The arc space of horospherical varieties and motivic integration

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Abstract

For an arbitrary connected reductive group G we consider the motivic integral over the arc space of an arbitrary $\mathbb{Q}$-Gorenstein horospherical G-variety X_Σ associated with a colored fan Σ and prove a formula for the stringy E-function of X_Σ which generalizes the one for toric varieties. We remark that in contrast to toric varieties the stringy E-function of a Gorenstein horospherical variety X_Σ may be not a polynomial if some cones in Σ have nonempty sets of colors. Using the stringy E-function, we can formulate and prove a new smoothness criterion for locally factorial horospherical varieties. We expect that this smoothness criterion holds for arbitrary spherical varieties.

Introduction

Throughout the paper, we consider algebraic varieties and algebraic groups over the ground field $\mathbb{C}$.

Let G be a connected reductive group and $H \subseteq G$ a closed subgroup. The homogeneous space G/H is called *horospherical* if H contains a maximal unipotent subgroup $U \subseteq G$. In this case, the normalizer $N_G(H)$ is a parabolic subgroup $P \subseteq G$ and P/H is an algebraic torus T. The horospherical homogeneous space G/H can be described as a principal torus bundle with the fiber T over the projective homogeneous space G/P. The dimension r of the torus T is called the *rank* of the horospherical homogeneous space G/H. Let M be the lattice of characters of the torus T, and $N = \mathrm{Hom}(M, \mathbb{Z})$ the dual lattice. According to the Luna-Vust theory [LV83], any G-equivariant embedding $G/H \hookrightarrow X$ of a horospherical homogeneous space G/H can be described combinatorially by a colored fan Σ in the r-dimensional vector space $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. In the case $H = U$, G-equivariant embeddings of G/U have been considered independently by Pauer [Pau81, Pau83]. Equivariant embeddings of horospherical homogeneous spaces are generalizations of the well-known toric varieties which are torus embeddings $T \hookrightarrow X$ ($G = T$, $H = \{e\}$).

Our paper is motivated by some known formulas for stringy invariants of toric varieties. Let X be a $\mathbb{Q}$-Gorenstein toric variety defined by a fan $\Sigma \subset N_\mathbb{R}$ and denote by $|\Sigma| \subset N_\mathbb{R}$ its support. Then there is a piecewise linear function $\omega_X : |\Sigma| \to \mathbb{R}$ such that its restriction to every cone $\sigma \in \Sigma$ is linear and ω_X has value -1 on all primitive lattice generators of 1-dimensional faces of σ. It was shown in [Ba98] that the stringy E-function of the toric variety X can be computed by the formula

$$E_{\mathrm{st}}(X; u, v) := \big(uv - 1\big)^r \sum_{n \in |\Sigma| \cap N} (uv)^{\omega_X(n)}. \tag{1}$$

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If X is smooth and projective, then the stringy E-function of X coincides with the usual E-function,

$$E(X;u,v) = \sum_{i=1}^{r} b_{2i}(X)(uv)^i\,,$$

where $b_{2i}(X)$ is the $2i$-th Betti number of X. Using the decomposition of X into torus orbits, we can compute $E(X;u,v)$ by the formula,

$$E(X;u,v) = \sum_{\sigma\in\Sigma}(uv-1)^{r-\dim\sigma} = \big(uv-1\big)^r \sum_{\sigma\in\Sigma}\frac{(-1)^{\dim\sigma}}{(1-uv)^{\dim\sigma}}\,.$$

Hence,

$$\sum_{n\in N}(uv)^{\omega_X(n)} = \sum_{\sigma\in\Sigma}\frac{(-1)^{\dim\sigma}}{(1-uv)^{\dim\sigma}} = (-1)^r P(R_\Sigma, uv) = (-1)^r\,\frac{\sum_{i=1}^r b_{2i}(X)(uv)^i}{(1-uv)^r}\,,$$

where $P(R_\Sigma,t) = \sum_{i\geqslant 0}\dim R_\Sigma^i\, t^i$ is the Poincaré series of the graded Stanley-Reisner ring $R_\Sigma = \bigoplus_{i\geqslant 0} R_\Sigma^i$ associated with the fan Σ.

Recall the definition of the Stanley-Reisner ring R_Σ. Let $e_1,\dots,e_s$ be the primitive integral generators of all 1-dimensional cones in Σ. We consider the polynomial ring $\mathbb{C}[z_1,\dots,z_s]$ whose the variables $z_1,\dots,z_s$ are in bijection to lattice vectors $e_1,\dots,e_s$. Then the *Stanley-Reisner ring* R_Σ is the quotient of $\mathbb{C}[z_1,\dots,z_s]$ by the ideal generated by those square free monomials $z_{i_1}\dots z_{i_k}$ such that the lattice vectors $e_{i_1}\dots e_{i_k}$ do not generate any k-dimensional cone in Σ. The cohomology ring $H^*(X,\mathbb{C})$ of the smooth projective toric variety X associated with Σ is isomorphic to the quotient of R_Σ modulo the ideal generated by a regular sequence $f_1,\dots,f_r$ in R_Σ^1 (see e.g. [D78, Theorem 10.8]).

In this paper, we prove a similar to (1) formula for any $\mathbb{Q}$-Gorenstein horospherical variety X defined by a colored fan Σ:

$$E_{\mathrm{st}}(X;u,v) := E(G/H;u,v)\sum_{n\in|\Sigma|\cap N}(uv)^{\omega_X(n)}\,, \tag{2}$$

where $\omega_X : |\Sigma| \to \mathbb{R}$ is a certain Σ-piecewise linear function (cf. Theorem 4.3). Let X be a complete and locally factorial horospherical variety defined by a colored cone Σ. Let $e_1,\dots,e_s$ be the primitive integral generators of all 1-dimensional cones in Σ. Consider the positive integers $a_i := -\omega_X(e_i)$ for $i\in\{1,\dots,s\}$ and define the *weighted Stanley-Reisner ring* R_Σ^w corresponding to the colored fan Σ by putting $\deg z_i = a_i$ in the standard Stanley-Reisner ring R_Σ (here we consider Σ as an uncolored fan). In Proposition 6.1, we prove that

$$\sum_{n\in N}(uv)^{\omega_X(n)} = (-1)^r P(R_\Sigma^w, uv) = (-1)^r\sum_{\sigma\in\Sigma}\frac{(-1)^{\dim\sigma}}{\prod_{e_i\in\sigma}\big(1-(uv)^{a_i}\big)}\,,$$

where $P(R_\Sigma^w,t)$ is the Poincaré series associated with the weighted Stanley-Reisner ring R_Σ^w. So we get

$$E_{\mathrm{st}}(X;u,v) = (-1)^r E(G/H;u,v)P(R_\Sigma^w,uv)\,.$$

In contrast to toric varieties, the stringy E-function of a locally factorial horospherical variety X needs not be a polynomial. If X is smooth, then $E_{\mathrm{st}}(X;u,v) = E(X;u,v)$ is polynomial and in particular the *stringy Euler number*, $e_{\mathrm{st}}(X) := E_{\mathrm{st}}(X;1,1)$, is equal to the usual Euler number $e(X) := E(X;1,1)$. If X is a locally factorial horospherical variety whose closed orbits are projective, then we show that $e_{\mathrm{st}}(X) \geqslant e(X)$ and that the equality holds if and only if X is smooth (cf. Theorem 5.3). We conjecture that the equality

$$e_{\mathrm{st}}(X) = e(X)$$

can be used as a smoothness criterion for arbitrary locally factorial spherical varieties (cf. Conjecture 6.7).

The key idea behind the formula (2) for toric varieties is the isomorphism

$$T(\mathcal{K})/T(\mathcal{O}) \simeq N,$$

where $\mathcal{O} := \mathbb{C}[[t]]$, $\mathcal{K} := \mathbb{C}((t))$ and $T(\mathcal{O})$ (resp. $T(\mathcal{K})$) denotes the set of $\mathcal{O}$-valued (resp. $\mathcal{K}$-valued) points in T. We remark that the *stringy motivic integral* over the arc space $X(\mathcal{O})$ of a toric variety X is equal to its restriction to the arc space $T(\mathcal{K})$. The latter contains countably many orbits of the maximal compact subgroup $T(\mathcal{O}) \subset T(\mathcal{K})$ that are parametrized by the elements n of the lattice N. The stringy motivic integral over a $T(\mathcal{O})$-orbit corresponding to an element $n \in N$ is equal to $(\mathbb{L}-1)^r \mathbb{L}^{\omega_X(n)}$ where $(\mathbb{L}-1)^r$ is the stringy motivic volume of the torus T and $\mathbb{L}$ is the class of the affine line in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ of algebraic varieties. Our approach in the proof of the formula (2) is to use a more general bijection

$$G(\mathcal{O}) \setminus (G/H)(\mathcal{K}) \simeq N$$

which holds for any horospherical homogeneous space G/H, see [LV83] and [GN10].

The paper is organized as follows.

Section 1 contains a review of known facts about the spaces of arcs of algebraic varieties and their relation to motivic integrals and stringy E-functions. In Section 2, we collect basic results on horospherical embeddings. In Section 3, we prove that there is a bijection between the quotient by $G(\mathcal{O})$ of the intersection $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$ and the set of lattice points $|\Sigma| \cap N$ for any horospherical G/H-embedding (cf. Theorem 3.1). Section 4 is devoted to the formula which expresses the stringy motivic volume of any $\mathbb{Q}$-Gorenstein horospherical variety as a sum over lattice points $n \in N \cap |\Sigma|$ (cf. Theorem 4.3). We use this formula to obtain a smoothness criterion for locally factorial horospherical embeddings in Section 5 (Theorem 5.3). Section 6 contains some applications, examples, open questions and a conjecture related to our results.

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1. Arc spaces, motivic integration and stringy motivic volumes

Interesting invariants of a singular algebraic variety X can be obtained via the nonarchimedean motivic integration over its space of arcs $\mathcal{J}_\infty(X)$.

Here we recall the basic definitions on the arc space of an algebraic variety and refer the reader to [DL99], [M01] or [EM05] for more details concerning this topic. Let X be an algebraic variety over $\mathbb{C}$. For any $m \geqslant 0$, we denote by $\mathcal{J}_m(X)$ the m-th *jet scheme* of X over $\mathbb{C}$ whose $\mathbb{C}$-valued points are all morphisms of schemes $\operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to X$. One has $\mathcal{J}_0(X) = X$ and

$\mathcal{J}_1(X) = \mathrm{T}X$ is the total space of the tangent bundle over X. For $m \geqslant n$, the natural ring homomorphism $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{n+1})$ induces truncation morphisms

$$\pi_{m,n} : \mathcal{J}_m(X) \longrightarrow \mathcal{J}_n(X).$$

The truncation morphisms form a projective system whose projective limit is an infinite dimensional scheme $\mathcal{J}_\infty(X)$ over $\mathbb{C}$. The scheme $\mathcal{J}_\infty(X)$ is called the *arc space* of X, and the $\mathbb{C}$-valued points of $\mathcal{J}_\infty(X)$ are all morphisms $\operatorname{Spec}\mathbb{C}[[t]] \to X$. For each m, there is a natural morphism

$$\pi_m : \mathcal{J}_\infty(X) \longrightarrow \mathcal{J}_m(X)$$

induced by the ring homomorphism $\mathbb{C}[[t]] \to \mathbb{C}[[t]]/(t^{m+1}) \simeq \mathbb{C}[t]/(t^{m+1})$.

The motivic integration over the arc space of a smooth variety is due to Kontsevich [Kon95]. One of its generalizations for singular varieties was suggested by Denef and Loeser in [DL99]. Another generalization motivated by stringy invariants was proposed in [Ba98]; see also [Cr04] and [V06].

Let $\mathrm{Var}_{\mathbb{C}}$ be the category of complex algebraic varieties and denote by $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ the Grothendieck ring of $\mathrm{Var}_{\mathbb{C}}$. For an element X in $\mathrm{Var}_{\mathbb{C}}$ we denote by $[X]$ its class in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$. The symbol $\mathbb{L}$ stands for the class of the affine line $\mathbb{A}^1$ and we denote by 1 the class of $\operatorname{Spec}\mathbb{C}$. For example,

$$[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

The map $X \mapsto [X]$ naturally extends to the category of constructible algebraic sets. There is a natural function, $\dim : \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z} \cup \{\infty\}$, which can be extended to the localization $\mathcal{M}_{\mathbb{C}} := \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ of $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ with respect to $\mathbb{L}$ simply by setting $\dim(\mathbb{L}^{-1}) := -1$. For any $m \in \mathbb{Z}$, set $F^m\mathcal{M}_{\mathbb{C}} := \{\tau \in \mathcal{M}_{\mathbb{C}} \mid \dim\tau \leqslant m\}$. Then $\{F^m\mathcal{M}_{\mathbb{C}}\}_{m\in\mathbb{Z}}$ is a decreasing filtration of $\mathcal{M}_{\mathbb{C}}$ and we denote by $\hat{\mathcal{M}}_{\mathbb{C}}$ the separated completion of $\mathcal{M}_{\mathbb{C}}$ with respect to this filtration.

Let X be a d-dimensional smooth variety.

Definition 1.1. A subset C in $\mathcal{J}_\infty(X)$ is called *a cylinder* if there are $m \in \mathbb{N}$ and a constructible subset $B_m \subseteq \mathcal{J}_m(X)$ such that $C = \pi_m^{-1}(B_m)$. Such a set B_m is called a *m-base* of C.

If $C \subseteq \mathcal{J}_\infty(X)$ is a cylinder with m-base $B_m \subseteq \mathcal{J}_m(X)$, we define its *motivic measure* $\mu_X(C)$ by

$$\mu_X(C) := [B_m]\mathbb{L}^{-md} = [\pi_m(C)]\mathbb{L}^{-md} \in \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}}).$$

This definition does not depends on m: Indeed, because X is smooth, the map

$$\pi_{n,m} : \pi_n(C) \to \pi_m(C)$$

is a locally trivial $\mathbb{A}^{(n-m)d}$-bundle for any $n \geqslant m$. The collection of cylinders forms an algebra of sets which means that $\mathcal{J}_\infty(X)$ is a cylinder and if C, C' are cylinders, then also are $\mathcal{J}_\infty(X) \smallsetminus C$ and $C \cap C'$. On the set on cylinders, the measure μ_X is additive on finite disjoint unions. Furthermore, for cylinders $C \subseteq C'$, one has $\dim\mu_X(C) \leqslant \dim\mu_X(C')$.

Definition 1.2. A subset $C \subset \mathcal{J}_\infty(X)$ is called *measurable* if for all $n \in \mathbb{N}$ there is a cylinder C_n and cylinders $D_{n,i}$ for $i \in \mathbb{N}$ such that

$$C \bigtriangleup C_n \subseteq \bigcup_{i\in\mathbb{N}} D_{n,i}$$

and $\dim\mu_X(D_{n,i}) \leqslant -n$ for all i. Here $C \bigtriangleup C_n = (C \smallsetminus C_n) \cup (C_n \smallsetminus C)$ denotes the symmetric difference of two sets.

If C is measurable, we define its *motivic measure* $\mu_X(C)$ by

$$\mu_X(C) := \lim_{n\to\infty} \mu_X(C_n).$$

This limit converges in $\hat{\mathcal{M}}_{\mathbb{C}}$ and is independent of the C_n's, cf. [Ba98, Theorem 6.18].

PROPOSITION 1.3 [Ba98, Prop. 6.19 and 6.22]. (i) *The measurable sets form an algebra of sets and the motivic measure μ_X is additive on finite disjoint unions. If $(C_i)_{i\in\mathbb{N}}$ is a disjoint sequence of measurable sets such that $\lim_{i\to\infty}\mu_X(C_i) = 0$, then $C := \bigcup_{i\in\mathbb{N}} C_i$ is measurable and*

$$\mu_X(C) = \sum_{i\in\mathbb{N}} \mu_X(C_i).$$

(ii) *Let $Y \subseteq X$ be a locally closed subvariety. Then $\mathcal{J}_\infty(Y)$ is a measurable subset of $\mathcal{J}_\infty(X)$ and if $\dim Y < \dim X$ then $\mu_X(\mathcal{J}_\infty(Y)) = 0$.*

DEFINITION 1.4. A function $F : \mathcal{J}_\infty(X) \to \mathbb{Z}\cup\{+\infty\}$ is called *measurable* if $F^{-1}(s)$ is measurable for all $s \in \mathbb{Z}\cup\{+\infty\}$.

Let $A \subseteq \mathcal{J}_\infty(X)$ be a measurable set and $F : \mathcal{J}_\infty(X) \to \mathbb{Z}\cup\{+\infty\}$ a measurable function such that $\mu_X(F^{-1}(+\infty)) = 0$. Then we set

$$\int_A \mathbb{L}^{-F} \mathrm{d}\mu_X := \sum_{s\in\mathbb{Z}} \mu_X(A\cap F^{-1}(s))\mathbb{L}^{-s}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$ whenever the right hand side converges in $\hat{\mathcal{M}}_{\mathbb{C}}$. In this case, we say that $\mathbb{L}^{-F}$ is *integrable on A*. To any subvariety Y of X, one associates the *order function*

$$\mathrm{ord}_Y : \mathcal{J}_\infty(X) \to \mathbb{N}\cup\{\infty\}$$

sending an arc $\nu \in \mathcal{J}_\infty(X)$ to the order of vanishing of ν along Y. An important example of an integrable function is the function $\mathbb{L}^{-\mathrm{ord}_Y}$ where Y is a smooth hypersurface in X.

We consider now the case where X is a singular normal irreducible variety. Let K_X be a canonical divisor of X. Assume that X is $\mathbb{Q}$-Gorenstein, that is mK_X is Cartier for some $m \in \mathbb{N}$. Let $f : X' \to X$ be a resolution of singularities of X such that the exceptional locus of f is a divisor whose irreducible components $D_1, \dots, D_l$ are smooth divisors with only normal crossings, and set

$$K_{X'/X} := K_{X'} - f^*K_X = \sum_{i=1}^{l} \nu_i D_i,$$

where the rational numbers ν_i $(1 \leqslant i \leqslant l)$ are called the discrepancies of f. The rational numbers ν_i $(1 \leqslant i \leqslant l)$ can be computed as follows. Since mK_X is Cartier, we can consider $f^*(mK_X)$ as a pullback of the Cartier divisor and write

$$mK_{X'} - f^*(mK_X) = \sum_{i=1}^{l} n_i D_i$$

with $n_i \in \mathbb{Z}$ for all i. Then $K_{X'/X}$ can be viewed as an abbreviation of the $\mathbb{Q}$-divisor $\sum\limits_{i=1}^{l} \nu_i D_i$ where $\nu_i := \frac{n_i}{m}$ for all i. Assume further that X has at worst log-terminal singularities, that is

$\nu_i > -1$ for all i (cf. [KMM87]). Set $I := \{1, \dots, l\}$ and for any subset $J \subseteq I$,

$$D_J := \begin{cases} \bigcap_{j \in J} D_J & \text{if } J \neq \varnothing \\ Y & \text{if } J = \varnothing \end{cases} \quad \text{and} \quad D_J^0 := D_J \smallsetminus \bigcup_{j \in I \smallsetminus J} D_j .$$

Definition 1.5. We define the *stringy motivic volume* $\mathcal{E}_{\rm st}(X)$ of X by

$$\mathcal{E}_{\rm st}(X) := \sum_{J \subseteq \{1,\dots,l\}} [D_J^0] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_j + 1} - 1} \in \hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{\frac{1}{m}}).$$

(In [V06], the element $\mathcal{E}_{\rm st}(X)$ is also called the *stringy $\mathcal{E}$-invariant* of X.)

The inequality $\nu_i > -1$ for any i implies that the function $\mathrm{ord}_{K_{X'/X}} := \sum_{i=1}^{l} \nu_i \,\mathrm{ord}_{D_i}$ is integrable on $\mathcal{J}_\infty(X')$, see [Ba98, Theorem 6.28]. So we can express $\mathcal{E}_{\rm st}(X)$ as a motivic integral:

Proposition 1.6.

$$\mathcal{E}_{\rm st}(X) = \int_{\mathcal{J}_\infty(X')} \mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}} \, \mathrm{d}\mu_{X'} \in \hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{\frac{1}{m}}).$$

The crucial point is that the above expressions of $\mathcal{E}_{\rm st}(X)$ do not depend on the chosen resolution, see [Ba98, Theorem 3.4]. This relevant fact essentially comes from the transformation rule for motivic integrals, see [DL99].

Recall that the *E-polynomial* of an arbitrary d-dimensional complex algebraic variety Z is defined by

$$E(Z; u, v) := \sum_{p,q=0}^{d} \sum_{i=0}^{2d} (-1)^i h^{p,q}(\mathrm{H}_c^i(Z; \mathbb{C})) u^p v^q ,$$

where $h^{p,q}(\mathrm{H}_c^i(Z;\mathbb{C}))$ $(0 \leqslant i \leqslant 2d)$ is the dimension of the (p,q)-type Hodge component in the i-th cohomology group $\mathrm{H}_c^i(Z;\mathbb{C})$ with compact support. The polynomial E has properties similar to the ones of the usual Euler characteristic. In particular, the map $Z \mapsto E(Z;u,v)$ factors through the ring $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$. The map $Z \mapsto E(Z;u,v)$ extends to $\mathcal{M}_{\mathbb{C}}$ by setting $E(\mathbb{L}^{-1};u,v) := (uv)^{-1}$. So, we get a map from $\mathcal{M}_{\mathbb{C}}$ to $\mathbb{Z}[u, v, (uv)^{-1}]$ which uniquely extends to $\hat{\mathcal{M}}_{\mathbb{C}}$. This extension will be again denoted by E.

Definition 1.7. The *stringy E-function* of X is given by (cf. [Ba98]):

$$E_{\rm st}(X; u, v) := \sum_{J \subseteq \{1,\dots,l\}} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{\nu_j + 1} - 1} .$$

Note that $E_{\rm st}(X;u,v) = E(\mathcal{E}_{\rm st}(X);u,v)$.

Remark 1.8. Whenever X is smooth, then $\mathcal{E}_{\rm st}(X) = \mu_X(\mathcal{J}_\infty(X)) = [X]$ and $E_{\rm st}(X;u,v) = E(X;u,v)$.

2. Horospherical varieties

In this section, we use our notations from the introduction: G is a connected reductive group over $\mathbb{C}$, $H \subset G$ is a closed horospherical subgroup, G/H is the corresponding horospherical homogeneous space, U is a maximal unipotent subgroup in G such that $U \subseteq H$, $B := N_G(U)$ is the corresponding Borel subgroup of G, $P := N_G(H)$ is a parabolic subgroup, $T := P/H$ a r-dimensional algebraic torus, M is the group of characters of T, and $N := \mathrm{Hom}(M, \mathbb{Z})$.

Let S be the set of simple roots of (G, B) with respect to a maximal torus of B. There is a bijective map $I \mapsto P_I$ sending a subset I of S to the parabolic subgroup P_I of G containing B such that $P_I = BW_IB$, where $W_I \subseteq W$ is the subgroup of the Weyl group $W = W_S$ generated by the reflections s_α $(\alpha \in I)$. In particular, one has $P_\varnothing = B$ and $P_S = G$. From now on, we denote by I the subset of S corresponding to $P := N_G(H)$. Let $U_0 \subset G/P$ be the open dense B-orbit. Then U_0 is isomorphic to an affine space and the Picard group of G/P is free generated by the classes $[\mathit{\Gamma}_\alpha]$ of irreducible components $\{\mathit{\Gamma}_\alpha \mid \alpha \in S \smallsetminus I\}$ in the complement $(G/P) \smallsetminus U_0$. The space of global sections $H^0(G/P, \mathcal{O}(\mathit{\Gamma}_\alpha))$ is an irreducible representation of the universal cover of the semisimple group $G' := [G, G]$ corresponding to the fundamental weight ϖ_α associated with $\alpha \in S \smallsetminus I$. Let $\phi : G/H \to G/P$ be the canonical surjective morphism whose fibers are isomorphic to the torus T. Then the divisors $\mathit{\Delta}_\alpha := \phi^{-1}(\mathit{\Gamma}_\alpha)$, for $\alpha \in S \smallsetminus I$, are exactly the irreducible components in the complement to the open dense B-orbit $\widetilde{U}_0 \simeq U_0 \times T$ in G/H. The lattice M can be identified with the group $\mathbb{C}[\widetilde{U}_0]^*/\mathbb{C}^*$ of invertible regular functions over $\widetilde{U}_0$ modulo nonzero constant functions.

Definition 2.1. A normal G-variety X is said to be *horospherical* if G has an open orbit in X isomorphic to the horospherical homogeneous space G/H. In that case, X is also called a *G/H-embedding*.

Horospherical varieties are special examples of spherical varieties. According to the Luna-Vust theory [LV83], any G/H-embedding X can be described by a colored fan Σ in the r-dimensional vector space $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Our basic reference for spherical varieties is [Kn91]. For recent accounts about horospherical varieties, see also [Pas07, Chap. 1] or [T11, Chap. 5].

Let X be a horospherical G/H-embedding. Each irreducible divisor D in X defines a valuation $v_D : \mathbb{C}(X)^* \to \mathbb{Z}$ on the function field $\mathbb{C}(X)$ which vanishes on $\mathbb{C}^*$. The restriction of v_D to the lattice $M \simeq \mathbb{C}[\widetilde{U}_0]^*/\mathbb{C}^*$ yields an element ϱ_D of the dual lattice N.

Let $\mathcal{X}(P)$ be the character group of the parabolic subgroup $P = P_I$. This group can be identified with the set of all characters $\chi \in \mathcal{X}(B)$ of the Borel subgroup B such that $\langle \chi, \check{\alpha} \rangle = 0$ for all $\alpha \in I$ where $\check{\alpha} \in \mathrm{Hom}(\mathcal{X}(B), \mathbb{Z})$ denotes the coroot corresponding to α. Since every character of P induces a line bundle over G/P, we get a homomorphism $\mathcal{X}(P) \to \mathrm{Pic}(G/P)$. Its composition with the monomorphism of character groups $M \to \mathcal{X}(P)$, induced by the epimorphism $P \to T = P/H$, gives a homomorphism $\delta : M \to \mathrm{Pic}(G/P)$. Let $\delta^* : \mathrm{Pic}(G/P)^* \to N$ be the dual map. Then, the lattice points $\{\varrho_{\mathit{\Delta}_\alpha} \mid \alpha \in S \smallsetminus I\} \subset N$ corresponding to the divisors $\mathit{\Delta}_\alpha \subset X$, $\alpha \in S \smallsetminus I$, are exactly the δ^*-images of the dual basis to $\{[\mathit{\Gamma}_\alpha] \mid \alpha \in S \smallsetminus I\}$ in $\mathrm{Pic}(G/P)^*$. For simplicity, we set $\varrho_\alpha := \varrho_{\mathit{\Delta}_\alpha}$ for any $\alpha \in S \smallsetminus I$. We note that ϱ_α is equal to the restriction to the sublattice $M \subseteq \mathcal{X}(B)$ of the corresponding coroot $\check{\alpha}$.

Let $\mathcal{D}_X = \{D_1, \ldots, D_t\}$ be the set of G-stable irreducible divisors of X. For any divisor D_i, we denote by ϱ_i the lattice point $\varrho_{D_i} \in N$. Thus, we get a map

$$\varrho : \{\mathit{\Delta}_\alpha \mid \alpha \in S \smallsetminus I\} \cup \mathcal{D}_X \to N$$

which sends $\mathit{\Delta}_\alpha$ $(\alpha \in S \smallsetminus I)$ to ϱ_α and $D_i \in \mathcal{D}_X$ $(1 \leqslant i \leqslant t)$ to ϱ_i. The restriction of ϱ to $\mathcal{D}_X$ is injective, but in general the restriction of ϱ to $\{\mathit{\Delta}_\alpha \mid \alpha \in S \smallsetminus I\}$ is not injective.

Let Z be a G-orbit in X. Denote by X_Z the union of all G-orbits in X which contain Z in their closure. Then X_Z is open in X. Moreover, X_Z is a G/H-embedding having Z as a unique closed G-orbit. Such a G/H-embedding is called *simple*. It is well-known that any simple embedding is quasi-projective. This fact follows from a result of Sumihiro [S74, Lemma 8] which states that any normal G-variety is covered by G-invariant quasi-projective open subsets (if X is a simple embedding of G/H with closed G-orbit Y, then any G-stable open neighborhood of Y in X is the whole X). The *colored cone corresponding to* Z is the pair $(\sigma_Z, \mathcal{F}_Z)$ where $\mathcal{F}_Z$ is the set $\{\alpha \in S \setminus I \mid \overline{\Delta_\alpha} \supset Z\}$ and σ_Z is the convex cone in $N_\mathbb{R}$ generated by $\{\varrho_\alpha \mid \alpha \in \mathcal{F}_Z\}$ and $\{\varrho_i \mid D_i \supset Z\}$. The *colored fan* Σ *of* X is the collection of the colored cones $(\sigma_Z, \mathcal{F}_Z)$ where Z runs through the set of G-orbits of X. We call $\mathcal{F} := \bigcup \mathcal{F}_Z$ the set of *colors of* X.

The set of colored cones in the colored fan Σ is a partially ordered set: We write $(\sigma', \mathcal{F}') \leqslant (\sigma, \mathcal{F})$ and call $(\sigma', \mathcal{F}')$ a *face* of $(\sigma, \mathcal{F})$ if σ' is a face of σ and $\mathcal{F}' = \{\alpha \in \mathcal{F} \mid \varrho_\alpha \in \sigma'\}$. On the other hand, we have a partial order on the set of orbits, $\big(Z \leqslant Z' \iff Z \subseteq \overline{Z'}\big)$, and the map $Z \mapsto (\sigma_Z, \mathcal{F}_Z)$ is an order-reversing bijection between the set of orbits of X and the set of colored cones, [Kn91]. Denote by $Z_{\sigma,\mathcal{F}}$ the G-orbit of X corresponding to $(\sigma, \mathcal{F})$. The open orbit G/H corresponds to the cone $(0, \varnothing)$.

A arbitrary pair $(\sigma, \mathcal{F})$ consisting of a convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$ and a subset $\mathcal{F} \subset S \setminus I$ is said to be a *strictly convex colored cone* if σ is strictly convex (i.e. $-\sigma \cap \sigma = 0$) and if ϱ_α is a nonzero element in σ for any $\alpha \in \mathcal{F}$. A *colored fan* $\Sigma \subset N_\mathbb{R}$ is a collection of strictly convex colored cones such that all faces of any colored cone $(\sigma, \mathcal{F}) \in \Sigma$ belong to Σ and the intersection of two colored cones is a common face of both cones, [Kn91, Section 3]. The following result was proved by Luna-Vust in a more general context, [LV83, Proposition 8.10] (see also [Kn91, Theorem 3.3]):

Theorem 2.2. *The correspondence $X \to \Sigma$ is a bijection between G-equivariant isomorphism classes of G/H-embeddings X and colored fans Σ in $N_\mathbb{R}$.*

We denote by X_Σ the G-equivariant G/H-embedding corresponding to a colored fan $\Sigma \subset N_\mathbb{R}$. For simplicity, we denote X_Σ by $X_{\sigma,\mathcal{F}}$ whenever Σ has only one maximal colored cone $(\sigma, \mathcal{F})$. The latter happens if and only if X has a unique closed G-orbit, i.e., X is simple.

A horospherical G/H-embedding X whose fan Σ has no colors is said to be *toroidal*. There is a simple method to construct a toroidal horospherical variety associated with the (uncolored) fan Σ. One considers the toric T-embedding Y_Σ with fan Σ. Using the canonical epimorphism $P \to T$ we can consider Y_Σ as a P-variety. Then X_Σ is isomorphic to the quotient space $(G \times Y_\Sigma)/P$ where the action of P on $G \times Y_\Sigma$ is given by $p(g, y) := (gp^{-1}, py)$ for any $p \in P$, $g \in G$ and $y \in Y_\Sigma$. One has a natural surjective morphism $\phi : X_\Sigma \to G/P$ whose fibers are isomorphic to the toric variety Y_Σ and $X \simeq X_\Sigma$. Over the open dense B-orbit U_0 in G/P the fibration $\phi : \phi^{-1}(U_0) \to U_0$ is trivial. Every toroidal horospherical variety is obtained as $(G_\Sigma \times Y)/P$ for a unique toric variety Y_Σ. Moreover, X_Σ is simple if and only if Y_Σ is affine.

Each horospherical variety is dominated by a toroidal variety in the following sense [Br91, §3.3]:

Proposition 2.3. *For any horospherical G-variety X, there is a toroidal G-variety $\tilde{X}$ and a proper birational G-equivariant morphism*

$$f : \tilde{X} \to X.$$

To obtain this toroidal variety $\tilde{X}$, we just need to remove all colors from all colored cones in the fan of X. It is worth mentioning that $\tilde{X} = (G \times Y)/P$, where Y denotes the closure of T in X.

In general, the toroidal variety $\tilde{X}$ is not smooth, but its singularities are locally isomorphic to toric singularities. In the sequel, it will useful to use a resolution of singularities $f' : X' \to X$, where f' if a proper birational G-equivariant morphism and where X' is a smooth toroidal G-equivariant embedding with (uncolored) fan Σ' obtained from Σ by removing colors in all colored cones of Σ and subdividing them into subcones generated by parts of $\mathbb{Z}$-bases of the lattice N. Note that the fans Σ' and Σ share the same support $|\Sigma|$.

Next proposition describes the stabilizer of G-orbits $Z_{\sigma,\mathcal{F}}$ in the horospherical case:

Proposition 2.4. *Let X be a horospherical G/H-embedding where $P := N_G(H) = P_I$ is the parabolic subgroup corresponding to a subset $I \subseteq S$. Consider a colored cone $(\sigma, \mathcal{F}) \in \Sigma$ ($\mathcal{F} \subseteq S \setminus I$). Define the sublattice $M_\sigma := M \cap \sigma^\perp$ consisted of all elements in M that are orthogonal to $\sigma \subset N_\mathbb{R}$. Then every element $m \in M_\sigma$ defines a character χ_m of the parabolic subgroup $P_{I\cup\mathcal{F}}$, and the closed G-orbit $Z_{\sigma,\mathcal{F}} \subseteq X_{\sigma,\mathcal{F}}$ is isomorphic to $G/H_{\sigma,\mathcal{F}}$ where*

$$H_{\sigma,\mathcal{F}} := \{g \in P_{I\cup\mathcal{F}} \mid \chi_m(g) = 1 \ \ \forall m \in M_\sigma\}.$$

In particular, one has:

$$\dim Z_{\sigma,\mathcal{F}} = \operatorname{rk} M_\sigma + \dim G/P_{I\cup\mathcal{F}}.$$

Proof. First of all we recall that the nonzero elements ϱ_α ($\alpha \in \mathcal{F}$) are the restrictions of the coroots $\check{\alpha}$ to the sublattice $M \subseteq \mathfrak{X}(B)$. Since $\varrho_\alpha \in \sigma$ for all $\alpha \in \mathcal{F}$, the restriction of the coroot $\check{\alpha}$ to M_σ is zero for all $\alpha \in \mathcal{F}$. The inclusions $M_\sigma \subseteq M \subseteq \mathfrak{X}(P_I)$ imply that the restriction of the coroot $\check{\alpha}$ to M_σ is zero for all $\alpha \in I$, too. Hence, we can consider the elements of M_σ as characters of B that extend to the parabolic subgroup $P_{I\cup\mathcal{F}}$.

Without loss of generality, we can assume that $X = X_{\sigma,\mathcal{F}}$ is the simple horospherical G/H-embedding corresponding to a colored cone $(\sigma, \mathcal{F})$. Consider the proper birational G-equivariant morphism $f : X_{\sigma,\varnothing} \to X_{\sigma,\mathcal{F}}$ where $X_{\sigma,\varnothing}$ is the simple toroidal variety associated with the uncolored cone $(\sigma, \varnothing)$, i.e., $X_{\sigma,\varnothing}$ is exactly the variety $\widetilde{X_{\sigma,\mathcal{F}}}$ in the notations of Proposition 2.3.

Then the toroidal simple horospherical variety $X_{\sigma,\varnothing}$ is a fibration over G/P with the affine toric fiber Y_σ. We remark that f induces a bijection between the set of G-orbits in $X_{\sigma,\varnothing}$ and the set of G-orbits in $X_{\sigma,\mathcal{F}}$. It immediately follows from the theory of toric varieties that the closed T-orbit Z_σ in Y_σ is isomorphic to T/T_σ where the subtorus $T_\sigma \subseteq T$ is the kernel of characters of T in the sublattice $M_\sigma = M \cap \sigma^\perp$ of M. Moreover, $Z_{\sigma,\varnothing} := f^{-1}(Z_{\sigma,\mathcal{F}})$ is the closed G-orbit in $X_{\sigma,\varnothing}$ which is isomorphic to $G \times_P (T/T_\sigma)$. This implies that the closed G-orbit $Z_{\sigma,\varnothing}$ is isomorphic to $G/H_{\sigma,\varnothing}$ where

$$H_{\sigma,\varnothing} := \{g \in P = P_I \mid \chi_m(g) = 1 \ \ \forall m \in M_\sigma\}.$$

Let $z_0 \in Z_{\sigma,\varnothing}$ be a point with the stabilizer $H_{\sigma,\varnothing}$. Then the stabilizer of $f(z_0) \in Z_{\sigma,\mathcal{F}}$ is a subgroup $H_{\sigma,\mathcal{F}} \subseteq G$ containing $H_{\sigma,\varnothing}$ so that we have the isomorphism $Z_{\sigma,\mathcal{F}} \cong G/H_{\sigma,\mathcal{F}}$. We remark that all fibers of the proper birational G-equivariant morphism $f : X_{\sigma,\varnothing} \to X_{\sigma,\mathcal{F}}$ are connected and proper. In particular, f induces a proper G-equivariant surjective morphism of the G-orbits $Z_{\sigma,\varnothing} \to Z_{\sigma,\mathcal{F}}$ whose fibers are connected proper algebraic varieties isomorphic to $H_{\sigma,\mathcal{F}}/H_{\sigma,\varnothing}$. Since the horospherical subgroup $H_{\sigma,\mathcal{F}}$ contains the horospherical subgroup $H_{\sigma,\varnothing}$, the normalizer $N_G(H_{\sigma,\mathcal{F}}) =: P_1$ contains the normalizer $N_G(H_{\sigma,\varnothing}) = P$. Indeed, we have that $H_{\sigma,\varnothing} \supseteq [P,P]$ since $P/H_{\sigma,\varnothing}$ is commutative. It follows that $P_1 = B[P_1,P_1] = BH_{\sigma,\mathcal{F}} = PH_{\sigma,\mathcal{F}} \supseteq P$. Let H' be the intersection $H_{\sigma,\mathcal{F}} \cap P$. The inclusions

$$H_{\sigma,\varnothing} \subseteq H' \subseteq H_{\sigma,\mathcal{F}}$$

enable to decompose the proper morphism $f : G/H_{\sigma,\varnothing} \to G/H_{\sigma,\mathcal{F}}$ into the composition of two proper morphisms with connected fibers:

$$f_1 : G/H_{\sigma,\varnothing} \to G/H', \quad f_2 : G/H' \to G/H_{\sigma,\mathcal{F}}.$$

The inclusions

$$[P,P] \subseteq H_{\sigma,\varnothing} \subseteq H' \subset P$$

imply that the fibers of f_1 are isomorphic to a diagonalisable subgroup $H'/H_{\sigma,\varnothing}$ in the torus $P/H_{\sigma,\varnothing}$. But $H'/H_{\sigma,\varnothing}$ is connected and proper only if it consists of one point, i.e., we get $H' := H_{\sigma,\mathcal{F}} \cap P = H_{\sigma,\varnothing}$. Let $M_1 \subset \mathcal{X}(P_1)$ be the sublattice of all characters of P_1 that vanish on $H_{\sigma,\mathcal{F}}$. Since $P/[P,P]$ is a torus with the group of characters $\mathcal{X}(P)$, it follows from the properties of diagonalisable groups that there exists one-to-one correspondence between the sublattices in the group of characters $\mathcal{X}(P)$ and the closed subgroups in P containing $[P,P]$. Therefore, the equality $H_{\sigma,\mathcal{F}} \cap P = H_{\sigma,\varnothing}$ and the injectivity of the restriction map $\mathcal{X}(P_1) \to \mathcal{X}(P)$ imply that M_1 is also the sublattice of all characters of P that vanish on $H_{\sigma,\varnothing}$, i.e., we get the equality $M_1 = M_\sigma$.

It remains to show that $P_1 = P_{I\cup\mathcal{F}}$. Since P_1 contains $P = P_I$, we get $P_1 = P_J$ for some subset $J \subseteq S$ containing I. Let $\alpha \in S \smallsetminus I$. By the definition of the set of colors $\mathcal{F}$, the simple root α belongs to $\mathcal{F}$ if and only if the closure of the B-invariant divisor $\Delta_\alpha := \phi^{-1}(\Gamma_\alpha) \subset G/H$ in $X_{\sigma,\mathcal{F}}$ contains the closed orbit $Z_{\sigma,\mathcal{F}} \subseteq X_{\sigma,\mathcal{F}}$. On the other hand, the horospherical homogeneous G-space $Z_{\sigma,\mathcal{F}}$ is a torus fibration over G/P_J, and the intersection of a closed B-invariant divisor $\overline{\Delta_\alpha} \subset X_{\sigma,\mathcal{F}}$ with the closed G-orbit $Z_{\sigma,\mathcal{F}}$ is either a closed B-invariant divisor in $Z_{\sigma,\mathcal{F}}$ (which projects to a B-invariant divisor in G/P_J), or the whole G-orbit $Z_{\sigma,\mathcal{F}}$. The latter implies that $\overline{\Delta_\alpha}$ contains $Z_{\sigma,\mathcal{F}}$ (i.e., $\alpha \in \mathcal{F}$) if and only if $\alpha \in J$. So we obtain $J = I \cup \mathcal{F}$.

□

3. Arcs spaces of horospherical varieties

Let $\mathcal{K} := \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathcal{O} := \mathbb{C}[[t]]$ be the ring of formal power series. If X is a scheme of finite type over $\mathbb{C}$, denote by $X(\mathcal{K})$ and $X(\mathcal{O})$ the sets of $\mathcal{K}$-valued points and $\mathcal{O}$-valued points of X respectively. Remark that the set $X(\mathcal{O})$ coincides with the set of $\mathbb{C}$-points of the scheme $J_\infty(X)$. If X is a normal variety admitting an action of an algebraic group A, then $X(\mathcal{K})$ and $X(\mathcal{O})$ both admit a canonical action of the group $A(\mathcal{O})$ induced from the A-action on X.

The following result can be viewed as a generalization in a slightly different context of [GN10, §8.2] (see also [LV83] or [D09]):

Theorem 3.1. *Let X be a horospherical G/H-embedding defined by a colored fan Σ. We consider the two sets $X(\mathcal{O})$ and $(G/H)(\mathcal{K})$ as subsets of $X(\mathcal{K})$. Then there is a surjective map*

$$\mathcal{V} : X(\mathcal{O}) \cap (G/H)(\mathcal{K}) \longrightarrow |\Sigma| \cap N$$

whose fiber over any $n \in |\Sigma| \cap N$ is precisely one $G(\mathcal{O})$-orbit. In particular, we obtain a one-to-one correspondence between the lattice points in $|\Sigma| \cap N$ and the $G(\mathcal{O})$-orbits in $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$.

In the special case where X is a toric T-embedding, Theorem 3.1 is due to Ishii, [I04, Theorem 4.1]. In more detail, by [I04, Theorem 4.1] (and its proof), we have:

Lemma 3.2. *Let $Y := Y_\Sigma$ be a toric T-embedding defined by a fan Σ. For any $\mathcal{K}$-rational point $\lambda \in T(\mathcal{K})$, we denote by λ^* the corresponding ring homomorphism $\lambda^* : \mathbb{C}[M] \to \mathcal{K}$ and define the element n_λ of the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ as the composition of $\lambda^*|_M : M \to \mathcal{K}^*$ and the standard valuation map* $\mathrm{ord} : \mathcal{K}^* \to \mathbb{Z}$. *Then the map*

$$\nu \ : T(\mathcal{K}) \to N,\ \lambda \mapsto n_\lambda$$

induces a canonical isomorphism $T(\mathcal{K})/T(\mathcal{O}) \cong N$ and one obtains a surjective map

$$\nu \ : \ Y(\mathcal{O}) \cap T(\mathcal{K}) \to |\Sigma| \cap N,\ \lambda \mapsto n_\lambda$$

whose fiber over any $n \in |\Sigma| \cap N$ is precisely one $T(\mathcal{O})$-orbit.

The above lemma will be used in the proof of Theorem 3.1:

Proof of Theorem 3.1. Consider the canonial surjective morphism $\phi \ : \ G/H \to G/P$ whose fibers are isomorphic to the algebraic torus $T := P/H$. We consider $p_0 := [P]$ as a distinguished $\mathbb{C}$-point of G/P such that the fiber $\phi^{-1}(p_0) = T$ is the closed subvariety $P/H \subseteq G/H$.

Since G/P is a projective variety, the valuative criterion of properness implies that the natural map $(G/P)(\mathcal{O}) \to (G/P)(\mathcal{K})$ from $\mathcal{O}$-points of G/P to $\mathcal{K}$-points of G/P is bijective. It follows from the local triviality of the map $G \to G/P$ that $(G/P)(\mathcal{O}) = G(\mathcal{O})/P(\mathcal{O})$. Thus, the group $G(\mathcal{O})$ transitively acts on $G(\mathcal{O})/P(\mathcal{O}) = (G/P)(\mathcal{O}) = (G/P)(\mathcal{K})$.

Let $\lambda \in (G/H)(\mathcal{K})$ be a $\mathcal{K}$-point of G/H. Then $\phi(\lambda) \in (G/P)(\mathcal{K}) = G(\mathcal{O})/P(\mathcal{O})$. So there exists an element $\gamma \in G(\mathcal{O})$ such that $\gamma(\phi(\lambda)) = p_0 \in (G/P)(\mathbb{C}) \subset (G/P)(\mathcal{K})$. Since the morphism $\phi \ : \ G/H \to G/P$ commutes with the left G-action, the equality $\gamma(\phi(\lambda)) = p_0 = [P]$ implies that $\gamma(\lambda) \in T(\mathcal{K}) = (P/H)(\mathcal{K}) \subset (G/H)(\mathcal{K})$.

Now we set $n_\lambda := \nu(\gamma(\lambda))$ where ν is the map $T(\mathcal{K}) \to N = \mathrm{Hom}(M, \mathbb{Z}) \cong T(\mathcal{K})/T(\mathcal{O})$ defined by Lemma 3.2. It is easy to see that the lattice point n_λ does no depend on the choice of the element $\gamma \in G(\mathcal{O})$. Indeed, if $\gamma' \in G(\mathcal{O})$ is another element such that $\gamma'(\phi(\lambda)) = p_0$ then the equality $\gamma'(\phi(\lambda)) = \gamma(\phi(\lambda)) = p_0$ implies that the element $\delta := \gamma'\gamma^{-1}$ belongs to $P(\mathcal{O})$ and its image under the homomorphism $P \to T = P/H$ is contained in $T(\mathcal{O})$. So, we obtain that the $\mathcal{K}$-points $\gamma'(\lambda), \gamma(\lambda) \in T(\mathcal{K})$ define the same element $n_\lambda \in N = T(K)/T(\mathcal{O})$. Finally, we get a map $\mathcal{V} : (G/H)(\mathcal{K}) \to N$, $\lambda \mapsto n_\lambda$ which is constant on $G(\mathcal{O})$-orbits.

Denote by $\widetilde{X}$ the toroidal embedding of G/H corresponding the decolorization $\widetilde{\Sigma}$ of Σ. Let $f : \widetilde{X} \to X$ be the proper birational G-equivariant morphism as in Proposition 2.3. The valuative criterion of properness for f implies the equality

$$\widetilde{X}(\mathcal{O}) \cap (G/H)(\mathcal{K}) = X(\mathcal{O}) \cap (G/H)(\mathcal{K}).$$

Since $|\Sigma| = |\widetilde{\Sigma}|$, it remains to prove the statement only for the toroidal horospherical variety $\widetilde{X}$.

Let $Y_{\widetilde{\Sigma}}$ be the closure of the torus $T = P/H \subset G/H$ in $\widetilde{X}$. Recall that the toroidal horospherical variety $\widetilde{X}$ is a homogeneous fiber bundle $G \times_P Y_{\widetilde{\Sigma}}$ over G/P with fiber isomorphic to the toric variety $Y_{\widetilde{\Sigma}}$ (see the discussion after the Theorem 2.2 for that point). This allows to consider the set $Y_{\widetilde{\Sigma}}(\mathcal{O}) \cap T(\mathcal{K})$ as a subset of $\widetilde{X}(\mathcal{O}) \cap (G/H)(\mathcal{K})$. The restriction of $\mathcal{V}$ to $Y_{\widetilde{\Sigma}}(\mathcal{O}) \cap T(\mathcal{K})$ is exactly the map $\nu \ : \ Y_{\widetilde{\Sigma}}(\mathcal{O}) \cap T(\mathcal{K}) \to |\widetilde{\Sigma}| \cap N$ from Lemma 3.2. So the image of $\mathcal{V}$ contains $|\widetilde{\Sigma}|$.

In the toric fibration $\phi \ : \ \widetilde{X} \to G/P$ the fiber $\phi^{-1}(p_0) \subset \widetilde{X}$ is exactly the toric variety $Y_{\widetilde{\Sigma}}$ and the intersection $Y_{\widetilde{\Sigma}} \cap G/H$ is exactly the torus $T = P/H$. Since the group $G(\mathcal{O})$ acts transitively on $(G/P)(\mathcal{O}) = (G/P)(\mathcal{K})$, for any $\lambda \in \widetilde{X}(\mathcal{O}) \cap (G/H)(\mathcal{K})$ there exists an element $\gamma \in G(\mathcal{O})$ such that $\gamma(\phi(\lambda)) = p_0$. This implies that $\gamma(\lambda) \in Y_{\widetilde{\Sigma}}(\mathcal{O}) \cap T(\mathcal{K})$ and $\mathcal{V}(\lambda) = \mathcal{V}(\gamma(\lambda))$. Therefore the images of ν and $\mathcal{V}$ are the same.

It remains only to show that the fibers of $\mathcal{V}$ are precisely the $G(\mathcal{O})$-orbits. The latter follows from the $G(\mathcal{O})$-action on $X(\mathcal{O})$ and from the canonical isomorphism $G(\mathcal{O}) \setminus (G/H)(\mathcal{K}) \simeq N$ induced by $\mathcal{V}$, see e.g. [GN10, §8.2] (or [LV83]), because the subset $X(\mathcal{O}) \cap (G/H)(\mathcal{K}) \subset (G/H)(\mathcal{K})$ is $G(\mathcal{O})$-invariant. □

We assume until the end of the section that X is a smooth toroidal G/H-embedding such that every closed orbit in X is projective. This means that X corresponds to an uncolored fan Σ such that every maximal cone of Σ is generated by a $\mathbb{Z}$-basis of N. Then X is a fibration over G/P with fiber isomorphic to the smooth toric T-embedding $Y := Y_\Sigma$ and the surjective map $\phi : X \to G/P$ induces, for $m \in \mathbb{N}$, surjective morphisms $\phi_m : \mathcal{J}_m(X) \to \mathcal{J}_m(G/P)$. For any $m \in \mathbb{N}$, denote by $\pi_m : \mathcal{J}_\infty(X) \to \mathcal{J}_m(X)$ and $\pi'_m : \mathcal{J}_\infty(Y) \to \mathcal{J}_m(Y)$ the canonical projection maps. For any $n \in |\Sigma| \cap N$, denote by $\mathcal{C}_{X,n}$ (resp. $\mathcal{C}_{Y,n}$) the $G(\mathcal{O})$-orbit (resp. $T(\mathcal{O})$-orbit) of $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$ (resp. $Y(\mathcal{O}) \cap T(\mathcal{K})$) corresponding to n (see Theorem 3.1 and Lemma 3.2). As a consequence of the above proof of Theorem 3.1, we get:

Corollary 3.3. *Let $n \in |\Sigma| \cap N$ and $m \in \mathbb{N}$. Then the restriction to $\pi_m(\mathcal{C}_{X,n})$ of ϕ_m is surjective onto $\mathcal{J}_m(G/P)$ and its fiber is isomorphic to $\pi'_m(\mathcal{C}_{Y,n})$.*

We aim to calculate the motivic measure (with respect to μ_X; cf. Definition 1.1) of the $G(\mathcal{O})$-orbits in $X(\mathcal{O}) \cap (G/H)(\mathcal{K})$, the other orbits having zero measure.

Let $n \in |\Sigma| \cap N$ and let σ be a r-dimensional cone of Σ such that $n \in \sigma$. Fix a basis $\{u_1, \dots, u_r\}$ of the semi-group $\sigma^\vee \cap M$.

Lemma 3.4. *Let $q \geqslant \max(\{\langle n, u_j\rangle \mid j = 1, \dots, r\})$. In the notations of Corollary 3.3, the set $\mathcal{C}_{Y,n}$ is a cylinder with q-basis $\pi'_q(\mathcal{C}_{Y,n}) \simeq (\mathbb{A} \smallsetminus 0)^r \times \mathbb{A}^{qr - \sum_{j=1}^r \langle n, u_j\rangle}$.*

Proof. By our choice of q, for any $\nu \in \pi'_q(\mathcal{C}_{Y,n})$, the truncated arc $\pi'_q(\nu)$ can be viewed as a r-tuple $(\nu^{(1)}, \dots, \nu^{(r)})$ where

$$\nu^{(j)} = \nu^{(j)}_{\langle n,u_j\rangle} t^{\langle n,u_j\rangle} + \nu^{(j)}_{\langle n,u_j\rangle+1} t^{\langle n,u_j\rangle+1} + \cdots + \nu^{(j)}_q t^q\,; \quad j \in 1, \dots, r\,,$$

for $\nu^{(j)}_{\langle n,u_j\rangle} \in \mathbb{C}^*$ and $(\nu^{(j)}_{\langle n,u_j\rangle+1}, \dots, \nu^{(j)}_q) \in \mathbb{C}^{q-\langle n,u_j\rangle}$. Indeed, the orbit $\mathcal{C}_{Y,n}$ is the set of all arcs $\nu \in Y_\sigma(\mathcal{O}) \cap T(\mathcal{K})$ such that $n_\nu = n$ (see Lemma 3.2). So, the space of the truncated arcs $\pi'_q(\nu)$ is isomorphic to

$$(\mathbb{A} \smallsetminus 0)^r \times \mathbb{A}^{\sum_{j=1}^r (q - \langle n,u_j\rangle)} = (\mathbb{A} \smallsetminus 0)^r \times \mathbb{A}^{qr - \sum_{j=1}^r \langle n,u_j\rangle}.$$

Moreover, if $\nu \in Y(\mathcal{O})$ lies in $\pi'^{-1}_q(\pi'_q(\mathcal{C}_{Y,n}))$ then $\nu \in \mathcal{C}_{Y,n}$. Hence $\mathcal{C}_{Y,n} = \pi'^{-1}_q(\pi'_q(\mathcal{C}_{Y,n}))$ and $\mathcal{C}_{Y,n}$ is a cylinder whose q-basis is the constructible set $\pi'_q(\mathcal{C}_{Y,n})$.

□

Theorem 3.5. *We have $\mu_X(\mathcal{C}_{X,n}) = [G/H]\, \mathbb{L}^{-\sum_{j=1}^r \langle n,u_j\rangle}$.*

Proof. By Corollary 3.3 and Definition 1.1, the motivic measure of the cylinder $\mathcal{C}_{X,n} = \pi_q^{-1}(\pi_q(\mathcal{C}_{X,n}))$ of $X(\mathcal{O})$, for $q \gg 0$, is expressed by the formula:

$$\mu_X(\mathcal{C}_{X,n}) = [\pi_q(\mathcal{C}_{X,n})]\mathbb{L}^{-qd} = [\mathcal{J}_q(G/P)]\,(\mathbb{L}-1)^r\,\mathbb{L}^{qr - \sum_{j=1}^r \langle n,u_j\rangle}\,\mathbb{L}^{-qd}\,.$$

Since $\mathcal{J}_q(G/P)$ is a locally trivial $\mathbb{A}^{q(d-r)}$-bundle over G/P and $[G/P](\mathbb{L}-1)^r = [G/H]$, we get $\mu_X(\mathcal{C}_{X,n}) = [G/H]\,\mathbb{L}^{-\sum_{j=1}^{r}\langle n,u_j\rangle}$.

□

4. The stringy motivic volume of horospherical varieties

The aim of this section is to prove a formula for $\mathcal{E}_{\mathrm{st}}(X)$ for any $\mathbb{Q}$-Gorenstein horospherical embedding $G/H \hookrightarrow X$, see Theorem 4.3.

For our purpose, we need to explain the canonical class of a horospherical variety. Let $G/H \hookrightarrow X$ be a $\mathbb{Q}$-Gorenstein d-dimensional horospherical embedding. For $\alpha \in S$, denote by ϖ_α the corresponding fundamental weight of S. Let ρ_S (resp. ρ_I) be the half sum of positive roots of S (resp. I). Note that $\rho_S = \sum_{\alpha\in S}\varpi_\alpha$. For any $\alpha \in S \setminus I$, we define the integers a_α by the equality:

$$2(\rho_S - \rho_I) = \sum_{\alpha\in S\setminus I} a_\alpha \varpi_\alpha.$$

We refer to [Br93, §4.1] or [Br97, Theorem 4.2] for the following result:

Proposition 4.1. *Let X be a G/H-embedding. Then*

$$K_X = \sum_{\alpha\in S\setminus I} -a_\alpha \overline{\Delta_\alpha} + \sum_{j=1}^{t} -D_j\,,$$

where $D_1,\dots,D_t$ are the irreducible divisors in the complement of X to the dense open G-orbit, and $\overline{\Delta_\alpha}$ ($\alpha \in S\setminus I$) is the closure of Δ_α in X.

Let $\Sigma \subset N_{\mathbb{R}}$ be the colored fan corresponding to X. The $\mathbb{Q}$-Gorenstein property is equivalent to the existence of a continuous function

$$\omega_X : |\Sigma| \to \mathbb{R}$$

satisfying the following conditions (cf. [Br93, Proposition 4.1]):

(P1) $\omega_X(e_\tau) = -1$ for a primitive integral generator e_τ of an uncolored ray τ of Σ;

(P2) $\omega_X(\varrho_\alpha) = -a_\alpha$ for a colored cone $(\sigma,\mathcal{F})$ of Σ and $\alpha\in\mathcal{F}$;

(P3) ω_X is linear on each cone $\sigma\in\Sigma$.

Let $f' : X' \to X$ be a proper birational G-equivariant morphism where X' is a smooth toroidal G-equivariant embedding with (uncolored) fan Σ' obtained from Σ by removing colors and subdividing (see the discussion after the proposition 2.3). Denote by

$$K_{X'/X} := K_{X'} - f'^{*}K_X$$

the discrepancy divisor of f'.

Let $\tau'_1,\dots,\tau'_q$ be the rays of Σ' which are not rays of Σ (this set may be empty), $e_{\tau'_1},\dots,e_{\tau'_q}$ the respective primitive integral generators, and $D'_1,\dots,D'_q$ the respective irreducible G-stable divisors of X'. Let also $\tau_1,\dots,\tau_t$ be the uncolored rays of Σ and $(\tau_{t+1},\mathcal{F}_{t+1}),\dots,(\tau_s,\mathcal{F}_s)$ the colored ones. Denote by $D_1,\dots,D_s$ the irreducible G-stable divisors of X' corresponding to the rays $\tau_1,\dots,\tau_s$ of Σ'. Thus,

$$\{D'_1,\dots,D'_m\}\cup\{D_1,\dots,D_s\}$$

is the set of irreducible G-stable divisors of X'. Let $e_{\tau_1}, \dots, e_{\tau_s}$ be primitive integral generators of the rays $\tau_1, \dots, \tau_s$ of Σ' respectively.

PROPOSITION 4.2. *Assume that X is $\mathbb{Q}$-Gorenstein. Then*

$$K_{X'/X} = \sum_{i=1}^{q}(-1-\omega_X(e_{\tau_i'}))D_i' + \sum_{j=t+1}^{s}(-1-\omega_X(e_{\tau_j}))D_j.$$

Moreover, $K_{X'/X}$ is a smooth simple normal crossings Cartier divisor and X has at worst log-terminal singularities.

Proof. Since X' is smooth, there is a continuous function, $\omega_{X'} : |\Sigma'| \to \mathbb{R}$, satisfying the following conditions:

(P1′) $\omega_{X'}(e_{\tau_i'}) = \omega_{X'}(e_{\tau_j}) = -1$ for all $i = 1, \dots, q$ and $j = 1, \dots, s$;

(P2′) $\omega_{X'}$ is linear on each cone of Σ'.

Define a function $\psi : |\Sigma'| \to \mathbb{R}$ by setting $\psi(n) := \omega_{X'}(n) - \omega_X(n)$ for any $n \in N_{\mathbb{R}}$. Then ψ is a continuous map which is linear on each cone of Σ' (use properties (P3) and (P2′)). By Proposition 4.1,

$$K_{X'} = \sum_{\alpha\in S\smallsetminus I} -a_\alpha\overline{\Delta_\alpha} + \sum_{i=1}^{q} -D_i' + \sum_{j=1}^{s} -D_j \quad \text{and} \quad K_X = \sum_{\alpha\in S\smallsetminus I} -a_\alpha\overline{\Delta_\alpha} + \sum_{j=1}^{t} -D_j\,.$$

So, by the conditions (P1), (P2) and (P1′), we get

$$K_{X'/X} = \sum_{i=1}^{q}(-1-\omega_X(e_{\tau_i'}))D_i' + \sum_{j=t+1}^{s}(-1-\omega_X(e_{\tau_j}))D_j\,.$$

Since X is $\mathbb{Q}$-Gorenstein, X has at worst log-terminal singularities, see [Br93, Theorem 4.1]. At last, X' being smooth and toroidal, $K_{X'/X}$ is a smooth simple normal crossings divisor.

□

We are now in the position to state the main result of this section:

THEOREM 4.3. *Let $G/H \hookrightarrow X$ be a $\mathbb{Q}$-Gorenstein d-dimensional horospherical embedding with colored fan $\Sigma \subset N_{\mathbb{R}}$, and ω_X as above. Then*

$$\mathcal{E}_{\mathrm{st}}(X) = [G/H] \sum_{n\in|\Sigma|\cap N} \mathbb{L}^{\omega_X(n)}\,.$$

The remaining of the section is devoted to the proof of Theorem 4.3: Theorem 4.3 will be a straightforward consequence of Lemma 4.4 and Lemma 4.5. Keep the above notations and denote by $\mathcal{C}_{X',n}$ the $G(\mathcal{O})$-orbit in $X'(\mathcal{O}) \cap (G/H)(\mathcal{K})$ corresponding to $n \in |\Sigma| \cap N$ (cf. Theorem 3.1).

LEMMA 4.4. *We have:*

$$\mathcal{E}_{\mathrm{st}}(X) = \sum_{n\in|\Sigma|\cap N} \int_{\mathcal{C}_{X',n}} \mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}} \mathrm{d}\mu_{X'}.$$

Proof. Since the $G(\mathcal{O})$-orbits in $X'(\mathcal{O})$ which are not contained in $(G/H)(\mathcal{K})$ have zero motivic measure, we get by Definition 1.5:

$$\mathcal{E}_{\mathrm{st}}(X) = \int_{X'(\mathcal{O})} \mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}} \mathrm{d}\mu_{X'} = \int_{X'(\mathcal{O})\cap(G/H)(\mathcal{K})} \mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}} \mathrm{d}\mu_{X'}\,.$$

In addition, by Theorem 3.1, $X'(\mathcal{O})\cap(G/H)(\mathcal{K})$ is a countable disjoint union of $G(\mathcal{O})$-orbits and each of these $G(\mathcal{O})$-orbits corresponds to a point $n\in|\Sigma|\cap N$:

$$X'(\mathcal{O})\cap(G/H)(\mathcal{K})=\bigsqcup_{n\in|\Sigma|\cap N}\mathcal{C}_{X',n}.$$

All $\mathcal{C}_{X',n}$ are cylinders whose union is a measurable set. The lemma is then a consequence of Proposition 1.3(i).

□

LEMMA 4.5. *For any lattice point $n\in|\Sigma|\cap N$, we have*

$$\int_{\mathcal{C}_{X',n}}\mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}}\,\mathrm{d}\mu_{X'}=[G/H]\,\mathbb{L}^{\omega_X(n)}.$$

Proof. Let $(\sigma,\mathcal{F})$ be a colored cone in Σ such that σ contains n. We remark that the statement of the lemma is local. So, it is enough to prove it in the case where X is the simple horospherical variety corresponding to $(\sigma,\mathcal{F})$. Furthermore, we can assume that σ has the maximal dimension r (i.e., the unique closed G-orbit in X is projective). Otherwise we can embed σ as a face into some r-dimensional cone $\hat{\sigma}$ such that the restriction of the linear function $\omega_{\hat{X}}$ to σ coincides with ω_X and the smooth subdivision of σ extends to a smooth subdivision of $\hat{\sigma}$. Here, $\hat{X}$ is the simple horospherical G/H-embedding corresponding to the r-dimensional colored cone $(\widehat{\sigma},\mathcal{F})$. Thus, it is enough to consider the case where every maximal cone of Σ' is generated by a $\mathbb{Z}$-basis of N.

For the sake of the simplicity, we set, in the notations of Proposition 4.2: $c'_i:=-1-\omega_X(e_{\tau'_i})$, for $i\in\{1,\dots,q\}$, and $c_j:=-1-\omega_X(e_{\tau_j})$, for $j\in\{t+1,\dots,s\}$. Thus,

$$K_{X'/X}=\sum_{i=1}^{q}c'_iD'_i+\sum_{j=t+1}^{s}c_jD_j\,.$$

Let $n\in|\Sigma|\cap N$. By the definition of motivic integrals,

$$\int_{\mathcal{C}_{X',n}}\mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}}\,\mathrm{d}\mu_{X'}=\sum_{\nu\in\mathbb{Q}}\mu_{X'}(\{\lambda\in\mathcal{C}_{X',n}\mid\mathrm{ord}_{K_{X'/X}}(\lambda)=\nu\})\,\mathbb{L}^{-\nu}\,.$$

Let σ be a r-dimensional cone of Σ' containing n and generated by a basis $\{e_1,\dots,e_r\}$ of N.

Its dual basis, $\{u_1,\dots,u_r\}$, is a basis of the semi-group $\sigma^\vee\cap M$. Possibly renumbering the vectors $e_1,\dots,e_r$, we can assume that there exist $l\in\{1,\dots,q\}$ and $k\in\{1,\dots,s\}$ such that, in the notations of Proposition 4.2, $\{e_1,\dots,e_l\}$ is a part of $\{e_{\tau'_1},\dots,e_{\tau'_q}\}$, $\{e_{l+1},\dots,e_{l+k}\}$ is a part of $\{e_{\tau_1},\dots,e_{\tau_t}\}$ and $\{e_{l+k+1},\dots,e_r\}$ is a part of $\{e_{\tau_{t+1}},\dots,e_{\tau_s}\}$.

It follows from the description of $\mathcal{C}_{X',n}$ (see the proof of Lemma 3.4) that, for any $\lambda\in\mathcal{C}_{X',n}$,

$$\mathrm{ord}_{K_{X'/X}}(\lambda)=\sum_{i=1}^{l}c'_i\langle n,u_i\rangle+\sum_{j=l+k+1}^{r}c_j\langle n,u_j\rangle.$$

As a result, we get:

$$\int_{\mathcal{C}_{X',n}}\mathbb{L}^{-\mathrm{ord}_{K_{X'/X}}}\,\mathrm{d}\mu_{X'}=\mu_{X'}(\mathcal{C}_{X',n})\,\mathbb{L}^{-\sum_{i=1}^{l}c'_i\langle n,u_i\rangle-\sum_{j=l+k+1}^{r}c_j\langle n,u_j\rangle}.$$

In addition, by Theorem 3.5,

$$\mu_{X'}(\mathcal{C}_{X',n}) = [G/H]\,\mathbb{L}^{-\sum\limits_{j=1}^{r}\langle n,u_j\rangle}\,.$$

So, it only remains to show that $\omega_X(n) = -\sum\limits_{j=1}^{r}\langle n,u_j\rangle - \sum\limits_{i=1}^{l} c_i'\langle n,u_i\rangle - \sum\limits_{j=l+k+1}^{r} c_j\langle n,u_j\rangle$. By the properties (P1), (P2) and (P3) of ω_X, one has:

$$\begin{aligned}\omega_X(n) = \omega_X(\sum_{j=1}^{r}\langle n,u_j\rangle e_j) &= \sum_{i=1}^{l}\langle n,u_i\rangle\omega_X(e_i) - \sum_{j=l+1}^{l+k}\langle n,u_j\rangle + \sum_{j=l+k+1}^{r}\langle n,u_j\rangle\omega_X(e_j)\\ &= -\sum_{j=1}^{r}\langle n,u_j\rangle - \sum_{i=1}^{l} c_i'\langle n,u_i\rangle - \sum_{j=l+k+1}^{r} c_j\langle n,u_j\rangle\,.\end{aligned}$$

Then, the expected expression for $\omega_X(n)$ follows.

□

As noticed before, Lemma 4.4 together with Lemma 4.5 complete the proof of Theorem 4.3.

EXAMPLE 4.6. Consider the case where $G = SL_3(\mathbb{C})$, B is the Borel subgroup of G consisted of upper triangular matrices of G, $S = \{\beta_1, \beta_2\}$ and $H = U$. Then G/H is a quasi-affine homogeneous horospherical variety whose affine closure is the 5-dimensional affine quadric

$$Q = \{(x_1,x_2,x_3,y_1,y_2,y_3) \in \mathbb{A}^6 \mid x_1y_1 + x_2y_2 + x_3y_3 = 0\}\,;$$

Q is the affine cone over the Grassmannian $G(2,4)$. Denote by $\check{\beta}_1$ and $\check{\beta}_2$ the coroots of β_1 and β_2 respectively. The representation of $SL_3(\mathbb{C})$ on $\mathbb{A}^6$ is the sum of two fundamental 3-dimensional irreducible representations with the dominant weights ϖ_{β_1}, ϖ_{β_2} and Q has for maximal colored cone $(\sigma, \{\beta_1,\beta_2\})$ where σ is the cone of $N_{\mathbb{R}}$ generated by $\check{\beta}_1|_M$ and $\check{\beta}_2|_M$. The quadric Q admits four G-orbits: 0, two copies of $\mathbb{A}^3 \setminus 0$, and the dense orbit G/U. We have $[G/U] = (\mathbb{L}^2-1)(\mathbb{L}^3-1)$. Using this decomposition into G-orbits of Q, one gets $[Q] = \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1)$. On the other hand, by Theorem 4.3,

$$\mathcal{E}_{\mathrm{st}}(Q) = [G/U]\left(\sum_{k\geqslant 0}\mathbb{L}^{-2k}\right)^2 = \frac{(\mathbb{L}^2-1)(\mathbb{L}^3-1)}{(1-\mathbb{L}^{-2})^2} = \frac{\mathbb{L}^4(\mathbb{L}^2+\mathbb{L}+1)}{\mathbb{L}+1}\,.$$

Let us show how this result can be obtained using resolutions of singularities of Q. We consider two different resolutions: the blowing-up of the point $0 \in Q$ and a decolorization of Q.

1) Let $p : \hat{Q} \to Q$ be the blowing-up of $0 \in Q$ and D the exceptional divisor. We have $K_{\hat{Q}} - p^*K_Q = 3D$ and

$$[\hat{Q} \setminus D] = [Q] - 1 = \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1) - 1.$$

On the other hand, $D \simeq G(2,4)$ and $[D]$ can be readily computed using the Betti numbers. Then by Definition 1.5, we get:

$$\mathcal{E}_{\mathrm{st}}(Q) = [\hat{Q} \setminus D] + [D]\left(\frac{\mathbb{L}-1}{\mathbb{L}^4-1}\right) = \frac{\mathbb{L}^4(\mathbb{L}^2+\mathbb{L}+1)}{\mathbb{L}+1}\,.$$

2) Let Q' be the smooth toroidal variety corresponding to the uncolored fan obtained from Σ and $f' : Q' \to Q$ the corresponding proper birational G-morphism. Note that Q' is the

homogeneous vector bundle on G/B associated with the representation of B on $\mathbb{A}^2$ with weights the fundamental weights ϖ_{β_1}, ϖ_{β_2}. The exceptional locus of f' has two irreducible components, D_1 and D_2, and $K_{Q'/Q} = D_1 + D_2$. The set $Q' \smallsetminus (D_1 \cup D_2)$ is isomorphic to the open orbit G/U and $D_1 \smallsetminus (D_1 \cap D_2)$ is a locally trivial fibration over $\mathbb{A}^3 \smallsetminus 0$ with fiber $\mathbb{P}^1$. Moreover, $D_1 \cap D_2$ is the unique closed G-orbit which is here isomorphic to G/B. Hence, by Definition 1.5,

$$\mathcal{E}_{\rm st}(Q) = [Q' \smallsetminus (D_1 \cup D_2)] + 2\,\frac{[D_1 \smallsetminus (D_1 \cap D_2)]}{\mathbb{L}+1} + \frac{[D_1 \cap D_2]}{(\mathbb{L}+1)^2} = \frac{\mathbb{L}^4(\mathbb{L}^2+\mathbb{L}+1)}{\mathbb{L}+1}\,.$$

5. Smoothness criterion

We obtain in this section (Theorem 5.3) a smoothness criterion for locally factorial horospherical embeddings in term of their stringy Euler numbers (cf. Definition 5.2). Since the smoothness condition is a local condition, we can restrict our study to the case of simple horospherical embeddings.

Recall that a normal variety is called *locally factorial* if any Weil divisor is a Cartier divisor. The following criterion for the locally factorial condition can be readily extracted from [Br89, Proposition 3.1] and [Br93, Proposition 4.2]:

Theorem 5.1. *Let X be a simple horospherical G/H-embedding with maximal cone $(\sigma, \mathcal{F})$. Then, X is locally factorial if and only if the following two conditions are satisfied:*

(L1) *the restriction to $\{\varDelta_\alpha \mid \alpha \in \mathcal{F}\}$ of the map ϱ is injective;*

(L2) *σ is generated by part of a basis of N which contains all ϱ_α for $\alpha \in \mathcal{F}$.*

Recall that the *usual Euler number* $e(V)$ of any complex algebraic variety V is defined by

$$e(V) := E(V; 1, 1).$$

Definition 5.2. Let X be a d-dimensional normal $\mathbb{Q}$-Gorenstein variety. Adopt the notations of Definition 1.5 and define the *stringy Euler number* $e_{\rm st}(X)$ of X by

$$e_{\rm st}(X) := \sum_{J \subseteq \{1,\dots,l\}} e(D_J^0) \prod_{j \in J} \frac{1}{\nu_j + 1}.$$

The stringy E-function of X was defined in Definition 1.7. Note that $e_{\rm st}(X)$ is nothing but $E_{\rm st}(X; 1, 1)$. We refer to [Ba98] of [Ba99] for more details about the stringy Euler numbers.

Theorem 5.3. *Let X be a simple locally factorial horospherical G/H-embedding. Assume that the maximal cone associated with X has dimension r. Then one has $e_{\rm st}(X) \geqslant e(X)$, and the equality holds if and only if X is smooth.*

Our assumption that the maximal cone associated with X has dimension r means that the closed orbit of X is projective. The proof of Theorem 5.3 will be achieved at the end of the section.

Example 5.4. The affine quadric Q introduced in Example 4.6 yields an example of horospherical variety which is locally factorial but not smooth,

$$e_{\rm st}(Q) = \frac{3}{2} > e(Q) = 1.$$

Example 5.5. Here we give an example of a singular horospherical variety X for which the stringy E-function is polynomial.

Consider the case where $G = SL_4(\mathbb{C})$, B is the set of upper triangular matrices of G and set $S = \{\beta_1, \beta_2, \beta_3\}$. The representation of G on $\mathbb{C}^4 \oplus \wedge^2\mathbb{C}^4$ is the sum of two fundamental representations with the dominant weights ϖ_{β_1} and ϖ_{β_2}. The stabilizer of $(e_1, e_1 \wedge e_2) \in \mathbb{C}^4 \oplus \wedge^2\mathbb{C}^4$ in G is the horospherical subgroup $H = P_{\{\beta_3\}} \cap (\ker \varpi_{\beta_1} \cap \ker \varpi_{\beta_2})$ where (e_1, e_2, e_3, e_4) is the canonical basis of $\mathbb{C}^4$. We have $\dim G/H = 7$ and $\operatorname{rk} G/H = 2$. Let $X \subset \wedge^2\mathbb{C}^5 \simeq \mathbb{C}^4 \oplus \wedge^2\mathbb{C}^4$ be the closure of the G-orbit of $(e_1, e_1 \wedge e_2)$ in $\mathbb{C}^4 \oplus \wedge^2\mathbb{C}^4$. Then X is the affine cone over the Grassmannian $G(2,5)$ and contains three more G-orbits: $(\wedge^2\mathbb{C}^4 \smallsetminus 0)$, $(\mathbb{C}^4 \smallsetminus 0)$ and 0. From this, we get: $[X] = \mathbb{L}^7 + \mathbb{L}^5 - \mathbb{L}^2$. The maximal colored cone corresponding to X is $(\sigma, \{\beta_1, \beta_2\})$ where σ is the cone of $N_{\mathbb{R}}$ generated by $\check{\beta}_1|_M$ and $\check{\beta}_2|_M$. We have $a_{\beta_1} = 2$ and $a_{\beta_2} = 3$. Hence, by Theorem 4.3,

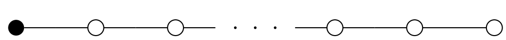
$$\mathcal{E}_{\rm st}(X) = \frac{(\mathbb{L}-1)^2\,(\mathbb{L}+1)\,(\mathbb{L}^2+1)\,(\mathbb{L}^2+\mathbb{L}+1)}{(1-\mathbb{L}^{-2})(1-\mathbb{L}^{-3})} = \mathbb{L}^5(\mathbb{L}^2+1)\,.$$

We have, $e_{\rm st}(X) = 2 > e(X) = 1$.

For $S' \subseteq S$, denote by $\Gamma_{S'}$ the Dynkin diagram corresponding to S'; the vertices of $\Gamma_{S'}$ are the elements of S'. In [Pau83, §3.5], Pauer gives a smoothness criterion for any G/H-embedding in the case where $H = U$; for the general case, see [Pas07, Theorem 2.6] or [T11, Theorem 28.10]. Recall here the criterion:

Proposition 5.6. *Let X be a simple locally factorial horospherical G/H-embedding with maximal colored cone $(\sigma, \mathcal{F})$ and let $I \subseteq S$ be such that $N_G(H) = P_I$. Then, X is smooth if and only if any connected component Γ of $\Gamma_{I \cup \mathcal{F}}$ verifies one of the following conditions:*

(C1) *Γ is a Dynkin diagram of type $\mathbf{A}_\ell$, $\ell \geqslant 1$, and Γ contains exactly one vertex in $\mathcal{F}$ which is extremal:*



(C2) *Γ is a Dynkin diagram of type $\mathbf{C}_\ell$, $\ell \geqslant 3$, and Γ contains exactly one vertex in $\mathcal{F}$ which is the simple extremal one:*

(C3) *Γ is any Dynkin diagram whose vertices are all in I.*

Example 5.7. 1) The standard representation $(\mathbb{C}^{\ell+1}, \varpi_1)$ of $G = SL_{\ell+1}(\mathbb{C})$ is a smooth affine horospherical variety corresponding to the situation (C1). Namely, the dense orbit $\mathbb{C}^{\ell+1} \smallsetminus 0$ of $\mathbb{C}^{\ell+1}$ is isomorphic to G/H where H is the kernel in the standard maximal parabolic P whose Levi part contains the α_j-root subgroups, for $j = 2, \ldots, \ell$, of the restriction to P of ϖ_1.

2) The standard representation $(\mathbb{C}^{2\ell}, \varpi_1)$ of $G = Sp_{2\ell}(\mathbb{C})$ is a smooth affine horospherical variety corresponding to the situation (C2). We have the same description of the dense orbit as in 1): The dense orbit $\mathbb{C}^{2\ell} \smallsetminus 0$ of $\mathbb{C}^{2\ell}$ is isomorphic to G/H where H is the kernel in the standard maximal parabolic P whose Levi part contains the α_j-root subgroups, for $j = 2, \ldots, \ell$, of the restriction to P of ϖ_1.

3) The case where $\mathcal{F}$ is empty (situation (C3)) corresponds to locally factorial toroidal embeddings which are known to be smooth.

We state several technical lemmas useful for the proof of Theorem 5.3. Our main reference for basics on Lie algebras and root systems is [OV90]. Assume that Γ_S is connected. Let I be a subset of S and let us introduce standard related notations.

- We denote by $\mathcal{R}$ the root system of G, by $\mathcal{R}^+$ the set of positive root of $\mathcal{R}$, by $\mathcal{R}_I$ the root subsystem of $\mathcal{R}$ generated by I and by $\mathcal{R}_I^+$ the set $\mathcal{R}_I \cap \mathcal{R}^+$.

- For any $\gamma \in \mathcal{R}$, we denote by $\check{\gamma}$ its coroot, and set $\check{S} := \{\check{\beta}\,;\, \beta \in S\}$.

- If Γ_I is connected, we denote by W_I the Weyl group associated with $\mathcal{R}_I$, that is the subgroup of $GL(V)$ where $V := \mathbb{Z}\mathcal{R}_I \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the reflections,

$$s_\alpha \,:\, V \to V,\; x \mapsto x - \langle x, \check{\alpha}\rangle\, \alpha, \qquad \alpha \in I.$$

- The exponents of S (or $\check{S}$) will be denoted by $m_1, \ldots, m_\ell$. We can assume that $m_1 \leqslant \cdots \leqslant m_\ell$. The integers $m_1 + 1, \ldots, m_\ell + 1$ are the degrees of the basic W_S-invariant polynomials and we have

$$|W_S| = \prod_{i=1}^{\ell} (m_i + 1).$$

In addition, $\sum_{i=1}^{\ell} m_i = |\mathcal{R}^+|$.

- For $\gamma \in \mathcal{R}^+$, the *height* of γ is $\mathrm{ht}(\gamma) := \sum_{\beta \in S} \langle \check{\varpi}_\beta, \gamma\rangle$ where for $\beta \in S$, $\check{\varpi}_\beta$ is the fundamental weight of $\check{S}$ corresponding to $\check{\beta}$. We denote by θ_S the highest root of S and by $\theta_{\check{S}}$ the highest root of $\check{S}$. One has $m_\ell = \mathrm{ht}(\theta_S) = \mathrm{ht}(\theta_{\check{S}})$.

- We denote by $\rho_I := \frac{1}{2} \sum_{\gamma \in \mathcal{R}_I^+} \gamma$ the half sum of positive roots of I. We have $\rho_S = \sum_{\beta \in S} \varpi_\beta$ and $\langle \rho_I, \check{\beta}\rangle = 1$ for any $\beta \in I$.

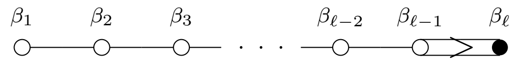
- Set $J := S \smallsetminus I$. The integers a_α, for $\alpha \in J$, are defined by:

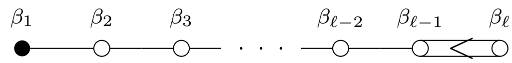
$$a_\alpha := 2\,\langle \rho_S - \rho_I, \check{\alpha}\rangle = 2 - 2\langle \rho_I, \check{\alpha}\rangle = 2 - \sum_{\gamma \in \mathcal{R}_I^+} \langle \gamma, \check{\alpha}\rangle\,.$$

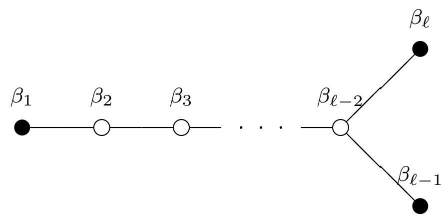
A dominant weight μ is called *minuscule* if $\langle \mu, \theta_{\check{S}}\rangle = 1$. If μ is minuscule then there is $\beta \in S$ such that $\mu = \varpi_\beta$, cf. [Bo68, Chapter VI, §2, exercise 24].

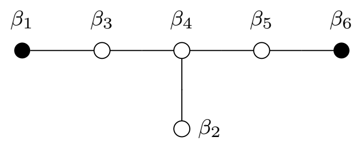
Lemma 5.8. *Let $\alpha \in J = S \smallsetminus I$. Then, $a_\alpha \in \{2, \ldots, m_\ell + 1\}$. Furthermore, the equality $a_\alpha = m_\ell + 1$ holds if and only if $J = \{\alpha\}$ and ϖ_α is minuscule, that is if α is one of the simple roots as described below:*

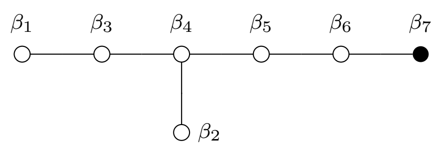
$\mathbf{A}_\ell$, $\ell \geqslant 1$:  $\alpha \in \{\beta_1, \ldots, \beta_\ell\}$;

$\mathbf{B}_\ell$, $\ell \geqslant 2$:  $\alpha = \beta_\ell$;

$\mathbf{C}_\ell$, $\ell \geqslant 3$:  $\alpha = \beta_1$;

$\mathbf{D}_\ell$, $\ell \geqslant 4$:  $\alpha \in \{\beta_1, \beta_{\ell-1}, \beta_\ell\}$;

$\mathbf{E}_6$:  $\alpha \in \{\beta_1, \beta_6\}$;

$\mathbf{E}_7$:  $\alpha = \beta_7$.

Proof. Let $\alpha \in J$. To begin with, since the coefficients of the Cartan matrix of S are nonpositive outside the diagonal, one has $a_\alpha \geqslant 2$. Moreover, $a_\alpha \leqslant 2 - 2\langle \rho_{S \smallsetminus \{\alpha\}}, \check{\alpha} \rangle$. Hence, we may assume that $J = \{\alpha\}$, i.e., $I = S \smallsetminus \{\alpha\}$. Consider now the two cases depending on whether ϖ_α is minuscule or not.

$*$ Assume that ϖ_α is not minuscule, i.e., $\langle \varpi_\alpha, \theta_{\check{S}} \rangle > 1$. Then we have

$$\begin{aligned} m_\ell + 1 = \mathrm{ht}(\theta_{\check{S}}) + 1 = \sum_{\beta \in S} \langle \varpi_\beta, \theta_{\check{S}} \rangle + 1 &= \langle \varpi_\alpha, \theta_{\check{S}} \rangle + \sum_{\beta \in I} \langle \varpi_\beta, \theta_{\check{S}} \rangle + 1 \\ &> 2 + \sum_{\beta \in I} \langle \varpi_\beta, \theta_{\check{S}} \rangle = 2 + \langle \rho_I, \theta_{\check{S}} - \langle \varpi_\alpha, \theta_{\check{S}} \rangle \check{\alpha} \rangle. \end{aligned}$$

Since ϖ_α is not minuscule, $\langle \varpi_\alpha, \theta_{\check{S}} \rangle \geqslant 2$. So, $\langle \rho_I, -\langle \varpi_\alpha, \theta_{\check{S}} \rangle \check{\alpha} \rangle \geqslant -2\langle \rho_I, \check{\alpha} \rangle$ because $-\langle \rho_I, \check{\alpha} \rangle \geqslant 0$. On the other hand, one has $\langle \rho_I, \theta_{\check{S}} \rangle \geqslant 0$. Otherwise there would be $\beta \in I$ such that $\langle \beta, \theta_{\check{S}} \rangle < 0$ which is impossible since $\theta_{\check{S}}$ is the highest root. In conclusion, we get $m_\ell + 1 > 2 - 2\langle \rho_I, \check{\alpha} \rangle = a_\alpha$ as desired.

$*$ Assume that ϖ_α is minuscule, i.e., $\langle \varpi_\alpha, \theta_{\check{S}} \rangle = 1$. Then, $a_\alpha = 2\langle \rho_S - \rho_I, \check{\alpha} \rangle = 2\langle \rho_S - \rho_I, \check{\alpha} + \sum_{\beta \in I} \langle \varpi_\beta, \theta_{\check{S}} \rangle \check{\beta} \rangle = 2\langle \rho_S - \rho_I, \theta_{\check{S}} \rangle$. Hence, we have

$$a_\alpha = 2\langle \rho_S - \rho_I, \theta_{\check{S}} \rangle = \mathrm{ht}(\theta_{\check{S}}) + \langle \rho_S - \rho_I, \theta_{\check{S}} \rangle - \langle \rho_I, \theta_{\check{S}} \rangle = m_\ell + 1 + \frac{1}{2}\Big(\sum_{\substack{\gamma \in \mathcal{R}^+ \smallsetminus \mathcal{R}_I \\ \check{\gamma} \neq \theta_{\check{S}}}} \langle \gamma, \theta_{\check{S}} \rangle - \sum_{\delta \in \mathcal{R}_I^+} \langle \delta, \theta_{\check{S}} \rangle \Big)$$

since $\langle \gamma, \theta_{\check{S}} \rangle = 2$ whenever $\check{\gamma} = \theta_{\check{S}}$. Then, our goal is to show that

$$\sum_{\substack{\gamma \in \mathcal{R}^+ \smallsetminus \mathcal{R}_I \\ \check{\gamma} \neq \theta_{\check{S}}}} \langle \gamma, \theta_{\check{S}} \rangle = \sum_{\delta \in \mathcal{R}_I^+} \langle \delta, \theta_{\check{S}} \rangle.$$

For any $\gamma \in \mathcal{R}^+$, we have $\langle \gamma, \theta_{\check{S}} \rangle \geqslant 0$ since $\theta_{\check{S}}$ is the highest root. Set $\mathcal{R}' := \{\gamma \in \mathcal{R}_S^+ \smallsetminus \mathcal{R}_I \mid \gamma \neq \theta_{\check{S}}$ and $\langle \gamma, \theta_{\check{S}} \rangle > 0\}$ and $\mathcal{R}'' := \{\delta \in \mathcal{R}_I^+ \mid \langle \delta, \theta_{\check{S}} \rangle > 0\}$. Then we have to show the equality:

$$\sum_{\gamma \in \mathcal{R}'} \langle \gamma, \theta_{\check{S}} \rangle = \sum_{\delta \in \mathcal{R}''} \langle \delta, \theta_{\check{S}} \rangle. \tag{3}$$

Let $\gamma \in \mathcal{R}'$. Since $\langle \gamma, \theta_{\check{S}} \rangle > 0$, $\check{\delta} = \theta_{\check{S}} - \check{\gamma}$ is a root of $\check{S}$ and $\theta_{\check{S}} - \check{\delta}$ is a root too. In particular, $\langle \delta, \theta_{\check{S}} \rangle > 0$. Next, show that $\delta \in \mathcal{R}_I^+$.

Since $\gamma \notin \mathcal{R}_I$, $\check{\gamma} \notin \mathcal{R}_{\check{I}}$. Moreover, since ϖ_α is minuscule, $\langle \varpi_\alpha, \check{\gamma} \rangle = \langle \varpi_\alpha, \theta_{\check{S}} \rangle = 1$. So, $\check{\delta} = \theta_{\check{S}} - \check{\gamma} \in \mathcal{R}_{\check{I}}^+$ and then $\delta \in \mathcal{R}_I^+$. Conversely, if $\delta \in \mathcal{R}''$, then $\check{\gamma} = \theta_{\check{S}} - \check{\delta}$ is a root and so $\langle \gamma, \theta_{\check{S}} \rangle > 0$. Moreover, γ is clearly an element of $\mathcal{R}_S^+ \smallsetminus \mathcal{R}_I^+$ which is different from $\theta_{\check{S}}$, that is $\gamma \in \mathcal{R}'$. Therefore, the map from $\mathcal{R}'$ to $\mathcal{R}''$ sending γ to δ, where $\check{\delta} = \theta_{\check{S}} - \check{\gamma}$, gives a bijection between the sets $\mathcal{R}'$ and $\mathcal{R}''$. So, in order to prove the equality (3), it remains to show that for any $\gamma \in \mathcal{R}'$, we have $\langle \gamma, \theta_{\check{S}} \rangle = \langle \delta, \theta_{\check{S}} \rangle$ where $\check{\delta} = \theta_{\check{S}} - \check{\gamma}$.

Let $\gamma \in \mathcal{R}'$ and set $p := \langle \gamma, \theta_{\check{S}} \rangle > 0$. Then the $\check{\gamma}$-string through $\theta_{\check{S}}$ is $\{\theta_{\check{S}}, \dots, \theta_{\check{S}} - p\check{\gamma}\}$. Since there is no minuscule weight in type $\mathbf{G}_2$, we have $p \in \{1, 2\}$. If $p = 1$, then $\theta_{\check{S}}$ and $\theta_{\check{S}} - \check{\gamma} = \check{\delta}$ are roots but not $\theta_{\check{S}} - 2\check{\gamma} = \check{\delta} - \check{\gamma} = -(\theta_{\check{S}} - 2\check{\delta})$. So, the $\check{\delta}$-string through $\theta_{\check{S}}$ is $\{\theta_{\check{S}}, \theta_{\check{S}} - \check{\delta}\}$ and $\langle \delta, \theta_{\check{S}} \rangle = 1$. If $p = 2$, then $\theta_{\check{S}}$, $\theta_{\check{S}} - \check{\gamma} = \check{\delta}$ and $\theta_{\check{S}} - 2\check{\gamma} = \check{\delta} - \check{\gamma} = -(\theta_{\check{S}} - 2\check{\delta})$ are roots. So $\langle \delta, \theta_{\check{S}} \rangle \geqslant 2$ and then $\langle \delta, \theta_{\check{S}} \rangle = 2$. Hence, in both cases, we have obtained that $\langle \delta, \theta_{\check{S}} \rangle = p = \langle \gamma, \theta_{\check{S}} \rangle$ and the equality (3) is proven.

In conclusion, if ϖ_α is minuscule, we have showed that $a_\alpha = m_\ell + 1$.

□

LEMMA 5.9. *Let S' be a subset of S such that $\Gamma_{S'}$ is connected and denote by $m'_1 \leqslant \dots \leqslant m'_l$ the exponents of S'. Then we have $m'_j \leqslant m_j$ for any $j \in \{1, \dots, l\}$. In particular,* $\mathrm{ht}(\theta_{S'}) \leqslant m_l$.

Proof. By a classical result, [Kos59], the partition of $|\mathcal{R}^+|$ formed by the exponents is the dual to that formed by the number of positive roots of each height. This easily implies the statement.

□

Let k be the cardinality of I, and $m'_1, \dots, m'_k$ the union of all the exponents of subsets S' such that $\Gamma_{S'}$ is a connected component of Γ_I. Order them so that $m'_1 \leqslant \dots \leqslant m'_k$. Number the roots $\alpha_{k+1}, \dots, \alpha_\ell$ of J so that $a_{\alpha_{k+1}} \leqslant \dots \leqslant a_{\alpha_\ell}$ and set for simplicity $a_j := a_{\alpha_j}$ for any $j \in \{k+1, \dots, \ell\}$.

LEMMA 5.10. (i) *For all $i \in \{1, \dots, k\}$, one has $m'_i \leqslant m_i$ and, for all $j \in \{k+1, \dots, \ell\}$, one has $a_j \leqslant m_j + 1$. In particular,*

$$|W_I|\, a_{k+1} \cdots a_\ell \leqslant |W_S|.$$

(ii) *Equality holds in the above inequality if and only if I and J are in one of the configurations* (C1), (C2) *or* (C3) *as described in Proposition 5.6 with $\mathcal{F} = J$.*

Proof. (i) By Lemma 5.9, for all $i \in \{1, \dots, k\}$, we have $m'_i \leqslant m_i$. Turn to the second statement. Set for $j \in \{k+1, \dots, \ell\}$, $I_j := I \cup \{\alpha_{k+1}, \dots, \alpha_j\}$. Let $j \in \{k+1, \dots, \ell\}$ and S_j the connected component of I_j containing α_j. We have $a_j = 2 - \langle \rho_I, \check{\alpha} \rangle = 2 - \langle \rho_{I \cap S_j}, \check{\alpha} \rangle = 2\langle \rho_{S_j} - \rho_{I \cap S_j}, \check{\alpha} \rangle$.

So, by Lemma 5.8, $a_j \leqslant \mathrm{ht}(\theta_{S_j}) + 1$. Hence, by Lemma 5.9, $a_j \leqslant m_j + 1$ since I_j has cardinality j. All this shows:

$$|W_I| \prod_{j=k+1}^{\ell} a_j = \prod_{i=1}^{k} (m'_i + 1) \prod_{j=k+1}^{\ell} a_j \leqslant \prod_{i=1}^{\ell} (m_i + 1) = |W_S|.$$

(ii) By the proof of (i), if equality holds in the above inequality then $|W_I| = \prod_{i=1}^{k} (m_i + 1)$ and for all $j \in \{k+1, \dots, \ell\}$, $a_j = m_j + 1$. In particular, $a_\ell = m_\ell + 1$. Therefore, we are in one of the situations of the Lemma 5.8 and we consider the six cases as described in it.

- Type $\mathbf{A}_\ell$, $\ell \geqslant 1$: The $\ell - 1$ smallest degrees of the basic invariants are $2, 3 \dots, \ell$. If α_ℓ is not an extremal vertex, then $|W_{S \smallsetminus \{\alpha_\ell\}}| < \ell\,!$ as we easily verify. So α_ℓ must be extremal and I and J are in the configuration (C1).
- Type $\mathbf{B}_\ell$, $\ell \geqslant 2$: The $\ell - 1$ smallest degrees of the basic invariants are $2, 4, \dots, 2(\ell - 1)$. So their product is strictly greater than $|W_{S \smallsetminus \{\beta_\ell\}}| = \ell!$ and the equality does not hold.
- Type $\mathbf{C}_\ell$, $\ell \geqslant 3$: I and J are in the configuration (C2).
- Type $\mathbf{D}_\ell$, $\ell \geqslant 4$: The degrees of the basic invariants of $\mathbf{D}_\ell$, for $\ell \geqslant 4$, are $2, 4, \dots, 2\ell - 2, \ell$. So, the $\ell - 1$ smallest are $2, 4, \dots, 2\ell - 4, \ell$ ($\ell \geqslant 4$) and their product is $2^{\ell-2}\ell$. But for any $i \in \{1, \dots, \ell\}$, $|W_{S \smallsetminus \{\beta_i\}}| \leqslant |W_{S \smallsetminus \{\beta_1\}}| = 2^{\ell-2}(\ell - 1)! < 2^{\ell-2}\ell$; so the equality does not hold.
- Type $\mathbf{E}_6$: The 5-th smallest exponents of $\mathbf{E}_6$ are $1, 4, 5, 7, 8$ and those of $S \smallsetminus \{\beta_1\}$ (or of $S \smallsetminus \{\beta_6\}$) are $1, 3, 4, 5, 7$; so, the equality does not hold.
- Type $\mathbf{E}_7$: The 6-th smallest exponents of $\mathbf{E}_7$ are $1, 5, 7, 9, 11, 13$ and those of $S \smallsetminus \{\beta_7\}$ are $1, 4, 5, 7, 8, 11$; so, the equality does not hold.

One has proven one implication. The converse implication is an easy computation, left to the reader.

□

Proposition 5.11. *Assume that X is a simple locally factorial G/H-embedding with maximal colored cone $(\sigma, \mathcal{F})$ of dimension r. Let I be the subset of S such that $N_G(H) = P_I$. Then,*

$$e_{\mathrm{st}}(X) = \frac{|W_S|}{|W_I| \prod_{\alpha \in \mathcal{F}} a_\alpha} \quad \textit{and} \quad e(X) = \frac{|W_S|}{|W_{I \cup \mathcal{F}}|}.$$

Proof. First of all, observe that the Euler number of G/B is the number of fixed points of a maximal torus, i.e., the order of the Weyl group W_S. More generally, for any $S' \subset S$, the Euler number of $G/P_{S'}$ is $|W_S|/|W_{S'}|$. Thus, we have to show:

$$e_{\mathrm{st}}(X) = \frac{e(G/P_I)}{\prod_{\alpha \in \mathcal{F}} a_\alpha} \quad \text{and} \quad e(X) = e(G/P_{I \cup \mathcal{F}}).$$

Now, we observe that the usual Euler number of a horospherical homogeneous space is nonzero if and only if it has rank zero. As a consequence, one has $e(X) = e(G/P_{I \cup \mathcal{F}})$, according to the description of G-orbits in X (see Proposition 2.4).

Turn to the formula for $e_{\mathrm{st}}(X)$. Let $e_1, \dots, e_r$ be the primitive generators of σ. Since X is locally factorial, $e_1, \dots, e_r$ is a $\mathbb{Z}$-basis of $\sigma \cap N$ (cf. Theorem 5.1). Then

$$\sum_{e_i \in \sigma \cap N} \mathbb{L}^{\omega_X(e_i)} = \prod_{i=1}^{r} \frac{1}{1 - \mathbb{L}^{\omega_X(e_i)}} = \frac{1}{(\mathbb{L} - 1)^r} \prod_{i=1}^{r} \frac{\mathbb{L}^{-\omega_X(e_i)}}{\mathbb{L}^{-\omega_X(e_i) - 1} + \dots + 1}.$$

Then, by Theorem 4.3, one has:

$$\mathcal{E}_{\mathrm{st}}(X) = [G/H] \sum_{e_i \in \sigma \cap N} \mathbb{L}^{\omega_X(e_i)} = [G/P]\,[T]\, \frac{1}{(\mathbb{L}-1)^r} \prod_{i=1}^{r} \frac{\mathbb{L}^{-\omega_X(e_i)}}{\mathbb{L}^{-\omega_X(e_i)-1} + \cdots + 1}$$
$$= [G/P] \prod_{i=1}^{r} \frac{\mathbb{L}^{-\omega_X(e_i)}}{\mathbb{L}^{-\omega_X(e_i)-1} + \cdots + 1}\,.$$

From this, we get

$$e_{\mathrm{st}}(X) = e(G/P) \prod_{i=1}^{r} \frac{1}{\big(-\omega_X(e_i)\big)} = \frac{e(G/P_I)}{\prod_{\alpha \in \mathcal{F}} a_\alpha}\,.$$

The last equality holds because the set of elements ϱ_α ($\alpha \in \mathcal{F}$) is a subset of the basis $\{e_1, \dots, e_r\}$ (cf. Theorem 5.1).

□

We are in a position to prove Theorem 5.3.

Proof of Theorem 5.3. We can assume without loss of generality that S is connected and $I \cup \mathcal{F} = S$. By Lemma 5.10 and Proposition 5.11, we have $e_{\mathrm{st}}(X) \geqslant e(X)$. This proves one part of the theorem. Moreover, the equality holds if and only if $(I, \mathcal{F})$ is in one of the configurations (C1), (C2) or (C3) as described in Proposition 5.6, that is to say if and only if X is smooth by Proposition 5.6.

□

Remark 5.12. As a matter of fact, we gave another proof for the first implication of Pauer's criterion (Proposition 5.6). Indeed, whenever $(I, \mathcal{F})$ is not in one of the configurations (C1), (C2) or (C3) of Proposition 5.6, we have shown that $e_{\mathrm{st}}(X) > e(X)$, and so X is not smooth.

6. Some applications and open questions

Let X be a complete locally factorial horospherical G/H-embedding with colored fan Σ. Let $e_1, \dots, e_s$ be the primitive integral generators of all 1-dimensional cones in Σ and set $a_i := -\omega_X(e_i)$ for all $i \in \{1, \dots, s\}$.

Consider the polynomial ring $\mathbb{C}[z_1, \dots, z_s]$ whose variables $z_1, \dots, z_s$ are in bijection with the lattice vectors $e_1, \dots, e_s$. Recall that the Stanley-Reisner ring R_Σ is the quotient of $\mathbb{C}[z_1, \dots, z_s]$ by the ideal generated by all square free monomials $z_{i_1} \dots z_{i_k}$ such that the lattice vectors $e_{i_1} \dots e_{i_k}$ do not generate any k-dimensional cone in Σ. Recall also that the *weighted Stanley-Reisner ring* R^w_Σ is defined by putting $\deg z_i = a_i$ in the standard Stanley-Reisner ring R_Σ.

Proposition 6.1. *Let X be a complete locally factorial horospherical G/H-embedding with colored fan Σ. Then, one has:*

$$\sum_{n \in N} (uv)^{\omega_X(n)} = P(R^w_\Sigma, (uv)^{-1}) = \sum_{\sigma \in \Sigma} \frac{(-1)^{\dim \sigma}}{\prod_{e_i \in \sigma} \big(1 - (uv)^{a_i}\big)}\,; \tag{4}$$
$$E_{\mathrm{st}}(X; u, v) = E(G/H; u, v)(-1)^r P(R^w_\Sigma, uv)\,, \tag{5}$$

where $P(R^w_\Sigma, t)$ denotes the Poincaré series of the weighted Stanley-Reisner ring R^w_Σ.

Proof. The ring R_Σ has a monomial basis over $\mathbb{C}$ whose elements are in one-to-one correspondence with N. Namely, any monomial $z_{i_1}^{k_1} \dots z_{i_t}^{k_t}$ in R_Σ corresponds to the lattice point $k_1 e_{i_1} + \cdots + k_t e_{i_t}$

and the weighted degree of $z_{i_1}^{k_1}\dots z_{i_t}^{k_t}$ is $-k_1\omega_X(e_{i_1})-\cdots-k_t\omega_X(e_{i_t})$. Thus, the k-homogeneous component of the weighted Stanley-Reisner ring R_Σ^w consists of all monomials $z_{i_1}^{k_1}\dots z_{i_t}^{k_t}$ corresponding to lattice points $n\in N$ such that $\omega_X(n)=-k$. This implies the first equality in (4). For any cone $\sigma\in\Sigma$, we denote by σ° the relative interior of σ. Since X is locally factorial, one has by Theorem 5.1:

$$\sum_{n\in N} t^{\omega_X(n)} = \sum_{\sigma\in\Sigma}\sum_{n\in\sigma^\circ} t^{\omega_X(n)} = \sum_{\sigma\in\Sigma}\prod_{e_i\in\sigma}\frac{t^{-a_i}}{1-t^{-a_i}} = \sum_{\sigma\in\Sigma}\prod_{e_i\in\sigma}\frac{(-1)^{\dim\sigma}}{1-t^{a_i}}\,. \tag{6}$$

This implies the second equality in (4).

Let us prove the equality (5). By Theorem 4.3 and (4), we have:

$$E_{\rm st}(X;u,v)=E(G/H;u,v)P(R_\Sigma^w,(uv)^{-1}).$$

By the Poincaré duality [Ba98, Theorem 3.7], we have

$$(uv)^{\dim X}E_{\rm st}(X;u^{-1},v^{-1})=E_{\rm st}(X;u,v),$$
$$(uv)^{\dim G/P}E(G/P;u^{-1},v^{-1})=E(G/P;u,v)\,.$$

The above equalities imply:

$$\begin{aligned}
E_{\rm st}(X;u,v) &= (uv)^{\dim X}E_{\rm st}(X;u^{-1},v^{-1})\\
&= (uv)^{\dim X}E(G/H;u^{-1},v^{-1})P(R_\Sigma^w,uv)\\
&= (uv)^{\dim G/P}E(G/P;u^{-1},v^{-1})(uv)^r((uv)^{-1}-1)^rP(R_\Sigma^w,uv)\\
&= E(G/P;u,v)(uv-1)^r(-1)^rP(R_\Sigma^w,uv)\\
&= E(G/H;u,v)(-1)^rP(R_\Sigma^w,uv)\,.
\end{aligned}$$

□

Example 6.2. 1) Consider the locally factorial completion $\overline{Q}$ of the affine 5-dimensional quadric Q in Example 4.6; $\overline{Q}$ is a singular projective quadric. The colored fan $\overline{\Sigma}$ of $\overline{Q}$ is represented in Figure 1 and the positive integer $a_i=-\omega_{\overline{Q}}(e_i)$ $(i=1,2,3)$ is written down near to the integral point e_i. The circles stand for the colors ϱ_α, $\alpha\in\mathcal{F}$.

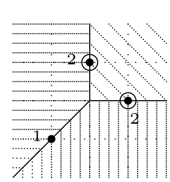


Figure 1. The colored fan $\overline{\Sigma}$ of $\overline{Q}$

The Stanley-Reisner ring is $R_{\overline{\Sigma}}\simeq\mathbb{C}[z_1,z_2,z_3]/(z_1z_2z_3)$ and we have

$$P(R_{\overline{\Sigma}}^w,t)=\frac{1-t^5}{(1-t)(1-t^2)^2}.$$

Hence, by Proposition 6.1, we get

$$E_{\rm st}(\overline{Q};u,v)=\frac{(1+uv+(uv)^2)(1+uv+(uv)^2+(uv)^3)}{(1+uv)}\,.$$

2) Consider the locally factorial completion $\overline{X}$ of the affine 7-dimensional cone X over the Grassmannian $G(2,5)$ from Example 5.5; $\overline{X}$ is the projective cone over the Grassmannian $G(2,5)$. The colored fan $\overline{\Sigma}$ of $\overline{X}$ is represented in Figure 2.

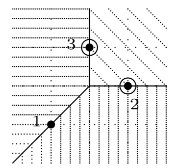


FIGURE 2. The colored fan $\overline{\Sigma}$ of $\overline{X}$

We have,

$$P(R^w_{\overline{\Sigma}}, t) = \frac{1-t^6}{(1-t)(1-t^2)(1-t^3)},$$

and

$$E_{\rm st}(\overline{X}; u, v) = (1+(uv)^2)(1+uv+(uv)^2+(uv)^3+(uv)^4+(uv)^5).$$

It would be interesting to compute the cohomology ring $H^*(X_\Sigma, \mathbb{C})$ of an arbitrary smooth projective horospherical variety X_Σ defined by a colored fan Σ. If X_Σ is a toroidal horospherical variety, then X_Σ is a toric bundle over G/P, and a general result of Sankaran and Uma [SU03, Theorem 1.2] implies the following description of the cohomology ring of X_Σ:

PROPOSITION 6.3. *Let X_Σ be a smooth projective toroidal horospherical variety defined by a (uncolored) fan Σ. Then the cohomology ring $H^*(X_\Sigma, \mathbb{C})$ is isomorphic to the quotient of $H^*(G/P, \mathbb{C}) \otimes_{\mathbb{C}} R_\Sigma$ by the ideal generated by the regular sequences $f_1, \ldots, f_r$ where f_i is given by*

$$f_i := \delta(m_i) \otimes 1 + 1 \otimes \sum_{j=1}^{s} \langle m_i, e_j \rangle \in \left(H^2(X, \mathbb{C}) \otimes R^0_\Sigma\right) \oplus \left(H^0(X, \mathbb{C}) \otimes R^1_\Sigma\right),$$

for some integral basis $\{m_1, \ldots, m_r\}$ of the lattice M.

Together with Proposition 6.3, our formula (5) in Proposition 6.1 motivates the following question:

QUESTION 6.4. Does there exist an analogous description of the cohomology ring of an arbitrary smooth projective horospherical variety defined by a colored fan Σ which involves the weighted Stanley-Reisner ring R^w_Σ?

Another interesting question is motivated by Theorem 4.3:

QUESTION 6.5. How to compute $E_{\rm st}(X; u, v)$ for an arbitrary $\mathbb{Q}$-Gorenstein spherical G/H-embedding?

REMARK 6.6. We hope that there is a formula for $E_{\rm st}(X; u, v)$ similar to the one in the horospherical case, e.g., which involves the summation of $(uv)^{\omega_X(n)}$ over all lattice points in the valuation cone $\mathcal{V}(G/H)$ of the spherical homogeneous space G/H.

A smoothness criterion for arbitrary spherical varieties was obtained by M. Brion in [Br91]. Unfortunately, this criterion is difficult to apply in practice. We expect that the smoothness criterion for locally factorial horospherical varieties (see Theorem 5.3) can be extended to arbitrary locally factorial spherical varieties:

Conjecture 6.7. Let X be a locally factorial spherical G/H-embedding whose closed orbits are projective. Then one has $e_{\rm st}(X) \geqslant e(X)$, and the equality holds if and only if X is smooth.

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