212	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,3,4,5,6,7\}}_{[0,0,0,1,0,1,0]}$	$\times$
$E_6(a_1)$	$[2,0,0,0,0,2,0,2]$	214	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5,3,1^6)}$	$\times$

Type $\mathbf{E_8}$ (cont'd).



$\mathcal{O}$		$\dim \mathcal{O}$	RC_1	RC_2	rigid
$D_5 + A_2$	$[0,0,0,0,2,0,0,2]$	214	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,3,4,5,6,7,8\}}_{(2^3,1^2)}$	$\times$
D_6	$[2,1,1,0,0,0,1,2]$	216	$\times$	?	$\times$
E_6	$[2,0,0,0,0,2,2,2]$	216	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(7,1^7)}$	$\times$
$D_7(a_2)$	$[1,0,0,1,0,1,0,1]$	216	$\times$	$\surd \leftarrow \mathcal{O}^{\{1,2,4,5,6,7,8\}}_{(1^2),(2^2,1^3)}$	$\times$
A_7	$[1,0,0,1,0,1,1,0]$	218	$\times$	?	$\times$
$E_6(a_1) + A_1$	$[1,0,0,1,0,1,0,2]$	218	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(4^2,2^2,1^2)}$	$\times$
$E_7(a_3)$	$[2,0,0,1,0,1,0,2]$	220	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5,3,2^2,1)}$	$\times$
$E_8(b_6)$	$[0,0,0,2,0,0,0,2]$	220	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(4^3,3,1^3)}$	$\times$
$D_7(a_1)$	$[2,0,0,0,2,0,0,2]$	222	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(4^2,3^2)}$	$\times$
$E_6 + A_1$	$[1,0,0,1,0,1,2,2]$	222	$\times$	?	$\times$
$E_7(a_2)$	$[0,1,1,0,1,0,2,2]$	224	$\times$	?	$\times$
$E_8(a_6)$	$[0,0,0,2,0,0,2,0]$	224	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5,3^3)}$	$\times$
D_7	$[2,1,1,0,1,1,0,1]$	226	$\times$	?	$\times$
$E_8(b_5)$	$[0,0,0,2,0,0,2,2]$	226	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5^2,2^2)}$	$\times$
$E_7(a_1)$	$[2,1,1,0,1,0,2,2]$	228	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(7,3,2^2)}$	$\times$
$E_8(a_5)$	$[2,0,0,2,0,0,2,0]$	228	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(5^2,3,1)}$	$\times$
$E_8(b_4)$	$[2,0,0,2,0,0,2,2]$	230	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(6^2,1^2)}$	$\times$
E_7	$[2,1,1,0,1,2,2,2]$	232	$\times$	?	$\times$
$E_8(a_4)$	$[2,0,0,2,0,2,0,2]$	232	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(7,5,1^2)}$	$\times$
$E_8(a_3)$	$[2,0,0,2,0,2,2,2]$	234	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(7^2)}$	$\times$
$E_8(a_2)$	$[2,2,2,0,2,0,2,2]$	236	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(9,5)}$	$\times$
$E_8(a_1)$	$[2,2,2,0,2,2,2,2]$	238	$\times$	$\surd \leftarrow \mathcal{O}^{\{2,3,4,5,6,7,8\}}_{(11,3)}$	$\times$

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