SHEETS AND ASSOCIATED VARIETIES OF AFFINE VERTEX ALGEBRAS

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Abstract. We show that sheet closures appear as associated varieties of affine vertex algebras. We also provide examples of associated varieties that are union of distinct sheet closures, which in particular shows that the associated varieties of vertex algebras need not to be irreducible. Further, we give new examples of non-admissible affine vertex algebras whose associated variety is contained in the nilpotent cone. We also prove some conjectures from our previous paper and give new examples of lisse affine W-algebras.

1. Introduction

It is known [Li05] that every vertex algebra V is canonically filtered and therefore it can be considered as a quantization of its associated graded Poisson vertex algebra $\operatorname{gr} V$. The generating subring R_V of $\operatorname{gr} V$ is called the *Zhu's C_2-algebra* of V [Z96] and has the structure of a Poisson algebra. Its spectrum

$$\tilde{X}_V = \operatorname{Spec} R_V$$

is called the *associated scheme* of V and the corresponding reduced scheme $X_V = \operatorname{Specm} R_V$ is called the *associated variety* of V ([Ar12, Ar16b]). Since it is Poisson, the coordinate ring of its arc space $J_\infty \tilde{X}$ has a natural structure of a Poisson vertex algebra ([Ar12]), and there is a natural surjective homomorphism $\mathbb{C}[J_\infty \tilde{X}_V] \to \operatorname{gr} V$, which is in many cases an isomorphism. We have [Ar12] $\dim \operatorname{Spec}(\operatorname{gr} V) = 0$ if and only if $\dim X_V = 0$, and in this case V is called *lisse* or *C_2-cofinite*.

In the case that V is the simple affine vertex algebra $V_k(\mathfrak{g})$ associated with a finite-dimensional simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{C}$, X_V is a Poisson subscheme of $\mathfrak{g}^*$ which is G-invariant and conic, where G is the adjoint group of $\mathfrak{g}$. Note that on the contrary to the associated variety of a primitive ideal of $U(\mathfrak{g})$, the variety $X_{V_k(\mathfrak{g})}$ is not necessarily contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. In fact, $X_{V_k(\mathfrak{g})} = \mathfrak{g}^*$ for a generic k. On the other hand $X_{V_k(\mathfrak{g})} = \{0\}$ if and only if $V_k(\mathfrak{g})$ is integrable, that is, k is a non-negative integer. Except for a few cases, the description of X_V is fairly open even for $V = V_k(\mathfrak{g})$, although this problem seems to be significant in connection with *four dimensional* superconformal field theory ([BLL$^+$15]).

In [Ar15a], the first named author showed that $X_{V_k(\mathfrak{g})}$ is the closure of some nilpotent orbit of $\mathfrak{g}^*$ in the case that $V_k(\mathfrak{g})$ is *admissible* [KW89].

In the previous article [AM15], we showed that $X_{V_k(\mathfrak{g})}$ is the minimal nilpotent orbit closure in the case that $\mathfrak{g}$ belongs to the Deligne exceptional series [De96] and $k = -h^\vee/6 - 1$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Note that the level $k = -h^\vee/6 - 1$ is not admissible for types D_4, E_6, E_7, E_8.

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In all the above cases $X_{V_k(\mathfrak{g})}$ is a closure of a nilpotent orbit $\mathbb{O} \subset \mathcal{N}$, or $(X_{V_k(\mathfrak{g})})_{\rm red} = \mathfrak{g}^*$. Therefore it is natural to ask the following.

Question 1. Are there cases when $X_{V_k(\mathfrak{g})} \not\subset \mathcal{N}$ and $X_{V_k(\mathfrak{g})}$ is a proper subvariety of $\mathfrak{g}^*$? For example, are there cases when $X_{V_k(\mathfrak{g})}$ is the closure of a non-nilpotent *Jordan class* (cf. §2)?

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through a non-degenerate bilinear form of $\mathfrak{g}$.

Given $m \in \mathbb{N}$, let $\mathfrak{g}^{(m)}$ be the set of elements $x \in \mathfrak{g}$ such that $\dim \mathfrak{g}^x = m$, with $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$. A subset $\mathbb{S} \subset \mathfrak{g}$ is called a *sheet* of $\mathfrak{g}$ if it is an irreducible component of one of the locally closed sets $\mathfrak{g}^{(m)}$. It is G-invariant and conic and by [ImH05], and it is smooth if $\mathfrak{g}$ is classical. The sheet closures are the closures of certain Jordan classes and they are parameterized by the G-conjugacy classes of pairs $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathbb{O}_{\mathfrak{l}}$ is a *rigid* nilpotent orbit of $\mathfrak{l}$, i.e., which cannot be properly induced in the sense of Lusztig-Spaltenstein [BK79, Bo81] (see also [TY05, §39]). The pair $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is called the *datum* of the corresponding sheet. When $\mathbb{O}_{\mathfrak{l}}$ is zero, the sheet is called *Dixmier*, meaning that it contains a semisimple element [Di75, Di76]. We will denote by $\mathbb{S}_{\mathfrak{l}}$ the sheet with datum $(\mathfrak{l}, \{0\})$. We refer to §2 for more details about this topic.

It is known that sheets appear in the representation theory of finite-dimensional Lie algebras, see, e.g., [BB82, BB85, BB89], and more recently of finite W-algebras, [PT14, Pr14].

Since the sheet closures are G-invariant, conic algebraic varieties which are not necessarily contained in $\mathcal{N}$, one may expect that there are simple affine vertex algebras whose associated variety is the closure of some sheet. This is indeed the case.

Theorem 1.1. (1) *For $n \geqslant 4$,*

$$\tilde{X}_{V_{-1}(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}_{\mathfrak{l}_1}}$$

as schemes, where $\mathfrak{l}_1$ is the standard Levi subalgebra of $\mathfrak{sl}_n$ generated by all simple roots except α_1. Moreover $V_{-1}(\mathfrak{sl}_n)$ is a quantization of the infinite jet scheme $J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}$ of $\overline{\mathbb{S}_{\mathfrak{l}_1}}$, that is,

$$\operatorname{gr} V_{-1}(\mathfrak{sl}_n) \cong \mathbb{C}[J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}]$$

as Poisson vertex algebras.

(2) *For $m \geqslant 2$,*

$$\tilde{X}_{V_{-m}(\mathfrak{sl}_{2m})} \cong \overline{\mathbb{S}_{\mathfrak{l}_0}}$$

as schemes, where $\mathfrak{l}_0$ is the standard Levi subalgebra of $\mathfrak{sl}_{2m}$ generated by all simple roots except α_m. Moreover, $V_{-m}(\mathfrak{sl}_{2m})$ is a quantization of $J_\infty \overline{\mathbb{S}_{\mathfrak{l}_0}}$, that is,

$$\operatorname{gr} V_{-m}(\mathfrak{sl}_{2m}) \cong \mathbb{C}[J_\infty \overline{\mathbb{S}_{\mathfrak{l}_0}}]$$

as Poisson vertex algebras.

Further, we show that the Zhu's algebras of the above vertex algebras are naturally embedded into the algebra of global differential operators on $Y = G/(P,P)$, where P is the connected parabolic subgroup of G corresponding to a parabolic subalgebra $\mathfrak{p}$ having the above Levi subalgebra–$\mathfrak{l}_1$ in case (1) and $\mathfrak{l}_0$ in case (2)– as Levi factor, see Theorem 7.14 and Theorem 8.13.

The vertex algebra $V_{-1}(\mathfrak{sl}_n)$ has appeared in the work of Adamović and Perše [AP14], where they studied the fusion rules and the complete reducibility of $V_{-1}(\mathfrak{sl}_n)$-modules. It would be very interesting to know whether the vertex algebra $V_{-m}(\mathfrak{sl}_{2m})$ has similar properties.

Now recall that the associated variety of primitive ideals of $U(\mathfrak{g})$ is irreducible [Jo85]. Hence it is also natural to ask the following question.

Question 2. Is $X_{V_k(\mathfrak{g})}$ always irreducible?

Theorem 1.2. *Let r be an odd integer, and let $\mathfrak{l}^I$ and $\mathfrak{l}^{II}$ be the Levi subalgebras of $\mathfrak{g} = \mathfrak{so}_{2r}$ generated by the simple roots $\alpha_1, \dots, \alpha_{r-2}, \alpha_r$ and $\alpha_1, \dots, \alpha_{r-2}, \alpha_{r-1}$ respectively. They are non G-conjugate and*

$$X_{V_{2-r}(\mathfrak{so}_{2r})} = \overline{\mathbb{S}_{\mathfrak{l}^I}} \cup \overline{\mathbb{S}_{\mathfrak{l}^{II}}}.$$

In particular the associated variety $X_{V_{2-r}(\mathfrak{so}_{2r})}$ is reducible.

We conjecture that $X_{V_k(\mathfrak{g})}$ is always equidimensional, and that $X_{V_k(\mathfrak{g})}$ is irreducible provided that $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$, see Conjecture 1.

The vertex algebra $V_{2-r}(\mathfrak{so}_{2r})$ has been studied by Perše [Pe13] for all r, and by Adamović and Perše [AP14] for odd r. The proof of Theorem 1.2 uses the fact proved in [AP14] that, for odd r, $V_{2-r}(\mathfrak{so}_{2r})$ has infinitely many simple objects in the category $\mathcal{O}$.

Remarkably, it turned out that the structure of the vertex algebra $V_{2-r}(\mathfrak{so}_{2r})$ substantially differs depending on the parity of r.

Theorem 1.3. *Let r be an even integer such that $r \geqslant 6$. Then*

$$\overline{\mathbb{O}_{min}} \subsetneq X_{V_{2-r}(\mathfrak{so}_{2r})} \subset \overline{\mathbb{O}_{(2^{r-2},1^4)}},$$

where $\mathbb{O}_{\min}$ is the minimal nilpotent orbit of $\mathfrak{so}_{2r}$ and $\mathbb{O}_{(2^{r-2},1^4)}$ is the nilpotent orbit of $\mathfrak{so}_{2r}$ associated with the partition $(2^{r-2}, 1^4)$ of $2r$.

In particular, $X_{V_{2-r}(\mathfrak{so}_{2r})}$ is contained in $\mathcal{N}$, and hence, there are only finitely many simple $V_{2-r}(\mathfrak{g})$-modules in the category $\mathcal{O}$.

The above theorem gives new examples of non-admissible affine vertex algebras whose associated varieties are contained in the nilpotent cone (cf. [AM15]). In fact we conjecture[1] that $X_{V_{2-r}(\mathfrak{so}_{2r})} = \overline{\mathbb{O}_{(2^{r-2},1^4)}}$. This conjecture is confirmed for $r = 6$, see Theorem 9.6. Notice that for $r = 4$, $X_{V_{-2}(\mathfrak{so}_8)} = \overline{\mathbb{O}_{min}} = \overline{\mathbb{O}_{(2^2,1^4)}}$ by [AM15]. So the conjecture also holds for $r = 4$.

Our proof of the above stated results is based on the analysis of singular vectors of degree 2 [AM15] and the theory of W-algebras [KRW03, KW04, Ar05, Gi09, Ar11, Ar15a]. This method works for some other types as well, in particular in types B and C, which will be studied in our subsequent paper.

We also take the opportunity of this note to clarify some points of [AM15] that are related to the present work. Let us state here the main results.

By [Ar15a, Theorem 4.23] (cf. Theorem 6.1) we know that the W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ associated with $(\mathfrak{g}, f)$ at level k ([KRW03]) is lisse if $X_{V_k(\mathfrak{g})} = \overline{G.f}$. For a minimal nilpotent element $f \in \mathbb{O}_{min}$, the converse is also true provided that $k \notin \mathbb{Z}_{\geqslant 0}$.

Theorem 1.4. *Suppose that $k \notin \mathbb{Z}_{\geqslant 0}$, and let $f \in \mathbb{O}_{min}$. Then the minimal W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $(X_{V_k(\mathfrak{g})})_{\rm red} = \overline{\mathbb{O}_{min}}$.*

[1]This conjecture is now confirmed in our paper arXiv:1608.03142.

We also positively answer some conjectures of [AM15]. In particular, we show the following.

Theorem 1.5. *Let $\mathfrak{g}$ of type G_2. Then $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse if and only if k is admissible with denominator 3, or an integer equal to or greater than -1.*

Thus, we obtain a new family of lisse minimal W-algebras $\mathcal{W}_k(G_2, f_\theta)$, for $k = -1, 0, 1, 2, 3 \dots$.

Theorem 1.6. *Suppose that $\mathfrak{g}$ is not of type A. Then Conjecture 2 of [AM15] holds. That is,*

$$X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$$

if and only if the one of the following conditions holds:

(1) *$\mathfrak{g}$ is of type C_r ($r \geqslant 2$), F_4, and k is admissible with denominator 2.*
(2) *$\mathfrak{g}$ is of type G_2, and k is admissible with denominator 3, or $k = -1$.*
(3) *$\mathfrak{g}$ is of type D_4, E_6, E_7, E_8 and k is an integer such that*
$$-\frac{h^\vee}{6} - 1 \leqslant k \leqslant -1.$$
(4) *$\mathfrak{g}$ is of type D_r with $r \geqslant 5$, and $k = -2, -1$.*

The rest of the paper is organized as follows. In §2 we recollect some results concerning sheets that will be needed later and give a description of Dixmier sheets of rank one. In §3, we use Slodowy slices and Ginzburg's results on finite W-algebras to state useful lemmas. In §4 we state results and conjectures on the associated variety of a vertex algebra. In §5 we recall some fundamental results on Zhu's algebras of vertex algebras. In §6 we recall and state some fundamental results on W-algebras. In §7 we study level -1 affine vertex algebras of type A_{n-1}, $n \geqslant 4$, and prove Theorem 1.1 (1). In §8 we study level $-m$ affine vertex algebras of type A_{2m-1}, $m \geqslant 2$, and prove Theorem 1.1 (2). In §9 we study level $2 - r$ affine vertex algebars of type D_{2r}, $r \geqslant 5$, and prove Theorem 1.2 and Theorem 1.3. In §10, we prove Theorem 1.4, Theorem 1.5 and Theorem 1.6 and some other results related to our previous work [AM15]. In particular we obtain a new family of lisse minimal W-algebras.

Notations. As a rule, for U a $\mathfrak{g}$-submodule of $S(\mathfrak{g})$, we shall denote by I_U the ideal of $S(\mathfrak{g})$ generated by U, and for I an ideal in $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$, we shall denote by $V(I)$ the zero locus of I in $\mathfrak{g}^*$.

Let
$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$
be the affine Kac-Moody Lie algebra associated with $\mathfrak{g}$ and the inner product $(\ |\ ) = 1/2h^\vee \times$ Killing form (see §4). For $\lambda \in \mathfrak{h}^*$ (resp. $\hat{\mathfrak{h}}^*$), $L_{\mathfrak{g}}(\lambda)$ (resp. $L(\lambda)$) denotes the irreducible highest weight representation of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) with highest weight λ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$.

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2. Jordan classes and sheets

Most of results presented in this section come from [BK79, Bo81] or [Kat82]. Our main reference for basics about Jordan classes and sheets is [TY05, §39].

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $(\ |\ ) = 1/2h^\vee \times$ Killing form, as in the introduction. We often identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via $(\ |\ )$.

For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$, denote by $\mathfrak{z}(\mathfrak{a})$ its center. For Y a subset of $\mathfrak{g}$, denote by Y^{reg} the set of $y \in Y$ for which $\mathfrak{g}^y$ has the minimal dimension with $\mathfrak{g}^y$ the centralizer of y in $\mathfrak{g}$. In particular, if $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$, then

$$\mathfrak{z}(\mathfrak{l})^{\mathrm{reg}} := \{y \in \mathfrak{g} \mid \mathfrak{z}(\mathfrak{g}^y) = \mathfrak{z}(\mathfrak{l})\},$$

and $\mathfrak{z}(\mathfrak{l})^{\mathrm{reg}}$ is a dense open subset of $\mathfrak{z}(\mathfrak{l})$. For $x \in \mathfrak{g}$, denote by x_s and x_n the semisimple and the nilpotent components of x respectively.

The *Jordan class* of x is

$$J_G(x) := G.(\mathfrak{z}(\mathfrak{g}^{x_s})^{\mathrm{reg}} + x_n).$$

It is a G-invariant, irreducible, and locally closed subset of $\mathfrak{g}$. To a Jordan class J, we associate its *datum* which is the pair $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ defined as follows. Pick $x \in J$. Then $\mathfrak{l}$ is the Levi subalgebra $\mathfrak{g}^{x_s}$ and $\mathbb{O}_\mathfrak{l}$ is the nilpotent orbit in $\mathfrak{l}$ of x_n. The pair $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ does not depend on $x \in J$ up to G-conjugacy, and there is a one-to-one correspondence between the set of pairs $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ as above, up to G-conjugacy, and the set of Jordan classes.

A *sheet* is an irreducible component of the subsets

$$\mathfrak{g}^{(m)} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = m\}, \qquad m \in \mathbb{N}.$$

It is a finite disjoint union of Jordan classes. So a sheet $\mathbb{S}$ contains a unique dense open Jordan class J and we can define the *datum* of $\mathbb{S}$ as the datum $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ of the Jordan class J. We have

$$\overline{\mathbb{S}} = \overline{J} \quad \text{ and } \quad \mathbb{S} = (\overline{J})^{\mathrm{reg}}.$$

A sheet is called *Dixmier* if it contains a semisimple element of $\mathfrak{g}$. A sheet $\mathbb{S}$ with datum $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ is Dixmier if and only if $\mathbb{O}_\mathfrak{l} = \{0\}$. We shall simply denote by $\mathbb{S}_\mathfrak{l}$ the Dixmier sheet with datum $(\mathfrak{l}, \{0\})$.

A nilpotent orbit is called *rigid* if it cannot be properly induced in the sense of Lusztig-Spaltenstein. A Jordan class with datum $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ is a sheet if and only if $\mathbb{O}_\mathfrak{l}$ is rigid in $\mathfrak{l}$. So we get a one-to-one correspondence between the set of pairs $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$, up to G-conjugacy, with $\mathfrak{l}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathbb{O}_\mathfrak{l}$ a rigid nilpotent orbit of $\mathfrak{l}$, and the set of sheets.

Each sheet contains a unique nilpotent orbit. Namely, if $\mathbb{S}$ is a sheet with datum $(\mathfrak{l}, \mathbb{O}_\mathfrak{l})$ then the induced nilpotent orbit $\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}(\mathbb{O}_\mathfrak{l})$ of $\mathfrak{g}$ from $\mathbb{O}_\mathfrak{l}$ in $\mathfrak{l}$ is the unique nilpotent orbit contained in $\mathbb{S}$. Note that a nilpotent orbit $\mathbb{O}$ is itself a sheet if and only if $\mathbb{O}$ is rigid. For instance, outside the type A, the minimal nilpotent orbit $\mathbb{O}_{min}$ is always a sheet.

The *rank* of a sheet $\mathbb{S}$ with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$ is by definition

$$\operatorname{rank}(\mathbb{S}) := \dim \mathbb{S} - \dim \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}}) = \dim \mathfrak{z}(\mathfrak{l}).$$

If $\mathbb{S} = \mathbb{S}_{\mathfrak{l}}$ is Dixmier, then $\mathbb{O}_{\mathfrak{l}} = 0$ and we have

$$\overline{\mathbb{S}} = G.[\mathfrak{p}, \mathfrak{p}]^{\perp} = G.(\mathfrak{z}(\mathfrak{l}) + \mathfrak{p}_u) \quad \text{ and } \quad \mathbb{S} = (G.[\mathfrak{p}, \mathfrak{p}]^{\perp})^{\mathrm{reg}}.$$

where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{p}_u$ (cf. [TY05, Proposition 39.2.4]).

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mathbb{S}$ a sheet with datum $(\mathfrak{l}, \mathbb{O}_{\mathfrak{l}})$, one can assume without loss of generality that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}$. In particular, $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h}$.

Lemma 2.1. (1) *Let $\mathbb{S}_{\mathfrak{l}}$ be a Dixmier sheet of rank one, that is, $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}\lambda$ with $\lambda \in \mathfrak{h} \setminus \{0\}$. Then*

$$\overline{\mathbb{S}_{\mathfrak{l}}} = \overline{G.\mathbb{C}^*\lambda} = G.(\mathbb{C}\lambda + \mathfrak{p}_u) = G.\mathbb{C}^*\lambda \cup \overline{\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)},$$

and

$$\mathbb{S}_{\mathfrak{l}} = G.\mathbb{C}^*\lambda \cup \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0).$$

(2) *Let $\mathbb{S}_{\mathfrak{l}_1}, \dots, \mathbb{S}_{\mathfrak{l}_n}$ be Dixmier sheets of rank one, that is, $\mathfrak{z}(\mathfrak{l}_i) = \mathbb{C}\lambda_i$ with $\lambda_i \in \mathfrak{h} \setminus \{0\}$, such that $\dim \operatorname{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(0) = \dim \operatorname{Ind}_{\mathfrak{l}_j}^{\mathfrak{g}}(0)$ for all i, j. Let X be a G-invariant, conic, Zariski closed subset of $\mathfrak{g}^*$ such that*

$$X \cap \mathfrak{h} = \bigcup_{i=1}^{n} \mathbb{C}\lambda_i, \quad X \cap \mathcal{N} \subset \bigcup_{i=1}^{n} \overline{\operatorname{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(0)}.$$

Then $X = \bigcup_{i=1}^{n} \overline{\mathbb{S}_{\mathfrak{l}_i}}$.

Part (1) of the lemma is probably well-known. We give a proof for the convenience of the reader.

Proof. (1) The equalities $\overline{\mathbb{S}_{\mathfrak{l}}} = \overline{G.\mathbb{C}^*\lambda} = G.(\mathbb{C}\lambda + \mathfrak{p}_u)$ are clear by [TY05, Corollaries 39.1.7 and 39.2.4]. Let us prove that $\overline{\mathbb{S}_{\mathfrak{l}}} = G.\mathbb{C}^*\lambda \cup \overline{\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)}$. The inclusion $G.\mathbb{C}^*\lambda \cup \overline{\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)} \subset \overline{\mathbb{S}_{\mathfrak{l}}}$ is known [TY05, Proposition 39.3.5] (and its proof). So it suffices to prove that $\mathbb{C}\lambda + \mathfrak{p}_u \subset G.\mathbb{C}^*\lambda \cup \overline{\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)}$ since $\overline{\mathbb{S}}$ is G-invariant.

Let $x = c\lambda + y \in \mathbb{C}\lambda + \mathfrak{p}_u$ with $c \in \mathbb{C}$ and $y \in \mathfrak{p}_u$. Assume that $c \in \mathbb{C}^*$. Then x_s and $c\lambda$ are G-conjugate. Since $x \in \overline{\mathbb{S}_{\mathfrak{l}}}$, $\dim \mathfrak{g}^x \geqslant \dim \mathfrak{g}^{\lambda}$. But $\dim \mathfrak{g}^x \geqslant \dim \mathfrak{g}^{\lambda}$ if and only if $x_n = 0$ since $\mathfrak{g}^x = (\mathfrak{g}^{x_s})^{x_n}$. Hence x is G-conjugate to $c\lambda$, and so $x \in G.\mathbb{C}^*\lambda$. If $c = 0$, then $x \in \mathfrak{p}_u$ and so x is nilpotent. But $(\mathbb{C}\lambda + \mathfrak{p}_u) \cap \mathcal{N} \subset \overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathcal{N} = \overline{\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)}$, whence the statement.

It remains to prove that $\mathbb{S}_{\mathfrak{l}} = G.\mathbb{C}^*\lambda \cup \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$. We have $\mathbb{S}_{\mathfrak{l}} = (G.(\mathbb{C}\lambda + \mathfrak{p}_u))^{\mathrm{reg}}$, and the inclusion $G.\mathbb{C}^*\lambda \cup \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0) \subset \mathbb{S}_{\mathfrak{l}}$ is clear. So it suffices to prove that $(\mathbb{C}\lambda + \mathfrak{p}_u)^{\mathrm{reg}} \subset G.\mathbb{C}^*\lambda \cup \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$ since $\overline{\mathbb{S}_{\mathfrak{l}}}$ and $\mathbb{S}_{\mathfrak{l}}$ are G-invariant. The above argument shows that for $x \in (\mathbb{C}\lambda + \mathfrak{p}_u)^{\mathrm{reg}} \setminus \mathfrak{p}_u$, $x \in G.\mathbb{C}^*\lambda$. And if $x \in (\mathfrak{p}_u)^{\mathrm{reg}}$, then $x \in (\mathfrak{p}_u)^{\mathrm{reg}} \cap \mathcal{N} \subset \mathbb{S}_{\mathfrak{l}} \cap \mathcal{N} = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$, whence the statement.

(2) The inclusion $\bigcup_i \overline{\mathbb{S}_{\mathfrak{l}_i}} \subset X$ is clear. Conversely, let $x \in X$. If x is nilpotent, then $x \in \bigcup_i \overline{\mathbb{S}_{\mathfrak{l}_i}}$ by the assumption. Assume that x is not nilpotent, that is $x_s \neq 0$. Since X is G-stable, we can assume that $x_s \in \mathfrak{h}$. If $x_n = 0$ then $x_s \in X \cap \mathfrak{h} \subset \bigcup_{i=1}^{n} \mathbb{S}_{\mathfrak{l}_i}$ by hypothesis.

Assume that $x_n \neq 0$ and let (e,h,f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ with $e = x_n$. We can assume that $h \in \mathfrak{h}$ so that $[h, x_s] = 0$. Let $\gamma : \mathbb{C}^* \to G$ be the one-parameter subgroup generated by h. Since X is G-invariant, for any $t \in \mathbb{C}^*$, the element

$$\gamma(t).x = x_s + t^2 x_n$$

belongs to X. Since X is closed, we deduce that $x_s \in X$. So, by the assumption, $x_s = c\lambda_i$ for some i and $c \in \mathbb{C}^*$. Therefore, because X is a cone, we can assume that $x_s = \lambda_i$. Thus $\mathfrak{l}_i = \mathfrak{g}^{x_s}$ and $x_n \in \mathfrak{l}_i$.

For any $t \in \mathbb{C}^*$, the element

$$t^2\gamma(t^{-1}).x = t^2(\lambda_i + t^{-2}x_n) = t^2\lambda_i + x_n$$

belongs to X. This shows that $\mathbb{C}^*\lambda_i + x_n \subset X$. Then

$$G.(\mathbb{C}^*\lambda_i + x_n) = G.(\mathfrak{z}(\mathfrak{l}_i)^{\mathrm{reg}} + x_n) = J_G(x) \subset X,$$

whence $\overline{J_G(x)} \subset X$ because X is closed.

Let $\mathbb{O}_{\mathfrak{l}_i,x_n}$ be the nilpotent orbit of x_n in $\mathfrak{l}_i$. One knows that $\mathrm{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}_i,x_n}) \subset \overline{J_G(x)}$ [Bo81]. So $\mathrm{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}_i,x_n}) \subset X$, and the assumption gives that $\mathrm{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}_i,x_n}) \subset \bigcup_j \mathrm{Ind}_{\mathfrak{l}_j}^{\mathfrak{g}}(0)$. In particular,

$$\dim \mathrm{Ind}_{\mathfrak{l}_i}^{\mathfrak{g}}(\mathbb{O}_{\mathfrak{l}_i,x_n}) = \dim \mathrm{Ind}_{\mathfrak{l}_j}^{\mathfrak{g}}(0)$$

for any j (this makes sense by our assumption on the Levi subalgebras $\mathfrak{l}_j$), whence $\mathrm{codim}_{\mathfrak{l}_i}(\mathbb{O}_{\mathfrak{l}_i,x_n}) = \mathrm{codim}_{\mathfrak{l}_j}(0) = \mathrm{codim}_{\mathfrak{l}_i}(0) = \dim \mathfrak{l}_i$ by the properties of induced nilpotent orbits. So $\mathbb{O}_{\mathfrak{l}_i,x_n} = \{0\}$, that is $x_n = 0$, and $x = \lambda_i \in \overline{\mathbb{S}_{\mathfrak{l}_i}}$. $\square$

Let P be the connected parabolic subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$. The G-action on

$$Y := G/(P,P),$$

where (P,P) is the commutator-subgroup of P, induces an algebra homomorphism

$$\psi_Y : U(\mathfrak{g}) \to \mathcal{D}_Y$$

from $U(\mathfrak{g})$ to the algebra $\mathcal{D}_Y$ of global differential operators on Y. Let

$$\mathscr{J}_Y := \ker \psi_Y$$

be the kernel of this homomorphism. It is a two-sided ideal of $U(\mathfrak{g})$. It has been shown by Borho and Brylinski [BB82, Corollary 3.11 and Theorem 4.6] that $\sqrt{\mathrm{gr}\,\mathscr{J}_Y}$ is the defining ideal of the Dixmier sheet closure determined by P, that is, $\overline{\mathbb{S}_{\mathfrak{l}}}$. Furthermore,

$$\mathscr{J}_Y = \bigcap_{\lambda \in \mathfrak{z}(\mathfrak{l})^*} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda.$$

Here, for $\lambda \in \mathfrak{p}^*$, $\mathbb{C}_\lambda$ stands for the one-dimensional representation of $\mathfrak{p}$ corresponding to λ, and we extend a linear form $\lambda \in \mathfrak{z}(\mathfrak{l})^*$ to $\mathfrak{p}^*$ by setting $\lambda(x) = 0$ for $x \in [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{p}_u$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\ |\ )$, $\mathfrak{z}(\mathfrak{l})^*$ identifies with $\mathfrak{z}(\mathfrak{l})$. In particular, if $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}\lambda$ for some nonzero semisimple element $\lambda \in \mathfrak{g}$, we get

$$\mathscr{J}_Y = \bigcap_{t \in \mathbb{C}} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\lambda}.$$

In fact

$$\mathscr{J}_Y = \bigcap_{t \in Z} \mathrm{Ann}\, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\lambda}. \tag{1}$$

for any Zariski dense subset Z of $\mathbb{C}$ ([BJ77]).

In this paper, we shall consider sheets in Lie algebras of classical types A_r and D_r. Let us introduce more specific notations. Let $n \in \mathbb{N}^*$, and denote by $\mathscr{P}(n)$ the set of partitions of n. As a rule, we write an element $\boldsymbol{\lambda}$ of $\mathscr{P}(n)$ as a decreasing sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ omitting the zeroes.

Case $\mathfrak{sl}_n$. According to [CMa93, Theorem 5.1.1], nilpotent orbits of $\mathfrak{sl}_n$ are parametrized by $\mathscr{P}(n)$. For $\boldsymbol{\lambda} \in \mathscr{P}(n)$, we denote by $\mathbb{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{sl}_n$. In $\mathfrak{sl}_n$, all sheets are Dixmier and each nilpotent orbit is contained in exactly one sheet. The Levi subalgebras of $\mathfrak{sl}_n$, and so the (Diximer) sheets, are parametrized by compositions of n. More precisely, if $\boldsymbol{\lambda} \in \mathscr{P}(n)$, then the (Dixmier) sheet associated with $\boldsymbol{\lambda}$ is the unique sheet containing $\mathbb{O}_{{}^t\boldsymbol{\lambda}}$ where ${}^t\boldsymbol{\lambda}$ is the dual partition of $\boldsymbol{\lambda}$.

Case $\mathfrak{so}_n$. Set

$$\mathscr{P}_1(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n)\ ;\ \text{number of parts of each even number is even}\}.$$

According to [CMa93, Theorem 5.1.2 and Theorem 5.1.4], nilpotent orbits of $\mathfrak{so}_n$ are parametrized by $\mathscr{P}_1(n)$, with the exception that each *very even* partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ (i.e., $\boldsymbol{\lambda}$ has only even parts) corresponds to two nilpotent orbits. For $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, not very even, we denote by $\mathbb{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{so}_n$. For very even $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, we denote by $\mathbb{O}_{\boldsymbol{\lambda}}^{I}$ and $\mathbb{O}_{\boldsymbol{\lambda}}^{II}$ the two corresponding nilpotent orbits of $\mathfrak{so}_n$. In fact, their union form a single O_n-orbit.

Contrary to the $\mathfrak{sl}_n$ case, it may happen in the $\mathfrak{so}_n$ case that a given nilpotent orbit belongs to different sheets, and not all sheets are Dixmier.

3. Some useful lemmas

Let f be a nilpotent element of $\mathfrak{g}$ that we embed into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$ and let

$$\mathscr{S}_f := \chi + (\mathfrak{g}^f)^*$$

be the *Slodowy slice associated with* (e, h, f) where

$$\chi := (f|\cdot) \in \mathfrak{g}^*.$$

Denote by $\mathfrak{g}(h, i)$ the i-eigenspace of $\mathrm{ad}(h)$ for $i \in \mathbb{Z}$. Choose a Lagrangian subspace $\mathfrak{L} \subset \mathfrak{g}(h, 1)$ and set

$$\mathfrak{m} := \mathfrak{L} \oplus \bigoplus_{i \geqslant 2} \mathfrak{g}(h, i), \qquad J_\chi := \sum_{x \in \mathfrak{m}} \mathbb{C}[\mathfrak{g}^*](x - \chi(x)).$$

Let M be the unipotent subgroup of G corresponding to $\mathfrak{m}$.

Let

$$\mu : \mathfrak{g}^* \to \mathfrak{m}^*$$

be the moment map for the M-action, which is just a restriction map. By [GG02], the adjoint action map gives the isomorphism

$$M \times \mathscr{S}_f \xrightarrow{\sim} \mu^{-1}(\chi),$$

and thus,

$$\mathscr{S}_f \cong \mu^{-1}(\chi)/M.$$

In particular,

$$\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[\mu^{-1}(\chi)]^M = (\mathbb{C}[\mathfrak{g}^*]/J_\chi)^M .$$

Let $\overline{\mathcal{HC}}$ be the category of finitely generated $(\mathbb{C}[\mathfrak{g}^*], G)$-modules, that is, the category of finitely generated $\mathbb{C}[\mathfrak{g}^*]$-modules K equipped with the G-module structure such that $g.(f.m) = (g(f)).g.m$ for $g \in G$, $f \in \mathbb{C}[\mathfrak{g}^*]$, $m \in K$.

Theorem 3.1 ([Gi09], see also [Ar15b])**.** (1) *The functor*

$$H_f \colon \overline{\mathcal{HC}} \to \mathbb{C}[\mathscr{S}_f]\text{-mod}, \quad K \mapsto (K/J_\chi K)^M,$$

is exact.

(2) *For any $K \in \overline{\mathcal{HC}}$, $\operatorname{supp}_{\mathbb{C}[\mathscr{S}_f]} H_f(K) = (\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} K) \cap \mathscr{S}_f$.*

Lemma 3.2. *Let $K \in \overline{\mathcal{HC}}$. Then $\overline{G.f} \subset \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} K$ if and only if $K \neq J_\chi K$.*

Proof. Since $\mathscr{S}_f$ admits a $\mathbb{C}^*$-action contracting to f, Theorem 3.1(2) implies that $\overline{G.f} \subset \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]} K$ if and only if $H_f(K) \neq 0$. However $H_f(K) \neq 0$ if and only if $K/J_\chi K \neq 0$ by [Gi09, Proposition 3.3.6]. □

Let I be an $\operatorname{ad}\mathfrak{g}$-invariant ideal of $\mathbb{C}[\mathfrak{g}^*]$, so that $\mathbb{C}[\mathfrak{g}^*]/I \in \overline{\mathcal{HC}}$. Applying Lemma 3.2 to $K = \mathbb{C}[\mathfrak{g}^*]/I$ we obtain the following assertion.

Lemma 3.3. *Let I be an $\operatorname{ad}\mathfrak{g}$-invariant ideal of $\mathbb{C}[\mathfrak{g}^*]$. Then $\overline{G.f} \not\subset V(I)$ if and only if*

$$\mathbb{C}[\mathfrak{g}^*] = I + J_\chi$$

where $V(I)$ is the zero locus of I in $\mathfrak{g}^*$.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$. Thus $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h}$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{p}_u$. Assume that $e \in (\mathfrak{p}_u)^{\mathrm{reg}}$ and $h \in \mathfrak{h}$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\ |\ )$, we get

$$\mathscr{S}_f \cong f + \mathfrak{g}^e,$$

and by [Kat82, Lemma 3.2] (see also [Bu11, Proposition 3.2]), we have

$$\mathbb{S}_{\mathfrak{l}} = G.(f + \mathfrak{z}(\mathfrak{l})).$$

Note that we have the following decomposition:

$$\mathfrak{g}(h,0) = [f, \mathfrak{g}(h,2)] \oplus (\mathfrak{g}(h,0) \cap \mathfrak{g}^e),$$

and since $\operatorname{ad}(f)$ induces a bijection from $\mathfrak{g}(h,2)$ to $[f, \mathfrak{g}(h,2)]$, for any $x \in \mathfrak{g}(h,0)$ there is a well-defined element $\eta(x) \in \mathfrak{g}(h,2)$ such that

$$x - [f, \eta(x)] \in \mathfrak{g}(h,0) \cap \mathfrak{g}^e.$$

Lemma 3.4. *Assume that $\mathfrak{z}(\mathfrak{l})$ is generated by a nonzero element λ of $\mathfrak{h}$ and that $\mathfrak{g}(h,i) = 0$ for $i > 2$.*

(1) *The set $\{\exp(\operatorname{ad}\eta(t\lambda))(f + t\lambda) \mid t \in \mathbb{C}\}$ is an irreducible component of $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f$. Moreover, if $\mathfrak{g}$ is classical, then $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f$ is irreducible and $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f = \overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f = \{\exp(\operatorname{ad}\eta(t\lambda))(f + t\lambda) \mid t \in \mathbb{C}\}$.*

(2) *If $\mathfrak{g}$ is classical then, as schemes, $\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f \cong \operatorname{Spec}\mathbb{C}[z]$. Here we endow $\overline{\mathbb{S}_{\mathfrak{l}}}$ with its natural structure of irreducible reduced scheme.*

Proof. (1) The first assertion results from [ImH05, Lemma 2.9] and the proof of [Bu11, Lemma 3.4] (which is a reformulation of [Kat82, Lemma 3.2 and Lemma 5.1]). In the case that $\mathfrak{g}$ is classical, $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f$ is irreducible (see [ImH05, Theorem 5.2 and Theorem 6.2]) and so $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f = \{\exp(\operatorname{ad}\eta(t\lambda))(f+t\lambda) \mid t \in \mathbb{C}\}$. On the other hand, by Lemma 2.1(1),

$$\overline{\mathbb{S}_{\mathfrak{l}}} = \overline{G.\mathbb{C}^*\lambda} = G.\mathbb{C}^*\lambda \cup \overline{G.f}$$

and

$$\mathbb{S}_{\mathfrak{l}} = G.\mathbb{C}^*\lambda \cup G.f.$$

So, $\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f = \{\exp(\operatorname{ad}\eta(t\lambda))(f+t\lambda) \mid t \in \mathbb{C}\}$ since $\overline{G.f} \cap \mathscr{S}_f = \{f\}$ and $f \in \{\exp(\operatorname{ad}\eta(t\lambda))(f+t\lambda) \mid t \in \mathbb{C}\}$. In particular, $\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f$ is an irreducible variety of dimension one.

(2) According to [Gi09, Corollary 1.3.8(1)], $\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f$ is a reduced complete intersection in $\overline{\mathbb{S}_{\mathfrak{l}}}$ since $\overline{\mathbb{S}_{\mathfrak{l}}}$ is reduced, whence $\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f \cong \operatorname{Spec}\mathbb{C}[z]$ as a scheme by (1). □

Remark 3.5. Assume that $\mathfrak{g}$ is classical. Define a one-parameter subgroup $\tilde{\gamma} : \mathbb{C}^* \to G$ by:

$$\forall\, t \in \mathbb{C}^*,\ \forall\, x \in \mathfrak{g}, \qquad \tilde{\gamma}(t).x := t^2\gamma(t).x$$

where $\gamma(t)$ is the one-parameter of G generated by $\operatorname{ad}(h)$. In the notation of Lemma 3.4, the sets $\mathbb{S}_{\mathfrak{l}}$ and $\mathscr{S}_f$ are both stabilized by $\tilde{\gamma}(t)$, and we have:

$$\overline{\mathbb{S}_{\mathfrak{l}}} \cap \mathscr{S}_f = \mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f = \overline{\{\tilde{\gamma}(t).\mu \mid t \in \mathbb{C}^*\}},$$

for any nonzero semisimple μ in $\mathbb{S}_{\mathfrak{l}} \cap \mathscr{S}_f$, for example

$$\mu = \exp(\operatorname{ad}\eta(z))(f+\lambda).$$

Let Ω be the Casimir element of $\mathfrak{g}$ and denote by I_Ω the ideal of $\mathbb{C}[\mathfrak{g}^*]$ generated by Ω.

Lemma 3.6. *Assume that $\mathfrak{g}$ is classical. Let I be a homogeneous $\operatorname{ad}\mathfrak{g}$-invariant ideal of $\mathbb{C}[\mathfrak{g}^*]$, $\mathbb{S}_{\mathfrak{l}}$ a Dixmier sheet of rank one, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{p}_u$. Let (e,f,h) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ such that $e \in (\mathfrak{p}_u)^{reg}$. Further, assume that the following conditions are satisfied:*

1. $\mathfrak{g}(h,i) = 0$ *for* $i > 2$,
2. $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(\mathbb{C}[\mathfrak{g}^*]/I) = \overline{\mathbb{S}_{\mathfrak{l}}}$,
3. $I + I_\Omega$ *is the defining ideal of* $\overline{G.f}$,
4. $H_f(\mathbb{C}[\mathfrak{g}^*]/I) \cong \mathbb{C}[z]$ *as algebras,*
5. $\Omega(\lambda) \neq 0$,

Then I is prime, that is, $I = \sqrt{I}$.

Condition (2) implies that $\sqrt{I}$ is defining ideal of $\overline{\mathbb{S}_{\mathfrak{l}}}$ since $\overline{\mathbb{S}_{\mathfrak{l}}}$ is irreducible. In particular, $\sqrt{I}$ is prime. Also, condition (3) means that $I + I_\Omega$ is prime. Note that conditions (2) and (3) imply that

$$\sqrt{I} \subset I + I_\Omega.$$

since $\overline{G.f} \subset \overline{\mathbb{S}_{\mathfrak{l}}}$.

Proof. Set $J := \sqrt{I}$. Then $J/I \in \overline{\mathcal{HC}}$. Since the sequence

$$0 \to J/I \to \mathbb{C}[\mathfrak{g}^*]/I \to \mathbb{C}[\mathfrak{g}^*]/J \to 0$$

is exact, we get an exact sequence

$$0 \to H_f(J/I) \to H_f(\mathbb{C}[\mathfrak{g}^*]/I) \to H_f(\mathbb{C}[\mathfrak{g}^*]/J) \to 0$$

by Theorem 3.1 (1). Furthermore by Theorem 3.1 (2),

$$\operatorname{Spec} H_f(\mathbb{C}[\mathfrak{g}^*]/J) = \operatorname{Spec}(\mathbb{C}[\mathfrak{g}^*]/J) \cap \mathscr{S}_f = \operatorname{Spec} \mathbb{C}[z].$$

The last equality comes from Lemma 3.4 (2) since J is the defining ideal of $\overline{\mathbb{S}_{\mathfrak{l}}}$. Hence by condition (4), we get

$$H_f(J/I) = 0.$$

By [Gi09, Proposition 3.3.6], $H_f(J/I) \neq 0$ if and only if $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(J/I) \supset \overline{G.f}$. However, $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(J/I) \subset \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(\mathbb{C}[\mathfrak{g}^*]/I) = \overline{\mathbb{S}_{\mathfrak{l}}}$ and any G-invariant closed cone of $\overline{\mathbb{S}_{\mathfrak{l}}}$ which strictly contains $G.f$ contains $\overline{G.f}$. Therefore,

$$\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(J/I) \subset G.f$$

since $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(J/I)$ is a G-invariant closed cone of $\mathfrak{g}$. In particular, $\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(J/I)$ is contained in the nilpotent cone $\mathcal{N}$. Since Ω is a nonzero homogeneous element in the defining ideal of $\mathcal{N}$, we deduce that Ω acts nilpotently on J/I. Hence for n sufficiently large,

$$\Omega^n J/I = 0. \tag{2}$$

We can now achieve the proof of the lemma. We have to show that $J \subset I$. Let $a \in J$. Since $J \subset I + I_\Omega$, for some $b_1 \in I$ and $f_1 \in \mathbb{C}[\mathfrak{g}^*]$, we have

$$a = b_1 + \Omega f_1.$$

Since J is prime and $\Omega \notin J$ by condition (5), $f_1 \in J$. Applying what foregoes to the element f_1 of J, we get that for some $b_2 \in I$ and $f_2 \in J$,

$$a = b_1 + \Omega(b_2 + \Omega f_2) = c_2 + \Omega^2 f_2,$$

with $c_2 := b_1 + \Omega b_2 \in I$. A rapid induction shows that for any $n \in \mathbb{Z}_{>0}$, there exist $c_n \in I$ and $f_n \in J$ such that

$$a = c_n + \Omega^n f_n.$$

But $c_n + \Omega^n f_n \in I$ for n big enough by (2), whence $a \in I$. □

4. Associated variety and singular support of affine vertex algebras

Let

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

be the affine Kac-Moody Lie algebra associated with $\mathfrak{g}$ and $(\mid)$, with the commutation relations

$$[x(m), y(n)] = [x, y](m+n) + m(x|y)\delta_{m+n,0}K, \quad [D, x(m)] = mx(m), \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for $m, n \in \mathbb{Z}$ and $x, y \in \mathfrak{g}$, where $x(m) = x \otimes t^m$. For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$ on which $\mathfrak{g}[t] \oplus \mathbb{C}D$ acts trivially and K acts as multiplication by k. The space $V^k(\mathfrak{g})$ is naturally a vertex algebra, and it is called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. By the PBW theorem, $V^k(\mathfrak{g}) \cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces.

The vertex algebra $V^k(\mathfrak{g})$ is naturally graded:

$$V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d, \quad V^k(\mathfrak{g})_d = \{a \in V^k(\mathfrak{g}) \mid Da = -da\},$$

Let $V_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$. As a $\widehat{\mathfrak{g}}$-module, $V_k(\mathfrak{g})$ is isomorphic to the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, where Λ_0 is the dual element of K.

A $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k.

As in the introduction, let X_V be the *associated variety* [Ar12] of a vertex algebra V, which is the maximum spectrum of the *Zhu's* C_2*-algebra*

$$R_V := V/C_2(V)$$

of V. In the case that V is a quotient of $V^k(\mathfrak{g})$, $V/C_2(V) = V/\mathfrak{g}[t^{-1}]t^{-2}V$ and we have a surjective Poisson algebra homomorphism

$$\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \twoheadrightarrow R_V = V/\mathfrak{g}[t^{-1}]t^{-2}V, \quad x \mapsto \overline{x(-1)} + \mathfrak{g}[t^{-1}]t^{-2}V,$$

where $\overline{x(-1)}$ denotes the image of $x(-1)$ in the quotient V. Then X_V is just the zero locus of the kernel of the above map in $\mathfrak{g}^*$. It is G-invariant and conic.

Conjecture 1. *Let* $V = \oplus_{d \geqslant 0} V_d$ *be a simple, finitely strongly generated (i.e.,* R_V *is finitely generated), positively graded conformal vertex operator algebra such that* $V_0 = \mathbb{C}$.

(1) X_V *is equidimensional.*
(2) *Assume that* X_V *has finitely many symplectic leaves. Then* X_V *is irreducible. In particular* $X_{V_k(\mathfrak{g})}$ *is irreducible if* $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$.

For a scheme X of finite type, let $J_m X$ be the m-th jet scheme of X, and $J_\infty X$ the infinite jet scheme of X (or the arc space of X). Recall that the scheme $J_m X$ is determined by its functor of points: for every $\mathbb{C}$-algebra A, there is a bijection

$$\operatorname{Hom}(\operatorname{Spec} A, J_m X) \cong \operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X).$$

If $m > n$, we have a natural morphism $J_m X \to J_n X$. This yields a projective system $\{J_m X\}$ of schemes, and the infinite jet scheme $J_\infty X$ is the projective limit $\varprojlim J_m X$ in the category of schemes. Let $\pi_m \colon J_m X \to X$, $m > 0$, and $\pi_\infty \colon J_\infty X \to X$ be the natural morphisms.

If X is an affine Poisson scheme then its coordinate ring $\mathbb{C}[J_\infty X]$ is naturally a Poisson vertex algebra ([Ar12]).

Let $V = F^0 V \supset F^1 V \supset \dots$ be the canonical decreasing filtration of the vertex algebra V defined by Li [Li05]. The associated graded algebra $\operatorname{gr} V = \bigoplus_{p \geqslant 0} F^p V / F^{p+1} V$ is naturally a Poisson vertex algebra. In particular, it has the structure of a commutative algebra. We have $F^1 V = C_2(V)$ by definition, and by restricting the Poisson vertex algebra structure of $\operatorname{gr} V$ we obtain the Poisson structure of $R_V = V/F^1 V \subset \operatorname{gr} V$. There is a surjection

$$\mathbb{C}[J_\infty X_V] \to \operatorname{gr} V$$

of Poisson vertex algebras ([Li05, Ar12]). By definition [Ar12], the *singular support* of V is the subscheme

$$SS(V) = \operatorname{Spec}(\operatorname{gr} V)$$

of $J_\infty X_V$.

Theorem 4.1. *Let V be a quotient of the vertex algebra $V^k(\mathfrak{g})$. Suppose that $X_V = \overline{G.\mathbb{C}^* x}$ for some $x \in \mathfrak{g}$. Then*

$$SS(V)_{\mathrm{red}} = J_\infty X_V = J_\infty \overline{G.\mathbb{C}^* x} = \overline{J_\infty G.\mathbb{C}^* x}.$$

Proof. By [Ar12, Lemma 3.3.1], $\tilde{X}_V = \pi_\infty(SS(V))$. We know that $SS(V) \subset J_\infty \tilde{X}_V$, so that $SS(V)_{\mathrm{red}} \subset J_\infty X_V$. Let us prove the other inclusion.

Set $X = X_V$ and $U = G.\mathbb{C}^* x$. Since U is an irreducible open dense subset of X, $\pi_m^{-1}(U) = J_m U$ for any $m > 0$ [EM09, Lemma 2.3], and $\overline{\pi_m^{-1}(U)} = \overline{\pi_m^{-1}(X_{\mathrm{reg}})}$ is an irreducible component of $J_m X$. Hence

$$J_\infty X = \overline{\pi_\infty^{-1}(U)} = \overline{J_\infty U}$$

because $J_\infty X$ is irreducible [Ko73] and closed. Therefore, it is enough to prove that $SS(V)_{\mathrm{red}}$ contains $J_\infty U$ since $SS(V)_{\mathrm{red}}$ is closed.

The map

$$\mu \colon G \times (\mathbb{C}^* x) \to G.\mathbb{C}^* x, \quad (g, tx) \mapsto g.(tx)$$

is a submersion at each point, so it is smooth (cf. [Ha76, Ch. III, Proposition 10.4]). Hence by [EM09, Remark 2.10], we get that the induced map

$$\mu_\infty \colon J_\infty G \times J_\infty \mathbb{C}^* x \to J_\infty G.\mathbb{C}^* x$$

is surjective and formally smooth, and so

$$\mu_\infty(J_\infty G \times J_\infty \mathbb{C}^*.x) \cong J_\infty G.\mathbb{C}^* x.$$

Since $SS(V)_{\mathrm{red}}$ is $J_\infty G$-invariant and $J_\infty \mathbb{C}^*$-invariant, we deduce that $J_\infty G.\mathbb{C}^* x = J_\infty U$ is contained in $SS(V)_{\mathrm{red}}$. $\square$

The closures of nilpotent orbits satisfy the conditions of Theorem 4.1, and also Dixmier sheets of rank one by Lemma 2.1 (1).

Corollary 4.2. *Let V be a quotient of the affine vertex algebra $V^k(\mathfrak{g})$.*

(1) *Suppose that $X_V = \overline{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O}$ of $\mathfrak{g}$. Then*

$$SS(V)_{\mathrm{red}} = J_\infty \overline{\mathbb{O}} = \overline{J_\infty \mathbb{O}}.$$

(2) *Suppose that $X_V = \overline{\mathbb{S}}$ for some Dixmier sheet $\mathbb{S}$ of $\mathfrak{g}$ of rank one. Then*

$$SS(V)_{\mathrm{red}} = J_\infty \overline{\mathbb{S}} = \overline{J_\infty \mathbb{S}}.$$

See [Ar15a] and [AM15] for examples of affine vertex algebras $V_k(\mathfrak{g})$ satisfying the condition of Corollary 4.2 (1).

5. Zhu's algebra of affine Vertex algebras

For a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra $V = \bigoplus_d V_d$, let $A(V)$ be the Zhu's algebra of V,

$$A(V) = V / V \circ V,$$

where $V \circ V$ is the $\mathbb{C}$-span of the vectors

$$a \circ b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-2)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$, and $V \to (\operatorname{End} V)[[z, z^{-1}]]$, $a \mapsto \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, denotes the state-field correspondence. The space $A(V)$ is a unital associative algebra with respect to the multiplication defined by

$$a * b := \sum_{i\geqslant 0} \binom{\Delta}{i} a_{(i-1)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$.

Let $M = \bigoplus_{d \geqslant d_0} M_d$, $M_{d_0} \neq 0$, be a positive energy representation of V. Then $A(V)$ naturally acts on its top weight space $M_{top} := M_{d_0}$, and the correspondence $M \mapsto M_{top}$ defines a bijection between isomorphism classes of simple positive energy representations of V and simple $A(V)$-modules ([Z96]).

The vertex algebra V is called a *chiralization* of an algebra A if $A(V) \cong A$.

For instance, consider the universal affine vertex algebra $V^k(\mathfrak{g})$. The Zhu's algebra $A(V^k(\mathfrak{g}))$ is naturally isomorphic to $U(\mathfrak{g})$ ([FZ92], see also [Ar16a, Lemma 2.3]), and hence, $V^k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})$.

Let $\widehat{\mathcal{J}}_k$ be the unique maximal ideal of $V^k(\mathfrak{g})$, so that

$$V_k(\mathfrak{g}) = V^k(\mathfrak{g})/\widehat{\mathcal{J}}_k.$$

We have an exact sequence $A(\widehat{\mathcal{J}}_k) \to U(\mathfrak{g}) \to A(V_k(\mathfrak{g})) \to 0$ since the functor $A(?)$ is right exact and thus $A(V_k(\mathfrak{g}))$ is the quotient of $U(\mathfrak{g})$ by the image $\mathcal{I}_k$ of $A(\widehat{\mathcal{J}}_k)$ in $U(\mathfrak{g})$:

$$A(V_k(\mathfrak{g})) = U(\mathfrak{g})/\mathcal{I}_k.$$

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. Then $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ is a Cartan subalgebra of $\widehat{\mathfrak{g}}$. A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called of level k if $\lambda(K) = k$. The top degree component of $L(\lambda)$ is $L_{\mathfrak{g}}(\bar\lambda)$, where $\bar\lambda$ is the restriction of λ to $\mathfrak{h}$. Hence, by Zhu's Theorem, the level k representation $L(\lambda)$ is a $V_k(\mathfrak{g})$-module if and only if $\mathcal{I}_k L_{\mathfrak{g}}(\bar\lambda) = 0$.

Set $U(\mathfrak{g})^{\mathfrak{h}} := \{u \in U(\mathfrak{g}) \mid [h, u] = 0 \text{ for all } h \in \mathfrak{h}\}$ and let

$$\Upsilon \colon U(\mathfrak{g})^{\mathfrak{h}} \to U(\mathfrak{h})$$

be the *Harish-Chandra projection map* which is the restriction of the projection map $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \to U(\mathfrak{h})$ to $U(\mathfrak{g})^{\mathfrak{h}}$. It is known that Υ is an algebra homomorphism. For a two-sided ideal I of $U(\mathfrak{g})$, the *characteristic variety of I* (without ρ-shift) is defined as

$$\mathcal{V}(I) = \{\lambda \in \mathfrak{h}^* \mid p(\lambda) = 0 \text{ for all } p \in \Upsilon(I^{\mathfrak{h}})\}$$

where $I^{\mathfrak{h}} = I \cap U(\mathfrak{g})^{\mathfrak{h}}$, cf. [Jo77]. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(\ |\ )$, and thus $\mathfrak{h}^*$ with $\mathfrak{h}$, we view $\mathcal{V}(I)$ as a subset of $\mathfrak{h}$.

Proposition 5.1 ([Ar16a, Proposition 2.5])**.** *For a level k weight $\lambda \in \widehat{\mathfrak{h}}^*$, $L(\lambda)$ is a $V_k(\mathfrak{g})$-module if and only if $\bar\lambda \in \mathcal{V}(\mathcal{I}_k)$.*

The Zhu's algebra $A(V)$ is related with the Zhu's C_2 algebra R_V as follows. The grading of V induces a filtration of $A(V)$ and the associated graded algebra $\operatorname{gr} A(V)$ is naturally a Poisson algebra ([Z96]). There is a natural surjective homomorphism

$$\eta_V : R_V \twoheadrightarrow \operatorname{gr} A(V)$$

of Poisson algebras ([DSK06, Proposition 2.17(c)], [ALY14, Proposition 3.3]). In particular, $\operatorname{Spec}(\operatorname{gr} A(V))$ is a subscheme of X_V.

For $V = V_k(\mathfrak{g})$ this means the following. We have

$$V(\operatorname{gr}\mathcal{I}_k) \subset X_{V_k(\mathfrak{g})},$$

where $V(\operatorname{gr}\mathcal{I}_k)$ is the zero locus of $\operatorname{gr}\mathcal{I}_k \subset \mathbb{C}[\mathfrak{g}^*]$ in $\mathfrak{g}^*$.

The map $\eta_{V_k(\mathfrak{g})}$ is not necessarily an isomorphism. However, conjecturally [Ar15b] we have $V(\operatorname{gr}\mathcal{I}_k) = X_{V_k(\mathfrak{g})}$.

6. Affine W-algebras

For a nilpotent element f of $\mathfrak{g}$, let $\mathcal{W}^k(\mathfrak{g}, f)$ be the W-algebra associated with $(\mathfrak{g}, f)$ at level k, defined by the generalized quantized Drinfeld-Sokolov reduction [FF90, KRW03]:

$$\mathcal{W}^k(\mathfrak{g}, f) = H_f^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})).$$

Here $H_f^{\frac{\infty}{2}+0}(M)$ is the corresponding BRST cohomology with coefficients in a $\widehat{\mathfrak{g}}$-module M.

Let (e, f, h) be an $\mathfrak{sl}_2$-triple associated with f. The W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is conformal provided that $k \neq -h^\vee$, which we assume in this paper, and the central charge $c_f(k)$ of $\mathcal{W}^k(\mathfrak{g}, f)$ is given by

$$c_f(k) = \dim \mathfrak{g}(h, 0) - \frac{1}{2}\dim \mathfrak{g}(h, 1) - 12\left|\frac{\rho}{\sqrt{k + h^\vee}} - \sqrt{k + h^\vee}\frac{h}{2}\right|^2,$$

where ρ is the half sum of the positive roots of $\mathfrak{g}$. We have

$$\mathcal{W}^k(\mathfrak{g}, f) = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}} \mathcal{W}^k(\mathfrak{g}, f)_\Delta,$$
$$\mathcal{W}^k(\mathfrak{g}, f)_0 = \mathbb{C}, \quad \mathcal{W}^k(\mathfrak{g}, f)_{1/2} = 0, \quad \mathcal{W}^k(\mathfrak{g}, f)_1 \cong \mathfrak{g}^\natural,$$

where $\mathfrak{g}^\natural$ is the centralizer in $\mathfrak{g}$ of the $\mathfrak{sl}_2$-triple (e, f, h).

We have [DSK06, Ar15a] a natural isomorphism $R_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$\tilde{X}_{\mathcal{W}^k(\mathfrak{g},f)} = \mathscr{S}_f.$$

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. Then $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant, Poisson subvariety of $\mathscr{S}_f$. Since it is $\mathbb{C}^*$-invariant, $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $X_{\mathcal{W}_k(\mathfrak{g},f)} = \{f\}$.

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k. We have a functor

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H_f^{\frac{\infty}{2}+0}(M),$$

where $\mathcal{W}^k(\mathfrak{g}, f)$-Mod denotes the category of $\mathcal{W}^k(\mathfrak{g}, f)$-modules.

Let KL_k be the full subcategory of $\mathcal{O}_k$ consisting of objects M on which $\mathfrak{g}$ acts locally finitely. Note that $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ are objects of KL_k.

Theorem 6.1 ([Ar15a], k, f arbitrary)**.** (1) *$H_f^{\frac{\infty}{2}+i}(M) = 0$ for all $i \neq 0$, $M \in \mathrm{KL}_k$. In particular, the functor $\mathrm{KL}_k \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod, $M \mapsto H_f^{\frac{\infty}{2}+0}(M)$, is exact.*

(2) *For any quotient V of $V^k(\mathfrak{g})$ we have*

$$R_{H_f^{\frac{\infty}{2}+0}(V)} \cong H_f(R_V),$$

where $H_f(R_V)$ is defined in Theorem 3.1. Hence, $\tilde{X}_{H_f^{\frac{\infty}{2}+0}(V)}$ is isomorphic to the scheme theoretic intersection $\tilde{X}_V \times_{\mathfrak{g}^*} \mathscr{S}_f$.

(3) $H_f^{\frac{\infty}{2}+0}(V) \neq 0$ *if and only if* $\overline{G.f} \subset X_V$.

(4) $H_f^{\frac{\infty}{2}+0}(V)$ *is lisse if* $X_V = \overline{G.f}$.

Let f_θ be a root vector of the highest root θ of $\mathfrak{g}$, which is a minimal nilpotent element so that $f \in \mathbb{O}_{\min}$ where $\mathbb{O}_{min}$ is the minimal nilpotent orbit of $\mathfrak{g}$.

Theorem 6.2 ([Ar05], $f = f_\theta$)**.** (1) $H_{f_\theta}^{\frac{\infty}{2}+i}(M) = 0$ *for all* $i \neq 0$, $M \in \mathcal{O}_k$. *In particular, the functor* $\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod, $M \mapsto H_f^{\frac{\infty}{2}+0}(M)$, *is exact.*

(2) $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda)) \neq 0$ *if and only if* $\lambda(\alpha_0^\vee) \notin \mathbb{N}_{\geqslant 0}$. *If this is the case,* $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ *is a simple* $\mathcal{W}^k(\mathfrak{g}, f)$*-module, where* $\alpha_0^\vee = -\theta^\vee + K$.

Recall that f is a *short* nilpotent element if

$$\mathfrak{g} = \mathfrak{g}(h, -2) \oplus \mathfrak{g}(h, 0) \oplus \mathfrak{g}(h, 2).$$

If this is the case, we have $\frac{1}{2}h \in P^\vee$, where $P^\vee$ is the coroot lattice of $\mathfrak{g}$, and $\frac{1}{2}h$ defines an element of the extended affine Weyl group $\widetilde{W} = W \ltimes P^\vee$ of $\widehat{\mathfrak{g}}$, which we denote by $t_{\frac{1}{2}h}$. Here W is the Weyl group of $\mathfrak{g}$. Let $\tilde{t}_{\frac{1}{2}h}$ be a Tits lifting of $t_{\frac{1}{2}h}$ to an automorphism of $\widehat{\mathfrak{g}}$.

Set

$$D_h = D + \frac{1}{2}h,$$

and put $M_{d,h} = \{m \in M \mid D_h m = dm\}$ for a $\widehat{\mathfrak{g}}$-module M. The operator D_h extends to the grading operator of $\mathcal{W}^k(\mathfrak{g}, f)$ ([KRW03, Ar05]).

The subalgebra $\tilde{t}_{\frac{1}{2}h}(\mathfrak{g})$ acts on each homogeneous component $M_{d,h}$ because $[D_h, t_{\frac{1}{2}h}(\mathfrak{g})] = 0$. Note that $\tilde{t}_{\frac{1}{2}h}(\mathfrak{g})$ is the subalgebra of $\widehat{\mathfrak{g}}$ generated by root vectors of roots $t_{\frac{1}{2}h}(\alpha) = \alpha - \frac{1}{2}\alpha(h)\delta$, $\alpha \in \Delta$, where $\delta \in \widehat{\mathfrak{h}}^*$ is the dual element of D. In particular, $\tilde{t}_{\frac{1}{2}h}(\mathfrak{g})$ contains

$$\mathfrak{l} := \mathfrak{g}(h, 0)$$

as a Levi subalgebra.

We regard $M_{d,h}$ as a $\mathfrak{g}$-module through the isomorphism $\tilde{t}_{\frac{1}{2}h}(\mathfrak{g}) \cong \mathfrak{g}$.

Since $\alpha(D_h) \geqslant 0$ for all positive roots α of $\widehat{\mathfrak{g}}$ by the assumption that f is short, we have

$$L(\lambda) = \bigoplus_{d \leqslant \lambda(D_h)} L(\lambda)_{d,h}.$$

Let $\mathcal{O}_k^{\mathfrak{l}}$ be the full subcategory of category $\mathcal{O}_k$ of $\widehat{\mathfrak{g}}$ consisting of objects on which $\mathfrak{l}$ acts locally finitely.

Theorem 6.3. *Let f be a short nilpotent element as above.*

(1) $H_f^{\frac{\infty}{2}+i}(M) = 0$ *for all* $i \neq 0$, $M \in \mathcal{O}_k^{\mathfrak{l}}$. *In particular, the functor* $\mathcal{O}_k^{\mathfrak{l}} \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod, $M \mapsto H_f^{\frac{\infty}{2}+0}(M)$, *is exact.*

(2) *Let* $L(\lambda) \in \mathcal{O}_k^{\mathfrak{l}}$. *Then* $H_f^{\frac{\infty}{2}+0}(L(\lambda)) \neq 0$ *if and only if* $\operatorname{Dim} L(\lambda)_{\lambda(D_h),h} = 1/2 \dim G.f (= \dim \mathfrak{g}(h,2))$, *where* $\operatorname{Dim} M$ *is the Gelfand-Kirillov dimension of the* $\mathfrak{g}$*-module* M. *If this is the case,* $H_f^{\frac{\infty}{2}+0}(L(\lambda))$ *is almost irreducible over* $\mathcal{W}^k(\mathfrak{g}, f)$, *that is, any nonzero submodule of* $H_f^{\frac{\infty}{2}+0}(L(\lambda))$ *intersects its top weight component non-trivially (cf.* [Ar11]*).*

(3) *Suppose that* $\mathfrak{g}$ *is of type* A. *Then, for* $L(\lambda) \in \mathcal{O}_k^{\mathfrak{l}}$, $H_f^{\frac{\infty}{2}+0}(L(\lambda))$ *is zero or irreducible.*

Proof. The above theorem is just a restatement of main results of [Ar11] since the functor $\mathcal{O}_k^{\mathfrak{l}} \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod, $M \mapsto H_f^{\frac{\infty}{2}+0}(M)$, is identical to the "$-$"-reduction functor $H_0^{\mathrm{BRST}}(?)$ studied in [Ar11] (in the case of short nilpotent elements). □

7. Level -1 affine vertex algebra of type A_{n-1}, $n \geqslant 4$

We assume in this section that $\mathfrak{g} = \mathfrak{sl}_n$ with $n \geqslant 4$.

Let
$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i, j = 1, \dots, n, i \neq j\}$$
be the root system of $\mathfrak{g}$. Fix the set of positive roots $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid i, j = 1, \dots, n, i < j\}$. Then the simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, n-1$. The highest root is $\theta = \varepsilon_1 - \varepsilon_n = \alpha_1 + \cdots + \alpha_{n-1}$. Denote by (e_i, h_i, f_i) the Chevalley generators of $\mathfrak{g}$, and fix the root vectors e_α, f_α, $\alpha \in \Delta_+$ as follows.
$$e_{\varepsilon_i - \varepsilon_j} = e_{i,j} \quad \text{and} \quad f_{\varepsilon_i - \varepsilon_j} = e_{j,i} \quad \text{for} \quad i < j,$$
where $e_{i,j}$ is the standard elementary matrix associated with the coefficient (i, j). For $\alpha \in \Delta_+$, denote by $h_\alpha = [e_\alpha, f_\alpha]$ the corresponding coroot. In particular,
$$h_i = e_{i,i} - e_{i+1,i+1} \quad \text{for} \quad i = 1, \dots, n-1.$$

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding triangular decomposition. Denote by $\varpi_1, \dots, \varpi_{n-1}$ the fundamental weights of $\mathfrak{g}$,
$$\varpi_i := (\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n).$$
Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\mid)$. Thus, the fundamental weights are viewed as elements of $\mathfrak{g}$.

Let θ_1 be the highest root of the root system generated by the simple roots perpendicular to θ, i.e.,
$$\theta_1 = \alpha_2 + \ldots + \alpha_{n-2} = \varepsilon_2 - \varepsilon_{n-1},$$
Set
$$\beta := \alpha_1 + \cdots + \alpha_{n-2} = \varepsilon_1 - \varepsilon_{n-2}, \qquad \gamma := \alpha_2 + \cdots + \alpha_{n-1} = \varepsilon_2 - \varepsilon_n,$$
and put
$$v_1 = e_\theta e_{\theta_1} - e_\beta e_\gamma \in S^2(\mathfrak{g}),$$
where $S^d(\mathfrak{g})$ denotes the component of degree d in $S(\mathfrak{g})$ for $d \geqslant 0$. Then v_1 is a singular vector with respect to the adjoint action of $\mathfrak{g}$ and generates an irreducible finite-dimensional representation W_1 of $\mathfrak{g}$ in $S^2(\mathfrak{g})$ isomorphic to $L_{\mathfrak{g}}(\theta + \theta_1)$.

Recall that we have a $\mathfrak{g}$-module embedding [AM15, Lemma 4.1],
$$\sigma_d \colon S^d(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g})_d, \quad x_1 \dots x_d \mapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} x_{\sigma(1)}(-1) \dots x_{\sigma(d)}(-1). \tag{3}$$

We will denote simply by σ this embedding for $d=2$.

Proposition 7.1 ([Ad03])**.** *For $l \geqslant 0$, The vector $\sigma(v_1)^{l+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $k=l-1$.*

Theorem 7.2. *The vecor $\sigma(v_1)$ generates the maximal submodule of $V^{-1}(\mathfrak{g})$, that is, $V_{-1}(\mathfrak{g}) = V^{-1}(\mathfrak{g})/U(\widehat{\mathfrak{g}})\sigma(v_1)$.*

Set

$$\tilde{V}_{-1}(\mathfrak{g}) = V^{-1}(\mathfrak{g})/U(\widehat{\mathfrak{g}})\sigma(v_1).$$

To prove Theorem 7.2, we shall use the minimal W-algebra $\mathcal{W}^k(\mathfrak{g}, f_\theta)$.

Let $\mathfrak{g}^\natural$ be the centralizer in $\mathfrak{g}$ of the $\mathfrak{sl}_2$-triple $(e_\theta, h_\theta, f_\theta)$. Then

$$\mathfrak{g}^\natural = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0$ is the one-dimensional center of $\mathfrak{g}^\natural$ and $\mathfrak{g}_1 = [\mathfrak{g}^\natural, \mathfrak{g}^\natural]$. Note that $\mathfrak{g}_0 = \mathbb{C}(h_1 - h_{n-1})$ and $\mathfrak{g}_1 = \langle e_{\alpha_i}, f_{\alpha_i} \mid i = 2, \dots, n-2\rangle \cong \mathfrak{sl}_{n-2}$.

There is an embedding, [KW04], [AM15, §7],

$$V^{k_0^\natural}(\mathfrak{g}_0) \otimes V^{k_1^\natural}(\mathfrak{g}_1) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f_\theta)$$

of vertex algebras, where $k_0^\sharp = k + n/2$ and $k_1^\natural = k+1$. Note that $V_0^{k_0^\natural}(\mathfrak{g}_0)$ is isomorphic to the rank one Heisenberg vertex algebra $M(1)$ provided that $k_0^\natural \neq 0$.

For $k=-1$, we have $k_1^\natural = 0$ and $V_{k_1^\natural}(\mathfrak{g}_1) = V^{k_1^\natural}(\mathfrak{g})/U(\widehat{\mathfrak{g}}_1)e_{\theta_1}(-1)\mathbf{1} = \mathbb{C}$.

By Theorem 6.2 the exact sequence $0 \to U(\widehat{\mathfrak{g}})\sigma(v_1) \to V^{-1}(\mathfrak{g}) \to \tilde{V}_{-1}(\mathfrak{g}) \to 0$ induces an exact sequence

$$0 \to H_{f_\theta}^{\frac{\infty}{2}+0}(U(\widehat{\mathfrak{g}})\sigma(v_1)) \to \mathcal{W}^{-1}(\mathfrak{g}, f_\theta) \to H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g})) \to 0. \tag{4}$$

Lemma 7.3. *The image of $\sigma(v_1)$ in $\mathcal{W}^{-1}(\mathfrak{g}, f_\theta)$ coincides with the image of the singular vector $e_{\theta_1}(-1)\mathbf{1}$ of $V^0(\mathfrak{g}_1) \subset \mathcal{W}^{-1}(\mathfrak{g}, f_\theta)$.*

Proof. Since it is singular, $\sigma(v_1)$ defines a singular vector of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$. Its image in $R_{\mathcal{W}^k(\mathfrak{g}, f_\theta)}$ is the image of v_1 in $\mathbb{C}[\mathscr{S}_{f_\theta}] = (\mathbb{C}[\mathfrak{g}^*]/J_\chi)^M$, where $\chi = (f_\theta | \cdot)$, see §3. Since $v_1 \equiv e_{\theta_1} \pmod{J_\chi}$, we have $\sigma(v_1) \equiv e_{\theta_1}(-1)\mathbf{1} \pmod{C_2(\mathcal{W}^k(\mathfrak{g}, f_\theta))}$. Then $\sigma(v_1)$ and $e_{\theta_1}(-1)\mathbf{1}$ must coincide since they both have the same weight and are singular vectors with respect to the action of $\widehat{\mathfrak{g}}_1$. $\square$

Theorem 7.4. *We have $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g})) \cong M(1)$, the rank one Heisenberg vertex algebra. In particular $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$ is simple, and hence, isomorphic to $\mathcal{W}_{-1}(\mathfrak{g}, f_\theta)$.*

Proof. By Theorem 6.2, $H_{f_\theta}^{\frac{\infty}{2}+0}(V_{-1}(\mathfrak{g}))$ is isomorphic to $\mathcal{W}_{-1}(\mathfrak{g}, f_\theta)$. Since $V_{-1}(\mathfrak{g})$ is a quotient of $\tilde{V}_{-1}(\mathfrak{g})$, $\mathcal{W}_{-1}(\mathfrak{g}, f_\theta)$ is a quotient of $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$ by the exactness result of Theorem 6.2. In particular, $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$ is nonzero.

Recall that $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is generated by fields $J^{\{a\}}(z)$, $a \in \mathfrak{g}^\natural$, $G^{\{v\}}(z)$, $v \in \mathfrak{g}_{-1/2}$, and $L(z)$ described in [KW04, Theorem 5.1]. Since $V^0(\mathfrak{g}_1)/U(\widehat{\mathfrak{g}}_1)e_{\theta_1}(-1)\mathbf{1} = \mathbb{C}$, Lemma 7.3 implies that $J^{\{a\}}(z)$, $a \in \mathfrak{g}_1$, are all zero in $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$. Then, since $[J_\lambda^{\{a\}} G^{\{v\}}] = G^{[a,v]}$ and $\mathfrak{g}_{-1/2}$ is a direct sum of non-trivial irreducible finite-dimensional representations of $\mathfrak{g}_1$, it follows that $G^{(v)}(z) = 0$ for all $v \in \mathfrak{g}_{-1/2}$.

Finally, [KW04, Theorem 5.1 (e)] implies that L coincides with the conformal vector of $V^{\frac{n-2}{2}}(\mathfrak{g}_1) \cong M(1)$ in $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$. We conclude that $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g}))$ is generated by $J^{\{a\}}(z)$, $a \in \mathfrak{g}_0$, which proves the assertion. $\square$

Proof of Theorem 7.2. Suppose that $\tilde{V}_{-1}(\mathfrak{g})$ is not simple. Then there is at least one singular vector v of weight, say μ. Then

$$\mu \equiv s\varpi_1 - \Lambda_0 \quad \text{or} \quad s\varpi_l - \Lambda_0 \quad \text{for some} \quad s \in \mathbb{N} \pmod{\mathbb{C}\delta}$$

by [AP08, Proposition 5.5]. Let M be the submodule of $\tilde{V}_{-1}(\mathfrak{g})$ generated by v. Since M is a $\tilde{V}_{-1}(\mathfrak{g})$-module, $H_{f_\theta}^{\frac{\infty}{2}+0}(M)$ is a module over $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g})) = M(1)$. Because $\mu(\alpha_0^\vee) < 0$, Theorem 6.2 implies that the image of v in $H_{f_\theta}^{\frac{\infty}{2}+0}(M)$ is nonzero, and thus, it generates the irreducible highest weight representation $M(1, \frac{\mu(h_1 - h_{n-1})}{\sqrt{2(n-2)}})$ of $M(1)$ with highest weight $\frac{\mu(h_1-h_{n-1})}{\sqrt{2(n-2)}}$. Now the exactness of the functor $H_{f_\theta}^{\frac{\infty}{2}+0}(?)$ shows that $H_{f_\theta}^{\frac{\infty}{2}+0}(M)$ is a submodule of $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_{-1}(\mathfrak{g})) = M(1)$, and therefore, so is $M(1, \frac{\mu(h_1-h_{n-1})}{\sqrt{2(n-2)}})$. But this contradicts the simplicity of $M(1)$. $\square$

By Theorem 7.2, we have

$$R_{V_{-1}(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_{W_1}, \quad \text{so} \quad \tilde{X}_{V_{-1}(\mathfrak{g})} = \operatorname{Spec}(\mathbb{C}[\mathfrak{g}^*]/I_{W_1}). \tag{5}$$

Corollary 7.5. *We have $H_f(\mathbb{C}[\mathfrak{g}^*]/I_{W_1}) = \mathbb{C}[z]$.*

Proof. By Theorem 6.1(2), we know that $R_{H_{f_\theta}^{\frac{\infty}{2}+0}(V_{-1}(\mathfrak{g}))} = H_f(R_{V_{-1}(\mathfrak{g})})$. The assertion follows since $R_{H_{f_\theta}^{\frac{\infty}{2}+0}(V_{-1}(\mathfrak{g}))} = R_{M(1)} \cong \mathbb{C}[z]$ by Theorem 7.4. $\square$

As in Theorem 1.1, let $\mathfrak{l}_1$ be the standard Levi subalgebra of $\mathfrak{sl}_n$ generated by all simple roots except α_1, that is,

$$\mathfrak{l}_1 = \mathfrak{h} + \langle e_{\alpha_i}, f_{\alpha_i} \mid i \neq 1 \rangle.$$

The center of $\mathfrak{l}_1$ is generated by ϖ_1. Thus, the Dixmier sheet closure $\overline{\mathbb{S}_{\mathfrak{l}_1}}$ is given by

$$\overline{\mathbb{S}_{\mathfrak{l}_1}} = \overline{G.\mathbb{C}^*\varpi_1},$$

see §2. The unique nilpotent orbit contained in $\mathbb{S}_{\mathfrak{l}_1}$ is $\operatorname{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(0) = \mathbb{O}_{min}$, that is, the G-orbit of f_θ.

Lemma 7.6. *We have $V(I_{W_1}) \cap \mathcal{N} \subset \overline{\mathbb{O}_{min}}$.*

Proof. First of all, we observe that $\mathbb{O}_{(2^2,1^{n-4})}$ it the smallest nilpotent orbit of $\mathfrak{g}$ which dominates $\mathbb{O}_{min} = \mathbb{O}_{(2,1^{n-2})}$. By this, it means that if $\boldsymbol{\lambda} \succ (2, 1^{n-2})$ then $\boldsymbol{\lambda} \succcurlyeq (2^2, 1^{n-4})$ where $\succcurlyeq$ is the Chevalley order on the set $\mathscr{P}(n)$. Therefore, it is enough to show that $V(I_{W_1})$ does not contain $\overline{\mathbb{O}_{(2^2,1^{n-4})}}$.

Let $f \in \mathbb{O}_{(2^2,1^{n-4})}$ that we embed into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. Denote by $\mathfrak{g}(h,i)$ the i-eigenspace of $\operatorname{ad}(h)$ for $i \in \mathbb{Z}$ and by $\Delta_+(h,i)$ the set of positive roots $\alpha \in \Delta_+$ such that $e_\alpha \in \mathfrak{g}(h,i)$. We have $f = \sum_{\alpha \in \Delta_+(h,2)} c_\alpha f_\alpha$ with $c_\alpha \in \mathbb{C}$. We call the set of $\alpha \in \Delta_+(h,2)$ such that $c_\alpha \neq 0$ the *support* of f. Choose a Lagrangian subspace $\mathfrak{L} \subset \mathfrak{g}(h,1)$ and set

$$\mathfrak{m} := \mathfrak{L} \oplus \bigoplus_{i \geqslant 2} \mathfrak{g}(h,i), \qquad J_\chi := \sum_{x \in \mathfrak{m}} \mathbb{C}[\mathfrak{g}^*](x - \chi(x)),$$

with $\chi = (f|\cdot) \in \mathfrak{g}^*$, as in §3. By Lemma 3.3, it is sufficient to show that

$$\mathbb{C}[\mathfrak{g}^*] = I_{W_1} + J_\chi.$$

To see this, we shall use the vector

$$v_1 = e_\theta e_{\theta_1} - e_\beta e_\gamma \in I_{W_1}.$$

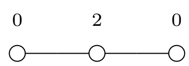
If $n > 4$, the weighted Dynkin diagram of the nilpotent orbit $G.f$ is

So we can assume that $h = \varpi_2 + \varpi_{n-2}$ and we can choose for f the element $f_\beta + f_\gamma$. We see that θ, θ_1, β and γ all belong to $\Delta_+(h,2)$. Since β and γ are in the support of f, but not θ and θ_1, for some nonzero complex number c, we have

$$e_\theta e_{\theta_1} - e_\beta e_\gamma = c \pmod{J_\chi}.$$

So $v_1 = c \pmod{J_\chi}$ and $I_{W_1} + J_\chi = \mathbb{C}[\mathfrak{g}^*]$, whence $\mathbb{C}[\mathfrak{g}^*]/(I_{W_1} + J_\chi) = 0$.

For $n = 4$, the weighted Dynkin diagram of the nilpotent orbit $G.f$ is



and we conclude similarly. $\square$

Lemma 7.7. *Let λ be a nonzero semisimple element of $\mathfrak{g}$. Then, $\lambda \in V(I_{W_1})$ if and only if $\lambda \in G.\mathbb{C}^*\varpi_1$.*

Proof. Set for $i, j \in \{1, \dots, n-2\}$, with $j - i \geqslant 2$,

$$p_{i,j} := h_i h_j,$$

and for $i \in \{2, \dots, n-2\}$,

$$q_i := h_i(h_{i-1} + h_i + h_{i+1}).$$

According to [AP08], the zero weight space of W_1 is generated by the elements $p_{i,j}$ and q_i. Clearly, $p_{i,j}(\varpi_1) = 0$ for $j - i \geqslant 2$ and $q_i(\varpi_1) = 0$ for any $i \in \{2, \dots, n-2\}$. So $\varpi_1 \in V(I_{W_1})$, and whence $G.\mathbb{C}^*\varpi_1 \subset V(I_{W_1})$ since $V(I_{W_1})$ is a G-invariant cone. This proves the converse implication.

For the first implication, let λ be a nonzero semisimple element of $\mathfrak{g}$, and assume that $\lambda \in V(I_{W_1})$. Since $V(I_{W_1})$ is G-invariant, we can assume that $\lambda \in \mathfrak{h}$. Then write

$$\lambda = \sum_{i=1} \lambda_i \varpi_i, \qquad \lambda_i \in \mathbb{C}.$$

Since $p_{i,j}(\lambda) = q_i(\lambda) = 0$ for all i, j, we get

$$\lambda_i \lambda_j = 0, \quad i, j \in \{1, \dots, n-2\},\ j - i \geqslant 2, \tag{6}$$

$$\lambda_i(\lambda_{i-1} + \lambda_i + \lambda_{i+1}) = 0, \quad i = 2, \dots, n-2. \tag{7}$$

Since λ is nonzero, $\lambda_k \neq 0$ for some $k \in \{1, \dots, n-1\}$.

If $k \notin \{1, n-1\}$, then by (6), $\lambda_j = 0$ for $|j-k| \geqslant 2$. So by (7), either $\lambda_{k-1}+\lambda_k = 0$ or $\lambda_k + \lambda_{k+1} = 0$ since $\lambda_{k-1}\lambda_{k+1} = 0$ by (6).

If $k = 1$, then by (6), $\lambda_j = 0$ for $j \geqslant 3$. So by (7), $\lambda_2(\lambda_1 + \lambda_2) = 0$.

If $k = n-1$, then by (6), $\lambda_j = 0$ for $j \leqslant n-3$. So by (7), $\lambda_{n-2}(\lambda_{n-2}+\lambda_{n-1}) = 0$. We deduce that

$$\begin{aligned}\lambda &\in \mathbb{C}^*\varpi_1 \cup \mathbb{C}^*\varpi_{n-1} \cup \bigcup_{1\leqslant j\leqslant n-2} \mathbb{C}^*(\varpi_i - \varpi_{i+1}) \\ &= \bigcup_{1\leqslant j\leqslant n-1} \mathbb{C}^*(\varepsilon_1 + \cdots + \varepsilon_{i-1} - (n-1)\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n)\end{aligned}$$

All the above weights are conjugate to $t\varpi_1$ for some $t \in \mathbb{C}^*$ under the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ which is the group of permutations of $\{\varepsilon_1, \ldots, \varepsilon_n\}$, whence the expected implication. □

The following assertion follows immediately from Lemma 2.1, Lemma 7.6 and Lemma 7.7.

Proposition 7.8. *We have $V(I_{W_1}) = \overline{\mathbb{S}_{\mathfrak{l}_1}}$.*

Remark 7.9. The zero locus of the Casimir element Ω in $V(I_{W_1})$ is $\overline{\mathbb{O}_{min}}$. Indeed, by Lemma 2.1 the zeros locus of Ω in $\overline{\mathbb{S}_{\mathfrak{l}_1}}$ is contained in the nilpotent cone since $\Omega(\varpi_1) \neq 0$. The statement follows since $\overline{\mathbb{O}_{min}}$ has codimension one in $\overline{\mathbb{S}_{\mathfrak{l}_1}}$.

As a consequence, the zero locus of the ideal generated by $\mathbb{C}\Omega \oplus W_1 \subset S^2(\mathfrak{g})$ is $\overline{\mathbb{O}_{min}}$. This latter fact is known by [Ga82] using a different approach.

Proposition 7.10. *The ideal I_{W_1} is prime, and therefore it is the defining ideal of $\overline{\mathbb{S}_{\mathfrak{l}_1}}$.*

Proof. We apply Lemma 3.6 to the ideal $I := I_{W_1}$. First of all, $\mathfrak{l}_1$ and $(e_\theta, h_\theta, f_\theta)$ satisfy the conditions of Lemma 3.4 since $\mathfrak{z}(\mathfrak{l}_1) = \mathbb{C}\varpi_1$ and $\mathfrak{g}(h_\theta, i) = 0$ for $i > 2$. It remains to verify that the conditions (1),(2),(3),(4) of Lemma 3.6 are satisfied.

Condition (1) is satisfied by Proposition 7.8. According to [Ga82, Corollary 2 and Theorem 1, Chap. V], the ideal $I + I_\Omega$ is the defining ideal of $\overline{\mathbb{O}_{min}}$. So condition (2) is satisfied. The condition (3) is satisfied too, by Corollary 7.5. At last, because $\Omega(\varpi_1) \neq 0$, condition (4) is satisfied. In conclusion, by Lemma 3.6, $I = I_{W_1}$ is prime. □

Proof of Theorem 1.1 (1). The first statement follows immediately from (4) and Proposition 7.10. The second statement follows from the inclusions

$$J_\infty\overline{\mathbb{S}_{\mathfrak{l}_1}} = SS(V_{-1}(\mathfrak{g}))_{\mathrm{red}} \subset SS(V_{-1}(\mathfrak{g})) \subset J_\infty X_{V_{-1}(\mathfrak{g})} = J_\infty\overline{\mathbb{S}_{\mathfrak{l}_1}},$$

where the first equality follows from Corollary 4.2. □

Remark 7.11. Let $V_k(\mathfrak{g})\text{-Mod}^{\mathfrak{g}[t]}$ be the full subcategory of the category of $V_k(\mathfrak{g})$-modules consisting of objects that belong to KL_k. Since $H^{\frac{\infty}{2}+0}_{f_\theta}(V_{-1}(\mathfrak{g})) \cong M(1)$, we have a functor

$$V_{-1}(\mathfrak{g})\text{-Mod}^{\mathfrak{g}[t]} \to M(1)\text{-Mod}, \quad M \mapsto H^{\frac{\infty}{2}+0}_{f_\theta}(M),$$

where $M(1)$-Mod denotes the category of the modules over the Heisenberg vertex algebra $M(1)$. The proof of Theorem 1.1 (1) and [AP14, Theorem 6.2] imply that this is a *fusion functor*.

Let $\mathfrak{p} = \mathfrak{l}_1 \oplus \mathfrak{p}_u$ be a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{p}_u$. It is a maximal parabolic subalgebra of $\mathfrak{g}$. Let P be the connected parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Set

$$Y := G/(P, P).$$

As explained in §2, we have $V(\operatorname{gr} \mathcal{J}_Y) = \overline{\mathbb{S}_{\mathfrak{l}_1}}$, where $\mathcal{J}_Y = \ker \psi_Y$ and ψ_Y is the algebra homomorphism $U(\mathfrak{g}) \to \mathcal{D}_Y$.

The variety Y is a quasi-affine variety, isomorphic to $\mathbb{C}^n \backslash \{0\}$. It follows that the natural embedding $Y = \mathbb{C}^n \backslash \{0\} \hookrightarrow \mathbb{C}^n$ induces an isomorphism $\mathcal{D}_{\mathbb{C}^n} \xrightarrow{\sim} \mathcal{D}_Y$. In this realization, the map ψ_Y is described as follows.

$$\begin{aligned} \psi_Y : U(\mathfrak{g}) &\to \mathcal{D}_{\mathbb{C}^n} = \langle z_1, \dots, z_n, \frac{\partial}{\partial z_1}, \dots \frac{\partial}{\partial z_n} \rangle, \\ e_{i,j} &\mapsto -z_j \frac{\partial}{\partial z_i} \quad (i \neq j), \\ h_i &\mapsto -z_i \frac{\partial}{\partial z_i} + z_{i+1} \frac{\partial}{\partial z_{i+1}}. \end{aligned}$$

where $e_{i,j}$ is as before the standard elementary matrix element and $h_i = e_{i,i} - e_{i+1,i+1}$.

This has the following chiralization: Let $\mathcal{D}^{ch}_{\mathbb{C}^n}$ be the $\beta\gamma$-system of rank n, generated by fields $\beta_1(z), \dots, \beta_n(z)$, $\gamma_1(z), \dots, \gamma_n(z)$, satisfying OPE

$$\gamma_i(z)\beta_j(w) \sim \frac{\delta_{ij}}{z-w}, \quad \gamma_i(z)\gamma_j(w) \sim 0, \qquad \beta_i(z)\beta_j(w) \sim 0,$$

see e.g. [Kac98]. We have

$$R_{\mathcal{D}^{ch}_{\mathbb{C}^n}} \cong \mathbb{C}[T^*\mathbb{C}^n], \qquad A(\mathcal{D}^{ch}_{\mathbb{C}^n}) \cong \mathcal{D}_{\mathbb{C}^n},$$

where $\mathcal{D}_{\mathbb{C}^n}$ denotes the Weyl algebra of rank n, which is identified with the algebra of differential operators on the affine space $\mathbb{C}^n$. In particular, $\mathcal{D}^{ch}_{\mathbb{C}^n}$ is a chiralization of $\mathcal{D}_{\mathbb{C}^n}$.

Lemma 7.12. *The following gives a vertex algebra homomorphism.*

$$\begin{aligned} \hat{\psi}_Y : V^{-1}(\mathfrak{g}) &\to \mathcal{D}^{ch}_{\mathbb{C}^n}, \\ e_{i,j}(z) &\mapsto - : \beta_j(z)\gamma_i(z) : \quad (i \neq j), \\ h_i(z) &\mapsto - : \beta_i(z)\gamma_i(z) : + : \beta_{i+1}(z)\gamma_{i+1}(z) : . \end{aligned}$$

Note that the map $\hat{\psi}_Y$ induces an algebra homomorphism $U(\mathfrak{g}) = A(V^{-1}(\mathfrak{g})) \to A(\mathcal{D}^{ch}_{\mathbb{C}^n}) = \mathcal{D}_{\mathbb{C}^n}$, which is identical to ψ_Y.

Theorem 7.13 ([AP14])**.** *The map $\hat{\psi}_Y$ factors through the vertex algebra embedding*

$$V_{-1}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_{\mathbb{C}^n}.$$

By Theorem 7.13, $\hat{\psi}_Y$ induces a homomorphism

$$A(V_{-1}(\mathfrak{g})) \to A(\mathcal{D}^{ch}_{\mathbb{C}^n}) = \mathcal{D}_{\mathbb{C}^n}. \tag{8}$$

Theorem 7.14. (1) *The ideal $\operatorname{gr} \mathscr{J}_Y \subset \mathbb{C}[\mathfrak{g}^*]$ is prime and hence it is the defining ideal of $\overline{\mathbb{S}_{\mathfrak{l}_1}}$.*

(2) *The natural homomorphism $R_{V_{-1}(\mathfrak{g})} \to \operatorname{gr} A(V_{-1}(\mathfrak{g}))$ is an isomorphism.*

(3) *The map $\widehat{\psi}_Y$ induces an embedding*

$$A(V_{-1}(\mathfrak{g})) \hookrightarrow \mathcal{D}_{\mathbb{C}^n}.$$

Proof. We have

$$A(V_{-1}(\mathfrak{g})) = U(\mathfrak{g})/\mathscr{J}_{W_1},$$

where $\mathscr{J}_{W_1}$ is the two-sided ideal of $U(\mathfrak{g})$ generated by W_1. By (8), we have the inclusion $\mathcal{J}_{W_1} \subset \mathcal{J}_Y = \ker \psi_Y$. Also, $\mathrm{gr}\, \mathscr{J}_{W_1} \supset I_{W_1}$ by definition, and $\sqrt{\mathscr{J}_Y} = I_{W_1}$ by Proposition 7.10. Thus,

$$I_{W_1} \subset \mathrm{gr}\, \mathscr{J}_{W_1} \subset \mathrm{gr}\, \mathscr{J}_Y \subset \sqrt{\mathrm{gr}\, \mathscr{J}_Y} = I_{W_1}.$$

It follows that all the above inclusions are equalities:

$$I_{W_1} = \mathrm{gr}\, \mathscr{J}_{W_1} = \mathrm{gr}\, \mathscr{J}_Y = \sqrt{\mathrm{gr}\, \mathscr{J}_Y}.$$

The statements (1), (2) and (3) follow from the third, the first and the second equality, respectively. ☐

Remark 7.15. Adamović and Perše [AP14] showed that $\mathcal{D}^{ch}_{\mathbb{C}^n}$ decomposes into a direct sum of simple highest weight representations of $V_{-1}(\mathfrak{g})$. From their results, it is possible to obtain an explicit character formula of $V_{-1}(\mathfrak{g})$ as in [KW01]. In view of Theorem 1.1, this gives the Hilbert series of $J_\infty\overline{\mathbb{S}_{\mathfrak{l}_1}}$.

8. Level $-m$ affine vertex algebra of type A_{2m-1}

Set

$$v_0 := \sum_{i=1}^{2m-1} \frac{m-i}{m} h_i e_\theta + \sum_{i=1}^{2m-2} e_{1,i+1}e_{i+1,2m} \in S^2(\mathfrak{g}).$$

Then v_0 is a singular vector and generates a finite-dimensional irreducible representation W_0 of $\mathfrak{g}$ in $S^2(\mathfrak{g})$ isomorphic to $L_{\mathfrak{g}}(\theta)$. We have

$$\begin{aligned}&\sigma(v_0)\\ &= \sum_{i=1}^{2m-1} \frac{m-i}{m} h_i(-1)e_\theta(-1) + \sum_{i=1}^{2m-2} e_{1,i+1}(-1)e_{i+1,2m}(-1) - (m-1)e_\theta(-2),\end{aligned}$$

where $\sigma = \sigma_2$ is defined in §7, equality (3).

Lemma 8.1. *The vector $\sigma(v_0)$ is a singular vector of $V_k(\mathfrak{g})$ if and only if $k = -m$.*

Proof. The verifications are identical to [Pe08, Lemma 4.2] and we omit the details[2]. ☐

Theorem 8.2. *The singular vector $\sigma(v_0)$ generates the maximal submodule of $V^{-m}(\mathfrak{g})$, that is, $V_{-m}(\mathfrak{g}) = V^{-m}(\mathfrak{g})/U(\widehat{\mathfrak{g}})\sigma(v_0)$.*

[2]In [Pe08], the author only deals with $\mathfrak{sl}_{r+1}$ for even r in order to have an admissible level $k = -\frac{1}{2}(r+1)$. Note that there is a change of sign because our Chevalley basis slightly differs from that of [Pe08].

Let (e, f, h) be the $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ defined by

$$e = \sum_{i=1}^{m} e_{i,m+i}, \qquad h = \sum_{i=1}^{m} e_{i,i} - \sum_{i=1}^{m} e_{m+i,m+i} = 2\varpi_m,$$
$$f = \sum_{i=1}^{m} e_{m+i,i} \in \mathfrak{g}.$$

Then f is a nilpotent element of $\mathfrak{g}$ corresponding to the partition (2^m). We have

$$\mathfrak{g} = \mathfrak{g}(h,-2) \oplus \mathfrak{g}(h,0) \oplus \mathfrak{g}(h,2),$$

where $\mathfrak{g}(h,2) = \mathrm{span}_{\mathbb{C}}\{e_{i,m+j} \mid 1 \leqslant i,j \leqslant m\}$, $\mathfrak{g}(h,-2) = \mathrm{span}_{\mathbb{C}}\{e_{m+i,j} \mid 1 \leqslant i,j \leqslant m\}$, $\mathfrak{g}(h,0) = \mathrm{span}_{\mathbb{C}}\{e_{i,j}, e_{m+i,m+j} \mid 1 \leqslant i,j \leqslant m\} \cap \mathfrak{g}$. Thus, f is a short nilpotent element.

Let $\mathfrak{g}^\natural \subset \mathfrak{g}(h,0)$ be the centralizer in $\mathfrak{g}$ of (e, f, h). Then

$$\mathfrak{g}^\natural \cong \mathfrak{sl}_m.$$

By [KW04, Theorem 2.1] we have a vertex algebra embedding

$$V^{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f),$$

where

$$k^\natural = 2k + 2m,$$

which restricts to the isomorphism of spaces of weight 1

$$\mathfrak{g}^\natural \cong V^{k^\natural}(\mathfrak{g}^\natural)_1 \xrightarrow{\sim} \mathcal{W}^k(\mathfrak{g}, f)_1.$$

Since

$$\mathfrak{g}(h,-2) \cong \mathfrak{g}^\natural \oplus \mathbb{C}$$

as $\mathfrak{g}^\natural$-modules, there exists [KW04] a $\mathfrak{g}^\natural$-submodule E of $\mathcal{W}^k(\mathfrak{g}, f)_2$ such that $\mathcal{W}^k(\mathfrak{g}, f)$ is generated by $\mathcal{W}^k(\mathfrak{g}, f)_1 \cong \mathfrak{g}^\natural$, E and the conformal vector $\omega_W \in \mathcal{W}^k(\mathfrak{g}, f)_2$, provided that $k \neq -2m$.

Theorem 8.3. *We have*

$$H_f^{\frac{\infty}{2}+0}(\tilde{V}_{-m}(\mathfrak{g})) \cong \mathrm{Vir}_1,$$

where Vir_1 *is the simple Virasoro vertex algebra of central charge* 1.

Let $e_{\theta^\natural} = e_{1,m} + e_{m+1,2m} \in \mathfrak{g}^\natural$ be a root vector of the highest root $\theta^\natural$ of $\mathfrak{g}^\natural$.

Lemma 8.4. *The image of* $\sigma(v_0)$ *in* $\mathcal{W}^{-m}(\mathfrak{g}, f_0)$ *coincides with the image of the singular vector* $e_{\theta^\natural}(-1)\mathbf{1}$ *of* $V^0(\mathfrak{g}^\natural)$ *up to nonzero constant multiplication.*

Proof. Let w be the image of $\sigma(v_0)$ in $\mathcal{W}^{-m}(\mathfrak{g}, f_0)$. One finds that

$$w \equiv e_{\theta^\natural}(-1)\mathbf{1} \pmod{C_2(\mathcal{W}^{-m}(\mathfrak{g}, f_0))},$$

and hence, it is nonzero and has the same weight as $e_{\theta^\natural}(-1)\mathbf{1}$. Since it is a singular vector, w is singular vector for $V^0(\mathfrak{g}^\natural)$. The assertion follows since the corresponding weight space is one-dimensional. □

Proposition 8.5. *Let* $\lambda = s\varpi_m - m\Lambda_0$ *with* $s \in \mathbb{Z}_{\geqslant 0}$. *Then* $H_f^{\frac{\infty}{2}+0}(L(\lambda)) \neq 0$.

Proof. Since $L(\lambda)$ belongs to $\mathcal{O}_k^{\mathfrak{l}_0}$, it is sufficient to show that the Gelfand-Kirillov dimension of $L(\lambda)_{\lambda(D_W),h}$ is maximal by Theorem 6.3. Observe that $L(\lambda)_{\lambda(D_W),h}$ is an irreducible highest weight representation of $\mathfrak{g}$ with highest weight $\mu = -(m+s)\varpi_m$. It follows from Jantzen's simplicity criterion [Ja77, Satz 4] (see also [Hu08, Theorem 9.13]) that $L_{\mathfrak{g}}(\mu)$ is isomorphic to the generalized Verma module $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}L_{\mathfrak{l}_0}(\mu)$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}_0$, and $L_{\mathfrak{l}_0}(\mu)$ is the irreducible finite-dimensional representation of $\mathfrak{l}_1$ with highest weight μ. This completes the proof. $\square$

Proof of Theorem 8.3. First, $H_f^{\frac{\infty}{2}+0}(\tilde{V}_{-m}(\mathfrak{g})) \neq 0$ since $H_f^{\frac{\infty}{2}+0}(V_{-m}(\mathfrak{g})) \neq 0$ by Proposition 8.5. Second, the exact sequence $0 \to N \to V^{-m}(\mathfrak{g}) \to \tilde{V}_{-m}(\mathfrak{g}) \to 0$, where $N = U(\widehat{\mathfrak{g}})\sigma(v_0) \subset V^{-m}(\mathfrak{g})$, induces an exact sequence

$$0 \to H_f^{\frac{\infty}{2}+0}(N) \to \mathcal{W}^{-m}(\mathfrak{g}, f) \to H_f^{\frac{\infty}{2}+0}(\tilde{V}_{-m}(\mathfrak{g})) \to 0.$$

By Lemma 8.4, the weight 1 subspace $\mathcal{W}^{-m}(\mathfrak{g}, f)_1 (\cong \mathfrak{g}^{\natural})$ is contained in $H_f^{\frac{\infty}{2}+0}(N)$. Hence, $x_{(0)}\mathcal{W}^{-m}(\mathfrak{g}, f) \subset H_f^{\frac{\infty}{2}+0}(N)$ for all $x \in \mathfrak{g}^{\natural}$. This gives that $E \subset H_f^{\frac{\infty}{2}+0}(N)$. It follows that $H^{\frac{\infty}{2}+0}(\tilde{V}_{-m}(\mathfrak{g}))$ is generated by the single element ω_W. Since ω_W has central charge 1, there is a vertex algebra homomorphism $\mathrm{Vir}_1 \twoheadrightarrow H^{\frac{\infty}{2}+0}(\tilde{V}_{-m}(\mathfrak{g}))$. The assertion follows as Vir_1 is simple. $\square$

Proof of Theorem 8.2. Suppose that $\tilde{V}_k(\mathfrak{g})$ is not simple. Then there exists at least one singular weight vector, say v, which generates a proper submodule of $\tilde{V}_k(\mathfrak{g})$. As $A(\tilde{V}_k(\mathfrak{g})) = U(\mathfrak{g})/\mathcal{I}_{W_0}$, Proposition 8.8 implies that the weight of v has the form $\mu = s\varpi_m - m\Lambda_0$ with $s \in \mathbb{N}$. Consider the submodule M of $\tilde{V}_k(\mathfrak{g})$ generated by v. Since $H_f^{\frac{\infty}{2}+0}(L(\mu)) \neq 0$ by Proposition 8.5, Theorem 6.3(1) implies that $H_f^{\frac{\infty}{2}+0}(M)$ is a nonzero submodule of $H_f^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$. But by Theorem 8.3, $H^0_{f_0}(\tilde{V}_k(\mathfrak{g})) \cong \mathrm{Vir}_1$ is simple. Contradiction. $\square$

By Theorem 8.2, we have

$$R_{\tilde{V}_{-1}(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_{W_0}, \quad \text{and so,} \quad X_{\tilde{V}_{-1}(\mathfrak{g})} = \mathrm{Spec}(\mathbb{C}[\mathfrak{g}^*]/I_{W_0}). \tag{9}$$

Thus we get the following assertion (see Corollary 7.5).

Corollary 8.6. *We have $H_f(\mathbb{C}[\mathfrak{g}^*]/I_{W_0}) = \mathbb{C}[z]$.*

As in Theorem 1.1, define the Levi subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}$ by

$$\mathfrak{l}_0 = \mathfrak{h} + \langle e_{\alpha_i}, f_{\alpha_i} \mid i \neq m \rangle.$$

The center of $\mathfrak{l}_0$ is spanned by ϖ_m. Thus,

$$\overline{\mathbb{S}_{\mathfrak{l}_0}} = \overline{G.\mathbb{C}^*\varpi_m},$$

see §2. We have $\mathrm{Ind}_{\mathfrak{l}_0}^{\mathfrak{g}}(0) = \mathbb{O}_{(2^m)}$, and hence, $\mathbb{O}_{(2^m)}$ is the unique nilpotent orbit contained in the Diximier sheet $\mathbb{S}_{\mathfrak{l}_0}$.

Lemma 8.7. $V(I_{W_0}) \cap \mathcal{N} \subset \overline{\mathbb{O}_{(2^m)}}$.

Lemma 8.7 is proven in the same way as Lemma 7.6, and we omit the proof. Note that here $\mathbb{O}_{(3,2^{m-2},1)}$ it the smallest nilpotent orbit of $\mathfrak{g}$ which dominates $\mathbb{O}_{(2^m)}$.

We now view W_0 as a submodule of $U(\mathfrak{g})$ through the identification $S(\mathfrak{g}) \cong U(\mathfrak{g})$ given by the symmetrization map, and shall determine the characteristic variety $\mathcal{V}(I_{W_0})$.

Set for $s \in \{1, \dots, 2m-1\}$,

$$\Lambda_s := \{(i_1, \dots, i_1) \in \{1, \dots, 2m-1\}^s \mid i_1 < \cdots < i_s \text{ and } \sum_{k=1}^{s} (-1)^k i_k = (-1)^s m\}.$$

For example, $\Lambda_1 = \{m\}$ and $\Lambda_2 = \{1 \leqslant i_1 < i_2 \leqslant 2m-1 \mid -i_1 + i_2 = m\}$.

Proposition 8.8. *The characteristic variety $\mathcal{V}(I_{W_0})$ of I_{W_0} is the set*

$$\hat{\Xi} := \bigcup_{1 \leqslant s \leqslant 2m-1} \bigcup_{(i_1, \dots, i_s) \in \Lambda_s} \{t\varpi_{i_1} + \sum_{j=2}^{s} (-1)^j (-t + c_{i_j}) \varpi_{i_j} \ ; \ t \in \mathbb{C}\}.$$

where for $j \in \{2, \dots, s\}$,

$$c_j := i_1 + 2 \sum_{k=2}^{j-1} (-1)^{k+1} i_k + (-1)^{j+1} i_j.$$

In particular, the only integral dominant weights which lie in $\mathcal{V}(I_{W_0})$ are those of the form $t\varpi_m$ with $t \in \mathbb{Z}_{\geqslant 0}$.

Proof. Set for $i \in \{1, \dots, 2m-1\}$,

$$\hat{p}_i := h_i \hat{q}_i,$$

where

$$\hat{q}_i := \sum_{j=1}^{i-1} \frac{-j}{m} h_j + \frac{m-i}{m} h_i + \sum_{j=i+1}^{2m-1} \frac{2m-j}{m} h_j + m - i.$$

According to [Pe08, Lemma 5.1], $\Upsilon(W_0^{\mathfrak{h}})$ is generated by $\hat{p}_1, \dots, \hat{p}_{2m-1}$. Indeed, this part of Perše's proof does not use the parity of the rank of $\mathfrak{sl}_n$ and all computations hold for n even.

We first verify that any $\lambda \in \hat{\Xi}$ is a solution of the system of equations $\hat{p}_1(\lambda) = 0, \dots, \hat{p}_r(\lambda) = 0$. The verifications are left to the reader.

Conversely, let $\lambda = \sum_{i=1}^{2m-1} \lambda_i \varpi_i \in \mathfrak{h}$ be such that $\hat{p}_i(\lambda) = 0$ for all $i = 1, \dots, 2m-1$. Assume that $\lambda \neq 0$. Let us show that $\lambda \in \hat{\Xi}$. Since $\lambda \neq 0$, there exist integers $i_1, i_2, \dots, i_s$ in $\{1, \dots, 2m-1\}$, with $i_1 < i_2 \cdots < i_s$, such that $\lambda_{i_j} \neq 0$ if $j \in \{1, \dots, s\}$ and $\lambda_k = 0$ for all $k \notin \{i_1, \dots, i_1\}$. Thus, $\hat{q}_{i_j}(\lambda) = 0$ for all $j \in \{1, \dots, s\}$.

Assume $s = 1$. Then

$$0 = \hat{q}_{i_1}(\lambda) = \frac{m - i_1}{m} \lambda_{i_1} + m - i_1.$$

Either $i_1 = m$ and then $\lambda = \lambda_m \varpi_m \in \hat{\Xi}$, or $i_1 \neq m$ and then

$$\lambda_{i_1} = -m \quad \text{so} \quad \lambda = -m\varpi_{i_1}.$$

Since $i_1 \neq m$, either $i_1 > m$ or $i_1 < m$. By symmetry, we can assume that $i_1 < m$. Indeed, if $i_1 > m$, then $2m - i_1 < m$, but if $\hat{p}_i(\lambda') = 0$ for all $i = 1, \dots, 2m-1$, then $\lambda' := \lambda_{2m-i_1} \varpi_{2m-i_1}$ verifies $\hat{p}_i(\lambda') = 0$ for all $i = 1, \dots, 2m-1$, too. Now one can choose $i_2 \in \{i_1 + 1, \dots, 2m-1\}$ such that $-i_1 + i_2 = m$. Then

$$\begin{aligned} \lambda = \lambda_{i_1} \varpi_{i_1} = & \lambda_{i_1} \varpi_{i_1} + (-\lambda_{i_1} + i_1 - i_2) \varpi_{i_2} \\ & \in \bigcup_{(i_1, i_2) \in \Lambda_2} \{t\varpi_{i_1} + (-t + i_1 - i_2) \varpi_{i_2} \ ; \ t \in \mathbb{C}\} \subset \hat{\Xi} \end{aligned}$$

since $\lambda_{i_1} = -m = i_1 - i_2$ and $c_{i_2} = i_1 - i_2$, and we are done.

Assume now $s \geqslant 2$. Since $\lambda_{i_j} \neq 0$ for $j = 1, \ldots, s$, we get $\hat{q}_{i_j}(\lambda) = 0$ for $j = 1, \ldots, s$. Using the equations $\hat{q}_{i_1}(\lambda) - \hat{q}_{i_2}(\lambda) = 0, \ldots, \hat{q}_{i_s}(\lambda) - \hat{q}_{i_s}(\lambda) = 0$, we get

$$\forall\, j \in \{1, \ldots, s-1\}, \qquad \lambda_{i_{j+1}} + \lambda_{i_j} - i_j + i_{j+1} = 0.$$

By induction on j we obtain that

$$\forall\, j \in \{2, \ldots, s\}, \qquad \lambda_{i_j} = (-1)^j(-\lambda_{i_1} + c_j) \tag{10}$$

where c_j is defined as in the proposition:

$$c_j := i_1 + 2\sum_{k=2}^{j-1}(-1)^{k+1} i_k + (-1)^{j+1} i_j, \quad j = 2, \ldots, s.$$

Using the equations $\hat{q}_{i_s}(\lambda) = 0$ and (10), we get

$$\begin{aligned} 0 = &\left(\frac{1}{m}\sum_{k=1}^{s}(-1)^k i_k - (-1)^s\right)\lambda_{i_1} \\ &- \frac{1}{m}\sum_{k=2}^{s}(-1)^k i_k c_k + \frac{m - i_s}{m}(-1)^s c_s + m - i_s. \end{aligned} \tag{11}$$

Either

$$\frac{1}{m}\sum_{k=1}^{s}(-1)^k i_k - (-1)^s = 0, \quad \text{ that is,} \quad \sum_{k=1}^{s}(-1)^k i_k = (-1)^s m$$

and then $\lambda \in \hat{\Xi}$, or

$$\sum_{k=1}^{s}(-1)^k i_k \neq (-1)^s m$$

and then λ_{i_1} is entirely determined by the equation (11), and so is $\lambda_{i_2}, \ldots, \lambda_{i_s}$ by (10).

Claim 8.1. Set $i_0 := 0$. There exist $l \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, 2m-1\} \setminus \{i_1, \ldots, i_s\}$ such that $i_{l-1} < j < i_l$ and

$$\sum_{k=1}^{s+1}(-1)^k i'_k = (-1)^{s+1} m$$

where $i'_k = i_k$ for $k = 1, \ldots, l-1$, $i'_l = j$ and $i'_{k+1} = i_k$ for $k = l, \ldots, s$.

Proof. First of all, we observe that the sequence $(1, 2, \ldots, 2m-1)$ always belongs to Λ_{2m-1}. Hence, if $\sum_{k=1}^{s}(-1)^k i_k \neq (-1)^s m$, then there exist $l \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, 2m-1\} \setminus \{i_1, \ldots, i_s\}$ such that $i_{l-1} < j < i_l$. We prove the statement by induction on s. The claim is known for $s = 1$. Let us prove it prove for $s = 2$. There are two cases:

a) First case: $-i_1 + i_2 < m$. Then set

$$j := m - i_1 + i_2.$$

Possibly replacing i_1, i_2 by $2m - i_2, 2m - i_2$, we can assume that $i_1 < m$. Hence $j > i_2$. In addition, $j < 2m$ since $-i_1 + i_2 < m$. So j suits the conditions of the claim.

b) Second case: $-i_1+i_2>m$. Then $i_2>m$, $i_1<m$ and we set:

$$j:=-m+i_1+i_2.$$

We have $i_1<j<i_2$ and j suits the conditions of the claim.

Assume now that $s\geqslant 2$ and that the claim is true for all strictly smaller integers. We assume that s is even. The case where s is odd is dealt similarly. There are two cases:

a) First case: $\sum_{k=1}^{s}(-1)^k i_k<m$. Either

$$m+\sum_{k=1}^{s-1}(-1)^k i_k>0,$$

and then the integer

$$j:=m+\sum_{k=1}^{s}(-1)^k i_k>i_s$$

suits the conditions of the claims. Or $m+\sum_{k=1}^{s-1}(-1)^k i_k\leqslant 0$, that is $i_{s-1}\geqslant \frac{r+1}{1}+\sum_{k=1}^{s-2}(-1)^k i_k$. Since $i_{s-1}<2m-\delta$, with $\delta:=i_s-i_{s-1}$, we get

$$\sum_{k=1}^{s-2}(-1)^k i_k<m-\delta.$$

Apply the induction hypothesis to the sequence $i_1,\dots,i_{s-1}$ and $2m-2\delta-1$, which is an odd integer. Then there exists $j\in\{1,\dots,2m-2\delta-1\}\setminus\{i_1,\dots,i_{s-2}\}$ such that $\sum_{k=1}^{s-1}(-1)^k i'_k=-(m-\delta)$ where the sequence $i'_1,\dots,i'_{s-1}$ is defined as in the claim with respect to $i_1,\dots,i_{s-2}$ and j. We easily verify that $j<i_{s_1}$, and we have

$$\sum_{k=1}^{s-1}(-1)^k i'_k+i_{s-1}-i_s=-(m-\delta)-\delta=-m.$$

So j suits the conditions of the claim.

b) Second case: $\sum_{k=1}^{s}(-1)^k i_k>m$. Then $\sum_{k=1}^{s-2}(-1)^k i_k-i_{s-1}+i_s>m$, that is, $\sum_{k=1}^{s-2}(-1)^k i_k>m-\delta$ with $\delta:=i_s-i_{s-1}$. Applying the induction hypothesis to the sequence $i_1,\dots,i_{s-1}$ and $2m-2\delta-1$, we conclude as in case a).

We illustrate in figures 1 and 2 the construction of j. In Figure 1, the positive integer $-\sum_{k=1}^{s}(-1)^k i_k$ corresponds to the sum of the lengths of the thick lines while the positive integer $2m+\sum_{k=1}^{s}(-1)^k i_k$ corresponds to the sum of the lengths of the thin lines. In Figure 2, the positive integer $\sum_{k=1}^{s}(-1)^k i_k$ corresponds to the sum of the lengths of the thick lines while the positive integer $2m-\sum_{k=1}^{s}(-1)^k i_k$ corresponds to the sum of the lengths of the thin lines.

$\square$

Let j and $i'_1,\dots,i'_{s+1}$ be as in Claim 8.1. Possibly replacing the sequence $i_1,\dots,i_s$ by $2m-i_s-1,\dots,2m-i_s-1$ we can assume that $l\neq 1$, that is, $j\neq i_1$. Set

$$\lambda_{i'_l}:=0$$

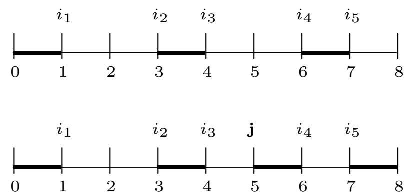


FIGURE 1. Construction of j for $m=4$, $s=5$ and $(i_1,i_2,i_3,i_4,i_5)=(1,3,4,6,7)$

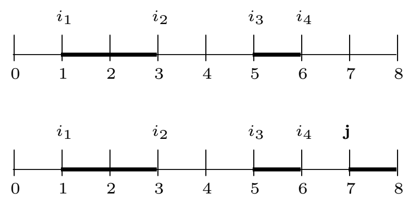


FIGURE 2. Construction of j for $m=4$, $s=4$ and $(i_1,i_2,i_3,i_4)=(1,3,5,6)$

so that

$$\lambda=\sum_{k=1}^{s}\lambda_{i_k}\varpi_{i_k}=\sum_{k=1}^{s+1}\lambda_{i'_k}\varpi_{i'_k}.$$

where for $j=1,\dots,l-1$, $\lambda_{i'_k}:=\lambda_{i_k}$ and for $k=l,\dots,s$, $\lambda_{i'_{k+1}}:=\lambda_{i_k}$. Then one can verify that for all $k\in\{1,\dots,s+1\}$,

$$(-1)^k(-\lambda_{i'_1}+c_{i'_k})=\lambda_{i'_k}.$$

The verifications are left to the reader. This proves that $\lambda\in\hat{\Xi}$ since $(i'_1,\dots,i'_{s+1})\in\Lambda_{s+1}$. $\square$

Lemma 8.9. *Let λ be a nonzero semisimple element of $\mathfrak{g}$ which lies in $V(I_{W_0})$. Then $\lambda\in G.\mathbb{C}^*\varpi_m$.*

Proof. Set $S(\mathfrak{g})^{\mathfrak{h}}:=\{x\in S(\mathfrak{g})\mid [h,x]=0 \text{ for all } h\in\mathfrak{h}\}$ and let

$$\Psi\colon S(\mathfrak{g})^{\mathfrak{h}}\to S(\mathfrak{h})$$

be the *Chevalley projection map* which is the restriction of the projection map $S(\mathfrak{g})=S(\mathfrak{h})\oplus(\mathfrak{n}_-+\mathfrak{n}_+)S(\mathfrak{g})\to S(\mathfrak{h})$ to $S(\mathfrak{g})^{\mathfrak{h}}$. It is known that Ψ is an algebra homomorphism. We have

$$V(I_{W_0})\cap\mathfrak{h}^*=\{\lambda\in\mathfrak{h}\mid p(\lambda)=0 \text{ for all } p\in\Psi(I_{W_0}\cap S(\mathfrak{g})^{\mathfrak{h}})\}\subset\mathfrak{h}^*\cong\mathfrak{h}.$$

Since $V(I_{W_0})$ is G-invariant, it is enough to prove the lemma for nonzero elements $\lambda\in V(I_{W_0})\cap\mathfrak{h}^*$.

It follows from the proof of Proposition 8.8 that $V(I_{W_0})\cap\mathfrak{h}^*$ is the zero locus in $\mathfrak{h}$ of $p_1,\dots,p_{2m-1}$ where for $i\in\{1,\dots,2m-1\}$,

$$p_i:=h_i\left(\sum_{j=1}^{i-1}\frac{-j}{m}h_j+\frac{m-i}{m}h_i+\sum_{j=i+1}^{2m-1}\frac{2m-j}{m}h_j\right).$$

Then it also follows from the proof of Proposition 8.8 that the zero locus in $\mathfrak{h}$ of $p_1, \dots, p_{2m-1}$ is the set

$$\Xi := \bigcup_{1\leqslant s\leqslant 2m-1} \bigcup_{(i_1,\dots,i_s)\in\Lambda_s} \mathbb{C}(\sum_{j=1}^{s}(-1)^j \varpi_{i_j}).$$

Let $\lambda \in \Xi$. We can assume that $\lambda = \sum_{j=1}^{s}(-1)^j\varpi_{i_j}$ for $s \in \{1,\dots,2m-1\}$ and $(i_1,\dots,i_s) \in \Lambda_s$. We have to show that λ is conjugate to ϖ_m under the Weyl group $W(\mathfrak{g},\mathfrak{h})$ of $(\mathfrak{g},\mathfrak{h})$ which is the group of permutations of $\{\varepsilon_1,\dots,\varepsilon_{2m}\}$. Observe that

$$\varpi_m = \frac{1}{2}(\sum_{i=1}^{m}\varepsilon_i - \sum_{i=m+1}^{2m}\varepsilon_i).$$

So it suffices to show that λ can be written as

$$\lambda = \sum_{i=1}^{2m}\sigma_i\varepsilon_i$$

with $\sigma_i \in \{-\frac{1}{2},\frac{1}{2}\}$ for all $i \in \{1,\dots,2m\}$ and

$$\mathrm{card}(\{i \mid \sigma_i = \frac{1}{2}\}) = \mathrm{card}(\{i \mid \sigma_i = -\frac{1}{2}\}) = m.$$

If s is even, we have

$$\begin{aligned}
\lambda = \sum_{j=1}^{s}(-1)^j\varpi_{i_j} =& \sum_{j=1}^{s/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}}) - \frac{\sum_{j=1}^{s}(-1)^j i_j}{2m}(\varepsilon_1+\cdots+\varepsilon_{2m})\\
=& \sum_{j=1}^{s/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}}) - \frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{2m})\\
=& \frac{1}{2}(-(\varepsilon_1+\cdots+\varepsilon_{i_1}) + \sum_{j=1}^{(s-2)/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}}) - (\varepsilon_{i_{2j}+1}+\cdots+\varepsilon_{i_{2j+1}})\\
&+ (\varepsilon_{i_{s-1}}+\cdots+\varepsilon_{i_s}) - (\varepsilon_{i_s+1}+\cdots+\varepsilon_{2m})).
\end{aligned}$$

since $(i_1,\dots,i_s) \in \Lambda_s$. We are done because $\sum_{j=1}^{s}(-1)^j i_j = m$.

If s is odd, we have

$$\begin{aligned}
\lambda = \sum_{j=1}^{s}(-1)^j\varpi_{i_j} =& \sum_{j=1}^{(s-1)/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}}) - (\varepsilon_1+\cdots+\varepsilon_{i_s}) - \frac{\sum_{j=1}^{s}(-1)^j i_j}{2m}(\varepsilon_1+\cdots+\varepsilon_{2m})\\
=& -(\varepsilon_1+\cdots+\varepsilon_{i_1}) - \sum_{j=1}^{(s-1)/2}(\varepsilon_{i_{2j}+1}+\cdots+\varepsilon_{i_{2j+1}}) + \frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{2m})\\
=& \frac{1}{2}(-(\varepsilon_1+\cdots+\varepsilon_{i_1}) + \sum_{j=1}^{(s-1)/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}}) - (\varepsilon_{i_{2j}+1}+\cdots+\varepsilon_{i_{2j+1}})\\
&+ (\varepsilon_{i_s+1}+\cdots+\varepsilon_{2m})).
\end{aligned}$$

since $(i_1,\dots,i_s) \in \Lambda_s$. We are done because $\sum_{j=1}^{s}(-1)^j i_j = -m$. $\square$

The following assertion follows immediately from Lemma 2.1, Lemma 8.7, and Lemma 8.9.

Proposition 8.10. *We have* $V(I_{W_0}) = \overline{\mathbb{S}_{\mathfrak{l}_0}}$.

Proposition 8.11. *The ideal* I_{W_0} *is prime, and therefore, it is the defining ideal of* $\overline{\mathbb{S}_{\mathfrak{l}_0}}$.

Proof. We apply Lemma 3.6 to the ideal $I := I_{W_0}$. First of all, $\mathfrak{l}_0$ and (e, h, f) satisfy the conditions of Lemma 3.4 for $f \in \mathbb{O}_{(2^m)}$. Indeed, $\mathfrak{z}(\mathfrak{l}_0) = \mathbb{C}\varpi_m$ and $\mathfrak{g}(h, i) = 0$ for $i > 2$. It remains to verify that the conditions (1),(2),(3),(4) of Lemma 3.6 are satisfied.

Condition (1) is satisfied by Proposition 8.10 (2). Let us show that the ideal $I + I_\Omega$ is the defining ideal of $\mathbb{O}_{(2^m)}$. According to Proposition 8.10, the zero locus of $I + I_\Omega$ in $\mathfrak{g}$ is $\overline{\mathbb{O}_{(2^m)}}$ since $\Omega(\varpi_m) \neq 0$. On the other hand, by [We02, Theorem 1], the defining ideal I_0 of $\overline{\mathbb{O}_{\boldsymbol{\lambda}_0}}$ is generated by the entries of the matrix X^2 as functions of $X \in \mathfrak{sl}_{2m}(\mathbb{C})$. In particular, I_0 is generated by homogeneous elements of degree 2. Assume that $I + I_\Omega$ is strictly contained in I_0. A contradiction is expected. Since $I_0 \supsetneq I + I_\Omega$, it results from the decomposition

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus W_0 \oplus L_{\mathfrak{g}}(0) \oplus W_1$$

then either I_0 contains a nonzero element of $L_{\mathfrak{g}}(2\theta)$, or I_0 contains an element of W_1. Since I_0 is $\mathfrak{g}$-invariant, either I_0 contains $L_{\mathfrak{g}}(2\theta)$ or I_0 contains W_1. The zero locus in $\mathfrak{g}$ of the ideal generated by $L_{\mathfrak{g}}(2\theta)$ is $\{0\}$ since $L_{\mathfrak{g}}(2\theta)$ is generated as a $\mathfrak{g}$-module by $(e_\theta)^2$. In addition, by Remark 7.9, the zero locus in $\mathfrak{g}$ of the ideal generated by W_1 and Ω is $\overline{\mathbb{O}_{min}}$. Hence in both cases we go to a contradiction since $\overline{\mathbb{O}_{(2^m)}}$ strictly contains $\overline{\mathbb{O}_{min}}$ and $\{0\}$. So $I + I_\Omega = I_0$ is prime. Finally, condition (2) is satisfied.

Condition (3) is satisfied too, by Corollary 8.6.

At last, because $\Omega(\varpi_m) \neq 0$, condition (4) is satisfied. In conclusion, by Lemma 3.6, $I = I_{W_0}$ is prime. □

Proof of Theorem 1.1 (2). The first statement follows from (9) and Proposition 8.11. The second statement follows from the inclusions

$$J_\infty \overline{\mathbb{S}_{\mathfrak{l}_0}} = SS(V_{-m}(\mathfrak{g}))_{\mathrm{red}} \subset SS(V_{-m}(\mathfrak{g})) \subset J_\infty X_{V_{-m}(\mathfrak{g})} = J_\infty \overline{\mathbb{S}_{\mathfrak{l}_0}}.$$

□

Remark 8.12. If n is odd, then the level $-n/2$ is admissible for $\widehat{\mathfrak{sl}}_n$, and we have

$$X_{V_{-n/2}(\mathfrak{sl}_n)} = \overline{\mathbb{O}_{(2^{(n-1)/2},1)}}$$

by [Ar15a].

Let $\mathfrak{p} = \mathfrak{l}_0 \oplus \mathfrak{p}_u$ be a parabolic subalgebra of $\mathfrak{g}$, and let P be the connected parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Let ψ_Y be the algebra homomorphism $U(\mathfrak{g}) \to \mathcal{D}_Y$ as in §2, where $Y := G/(P, P)$. Recall that $V(\mathrm{gr}\, \mathcal{J}_Y) = \overline{\mathbb{S}_{\mathfrak{l}_0}}$, where $\mathcal{J}_Y = \ker \psi_Y$.

Theorem 8.13. (1) *The ideal* $\mathrm{gr}\, \mathscr{J}_Y \subset \mathbb{C}[\mathfrak{g}^*]$ *is prime and hence it is the defining ideal of* $\overline{\mathbb{S}_{\mathfrak{l}_0}}$.

(2) *The natural homomorphism* $R_{V_{-m}(\mathfrak{g})} \to \mathrm{gr}\, A(V_{-m}(\mathfrak{g}))$ *is an isomorphism.*

(3) *The map* ψ_Y *induces an embedding*

$$A(V_{-m}(\mathfrak{g})) \hookrightarrow \mathcal{D}_Y.$$

Proof. We have

$$A(V_{-m}(\mathfrak{g})) = U(\mathfrak{g})/\mathscr{J}_{W_0},$$

where $\mathscr{J}_{W_0}$ is the two-sided ideal of $U(\mathfrak{g})$ generated by W_0.

Recall that

$$\mathscr{J}_Y = \bigcap_{t\in Z} \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\varpi_m}$$

for any Zariski dense subset Z of $\mathbb{C}$, see §2, equation (1). On the other hand, for a generic point t of $\mathbb{C}$ we have $L_{\mathfrak{g}}(t\varpi_m) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{t\varpi_m}$, and thus, $\mathcal{J}_{W_0} \subset \operatorname{Ann} L_{\mathfrak{g}}(t\varpi_m)$ by Proposition 5.1 and Proposition 8.8. Therefore $\mathcal{J}_{W_0} \subset \mathcal{J}_Y$. This gives

$$I_{W_0} \subset \operatorname{gr} \mathcal{J}_{W_0} \subset \operatorname{gr} \mathcal{J}_Y \subset \sqrt{\operatorname{gr} \mathcal{J}_Y}.$$

Since $\sqrt{\operatorname{gr} \mathcal{J}_Y} = I_{W_0}$, all inclusions above are equality. □

Since $H_f^{\frac{\infty}{2}+0}(V_{-m}(\mathfrak{g})) \cong \mathrm{Vir}_1$ by Theorem 8.3, we have a functor

$$V_{-m}(\mathfrak{g})\text{-Mod}^{\mathfrak{g}[t]} \to \mathrm{Vir}_1\text{-Mod}, \quad M \mapsto H_f^{\frac{\infty}{2}+0}(M), \tag{12}$$

where Vir_1-Mod denotes the category of Vir_1-modules. By Proposition 8.8, the simple objects of $V_{-m}(\mathfrak{g})\text{-Mod}^{\mathfrak{g}[t]}$ are $L(t\varpi_m - m\Lambda_0)$, $t \in \mathbb{Z}_{\geqslant 0}$. From Theorem 6.3 it follows that

$$H_f^{\frac{\infty}{2}+0}(L(t\varpi_m - m\Lambda_0)) \cong L(1, \frac{t(t+m+1)}{4}),$$

where $L(c,h)$ denotes the irreducible highest weight representation of the Virasoro algebra of central charge c and lowest weight h.

Question 3. Is the functor (12) fusion?

9. Level $-(r-2)$ affine vertex algebra of type D_r, $r \geqslant 5$

We assume in this section that $\mathfrak{g} = \mathfrak{so}_{2r}$ with $r \geqslant 5$. Let

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \mid 1 \leqslant i < j \leqslant r\}$$

be the root system of $\mathfrak{g}$ and take

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, \mid 1 \leqslant i < j \leqslant r\}$$

for the set of positive roots.

Denote by (e_i, h_i, f_i) the Chevalley generators of $\mathfrak{g}$, and fix the root vectors e_α, f_α, $\alpha \in \Delta_+$ so that $(h_i, i = 1, \ldots, r) \cup (e_\alpha, f_\alpha, \alpha \in \Delta_+)$ is a Chevalley basis satisfying the conditions of [Ga82, Chapter IV, Definition 6]. For $\alpha \in \Delta_+$, denote by $h_\alpha = [e_\alpha, f_\alpha]$ the corresponding coroot. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding triangular decomposition.

The fundamental weights are:

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leqslant i \leqslant r-2),$$

$$\varpi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-2} + \varepsilon_{r-1} - \varepsilon_r), \qquad \varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-2} + \varepsilon_{r-1} + \varepsilon_r).$$

Let

$$w_1 := \sum_{i=2}^{r} e_{\varepsilon_1 - \varepsilon_i} e_{\varepsilon_1 + \varepsilon_i} \in S^2(\mathfrak{g}).$$

Then w_1 is a singular vector with respect to the adjoint action of $\mathfrak{g}$ and generates an irreducible finite-dimensional representation W_1 of $\mathfrak{g}$ in $S^2(\mathfrak{g})$ isomorphic to $L_{\mathfrak{g}}(\theta+\theta_1)$.

Proposition 9.1 ([Pe13, Theorem 3.1])**.** *The vector $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $k=n-r+2$.*

Let $\tilde{V}_{2-r}(\mathfrak{g})=V^{2-r}(\mathfrak{g})/U(\widehat{\mathfrak{g}})\sigma(w_1)$. By definition, we have

$$X_{\tilde{V}_{2-r}(\mathfrak{g})}=V(I_{W_1}). \tag{13}$$

Let $\mathfrak{l}^I$ and $\mathfrak{l}^{II}$ be the standard Levi subalgebras of $\mathfrak{g}$ generated by the simple roots $\alpha_1,\dots,\alpha_{r-2},\alpha_r$ and $\alpha_1,\dots,\alpha_{r-2},\alpha_{r-1}$, respectively. Note that $\mathfrak{l}^I$ and $\mathfrak{l}^{II}$ are not G-conjugate [CMa93, Lemma 7.3.2]. Denote by $\mathbb{S}_{\mathfrak{l}^I}$ and $\mathbb{S}_{\mathfrak{l}^{II}}$ the corresponding Diximier sheets.

If r is odd, then $\mathbb{O}_{(2^{r-1},1^2)}$ is the unique nilpotent orbit contained in both $\mathbb{S}_{\mathfrak{l}^I}$ and $\mathbb{S}_{\mathfrak{l}^{II}}$. Indeed, $\mathbb{O}_{(2^{r-1},1^2)}$ is the unique nilpotent orbit of $\mathfrak{g}$ of dimension $r^2=\dim\mathfrak{l}^I=\dim\mathfrak{l}^{II}$.

If r is even, possible changing the numbering, we can assume that $\mathbb{O}^I_{(2^r)}$ is the unique nilpotent orbit contained in $\mathbb{S}_{\mathfrak{l}^I}$ and that $\mathbb{O}^{II}_{(2^r)}$ is the unique nilpotent orbit contained in $\mathbb{S}_{\mathfrak{l}^{II}}$. To see this, first observe that $\mathbb{O}^I_{(2^r)}$ and $\mathbb{O}^{II}_{(2^r)}$ are the unique nilpotent orbits of $\mathfrak{g}$ of dimension $r^2=\dim\mathfrak{l}^I=\dim\mathfrak{l}^{II}$. One knows that $\mathbb{S}_{\mathfrak{l}^I}$ contains a unique nilpotent orbit; so it is either $\mathbb{O}^I_{(2^r)}$, or $\mathbb{O}^{II}_{(2^r)}$. Assume for instance that it is $\mathbb{O}^I_{(2^r)}$. The nilpotent orbits $\mathbb{O}^I_{(2^r)}$ and $\mathbb{O}^{II}_{(2^r)}$ are induced only from a zero orbit in a Levi subalgebra of type $\mathfrak{l}^I$ or $\mathfrak{l}^{II}$ (see e.g. [CMa93, Corollary 7.3.4]). So the only possible sheets containing $\mathbb{O}^I_{(2^r)}$ and $\mathbb{O}^{II}_{(2^r)}$ are $\mathbb{S}_{\mathfrak{l}^I}$ and $\mathbb{S}_{\mathfrak{l}^{II}}$. This implies that $\mathbb{O}^{II}_{(2^r)}$ must be contained in $\mathbb{S}_{\mathfrak{l}^{II}}$. If we had assumed that $\mathbb{O}^{II}_{(2^r)}$ is contained in $\mathbb{S}_{\mathfrak{l}^I}$, we get in the same way that $\mathbb{O}^I_{(2^r)}$ must be contained in $\mathbb{S}_{\mathfrak{l}^{II}}$.

Lemma 9.2. (1) *If r is odd, $V(I_{W_1})\cap\mathcal{N}\subset\overline{\mathbb{O}_{(2^{r-1},1^2)}}$.*
(2) *If r is even, $V(I_{W_1})\cap\mathcal{N}\subset\overline{\mathbb{O}^I_{(2^r)}}\cup\overline{\mathbb{O}^{II}_{(2^r)}}$.*

Proof. Let $\boldsymbol{\lambda}$ be the element of $\mathscr{P}_1(2r)$ defined by

$$\boldsymbol{\lambda}:=(3,2^{r-3},1^3)\text{ if } r \text{ is odd,}$$
$$\boldsymbol{\lambda}:=(3,2^{r-2},1)\text{ if } r \text{ is even.}$$

We observe that $\mathbb{O}_{\boldsymbol{\lambda}}$ it the smallest nilpotent orbit of $\mathfrak{g}$ which dominates $\mathbb{O}_{(2^{r-1},1^2)}$ if r is odd, and both $\mathbb{O}^I_{(2^r)}$ and $\mathbb{O}^{II}_{(2^r)}$ if r is even. By this, it means that if for $\boldsymbol{\mu}\in\mathscr{P}_1(2r)$, $\boldsymbol{\mu}\succ(2^{r-1},1^2)$ if r is odd, and $\boldsymbol{\mu}\succ(2^r)$ if r is even, then $\boldsymbol{\mu}\succcurlyeq\boldsymbol{\lambda}$ where $\succcurlyeq$ is the partial order on the set $\mathscr{P}_1(n)$ induced by the Chevalley order. Therefore, it is enough to show that $V(I_{W_2})$ do not contain $\overline{\mathbb{O}_{\boldsymbol{\lambda}}}$.

Let $f\in\mathbb{O}_{\boldsymbol{\lambda}}$ that we embed into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. For $i\in\mathbb{Z}$, denote by $\mathfrak{g}(h,i)$ the i-eigenspace of $\mathrm{ad}(h)$ and by $\Delta_+(h,i)$ the set of positive roots $\alpha\in\Delta_+$ such that $e_\alpha\in\mathfrak{g}(h,i)$. Choose a Lagrangian subspace $\mathfrak{L}\subset\mathfrak{g}(h,1)$ and set

$$\mathfrak{m}:=\mathfrak{L}\oplus\bigoplus_{i\geqslant 2}\mathfrak{g}(h,i),\qquad J_\chi:=\sum_{x\in\mathfrak{m}}\mathbb{C}[\mathfrak{g}^*](x-\chi(x)),$$

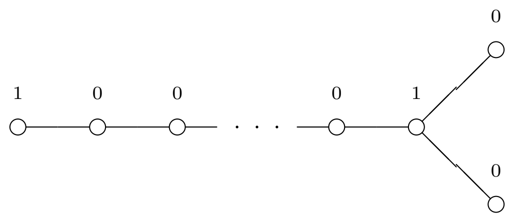
with $\chi=(f|\cdot)\in\mathfrak{g}^*$, as in §3. By Lemma 3.3, it is sufficient to show that

$$\mathbb{C}[\mathfrak{g}^*]=I_{W_1}+J_\chi.$$

To see this, we shall use the vector

$$w_1 = \sum_{i=2}^{r} e_{\varepsilon_1-\varepsilon_i} e_{\varepsilon_1+\varepsilon_i}.$$

Assume first that r is odd. It follows from [CMa93, Lemma 5.3.5] that the weighted Dynkin diagram of the nilpotent orbit $G.f$ is



So we can choose for h the element

$$h = \varpi_1 + \varpi_{r-2} = 2\alpha_1 + \sum_{j=2}^{r-2}(j+1)\alpha_j + \frac{r}{r-1}(\alpha_{r-1}+\alpha_r).$$

We see from the above diagram that $\varepsilon_1+\varepsilon_i \in \Delta_+(h,3)$ for $i \in \{2,\dots,r-2\}$, and

$$\Delta_+(h,2) = \{\varepsilon_1 \pm \varepsilon_{r-1},\, \varepsilon_1 \pm \varepsilon_r,\, \varepsilon_i+\varepsilon_j,\; 2 \leqslant i < j < r-2\}.$$

We can choose e, f so that

$$e = \sum_{\alpha\in\Delta_+(h,2)} a_\alpha e_\alpha \quad \text{and} \quad f = \sum_{\alpha\in\Delta_+(h,2)} b_\alpha f_\alpha$$

with $a_\alpha, b_\alpha \in \mathbb{C}$ for all $\alpha \in \Delta_+(h,2)$. Set for $\alpha \in \Delta_+(h,2)$,

$$c_\alpha := a_\alpha b_\alpha.$$

From the relation $[e,f]=h$, we obtain the equations:

$$c_{\varepsilon_1-\varepsilon_{r-1}} + c_{\varepsilon_1+\varepsilon_{r-1}} + c_{\varepsilon_1-\varepsilon_r} + c_{\varepsilon_1+\varepsilon_r} = 2 \tag{14}$$

$$c_{\varepsilon_1+\varepsilon_{r-1}} + c_{\varepsilon_1-\varepsilon_r} + \sum_{2\leqslant i<j<r-2} c_{\varepsilon_i+\varepsilon_j} = \frac{r-1}{2} \tag{15}$$

$$c_{\varepsilon_1+\varepsilon_{r-1}} + c_{\varepsilon_1+\varepsilon_r} + \sum_{2\leqslant i<j<r-2} c_{\varepsilon_i+\varepsilon_j} = \frac{r-1}{2} \tag{16}$$

$$c_{\varepsilon_1-\varepsilon_{r-1}} + c_{\varepsilon_1+\varepsilon_{r-1}} + c_{\varepsilon_1-\varepsilon_r} + c_{\varepsilon_1+\varepsilon_r} + 2\sum_{2\leqslant i<j<r-2} c_{\varepsilon_i+\varepsilon_j} = r-1 \tag{17}$$

by considering the coefficients of h in α_1, α_{r-1}, α_r and α_{r-2} respectively. By (15) and (16) we get that

$$c_{\varepsilon_1-\varepsilon_r} = c_{\varepsilon_1+\varepsilon_r}.$$

Then by (17), (15) and (16) we get that

$$c_{\varepsilon_1-\varepsilon_{r-1}} = c_{\varepsilon_1+\varepsilon_{r-1}}.$$

So from (14), we obtain:

$$c_{\varepsilon_1-\varepsilon_r} + c_{\varepsilon_1-\varepsilon_{r-1}} = 1.$$

In particular, $c_{\varepsilon_1-\varepsilon_r} \neq -c_{\varepsilon_1-\varepsilon_{r-1}}$ and $c_{\varepsilon_1-\varepsilon_{r-1}}$ and $c_{\varepsilon_1-\varepsilon_r}$ cannot be both zero.

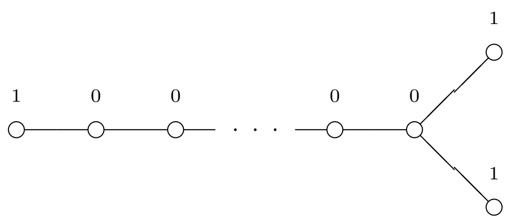
Hence for some nonzero complex number c, we have

$$w_1 = \sum_{i=2}^{r-2} e_{\varepsilon_1-\varepsilon_i} e_{\varepsilon_1+\varepsilon_i} + e_{\varepsilon_1-\varepsilon_{r-1}} e_{\varepsilon_1+\varepsilon_{r-1}} + e_{\varepsilon_1-\varepsilon_r} e_{\varepsilon_1+\varepsilon_r} = c \pmod{J_\chi}$$

and so $I_{W_2} + J_\chi = \mathbb{C}[\mathfrak{g}^*]$.

Assume that r is even.

It follows from [CMa93, Lemma 5.3.5] that the weighted Dynkin diagram of the nilpotent orbit $G.f$ is



So we can choose for h the element

$$h = \varpi_1 + \varpi_{r-1} + \varpi_r = 2\alpha_1 + \sum_{j=2}^{r-2}(j+1)\alpha_j + \frac{r}{2}(\alpha_{r-1} + \alpha_r).$$

We see from the above diagram that $\varepsilon_1 + \varepsilon_i \in \Delta_+(h,3)$ for $i \in \{2, \dots, r-1\}$, and

$$\Delta_+(h,2) = \{\varepsilon_1 \pm \varepsilon_r,\, \varepsilon_i + \varepsilon_j,\ 2 \leqslant i < j \leqslant r-1\}.$$

We can choose e, f so that

$$e = \sum_{\alpha \in \Delta_+(h,2)} a_\alpha e_\alpha \quad \text{and} \quad f = \sum_{\alpha \in \Delta_+(h,2)} b_\alpha f_\alpha$$

with $a_\alpha, b_\alpha \in \mathbb{C}$ for all $\alpha \in \Delta_+(h,2)$. Set for $\alpha \in \Delta_+(h,2)$,

$$c_\alpha := a_\alpha b_\alpha.$$

From the relation $[e,f] = h$, we obtain the equations:

$$c_{\varepsilon_1-\varepsilon_r} + c_{\varepsilon_1+\varepsilon_r} = 2 \tag{18}$$

$$c_{\varepsilon_1-\varepsilon_r} + \sum_{2 \leqslant i < j \leqslant r-1} c_{\varepsilon_i+\varepsilon_j} = \frac{r}{2} \tag{19}$$

$$c_{\varepsilon_1+\varepsilon_r} + \sum_{2 \leqslant i < j \leqslant r-1} c_{\varepsilon_i+\varepsilon_j} = \frac{r}{2} \tag{20}$$

by considering the coefficients of h in α_1, α_{r-1}, and α_r respectively. By (19) and (20) we get that $c_{\varepsilon_1-\varepsilon_r} = c_{\varepsilon_1+\varepsilon_r}$. So from (18), we get that

$$c_{\varepsilon_1-\varepsilon_r} = c_{\varepsilon_1+\varepsilon_r} = 1.$$

Hence for some nonzero complex number c, we have

$$w_1 = \sum_{i=2}^{r-1} e_{\varepsilon_1-\varepsilon_i} e_{\varepsilon_1+\varepsilon_i} + e_{\varepsilon_1-\varepsilon_r} e_{\varepsilon_1+\varepsilon_r} = c \pmod{J_\chi}$$

and so $I_{W_2} + J_\chi = \mathbb{C}[\mathfrak{g}^*]$. □

Lemma 9.3. *Let λ be a nonzero semisimple element of $\mathfrak{g}$ which belongs to $V(I_{W_1})$. Then $\lambda \in G.\mathbb{C}^*\varpi_{r-1}$ or $\lambda \in G.\mathbb{C}^*\varpi_r$.*

Proof. Arguing as in the beginning of the proof of Lemma 8.9, we see that it is enough to prove the lemma for a nonzero element λ in $V(I_{W_1}) \cap \mathfrak{h}$.

The image $\Upsilon(I_{W_1}^{\mathfrak{h}})$ of $I_{W_1}^{\mathfrak{h}}$, viewed as an ideal of $U(\mathfrak{g})$, was determined by Perše in the proof of [Pe13, Theorem 3.4]. From Perše's proof, we easily deduce that $\Psi(I_{W_1} \cap S(\mathfrak{g})^{\mathfrak{h}})$ is generated by the elements

$$p_i := h_i(h_i + 2h_{i+1} + \cdots + 2h_{r-2} + h_{r-1} + h_r), \qquad i = 1, \ldots, r,$$

and that

$$V(I_{W_1}) \cap \mathfrak{h} = \bigcup_{\substack{\{i_1,\ldots,i_k\} \subset \{1,\ldots,r-2\} \\ i_1 < \cdots < i_k}} (\mathbb{C}\left(\sum_{j=1}^{k} (-1)^{k-j+1}\varpi_{i_j} + \varpi_{r-1}\right) \cup \mathbb{C}\left(\sum_{j=1}^{k} (-1)^{k-j+1}\varpi_{i_j} + \varpi_r\right)).$$

Thus we are lead to show that for any nonempty sequence $(i_1, \ldots, i_k)$ in $\{1, \ldots, r-2\}$, with $i_1 < \cdots < i_k$, the element $\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j} + \varpi_{r-1}$ (respectively $\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j} + \varpi_r$) is either $W(\mathfrak{g}, \mathfrak{h})$-conjugate to ϖ_{r-1}, or $W(\mathfrak{g}, \mathfrak{h})$-conjugate to ϖ_r. Here, $W(\mathfrak{g}, \mathfrak{h}) \cong \mathfrak{S}_r \rtimes (\mathbb{Z}/2\mathbb{Z})^{r-1}$ denotes the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ which is the group of permutations and sign changes involving only even numbers of signs of the set $\{\varepsilon_1, \ldots, \varepsilon_r\}$, see, e.g., [Hu72, §12.1]. (If the sequence is empty, the statement is obvious.)

Let $(i_1, \ldots, i_k)$ in $\{1, \ldots, r-2\}$, with $i_1 < \cdots < i_k$. We prove the statement for $\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j} + \varpi_r$. Similar arguments holds for $\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j} + \varpi_{r-1}$.

If k is even, we have

$$\sum_{j=1}^{k} (-1)^{k-j+1}\varpi_{i_j} = \sum_{j=1}^{k-1} (\varpi_{i_j} - \varpi_{i_{j+1}}) = -\sum_{j=1}^{k/2} (\varepsilon_{i_{2j-1}+1} + \cdots + \varepsilon_{i_{2j}}).$$

Hence

$$\sum_{j=1}^{k} (-1)^{k-j+1}\varpi_{i_j} + \varpi_r = \frac{1}{2}((\varepsilon_1 + \cdots + \varepsilon_{i_1}) - (\varepsilon_{i_1} + \cdots + \varepsilon_{i_2}) \\ + \cdots + (\varepsilon_{i_{k-2}} + \cdots + \varepsilon_{i_{k-1}}) - (\varepsilon_{i_{k-1}+1} + \cdots + \varepsilon_{i_k}) + \varepsilon_{i_k+1} + \cdots + \varepsilon_{i_r})),$$

and this element is $W(\mathfrak{g}, \mathfrak{h})$-conjugate to ϖ_r of ϖ_{r-1} depending on the parity of $(i_2 - i_1) + \cdots + (i_k - i_{k-1})$.

If k is odd, then

$$\begin{aligned}\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j} &= (-\varpi_{i_1}+\varpi_{i_2})+(-\varpi_{i_3}+\varpi_{i_4})+\cdots+(-\varpi_{i_{k-2}}+\varpi_{i_{k-1}})-\varpi_{i_k}\\ &= \sum_{j=1}^{(k-1)/2}(\varepsilon_{i_{2j-1}+1}+\cdots+\varepsilon_{i_{2j}})-(\varepsilon_1+\cdots+\varepsilon_{i_k})\\ &= -(\varepsilon_1+\cdots+\varepsilon_{i_1})-\sum_{j=1}^{(k-1)/2}(\varepsilon_{i_{2j}+1}+\cdots+\varepsilon_{i_{2j}+1})\end{aligned}$$

and we conclude as in the case where k is even that $\sum_{j=1}^{k}(-1)^{k-j+1}\varpi_{i_j}+\varpi_r$ is conjugate to ϖ_r of ϖ_{r-1}, depending here on the parity of $i_1+(i_3-i_2)+\cdots+(i_k-i_{k-1})$. $\square$

The following assertion follows immediately from Lemma 2.1, Lemma 9.2 and Lemma 9.3 .

Proposition 9.4. *We have $V(I_{W_1})=\overline{\mathbb{S}_{\mathfrak{l}^I}}\cup\overline{\mathbb{S}_{\mathfrak{l}^{II}}}$, and hence, $X_{\tilde{V}_{2-r}(\mathfrak{g})}=\overline{\mathbb{S}_{\mathfrak{l}^I}}\cup\overline{\mathbb{S}_{\mathfrak{l}^{II}}}$.*

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Since $V_{2-r}(\mathfrak{g})$ is a quotient of $\tilde{V}_{2-r}(\mathfrak{g})$,

$$X_{V_{2-r}(\mathfrak{g})}\subset X_{\tilde{V}_{2-r}(\mathfrak{g})}=\overline{\mathbb{S}_{\mathfrak{l}^I}}\cup\overline{\mathbb{S}_{\mathfrak{l}^{II}}} \tag{21}$$

by Proposition 9.4. On the other hand, by [AP14, Theorem 7.2], one knows that $V_{2-r}(\mathfrak{g})$ has infinitely many simple modules in the category $\mathcal{O}$. This gives

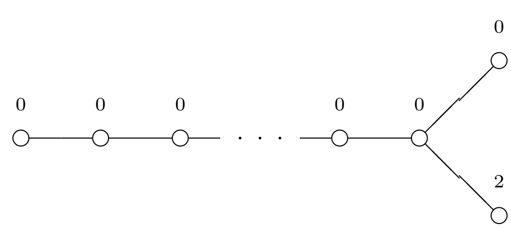
$$X_{V_{2-r}(\mathfrak{g})}\not\subset\mathcal{N} \tag{22}$$

by [AM15, Corollary 5.3]. Therefore $X_{V_{r-2}(\mathfrak{g})}\cap\mathfrak{h}\neq 0$ as $X_{V_{r-2}(\mathfrak{g})}$ is closed and G-invariant. Hence, by (21), either $\mathbb{C}\varpi_{r-1}\subset X_{V_{2-r}(\mathfrak{g})}$, or $\mathbb{C}\varpi_r\subset X_{V_{2-r}(\mathfrak{g})}$. Thus, $\overline{\mathbb{S}_{\mathfrak{l}^I}}\subset X_{V_{2-r}(\mathfrak{g})}$ or $\overline{\mathbb{S}_{\mathfrak{l}^{II}}}\subset X_{V_{2-r}(\mathfrak{g})}$. On the other hand $V_{2-r}(\mathfrak{g})$ is invariant under the Dynkin automorphism (cf. [Ar16a, Proof of Lemma 2.7]), and hence, so is $X_{V_{2-r}(\mathfrak{g})}$. This completes the proof. $\square$

We now wish to prove Theorem 1.3.

Theorem 9.5. *Assume r is even, and let $f\in\mathbb{O}^I_{(2^r)}\cup\mathbb{O}^{II}_{(2^r)}$. Then $H_f^{\frac{\infty}{2}+0}(V_{2-r}(\mathfrak{g}))=0$.*

Proof. By symmetry, we may assume that the weighted Dynkin diagram of f is given by



and $h=2\varpi_r$. Set $\mathfrak{l}=\mathfrak{l}^{II}=\mathfrak{g}(h,0)$.

Since f is a short nilpotent element, it is sufficient to show that

$$\dim V_{2-r}(\mathfrak{g})_{0,h} < \frac{1}{2}\dim \mathbb{O}^{II}_{(2^r)}$$

by Theorem 6.3. Observe that, as a $\mathfrak{g}$-module, $V_{2-r}(\mathfrak{g})_{0,h}$ is isomorphic to $L_{\mathfrak{g}}((2-r)\varpi_r)$, which is a quotient of the *scalar generalized Verma module* $M_{\mathfrak{l}}((2-r)\varpi_r)$. Here

$$M_{\mathfrak{l}}(a\varpi_r) = U(\mathfrak{g})\otimes_{U(\mathfrak{l}\,\oplus\,\mathfrak{g}(h,2))}\mathbb{C}_{a\varpi_r}$$

for $a\in\mathbb{C}$, where $\mathbb{C}_{a\varpi_r}$ is a one-dimensional representation of $\mathfrak{l}\oplus\mathfrak{g}(h,2)$ on which $[\mathfrak{l},\mathfrak{l}]\oplus\mathfrak{g}(h,2)$ acts trivially and $h_r\in\mathfrak{z}(\mathfrak{l})$ acts as multiplication by a. By [Mat06, Theorem 3.2.3 (2c)], one knows that there is an embedding $M_{\mathfrak{l}}((-a-2r-2)\varpi_r)\hookrightarrow M_{\mathfrak{l}}(a\varpi_r)$ for $a\in\mathbb{Z}$, $a\geqslant 2-r$. In particular, $M_{\mathfrak{l}}(-r\varpi_r)\subset M_{\mathfrak{l}}((2-r)\varpi_r)$. Therefore,

$$\begin{aligned}\operatorname{Dim} V_{2-r}(\mathfrak{g})_{0,h} = \operatorname{Dim} L_{\mathfrak{g}}((2-r)\varpi_r) &\leqslant \operatorname{Dim}(M_l((2-r)\varpi_r)/M_{\mathfrak{l}}(-r\varpi_r))\\ &< \operatorname{Dim}(M_{\mathfrak{l}}((2-r)\varpi_r) = \frac{1}{2}\dim \mathbb{O}^{II}_{(2^r)}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. First, we have $X_{V_{2-r}(\mathfrak{g})}\subset \overline{\mathbb{S}_{\mathfrak{l}^I}}\cup\overline{\mathbb{S}_{\mathfrak{l}^{II}}}$ by Proposition 9.4. Suppose that $X_{V_{2-r}(\mathfrak{g})}\subsetneqq\mathcal{N}$. Then it follows that $X_{V_{2-r}(\mathfrak{g})}=\overline{\mathbb{S}_{\mathfrak{l}^I}}\cup\overline{\mathbb{S}_{\mathfrak{l}^{II}}}$, see the proof of Theorem 1.2. But we have $\mathbb{O}^{I}_{(2^r)},\mathbb{O}^{II}_{(2^r)}\not\subset X_{V_{2-r}(\mathfrak{g})}$ by Theorem 6.1 and Theorem 9.5. Since this is a contradiction we get that $X_{V_{2-r}(\mathfrak{g})}\subset\mathcal{N}$, and hence,

$$X_{V_{2-r}(\mathfrak{g})}\subsetneqq\overline{\mathbb{O}^{I}_{(2^r)}},\ \overline{\mathbb{O}^{II}_{(2^r)}}.$$

We conclude that $X_{V_{2-r}(\mathfrak{g})}\subset\overline{\mathbb{O}_{(2^{r-2},1^4)}}=\overline{\mathbb{O}^{I}_{(2^r)}}\cap\overline{\mathbb{O}^{II}_{(2^r)}}$.

On the other hand, Theorem 6.2 gives that $H^{\frac{\infty}{2}+0}_{f_\theta}(V_{r-2}(\mathfrak{g}))=\mathcal{W}_{r-2}(\mathfrak{g},f_\theta)$, which is not lisse by [AM15, Theorem 7.1] unless $r=4$. Therefore, $\overline{\mathbb{O}_{min}}\subsetneqq X_{V_{2-r}(\mathfrak{g})}$ by Theorem 6.1.

The last assertion follows from [AM15, Corollary 5.3]. □

Conjecture 2. *Let r be even. Then $X_{V_{2-r}(\mathfrak{g})}=\overline{\mathbb{O}_{(2^{r-2},1^4)}}$. Therefore, the W-algebra $\mathcal{W}_{2-r}(\mathfrak{g},f)$ is lisse for $f\in\mathbb{O}_{(2^{r-2},1^4)}$.*

Theorem 9.6. *Conjecture 2 is true for $r=6$.*

Proof. For $r=6$, $\mathbb{O}_{min}$ is the only nilpotent orbit that is strictly contained in $\overline{\mathbb{O}_{(2^{r-2},1^4)}}$. □

10. On associated varieties and minimal W-algebras

We take the opportunity of this note to clarify some points of [AM15] that are related to the present work. Assume that $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.

Let $f\in\mathcal{N}$ and let $\mathcal{W}^k(\mathfrak{g},f)$ be the affine W-algebra associated with $(\mathfrak{g},f)$ at level k, and let $\mathcal{W}_k(\mathfrak{g},f_\theta)$ be its unique simple quotient as in §6. By Theorem 6.1, we know that $\mathcal{W}_k(\mathfrak{g},f)$ is lisse if $X_{V_k(\mathfrak{g})}=\overline{G.f}$. We now prove Theorem 1.4 which says that for the minimal nilpotent element $f=f_\theta$, the converse is also true provided that $k\notin\mathbb{Z}_{\geqslant 0}$.

Proof of Theorem 1.4. First, if $k \notin \mathbb{Z}_{\geqslant 0}$, then $\mathcal{W}_k(\mathfrak{g}, f_\theta) = H^{\frac{\infty}{2}+0}(V_k(\mathfrak{g}))$ by Theorem 6.2. Assume that $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse. Then by [Ar15a, Theorem 4.21 and Proposition 4.22],

$$\overline{\mathbb{O}_{min}} \subset X_{V_k(\mathfrak{g})} \quad \text{and} \quad \dim(X_{V_k(\mathfrak{g})} \cap \mathscr{S}_{f_\theta}) = 0. \tag{23}$$

Let x be a closed point of $\tilde{X}_{V_k(\mathfrak{g})}$. We need to show that $x \in \overline{\mathbb{O}_{min}}$. Since $X_{V_k(\mathfrak{g})}$ is a G-invariant closed cone, it contains the G-invariant cone $C(x)$ generated by x, and its closure $\overline{C(x)}$. By [CMo10, Theorem 2.9], $\overline{C(x)} \cap \mathcal{N}$ is the closure of the nilpotent orbit $\mathbb{O}$ induced from the nilpotent orbit of x_n in $\mathfrak{g}^{x_s}$. By [Gi09, Corollary 1.3.8(iii)], the nilpotent orbit $\mathbb{O}$ is $\mathbb{O}_{min}$ or 0; otherwise, $\dim(\overline{\mathbb{O}} \cap \mathscr{S}_{min}) > 0$ which would contradict (23). If $\mathbb{O} = \mathbb{O}_{min}$ then x_s must be zero; otherwise, again by [Gi09, Corollary 1.3.8(iii)], $\dim(\overline{C(x)} \cap \mathscr{S}_{min}) > 0$ which would contradict (23). If $\mathbb{O} = 0$, then $x_s = x_n = 0$ since $\operatorname{codim}_{\mathfrak{g}}(0) = \dim \mathfrak{g}$ is the codimension of the nilpotent orbit of x_n in $\mathfrak{g}^{x_s}$.

In both cases, $x = x_n$ and $x \in \overline{C(x)} \subset \overline{\mathbb{O}_{min}}$, whence the statement. □

Remark 10.1. Let $k \in \mathbb{Z}_{\geqslant 0}$. Then $\mathcal{W}_k(\mathfrak{g}, f_\theta) = H^{\frac{\infty}{2}+0}(L(s_0 \circ k\Lambda_0))$ where s_0 is the reflection corresponding to the simple root $\alpha_0 = -\theta + \delta$, see [Ar05]. Thus [Ar15a]

$$X_{\mathcal{W}_k(\mathfrak{g},f)} = X_{L(s_0 \circ \lambda)} \cap \mathscr{S}_{f_\theta},$$

where $X_{L(\lambda)} = \operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(L(\lambda)/C_2(L(\lambda)))$, $C_2(L(\lambda)) = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}v \mid a \in V^k(\mathfrak{g}), v \in L(\lambda)\}$. Therefore the proof of Theorem 1.4 implies that, for $k \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse if and only if $X_{L(s_0 \circ \lambda)} = \overline{\mathbb{O}_{min}}$.

For a general $f \in \mathcal{N}$, we have the following result.

Proposition 10.2. *Let $f \in \mathcal{N}$ and $k \in \mathbb{C}$. Assume that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Then $\overline{G.f}$ is an irreducible component of $X_{V_k(\mathfrak{g})}$.*

Proof. By hypothesis and [Ar15a, Theorem 4.21 and Proposition 4.22], we have

$$\overline{G.f} \subset X_{V_k(\mathfrak{g})} \quad \text{and} \quad \dim(X_{V_k(\mathfrak{g})} \cap \mathscr{S}_f) = 0.$$

Let Y be an irreducible component of $X_{V_k(\mathfrak{g})}$ which contains $\overline{G.f}$. By [Gi09, Corollary 1.3.8(iii)],

$$\dim(Y \cap \mathscr{S}_f) = \dim Y - \dim \overline{G.f}$$

which forces $Y = \overline{G.f}$, whence the statement. □

Let $\mathfrak{g}^\natural$ be the centralizer in $\mathfrak{g}$ of the $\mathfrak{sl}_2$-triple $(e_\theta, f_\theta, h_\theta)$, and let $\mathfrak{g}^\sharp = \bigoplus_{i\geqslant 0} \mathfrak{g}_i^\natural$ be the decomposition into the sum of its center $\mathfrak{g}_0^\natural$ and the simple summands $\mathfrak{g}_i^\natural$, $i \geqslant 1$. Then we have a vertex algebra embedding

$$\bigotimes_{i\geqslant 0} V^{k_i^\natural}(\mathfrak{g}_0) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f_\theta),$$

where $k_i^\natural$ is given in [AM15, Tables 3 and 4]. Let θ_i, $i \geqslant 1$, be the highest root of $\mathfrak{g}_i^\natural$.

Lemma 10.3. *Suppose that $k_i^\natural \in \mathbb{Z}_{\geqslant 0}$ for some $i \geqslant 1$. Then the image of the singular vector $e_{\theta_i}(-1)^{k_i^\natural+1}\mathbf{1}$ of $V^{k_i^\natural}(\mathfrak{g}_0)$ is a nonzero singular vector of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$.*

Proof. It is nonzero because its image in $R_{\mathcal{W}^k(\mathfrak{g},f_\theta)} = \mathbb{C}[\mathscr{S}_{f_\theta}]$ is nonzero. The rest follows from the commutation relations described in [KW04, Theorem 5.1]. □

Theorem 10.4. *Suppose that $\mathfrak{g}$ is not of type A. Then the following conditions are equivalent:*

(1) *$\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse,*
(2) *$k_i^\natural \in \mathbb{Z}_{\geqslant 0}$ for all $i \geqslant 1$.*

Proof. We have already showed the implication (1) $\Rightarrow$ (2) in our previous paper [AM15]. Let us show the implication (2) $\Rightarrow$ (1). Write $R_{\mathcal{W}_k(\mathfrak{g},f_\theta)} = \mathbb{C}[\mathcal{S}_{f_\theta}]/I$ for some Poisson ideal I. Note that the radical $\sqrt{I}$ is also a Poisson ideal of $\mathbb{C}[\mathcal{S}_{f_\theta}]$, see e.g. [Ar12, Lemma 2.4.1]. We identify $\mathbb{C}[\mathcal{S}_{f_\theta}]$ with $S(\mathfrak{g}^e)$.

By assumption $e_{\theta_i}(-1)^{k_i^\natural+1}\mathbf{1}$ is zero in the simple quotient $\mathcal{W}_k(\mathfrak{g}, f_\theta)$. Hence $e_{\theta_i} \in \sqrt{I}$ for all i. Since the restriction of the Poisson bracket to $S(\mathfrak{g}_i) \subset S(\mathfrak{g}^e) = \mathbb{C}[\mathcal{S}_{f_\theta}]$ coincides with the Kirillov-Kostant Poisson structure of $\mathfrak{g}_i^*$, $\mathfrak{g}_i \subset \sqrt{I}$ for all i, because $\mathfrak{g}_i$ is simple (note that $\mathfrak{g}_0 = 0$). Further, we have

$$\{x, \bar{G}^{\{a\}}\} = G^{\{[x,a]\}} \quad \text{for } a \in \mathfrak{g}_{-1/2}$$

by [KW04, Theorem 5.1 (d)], where $\bar{G}^{\{a\}}$ is the image of $G^{\{a\}}$ in $R_{W^k(\mathfrak{g},f_\theta)} = \mathbb{C}[\mathcal{S}_{f_\theta}]$. Since $\mathfrak{g}_{-1/2}$ is a direct sum of non-trivial representations of $\mathfrak{g}^\natural$ by [KW04, Table 1], we get that $\bar{G}^{\{a\}} \in \sqrt{I}$ for all $a \in \mathfrak{g}_{-1/2}$. Finally, [KW04, Theorem 5.1 (e)] implies that

$$\{\bar{G}^{\{u\}}, \bar{G}^{\{v\}}\} \equiv -2(k+h^\vee))(e|[u,v])\bar{L} \pmod{\sqrt{I}},$$

where $\bar{L}$ is the image of L in $R_{W^k(\mathfrak{g},f_\theta)} = \mathbb{C}[\mathcal{S}_{f_\theta}]$. Thus, $\bar{L} \in \sqrt{I}$, and we conclude that all generators of $S(\mathfrak{g}^e)$ belong to $\sqrt{I}$. $\square$

We are now in a position to prove Theorem 1.5 which was conjectured in [AM15], and Theorem 1.6.

Proof of Theorem 1.5. By Theorem 10.4 and [AM15, Table 4], $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse if and only if $3k+5 \in \mathbb{Z}_{\geqslant 0}$. $\square$

Proof of Theorem 1.6. We know that $X_{V_k(\mathfrak{g})} = \{0\}$ if $k \in \mathbb{Z}_{\geqslant 0}$ since $V_k(\mathfrak{g})$ is lisse in this case. Thus, the assertion follows from Theorem 1.4, Theorem 1.5 and [AM15, Theorem 7.1]. $\square$

By Theorem 1.5, we have thus obtained a new family of lisse minimal W-algebras $\mathcal{W}_k(G_2, f_\theta)$, for $k = -1, 0, 1, 2, 3 \dots$.

Finally we remark that the following assertion follows from Remark 10.1 and Theorem 10.4.

Theorem 10.5. *We have $X_{L(s_0 \circ k\Lambda_0)} = \overline{\mathbb{O}_{min}}$ for $k \in \mathbb{Z}_{\geqslant 0}$ if $\mathfrak{g}$ is of type D_r, $r \geqslant 4$, G_2, F_4, E_6, E_7 and E_8.*

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