COADJOINT ORBITS OF REDUCTIVE TYPE OF SEAWEED LIE ALGEBRAS

ANNE MOREAU AND OKSANA YAKIMOVA

ABSTRACT. A connected algebraic group Q defined over a field of characteristic zero is *quasi-reductive* if there is an element of $\mathfrak{q}^*$ of reductive type, that is such that the quotient of its stabiliser by the centre of Q is a reductive subgroup of $GL(\mathfrak{q})$. Such groups appear in harmonic analysis when unitary representations are studied. In particular, over the field of real numbers they turn out to be the groups with discrete series and their irreducible unitary square integrable representations are parameterised by coadjoint orbits of reductive type. Due to results of M. Duflo, coadjoint representation of a quasi-reductive Q possesses a so called *maximal reductive stabiliser* and knowing this subgroup, defined up to a conjugation in Q, one can describe all coadjoint orbits of reductive type.

In this paper, we consider quasi-reductive parabolic subalgebras of simple complex Lie algebras as well as all seaweed subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ and describe the classes of their maximal reductive stabilisers.

1. INTRODUCTION

Suppose that $\Bbbk$ is a field of characteristic zero and Q a connected algebraic (or Lie) group defined over $\Bbbk$. Let $\mathfrak{q} = \operatorname{Lie} Q$ be the Lie algebra of Q. Let Z denote the centre of Q. A linear function $\gamma \in \mathfrak{q}^*$ is said to be of *reductive type* if the quotient Q_γ/Z of the stabiliser $Q_\gamma \subset Q$ for the coadjoint action is a reductive subgroup of $GL(\mathfrak{q}^*)$ (or, what is equivalent, of $GL(\mathfrak{q})$). Whenever it makes sense, we will also say that γ is of *compact type* if Q_γ/Z is compact. Further, Q and $\mathfrak{q}$ are said to be *quasi-reductive* if there is $\gamma \in \mathfrak{q}^*$ of reductive type.

The notions go back to M. Duflo, who initiated the study of such Lie algebras because of applications in harmonic analysis, see [Du82]. In order to explain his (and our) motivation let us assume for a while that $\Bbbk = \mathbb{R}$.

A classical problem is to describe the unitary dual $\widehat{Q}$ of Q, i.e., the equivalence classes of unitary irreducible representations of Q. In this context, the coadjoint orbits play a fundamental rôle. When Q is a connected simply connected nilpotent Lie group, Kirillov's "orbit method" [Ki68] provides a bijection between $\widehat{Q}$ and $\mathfrak{q}^*/Q$. Great efforts have been made to extend the orbit method to arbitrary groups by Kostant, Duflo, Vogan and many others. In case of square integrable representations the extension is particularly successful.

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Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of Q. Then central elements of Q act on $\mathcal{H}$ as scalar multiplications. Hence for $v, w \in \mathcal{H}$ and $z \in Z$, $q \in Q$ the norm $|\langle v, \pi(qz)w\rangle|^2$ does not depend on z. Therefore $|\langle v, \pi(qz)w\rangle|^2$ is a function on Q/Z and the following definition makes sense:

Definition 1.1. An irreducible unitary representation $(\pi, \mathcal{H})$ is said to be *square integrable modulo Z* if there exists a non-zero matrix coefficient such that the integral of the function $q \mapsto |\langle v, \pi(qz)w\rangle|^2$ over a left invariant Haar measure on Q/Z is finite.

Let $\widehat{Q}_2$ denote the set of the equivalence classes of irreducible square integrable modulo Z representations. In case of a nilpotent Lie group N we have a result of C. Moore and J. Wolf [MW73] stating that $\widehat{N}_2 \neq \varnothing$ if and only if there is $\gamma \in \mathfrak{n}^*$ of compact type, where $\mathfrak{n} = \mathrm{Lie}\, N$. This also can be formulated as γ is of reductive type or $N_\gamma = Z$. In case of a reductive group G, Harish-Chandra's theorems [HC65], [HC66] give a bijection between $\widehat{G}_2$ and certain forms of compact type. Square integrable representations of unimodular groups were described by Ahn [A76], [A78], and the existence of such representations leads to a rather restrictive conditions on Q. For an arbitrary Lie group Q, description of $\widehat{Q}_2$ was obtained by Duflo [Du82, Chap. III, Théorème 14].

If $\widehat{Q}_2 \neq \varnothing$, then Q is quasi-reductive. In the other direction, having $\gamma \in \mathfrak{q}^*$ of compact type, there is a way to construct a square integrable unitary representation [Du82, Chap. III, §§14 and 16].

In this paper we consider complex Lie groups and linear forms γ of reductive type. If $\gamma \in \mathfrak{q}^*$ is of compact type, then γ extends to a function on $\mathfrak{q}\otimes\mathbb{C}$ and the extension is of reductive type. Going in the other direction is more difficult. One has to find a suitable real form of $\mathfrak{q}\otimes\mathbb{C}$ and prove that the stabiliser in it is compact. We do not address these problems here.

The coadjoint action of a quasi-reductive linear Lie group has quite remarkable properties. In Section 2, we recall relevant results of [DKT] and, since that article cannot be accessed yet, give independent proofs of them under the assumption that the centre Z of Q consists of semisimple elements of Q. These results can be also traced back to [Du82]. One if them is that there is a unique coadjoint orbit $Q\gamma$ of reductive type such that the restriction of γ to its stabiliser $\mathfrak{q}_\gamma$ is zero. Thus, the stabilisers of such γ are all conjugate by the group Q. Moreover, if $\beta \in \mathfrak{q}^*$ is of reductive type, then Q_β is contained in Q_γ up to conjugation. Therefore the following definition is justified.

Definition 1.2. Assume that Z consists of semisimple elements. If γ is a linear form of reductive type such that the restriction of γ to $\mathfrak{q}_\gamma$ is zero, then its stabiliser Q_γ is called a *maximal reductive stabiliser*, MRS for short. Let $M_*(\mathfrak{q})$ denote the Lie algebra of a MRS. The reader shall keep in mind that both MRS and $M_*(\mathfrak{q})$ are defined up to conjugation.

It also follows from Duflo's results [Du82] ([DKT] or see Corollary 2.10 here), that the coadjoint orbits of Q of reductive type are parametrised by the coadjoint orbits of reductive type of MRS. The latter are in one-to-one correspondence with the semisimple adjoint orbits of MRS, that is the closed adjoint orbits of MRS. Thus, the set $\mathfrak{q}^*_{\rm red}$ of linear forms of reductive type has an affine geometric quotient with respect to the action of Q and $\mathfrak{q}^*_{\rm red}/Q \cong M_*(\mathfrak{q})/\!\!/MRS$.

It is worth mentioning that finite-characteristic analogues of quasi-reductive Lie algebras have some nice properties. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$ and Q an algebraic group defined over $\mathbb{Z}$ such that $Q(\mathbb{C})$ is strongly quasi-reductive. Reducing scalars modulo p one may view $\mathfrak{q}$ as a Lie algebra over $\mathbb{F}$. Identifying $\mathfrak{q}$ with the Q-invariant derivations of $\mathbb{F}[Q]$ and taking the p'th power of a derivation, we get a p-operation: $x \to x^{[p]}$ on $\mathfrak{q}$ turning it into a restricted Lie algebra $\widetilde{\mathfrak{q}}$. Since $Q(\mathbb{C})$ is strongly quasi-reductive, for almost all p there is $\gamma \in \widetilde{\mathfrak{q}}^*$ such that $\widetilde{\mathfrak{q}}_\gamma$ is a toral subalgebra of $\widetilde{\mathfrak{q}}$. This implies that $\widetilde{\mathfrak{q}}$ satisfies a Kac-Weisfeiler conjecture (the so called KW1 conjecture) on the maximal dimension of irreducible $\widetilde{\mathfrak{q}}$-modules, [PS99, Section 4].

For an arbitrary Lie algebra $\mathfrak{q}$ there remain two important problems to solve.

(a) Does $\mathfrak{q}$ have linear forms of reductive type? In other words, is $\mathfrak{q}$ quasi-reductive?

(b) If so, what is the conjugation class of a maximal reductive stabiliser?

Classification of quasi-reductive Lie algebras seems to be a wild problem. (Easy examples of them are reductive algebras and Abelian algebras.) However, it is reasonable to look on specific subalgebras of semisimple Lie algebras, in particular on parabolic subalgebras.

By a classical result of Dynkin, a maximal subgroup of a semisimple group is either reductive or a parabolic subgroup. When studying branching rules one naturally is tempted to restrict first to a maximal subgroup. This is one of the instances where parabolic subgroups come into play in harmonic analysis. Of particular interest are branching of square integrable representations with finite multiplicities. In that case a subgroup must have an irreducible square integrable representation, in other words, it must be quasi-reductive.

More generally, the algebras of seaweed type, first introduced by V. Dergachev and A. Kirillov [DK00] in the $\mathfrak{sl}_n$ case, form a very interesting class of non-reductive subalgebras of semisimple Lie algebras. They naturally extend both the classes of parabolic subalgebras and of Levi subalgebras. A *seaweed subalgebra* of a semisimple Lie algebra is the intersection of two parabolic subalgebras whose sum is the total Lie algebra. They have been intensively studied these last years, see e.g. [Pa01], [Dv03], [TY04], [Pa05], [J06], [J07], [BM].

Observe that the centre of a seaweed subalgebra always consists of semisimple elements of the total Lie algebra. Therefore if a seaweed is quasi-reductive, it is *strongly* quasi-reductive, see Definition 2.1. The seaweed subalgebras in $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ are quasi-reductive by a result of D. Panyushev [Pa05]. The classification of quasi-reductive parabolic subalgebras of reductive Lie algebras has been recently completed in [DKT] and [BM]. In this paper, we focus on the above question (b) for quasi-reductive parabolic subalgebras of a reductive Lie algebra. The problem clearly reduces to the case of quasi-reductive parabolic subalgebras of simple Lie algebras.

Let $\mathfrak{q} \subset \mathfrak{g}$ be a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ or a seaweed in $\mathfrak{g} = \mathfrak{sl}_n$ (or $\mathfrak{gl}_n$). In this note, we describe $M_*(\mathfrak{q})$ and also specify an embedding $M_*(\mathfrak{q}) \subset \mathfrak{q}$. This allows to get back MRS and $\mathfrak{q}^*_{\rm red}$ in the following way. Set

$$\Upsilon := (\mathfrak{q}^*)^{M_*(\mathfrak{q})} \cap \operatorname{Ann}(M_*(\mathfrak{q})) = \{\xi \in \mathfrak{q}^* \mid \xi([\mathfrak{q}, M_*(\mathfrak{q})] = 0, \xi(M_*(\mathfrak{q})) = 0\}.$$

Clearly, if Q_γ is a maximal reductive stabiliser, then $\gamma \in Q\Upsilon$. Since Q-orbits of the maximal dimension form an open subset of $Q\Upsilon$, for generic $\xi \in \Upsilon$ we have $\mathfrak{q}_\xi = M_*(\mathfrak{q})$. Thus generic $\xi \in \Upsilon$ is of nilpotent and reductive type at the same time (see Section 2 for the definitions). By Proposition 2.9(i), Q_ξ is also a maximal reductive stabiliser. Identifying a Q_ξ-invariant complement of $\mathrm{Ann}(\mathfrak{q}_\xi) \subset \mathfrak{q}^*$ in $\mathfrak{q}^*$ with $\mathfrak{q}_\xi^*$ we get $(\mathfrak{q}^*)_{\rm red} = Q(\xi + (\mathfrak{q}_\xi^*)_{\rm red})$ (see proof of Proposition 2.9(ii)). We do not describe explicitly the component group of MRS. However, note that for seaweeds in GL_n, the maximal reductive stabilisers are connected.

Roughly speaking, we have two methods for calculating $M_*(\mathfrak{q})$. The first uses root system of $\mathfrak{g}$ and related objects, e.g. Kostant's cascade (see Section 3 for more details). It allows us to prove the crucial "additivity" property (Theorem 3.6) and provides the first proof of the so called "highest root reduction" (Theorem 3.4). Both are particularly useful in the exceptional case. For instance, the "additivity" property assures that it suffices to consider parabolic subalgebras with simple semisimple parts.

The second method uses algebraic Levi decomposition $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$ and allows us to "cut" an Abelian ideal in $\mathfrak{n}$ (Lemma 4.2). In principle, this reduction can be applied to any algebraic linear Lie algebra such that its centre consists of semisimple elements and at the end establish whether the algebra is quasi-reductive or not. In particular nicely the method works for $\mathbb{Z}$-graded Lie algebras with 2 or 3 graded components (Lemmas 4.3, 4.6). Similar gradings were used before by Panyushev [Pa01], [Pa05] to calculate the index of a seaweed and its generic stabiliser. This approach is used in Section 5 to deal with the classical Lie algebras.

For seaweeds in GL_n our answer is given in terms of the meander graph, used in [DK00] to express the index of $\mathfrak{q}$. Recall that a seaweed is an intersection of two complementary parabolic subalgebras. Hence, up to conjugation, it is defined by two compositions $\bar{a}, \bar{b}$ of n. To these objects one can attach a certain graph $\Gamma = \Gamma(\bar{a}|\bar{b})$ with n vertices and at most n edges. For example $\Gamma(5,2,2|2,4,3)$ has $\bar{a}$-edges $(1,5)$, $(2,3)$, $(6,7)$, $(8,9)$ and $\bar{b}$-edges $(1,2)$, $(3,6)$, $(4,5)$, $(7,9)$ (for a picture see Section 5.1). A cycle of Γ is said to be maximal if it does not lie inside any other cycles in the planar embedding of Γ. To a maximal cycle one can adjust a number, r, its dimension, and a subgroup $GL_r \subset GL_n$. Our result states that MRS is equal to the product of GL_r over all maximal cycles in Γ. This confirms a prediction of Duflo. Here, we furthermore describe an explicit embedding for MRS.

Finally, the exceptional Lie algebras are dealt with in Section 6 and the results are stated in Tables 4, 5, 6, 7.

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Contents

2. Strongly quasi-reductive Lie algebras

From now on, $\Bbbk = \mathbb{C}$. Let Q be a linear algebraic group and $\mathfrak{q} = \operatorname{Lie} Q$ its Lie algebra. Keep the notation of the introduction. Set $\mathfrak{z} := \operatorname{Lie} Z$. Recall that a linear form $\gamma \in \mathfrak{q}^*$ is of *reductive type* if Q_γ/Z is a reductive Lie subgroup of $GL(\mathfrak{q})$. Let $\mathfrak{q}^*_{\rm red} \subset \mathfrak{q}^*$ denote the set of linear forms of reductive type. The group Q acts on $\mathfrak{q}^*_{\rm red}$ and we denote by $\mathfrak{q}^*_{\rm red}/Q$ the set of coadjoint orbits of reductive type. Recall also that $\mathfrak{q}$ is called *quasi-reductive* if it has linear forms of reductive type. Most results of this section are due to Duflo et al. and are contained in [DKT]. Since [DKT] is out of reach yet, we give here independent proofs for the convenience of the reader. For our purpose, the following definition will be very useful as well.

Definition 2.1. If $\mathfrak{q}$ is quasi-reductive and $\mathfrak{z}$ consists of semisimple elements of $\mathfrak{q}$, then $\mathfrak{q}$ is said to be *strongly quasi-reductive*.

In this section we concentrate on strongly quasi-reductive Lie algebras. These algebras can be characterised by the property that there exists a linear form $\gamma \in \mathfrak{q}^*$ such that $\mathfrak{q}_\gamma$ is a reductive subalgebra of $\mathfrak{q}$. However, the reader can keep in mind that most of results stated in this section are still true for an arbitrary $\mathfrak{q}$ ([DKT]). The statements have then to be slightly modified accordingly.

Definition 2.2. Suppose that Q acts on an irreducible affine variety Y. Then a subgroup Q_y (with $y \in Y$) is called a *generic stabiliser* of this action if there is an open subset $U \subset Y$ such that Q_y and Q_w are conjugate in Q for all $w \in U$.

By a deep result of Richardson [R72], a generic stabiliser exists for any action of a reductive algebraic group on a smooth affine variety. We will say that a Lie algebra of a generic stabiliser is a generic stabiliser as well. By [R72, Prop. 4.1], if a generic stabiliser exists on the Lie algebra level, it also exists on the group level.

In case of a coadjoint representation, the following lemma can be deduced from [Du83, III] and [Du78]. We give here an independent proof applicable in a more general setting.

Lemma 2.3. *Suppose that Q acts on a linear space V and there is $v \in V$ such that Q_v is reductive. Then the action of Q on V has a generic stabiliser. In particular, if $\mathfrak{q}$ is strongly quasi-reductive, then the coadjoint action of Q possesses a generic stabiliser T such that T is reductive and $\mathfrak{t} = \mathrm{Lie}\, T$ is Abelian.*

Proof. This can be considered as a version of the Luna slice theorem [L73], see also [PY06, proof of Prop. 1.1]. Since Q_v is reductive, there is a Q_v-stable complement of $\mathrm{T}_v(Qv)$ in V, say N_v. Let us consider the associated fibre bundle $X_v := Q *_{Q_v} N_v$ and recall that it is the (geometric) quotient of $Q \times N_v$ by the Q_v-action defined by $Q_v \times Q \times N_v \to Q \times N_v$, $(s, q, n) \mapsto (qs^{-1}, sn)$. The image of $(q, n) \in Q \times N_v$ in X_v is denoted by $q * n$. The natural Q-equivariant morphism $\psi : X_v \to V$, $\psi(q * n) = q(v + n)$ is étale in $e * 0 \in X_v$ by construction. It follows that there is an open Q-stable neighbourhood U of Qv such that for all $y \in U$ the identity component Q_y° is conjugate to $(Q_v)_x^\circ$ with $x \in N_v$. Therefore on the Lie algebra level a generic stabiliser of the action $Q_v \times N_v \to N_v$ is also a generic stabiliser of $Q \times V \to V$. (The statement also follows from [R72, Prop. 3.3].) By [R72, Prop. 4.1] generic stabiliser exists also on the group level.

In case $V = \mathfrak{q}^*$, we have $\mathrm{T}_v(Qv) \cong \mathrm{Ann}(\mathfrak{q}_v)$ and $N_v \cong \mathfrak{q}_v^*$. Therefore a maximal torus $\mathfrak{t}$ in $\mathfrak{q}_v$ is a generic stabiliser for the coadjoint action of $\mathfrak{q}$. Since all stabilisers Q_y are algebraic groups, generic stabilisers in Q are reductive as well. □

Recall that the *index* of a Lie algebra is the minimal dimension of stabilisers for the coadjoint representation,

$$\mathrm{ind}\,\mathfrak{q} = \min\{\dim \mathfrak{q}_\gamma \mid \gamma \in \mathfrak{q}^*\}.$$

A linear form $\gamma \in \mathfrak{g}^*$ is *regular* if $\dim \mathfrak{q}_\gamma = \mathrm{ind}\,\mathfrak{q}$. Let $\mathfrak{q}^*_{\mathrm{reg}}$ denote the set of regular linear forms; it is an open dense subset of $\mathfrak{q}^*$. For reductive Lie algebras, the index is equal to the rank. For Abelian Lie algebras, the index is equal to the dimension. Both instances are easy examples of quasi-reductive Lie algebras.

Corollary 2.4. *Suppose that $\mathfrak{t} = \mathfrak{q}_\alpha$ is a reductive torus for $\alpha \in \mathfrak{q}^*$. Then $\mathfrak{t}$ is a generic stabiliser for the coadjoint action.*

Proof. The statement immediately follows from the description of a slice associated with $Q\alpha$. □

Suppose that $\mathfrak{t} = \mathfrak{q}_\alpha$ is reductive and $\alpha \in \mathfrak{q}^*_{\mathrm{reg}}$. In this case $\mathfrak{t}$ is a torus. Set $\mathfrak{h} := \mathfrak{z}_{\mathfrak{q}}(\mathfrak{t})$. Since $\mathfrak{q}$ is algebraic, it possesses a Levi decomposition $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent radical and $\mathfrak{l}$ is a maximal reductive subalgebra. We may assume that $\mathfrak{t} \subset \mathfrak{l}$. Let us fix a $\mathfrak{q}$-invariant bilinear form B on $\mathfrak{q}$ such that it is non-degenerate on $\mathfrak{t}$. Note that $\mathfrak{n}$ lies in the kernel of B. The set $\mathfrak{m}_0 := \mathfrak{t}^\perp \cap \mathfrak{h}$ is a subalgebra of $\mathfrak{q}$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}_0$. We have also

$$\mathfrak{q} = \mathfrak{t} \oplus \mathfrak{m}_0 \oplus [\mathfrak{t}, \mathfrak{q}], \text{ where } \mathfrak{m}_0 \oplus [\mathfrak{t}, \mathfrak{q}] = \mathfrak{t}^\perp \text{ and } \mathfrak{n} \subset \mathfrak{m}_0 \oplus [\mathfrak{t}, \mathfrak{q}]. \tag{1}$$

Since $\mathrm{ad}^*(\mathfrak{t})\alpha = 0$, we see that $\alpha([\mathfrak{t}, \mathfrak{q}]) = 0$. One can write α as $\alpha = s + \gamma + 0$, where γ is zero on $\mathfrak{t}$ and $[\mathfrak{t}, \mathfrak{q}]$ (and s is zero on $\mathfrak{m}_0$ and $[\mathfrak{t}, \mathfrak{q}]$). Clearly $\mathfrak{t} \subset \mathfrak{q}_\gamma$. One the most important properties of γ is that it is of reductive type. This is a result of [DKT] and we also give a proof below.

Example 2.5. Suppose that $\mathfrak{q}$ is reductive. Then the above construction gives $\gamma = 0$ and $\mathfrak{q}_\gamma = \mathfrak{q}$.

Example 2.6. Suppose that $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is a Heisenberg Lie algebra and $\mathfrak{l}$ is reductive. Let $\alpha \in \mathfrak{q}^*_{\rm reg}$ with $\mathfrak{q}_\alpha = \mathfrak{t}$ be of reductive type and $\mathfrak{u}$ the centre of $\mathfrak{n}$. Then $\alpha|_{\mathfrak{u}} \ne 0$. Consider the function $\tilde\alpha = \alpha|_{\mathfrak{u}}$ on $\mathfrak{u}$. Let $\mathfrak{v} \subset \mathfrak{n}$ be the kernel of $\alpha|_{\mathfrak{n}}$. Note that $[\mathfrak{t}, \mathfrak{v}] \subset \mathfrak{v}$. Since $[\mathfrak{q}, \mathfrak{v}] \subset \mathfrak{v}\oplus\mathfrak{u}$, the normaliser of $\mathfrak{v}$ in $\mathfrak{q}$ is a subalgebra of dimension $\dim \mathfrak{l} + 1$ and its intersection with $\mathfrak{n}$ is equal to $\mathfrak{u}$. Replacing $\mathfrak{l}$ by a conjugate subalgebra of $\mathfrak{q}$ (a reductive part of the normaliser of $\mathfrak{v}$) we may assume that $\mathfrak{v}$ is also $\mathfrak{l}$-stable. Since $\dim \mathfrak{u} = 1$, the stabiliser $\mathfrak{l}_{\tilde\alpha}$ is reductive and there is an orthogonal decomposition $\mathfrak{l} = \mathfrak{l}_{\tilde\alpha} \oplus \mathbb{C}\eta$ with η being a central (in $\mathfrak{l}$) element. Note that $\mathfrak{t}$ is a maximal torus in $\mathfrak{l}_{\tilde\alpha}$. The above construction produces a function γ such that $\gamma|_{\mathfrak{u}} = \alpha|_{\mathfrak{u}}$ and $\gamma(\mathfrak{l}_{\tilde\alpha}) = 0$, $\gamma(\mathfrak{v}) = 0$. Thereby $\mathfrak{q}_\gamma = \mathfrak{l}_{\tilde\alpha}$ is reductive.

The general case is proved by induction on $\dim \mathfrak{q}$.

Lemma 2.7 (Duflo-Khalgui-Torasso)**.** *Suppose that $\mathfrak{q}$ is strongly quasi-reductive, $\mathfrak{t} = \mathfrak{q}_\alpha$ is a generic (reductive) stabiliser, and $\gamma \in \mathfrak{q}^*$ is obtained from α by means of the decomposition (1) as above. Then $\mathfrak{q}_\gamma$ is reductive. Moreover, $\gamma(\mathfrak{q}_\gamma) = 0$.*

Proof. Note first that $[\mathfrak{m}_0, \mathfrak{q}] \subset \mathfrak{t}^\perp$ and hence γ is equal to α on $[\mathfrak{m}_0, \mathfrak{q}]$. Thereby $\mathfrak{q}_\gamma \cap \mathfrak{m}_0 = 0$ and since $\mathfrak{t} \subset \mathfrak{q}_\gamma$, we get $\mathfrak{q}_\gamma \subset \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{q}]$. In particular, $\gamma(\mathfrak{q}_\gamma) = 0$. Recall also that $\mathfrak{n} \subset \mathfrak{t}^\perp$ and hence $\alpha|_{\mathfrak{n}} = \gamma|_{\mathfrak{n}}$.

We prove that $\mathfrak{q}_\gamma$ is reductive by induction on $\dim \mathfrak{q}$. Let A be a bilinear form on $\mathfrak{n}\times\mathfrak{q}$ defined by $A(\eta, \xi) := \alpha([\eta, \xi]) = \gamma([\eta, \xi])$. Set $\ker A := \{\xi \in \mathfrak{q} \mid A(\mathfrak{n}, \xi) = 0\}$ and $\mathfrak{n}(A) := \mathfrak{n} \cap \ker A$. Let $\mathfrak{w} \subset \mathfrak{n}$ be a complement of $\mathfrak{n}(A)$ in $\mathfrak{n}$. Then A is non-degenerate on $\mathfrak{w}\times\mathfrak{w}$. Since $\mathfrak{q}_\alpha \cap \mathfrak{n} = 0$, there is a subspace $\tilde{\mathfrak{v}} \subset \mathfrak{q}$ of dimension $\dim \mathfrak{n}(A)$ such that the pairing $A(\tilde{\mathfrak{v}}, \mathfrak{n}(A))$ is non-degenerate. Set $\mathfrak{v} := \{v \in \tilde{\mathfrak{v}}\oplus\mathfrak{w} \mid A(v, \mathfrak{w}) = 0\}$. Then $\dim \mathfrak{v} = \dim \tilde{\mathfrak{v}}$ and $\mathfrak{v} \cap \mathfrak{w} = 0$, because A is non-degenerate on $\mathfrak{w}\times\mathfrak{w}$. Since also $A(\mathfrak{n}(A), \mathfrak{w}) = 0$, we get that A is non-degenerate on $\mathfrak{n}(A)\times\mathfrak{v}$.

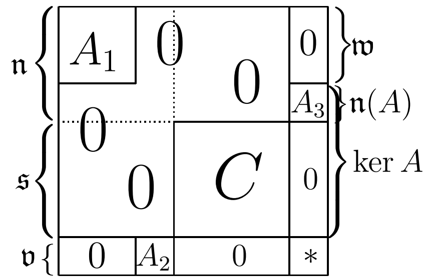
Finally in $\ker A$ we fix a decomposition $\ker A = \mathfrak{n}(A) \oplus \mathfrak{s}$, where $\mathfrak{s} = \mathfrak{s}(\alpha)$ or $\mathfrak{s} = \mathfrak{s}(\gamma)$ has the property that $\alpha([\mathfrak{s}, \mathfrak{v}]) = 0$ or $\gamma([\mathfrak{s}, \mathfrak{v}]) = 0$, respectively. Let us choose a basis of $\mathfrak{q}$ according to the inclusions

$$\mathfrak{w} \subset \mathfrak{w} \oplus \mathfrak{n}(A) \subset \mathfrak{n} \oplus \mathfrak{s} \subset \mathfrak{n} + \ker A \subset \mathfrak{w} \oplus \ker A \oplus \mathfrak{v} = \mathfrak{n} \oplus \mathfrak{s} \oplus \mathfrak{v}.$$

Then the matrices of the forms $\hat\alpha(\xi_1, \xi_2) = \alpha([\xi_1, \xi_2])$ and $\hat\gamma$ look as shown in Picture 1.
Since $\mathfrak{q}_\gamma = \ker \hat\gamma$, Picture 1 tells us that $\mathfrak{q}_\gamma \subset \ker A$ and $\mathfrak{q}_\gamma \cap \mathfrak{n}(A) = 0$. Moreover, $\mathfrak{q}_\gamma$ coincides with the kernel of a skew-symmetric form defined by the matrix $C = C(\gamma)$ in $\mathfrak{s} = \mathfrak{s}(\gamma)$. The same holds for $\mathfrak{q}_\alpha$ if we replace $C(\gamma)$ with $C(\alpha)$. In order to formalise this we set

$$\mathfrak{q}(A) := \{\xi \in \mathfrak{q} \mid A(\,.\,, \xi) \in \mathbb{C}\alpha|_{\mathfrak{n}}\}/\ker \alpha \cap \mathfrak{n}(A).$$

Note that $\mathfrak{q}(A)$ is a Lie algebra, $\mathfrak{t} \subset \mathfrak{q}(A)$, and B canonically induces an invariant bilinear form on $\mathfrak{q}(A)$ preserving the condition $\mathfrak{t}^\perp \cap \mathfrak{t} = 0$. Both α and γ can be restricted to $\mathfrak{q}(A)$, we keep the same letters for all the restrictions.



Here matrices A_i are non-degenerate.

Pic. 1

Equality $\dim \mathfrak{q}(A) = \dim \mathfrak{q}$ is possible only in two cases: $\mathfrak{q}$ is reductive, this is treated in Examples 2.5, or $\dim \mathfrak{n} = 1$, which is a particular case of Example 2.6. In the following we assume that $\dim \mathfrak{q}(A) < \dim \mathfrak{q}$.

The centre of $\mathfrak{n}$ has zero intersection with $\mathfrak{q}_\alpha$ and is obviously contained in $\mathfrak{n}(A)$. Hence there exists $\eta \in \mathfrak{n}(A)$ such that $\alpha(\eta) \neq 0$. Let $\bar{\eta} = \eta + \ker\alpha \cap \mathfrak{n}(A)$ be an element of $\mathfrak{q}(A)$. Next $\dim\{A(\,.\,,\xi) \mid \xi \in \mathfrak{q}\} = \dim \mathfrak{n}$ and therefore this set is isomorphic to $\mathfrak{n}^*$. Hence there are elements $l \in \mathfrak{q}$ multiplying $\alpha|_{\mathfrak{n}}$ by a non-zero constant. We also consider l as an element of $\mathfrak{q}(A)$. Then

$$\gamma([l,\bar{\eta}]) = \gamma([l, \eta + \ker\alpha \cap \mathfrak{n}(A)]) = \mathrm{ad}^*(l)\gamma(\eta + \ker\alpha \cap \mathfrak{n}(A)) = c\gamma(\eta) = c\alpha(\eta) \neq 0,$$

because c is a non-zero constant. Therefore $\bar{\eta}$, $l \notin \mathfrak{q}(A)_\gamma$ and one can conclude that $\mathfrak{q}(A)_\gamma = \mathfrak{q}_\gamma$. By the same reason, $\mathfrak{q}(A)_\alpha = \mathfrak{t}$. In $\mathfrak{q}(A)^*$, γ is obtained from α by the same procedure as in $\mathfrak{q}^*$. By the inductive hypothesis $\mathfrak{q}_\gamma$ is reductive. $\square$

Linear functions $\gamma \in \mathfrak{q}^*$ such that $\gamma(\mathfrak{q}_\gamma) = 0$ are said to be of *nilpotent type*. The name can be justified by the following observation.

Lemma 2.8. *Suppose that* $\mathrm{char}\,\Bbbk = 0$ *and* $\gamma \in \mathfrak{q}^*$ *is of nilpotent type. Then* $\overline{Q\gamma}$ *contains zero.*

Proof. Recall that $\mathrm{ad}(\mathfrak{q})^*\gamma = \mathrm{Ann}(\mathfrak{q}_\gamma)$. Since $\gamma(\mathfrak{q}_\gamma) = 0$, there is $\xi \in \mathfrak{q}$ such that $\mathrm{ad}(\xi)\gamma = \gamma$ or better $\mathrm{ad}(\xi)\gamma = -\gamma$. Take $t \in \mathbb{N} \subset \Bbbk$ and consider

$$\begin{gathered}\exp(t\mathrm{ad}(\xi))\gamma = \gamma + t\mathrm{ad}(\xi)\gamma + t^2\mathrm{ad}(\xi)^2\gamma/2 + \ldots + t^k\mathrm{ad}(\xi)^k\gamma/k! + \ldots = \\ \gamma - t\gamma + t^2\gamma/2 - t^3\gamma/3! + \ldots + (-1)^k t^k\gamma/k! + \ldots = \exp(-t)\gamma.\end{gathered}$$

Since $\exp(-t)$ goes to zero when t goes to infinity, $0 \in \overline{Q\gamma}$. $\square$

Importance of linear forms, which are simultaneously of reductive and nilpotent type is illustrated by the following proposition.

Proposition 2.9 (Duflo-Khalgui-Torasso)**.** *Suppose that* $\mathfrak{q}$ *is strongly quasi-reductive. Then*

(i) *there is a unique orbit* $Q\gamma \subset \mathfrak{q}^*$ *such that* γ *is of reductive and nilpotent type;*
(ii) *if* $\mathfrak{q}_\beta$ *is reductive, then* $\mathfrak{q}_\beta$ *is conjugate under* Q *to a subalgebra* $(\mathfrak{q}_\gamma)_s$ *with* $s \in \mathfrak{q}_\gamma^*$*.*

Proof. Existence of γ was shown in Lemma 2.7. Suppose we have two linear functions γ, γ' of reductive and nilpotent type. Replacing γ' by an element of $Q\gamma'$ we may (and will) assume that there is a generic stabiliser $\mathfrak{t} = \mathfrak{q}_\alpha$ such that $\mathfrak{t} \subset (\mathfrak{q}_\gamma \cap \mathfrak{q}_{\gamma'})$. Let $\mathfrak{m}_0$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}_0$ be as in (1). Set $\mathfrak{v} := [\mathfrak{t}, \mathfrak{q}]$. We have $\gamma(\mathfrak{v}) = \gamma'(\mathfrak{v}) = 0$ and both functions can be viewed as elements of $\mathfrak{h}^*$. Since $\mathfrak{t}$ is a maximal torus of $\mathfrak{q}_\gamma$, we have $\mathfrak{h} \cap \mathfrak{q}_\gamma = \mathfrak{t}$. Now let $y \in \mathfrak{h}$. Then $[y, \mathfrak{v}] \subset \mathfrak{v}$ and $\gamma([y, \mathfrak{v}]) = 0$. Thereby $\mathfrak{h}_\gamma = \mathfrak{h} \cap \mathfrak{q}_\gamma = \mathfrak{t}$. The same holds for γ' and both functions are zero on $\mathfrak{t}$.

Let $H \subset Q$ be a connected subgroup with $\operatorname{Lie} H = \mathfrak{h}$. Note that $H\gamma$ is a dense open subset of $Y := \{\xi \in \mathfrak{h}^* \mid \xi(\mathfrak{t}) = 0\}$ and the same holds for $H\gamma'$. Since Y is irreducible, $H\gamma \cap H\gamma' \neq \varnothing$ and the orbits coincide. Therefore $\gamma' \in Q\gamma$.

Now part (i) is proved and we pass to (ii). Let $\mathfrak{t} = \mathfrak{q}_\alpha$ be a maximal torus of $\mathfrak{q}_\beta$ and γ a linear form of reductive and nilpotent type constructed from α by means of (1). Since B is non-degenerate on $\mathfrak{t}$, it is also non-degenerate on $\mathfrak{q}_\beta$ and $\mathfrak{q} = \mathfrak{q}_\beta \oplus \mathfrak{w}$, where $\mathfrak{w} = \mathfrak{q}_\beta^\perp$. We have $\mathfrak{q}_\alpha \cap \mathfrak{h} = \mathfrak{t} = \mathfrak{q}_\beta \cap \mathfrak{h}$ and $H\beta$, $H\alpha$ are dense open subsets in $\beta + \mathfrak{m}_0^*$, $\alpha + \mathfrak{m}_0^*$, respectively. Replacing α by a conjugate function we may (and will) assume that $\alpha|_{\mathfrak{m}_0} = \beta|_{\mathfrak{m}_0}$. This implies that β and γ are equal on $\mathfrak{m}_0 \oplus \mathfrak{v}$.

Let $x \in \mathfrak{q}_\beta$. Then $[x, \mathfrak{q}_\beta] \subset \mathfrak{q}_\beta \subset \mathfrak{t} \oplus \mathfrak{v}$ and $\gamma([x, \mathfrak{q}_\beta]) = 0$. In addition, $[x, \mathfrak{w}] \subset \mathfrak{w} \subset \mathfrak{m}_0 \oplus \mathfrak{v}$ and $\gamma([x, \mathfrak{w}]) = \beta([x, \mathfrak{w}]) = 0$. This proves the inclusion $\mathfrak{q}_\beta \subset \mathfrak{q}_\gamma$. By a similar reason, $\beta([y, \mathfrak{q}_\gamma^\perp]) = 0$ for $y \in \mathfrak{q}_\gamma$. Therefore $\mathfrak{q}_\beta = (\mathfrak{q}_\gamma)_s$ for $s = \beta|_{\mathfrak{q}_\gamma}$. To conclude, note that by part (i) there is only one orbit of both reductive and nilpotent type. □

Proposition 2.9 can be also deduced from [Du82, Chap. I].

Corollary 2.10. *Suppose that $\mathfrak{q}$ is strongly quasi-reductive. Let $\gamma \in \mathfrak{q}^*$ be both of reductive and nilpotent type. Then the coadjoint orbits of reductive type in $\mathfrak{q}^*$ are in bijection with the closed (co)adjoint orbits of Q_γ. In other words, the geometric quotient $\mathfrak{q}^*_{\rm red}/Q$ exists and coincides with $\mathfrak{q}_\gamma /\!\!/ Q_\gamma$.*

Proof. By the proof of Proposition 2.9(ii), each Q-orbit of reductive type contains a point $\beta = \gamma + s$ with $s \in \mathfrak{q}_\gamma^*$ lying in the slice to $Q\gamma$. Let ψ and X_γ be as in the proof of Lemma 2.3, with $V = \mathfrak{q}^*$. We get that $Q\beta$ meets $\psi(X_\gamma)$. Each connected component of Q_β contains an element g preserving the maximal torus $\mathfrak{t}$ and hence the decomposition (1). Therefore $gs = s$ and $g\gamma = \gamma$. This proves that $Q_\beta = (Q_\gamma)_s$ and the result follows. □

Corollary 2.10 justifies the following definition:

Definition 2.11. Suppose that $\mathfrak{q}$ is strongly quasi-reductive and $\gamma \in \mathfrak{q}^*$ is a linear form of reductive type such that $\gamma(\mathfrak{q}_\gamma) = 0$. Then Q_γ is called a *maximal reductive stabiliser* of $\mathfrak{q}$. As an abbreviation, we will shortly write MRS for a maximal reductive stabiliser.

Note that a maximal reductive stabiliser, as well as a generic stabiliser, is defined up to conjugation. Let $M_*(\mathfrak{q})$ denote the Lie algebra of a representative of the conjugation class of a MRS for a strongly quasi-reductive Lie algebra $\mathfrak{q}$. For convenience, we set $M_*(\mathfrak{q}) = \varnothing$ whenever $\mathfrak{q}$ is not strongly quasi-reductive. Note that also $M_*(\mathfrak{q})$ is defined up to conjugation.

Remark 2.12. If $\mathfrak{q}$ is strongly quasi-reductive, then the index of $\mathfrak{q}$ is equal to the rank of $M_*(\mathfrak{q})$.

In this paper we deal with $M_*(\mathfrak{q})$ and leave the description of the MRS on the group level for further investigation. Following examples show that this problem is not entirely trivial.

Example 2.13. Let $Q = \mathbb{C}^\times \ltimes \exp(\mathbb{C}^2 \oplus \mathbb{C})$ be a semi-direct product of a one-dimensional torus and a Heisenberg Lie group. Assume that $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ with characters $(1,1)$, hence on the derived algebra of the Heisenberg algebra with character 2. Then $\operatorname{ind}\mathfrak{q} = 0$. Here MRS is equal to $\{1,-1\}$. Since the centre of Q is trivial, MRS will stay disconnected after taking a quotient by Z.

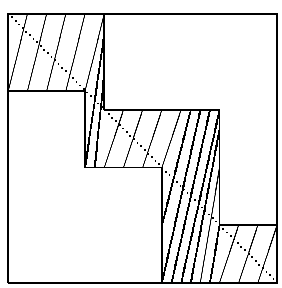
Example 2.14. Consider a semi-direct product $Q = (\mathbb{C}^\times \times SO_9(\mathbb{C})) \ltimes \exp(\mathbb{C}^9)$, where the central torus of the reductive part acts on $\mathbb{C}^9$ with character 1. Then MRS of $\mathfrak{q}$ is equal to $O_8(\mathbb{C})$ and the component group acts non-trivially on the set of coadjoint orbits of $M_*(\mathfrak{q})$.

Motivated by the assertion of Corollary 2.10, one is interested in a more precise description of $M_*(\mathfrak{q})$ for (strongly) quasi-reductive $\mathfrak{q}$.

3. On quasi-reductive seaweed subalgebras

In this section, $\mathfrak{g} = \operatorname{Lie} G$ is a complex finite dimensional semisimple Lie algebra. The dual of $\mathfrak{g}$ is identified with $\mathfrak{g}$ through the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$. For $u \in \mathfrak{g}$, we denote by φ_u the corresponding element of $\mathfrak{g}^*$; the restriction of φ_u to a subalgebra of $\mathfrak{g}$ will be also denoted by φ_u. When the orthogonality in $\mathfrak{g}$ refers to $\langle\ ,\ \rangle$, we use the symbol $\perp$.

Recall that a *seaweed subalgebra* of $\mathfrak{g}$ is defined to be the intersection of two parabolic subalgebras whose sum is $\mathfrak{g}$. The terms "seaweed" comes from their shape in the case of $\mathfrak{sl}_n$, see Picture 2. We study in this section properties of quasi-reductive seaweed subalgebras of $\mathfrak{g}$.



Pic. 2

3.1. In this section, $\mathfrak{q}$ is a seaweed subalgebra of $\mathfrak{g}$ and we assume that $\mathfrak{q}$ is quasi-reductive. Let Q be the connected Lie subgroup of G with Lie algebra $\mathfrak{q}$. As it has been noticed in the introduction, the centre of $\mathfrak{q}$ consists of semisimple elements of $\mathfrak{g}$. So, $\mathfrak{q}$ is strongly quasi-reductive and results of Section 2 apply.

The Killing form enables to identify the dual of $\mathfrak{q}$ to a seaweed subalgebra $\mathfrak{q}^-$ of $\mathfrak{g}$. Thus, to $\mathfrak{q}^*_{\mathrm{red}}$ and $\mathfrak{q}^*_{\mathrm{reg}}$ correspond subspaces of $\mathfrak{q}^-$:

$$\mathfrak{q}^-_{\mathrm{reg}} := \{u \in \mathfrak{q}^- ;\ \varphi_u \in \mathfrak{q}^*_{\mathrm{reg}}\};$$
$$\mathfrak{q}^-_{\mathrm{red}} := \{u \in \mathfrak{q}^- ;\ \varphi_u \in \mathfrak{q}^*_{\mathrm{red}}\}.$$

By Lemma 2.3, there is x in $\mathfrak{q}^-_{\mathrm{reg}} \cap \mathfrak{q}^-_{\mathrm{red}}$. The stabiliser $\mathfrak{t} = \mathfrak{q}_{\varphi_x}$ is a torus of $\mathfrak{g}$ and we have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$. Let $x_{\mathfrak{t}}$ and $x_{\mathfrak{t}^\perp}$ denote the components of x on $\mathfrak{t}$ and $\mathfrak{t}^\perp$ respectively. Set $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{t})^* := \{\xi \in \mathfrak{q}^* \mid \mathfrak{t} \subset \mathfrak{q}_\xi\}$. Obviously, $\varphi_{x_{\mathfrak{t}}}, \varphi_{x_{\mathfrak{t}^\perp}} \in \mathfrak{z}_{\mathfrak{q}}(\mathfrak{t})^*$.

Repeating proofs of Lemma 2.7 and Proposition 2.9, we obtain the first result of this section:

Theorem 3.1. *Suppose that $\mathfrak{q}$ is quasi-reductive and let $\mathfrak{t} = \mathfrak{q}_{\varphi_x}$ be a generic stabiliser. Then, the linear form $\varphi_{x_{\mathfrak{t}^\perp}}$ is of nilpotent and reductive type for $\mathfrak{q}$. In particular, its stabiliser in Q is a MRS. Moreover, the stabiliser of $\varphi_{x_{\mathfrak{t}^\perp}}$ in $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{t})$ is $\mathfrak{t}$.*

Remark 3.2. Whenever the generic reductive stabiliser $\mathfrak{t}$ can be explicitly computed, Theorem 3.1 provides a procedure to describe $M_*(\mathfrak{q})$. Indeed, a computer programme as `GAP` gives the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{g}$ and then, can compute the stabiliser of $\varphi_{x_{\mathfrak{t}^\perp}}$ in $\mathfrak{q}$.

In [BM, Appendix A], it is explained how to use `GAP` to compute $\mathfrak{t}$ in many of quasi-reductive parabolic subalgebras of simple Lie algebras of exceptional type. In this paper, we do not use `GAP` to describe the maximal reductive stabilisers. However, some of the results in the exceptional case may be verified thank to it.

3.2. Denote by Π the set of simple roots with respect to a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

of $\mathfrak{g}$, and by Δ (respectively Δ^+, Δ^-) the corresponding root system (respectively positive root system, negative root system). If π is a subset of Π, denote by Δ_π the root subsystem of Δ generated by π and set $\Delta^\pm_\pi := \Delta_\pi \cap \Delta^\pm$. For $\alpha \in \Delta$, denote by $\mathfrak{g}_\alpha$ the α-root subspace of $\mathfrak{g}$ and let h_α be the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$. For each $\alpha \in \Delta$, fix $x_\alpha \in \mathfrak{g}_\alpha$ so that the family $\{e_\alpha, h_\beta\ ;\ \alpha \in \Delta, \beta \in \Pi\}$ is a Chevalley basis of $\mathfrak{g}$.

We briefly recall a classical construction due to B. Kostant. It associates to a subset of Π a system of strongly orthogonal positive roots in Δ. This construction is known to be very helpful to obtain regular forms on seaweed subalgebras of $\mathfrak{g}$. For a recent account about the *cascade* construction of Kostant, we refer to [TY04, §1.5] or [TY05, §40.5].

Recall that two roots α and β in Δ are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is in Δ. Let π be a subset of Π. The *cascade* $\mathcal{K}_\pi$ of π is defined by induction on the cardinality of π as follows:

(1) $\mathcal{K}(\varnothing) = \varnothing$,
(2) If $\pi_1,\ldots,\pi_r$ are the connected components of π, then $\mathcal{K}_\pi = \mathcal{K}_{\pi_1} \cup \cdots \cup \mathcal{K}_{\pi_r}$,

(3) If π is connected, then $\mathcal{K}_\pi = \{\pi\} \cup \mathcal{K}_T$ where T is the set of simple roots which are orthogonal to the highest positive root θ_π of Δ_π^+.

For π a subset of Π, denote by $\mathcal{E}_\pi$ the set of the highest roots θ_K where K runs over the elements of the cascade of π. The cardinality of $\mathcal{K}_\Pi$ only depends on $\mathfrak{g}$; it is independent of the choices of $\mathfrak{h}$ and Π. Denote it by $\mathrm{k}_\mathfrak{g}$. The values of $\mathrm{k}_\mathfrak{g}$ for the different types of simple Lie algebras are given in Table 1; in this table, for a real number x, we denote by $[x]$ the largest integer $\leqslant x$.

$\mathrm{A}_\ell, \ell \geqslant 1$	$\mathrm{B}_\ell, \ell \geqslant 2$	$\mathrm{C}_\ell, \ell \geqslant 3$	$\mathrm{D}_\ell, \ell \geqslant 4$	G_2	F_4	E_6	E_7	E_8
$\left[\dfrac{\ell+1}{2}\right]$	ℓ	ℓ	$2\left[\dfrac{\ell}{2}\right]$	2	4	4	7	8

TABLE 1. $\mathrm{k}_\mathfrak{g}$ for the simple Lie algebras.

We denote by $\mathfrak{p}_\pi^+$ the standard parabolic subalgebra of $\mathfrak{g}$ which is the subalgebra generated by $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and by $\mathfrak{g}_{-\alpha}$, for $\alpha \in \pi$. It is well-known that any parabolic subalgebra of $\mathfrak{g}$ are conjugated to a standard one. We denote by $\mathfrak{p}_\pi^-$ the "opposite parabolic subalgebra" generated by $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ and by $\mathfrak{g}_\alpha$, for $\alpha \in \pi$. Let $\mathfrak{l}_\pi = \mathfrak{p}_\pi^+ \cap \mathfrak{p}_\pi^-$ be the standard Levi factor of both $\mathfrak{p}_\pi^+$ and $\mathfrak{p}_\pi^-$. We denote by $\mathfrak{g}_\pi$ its derived Lie algebra. Let $\mathfrak{m}_\pi^+$ (respectively $\mathfrak{m}_\pi^-$) be the nilradical of $\mathfrak{p}_\pi^+$ (respectively $\mathfrak{p}_\pi^-$). In the sequel, we will make use of the following element of $\mathfrak{p}_\pi^-$:

$$u_\pi^- = \sum_{\theta \in \mathcal{E}_\Pi,\, \theta \notin \Delta_\pi^+} e_{-\theta}.$$

Remark 3.3. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is not simple, then $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2$ with $\mathfrak{q}_i \subset \mathfrak{g}_i$ and $M_*(\mathfrak{q}) = M_*(\mathfrak{q}_1) \oplus M_*(\mathfrak{q}_2)$. Therefore for the description of $M_*(\mathfrak{q})$ it suffices to consider only simple Lie algebras.

3.3. We assume that $\mathfrak{g}$ is simple. We give in this subsection consequences of Theorem 3.1 for (strongly) quasi-reductive standard parabolic subalgebras of $\mathfrak{g}$. As a first consequence, one obtains the "highest root reduction" (Theorem 3.4).

Let $\widetilde{\Pi}$ be the subset of Π defined by $\mathcal{K}_\Pi = \{\Pi\} \cup \mathcal{K}_{\widetilde{\Pi}}$.

Theorem 3.4 (Highest root reduction)**.** *Let π be a subset of $\widetilde{\Pi}$. Then $\mathfrak{p}_\pi^+$ is quasi-reductive if and only if $\mathfrak{p}_\pi^+ \cap \mathfrak{l}_{\widetilde{\Pi}}$ is. Moreover, if so, the derived algebras of $M_*(\mathfrak{p}_\pi^+ \cap \mathfrak{l}_{\widetilde{\Pi}})$ and $M_*(\mathfrak{p}_\pi^+)$ coincide.*

Proof. The first assertion results from [BM, Theorem 2.1]. Turn to the second one. Assume that $\mathfrak{p}_\pi^+$ is quasi-reductive. Then it is strongly quasi-reductive. By the proof of [BM, Theorem 2.1], we may choose $\widetilde{w} \in \mathfrak{l}_{\widetilde{\Pi}}$ so that $\varphi_{\widetilde{w}}$ and φ_w, with $w = \widetilde{w} + e_{-\theta_\Pi}$, are regular and of reductive type for $\mathfrak{p}_\pi^+ \cap \mathfrak{l}_{\widetilde{\Pi}}$ and $\mathfrak{p}_\pi^+$ respectively. Let $\widetilde{\mathfrak{t}}$ and $\mathfrak{t}$ be the stabilisers of $\varphi_{\widetilde{w}}$ and φ_w in $\mathfrak{p}_\pi^+ \cap \mathfrak{l}_{\widetilde{\Pi}}$ and $\mathfrak{p}_\pi^+$ respectively. Still by the proof of [BM, Theorem 2.1], we have $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}h_{\theta_\Pi}$. Moreover $\mathfrak{t}^\perp = \widetilde{\mathfrak{t}}^\perp \oplus \mathbb{C}h_{\theta_\Pi}$. In the notations of Theorem 3.1, we have $w_{\mathfrak{t}^\perp} = \widetilde{w}_{\mathfrak{t}^\perp} + e_{-\theta_\Pi}$. From that, it is easy to show that the stabiliser of $\varphi_{\widetilde{w}_{\widetilde{\mathfrak{t}}^\perp}}$ in $\mathfrak{p}_\pi^+ \cap \mathfrak{l}_{\widetilde{\Pi}}$ is precisely $\mathfrak{r} \oplus \mathbb{C}h_{\theta_\Pi}$ where $\mathfrak{r}$ is the stabiliser of $\varphi_{w_{\mathfrak{t}^\perp}}$ in $\mathfrak{p}_\pi^+$. Hence, the statement is a consequence of Theorem 3.1. □

We now turn to an additivity property. In [BM, Section 2], the authors described an additivity property for the quasi-reductivity of certain parabolic subalgebras of $\mathfrak{g}$. The additivity property runs particularly smoothly when $\mathrm{rk}\,\mathfrak{g} = \mathrm{k}_\mathfrak{g}$. For maximal reductive stabilisers as well, we are going to prove an additivity property for certain quasi-reductive parabolic subalgebras of $\mathfrak{g}$.

We say that π' is *not connected* to π'' if α' is orthogonal to α'', for all (α', α'') in $\pi' \times \pi''$. Denote by $\mathfrak{z}(\mathfrak{l}_{\pi_1\cup\pi_2})$ the center of $\mathfrak{l}_{\pi_1\cup\pi_2}$.

Lemma 3.5. *Assume that $\mathfrak{g}$ is simple of exceptional type and that $\mathrm{rk}\,\mathfrak{g} = \mathrm{k}_\mathfrak{g}$. Let π_1, π_2 be to subsets of Π which are not connected to each other and assume that $\mathfrak{p}_{\pi_1}^+$ and $\mathfrak{p}_{\pi_2}^+$ are both quasi-reductive.*

(i) *There is $(h, w_1, w_2) \in \mathfrak{z}(\mathfrak{l}_{\pi_1\cup\pi_2}) \times \mathfrak{g}_{\pi_1} \times \mathfrak{g}_{\pi_2}$ such that, for $i = 1, 2$, φ_v and φ_{v_i}, with $v = h + w_1 + w_2 + u_{\pi_1\cup\pi_2}^-$ and $v_i = h + w_i + u_{\pi_1\cup\pi_2}^-$, are regular and of reductive type for $\mathfrak{p}_{\pi_1\cup\pi_2}^+$ and $\mathfrak{p}_{\pi_i}^+$ respectively.*

Fix such a triple (h, w_1, w_2) and denote, for $i = 1, 2$, by $\mathfrak{t}_i$ the stabiliser of φ_{v_i} in $\mathfrak{p}_{\pi_i}^+$.

(ii) *The stabiliser of $\varphi_{v_{i,\mathfrak{t}_i^\perp}}$ in $\mathfrak{p}_{\pi_i}^+$ is contained in $\mathfrak{g}_{\pi_i} \oplus \mathfrak{m}_{\pi_1\cup\pi_2}^+$.*

Proof. (i) result from [BM, Theorem 2.11] and its proof.

(ii) Let $i = 1, 2$. To begin with, we claim that $\mathfrak{t}_i$ is contained in $\mathfrak{g}_{\pi_i} \oplus \mathfrak{m}_{\pi_1\cup\pi_2}^+$; this comes from the proof of [BM, Lemma 2.9]. Denote by $\mathfrak{r}_i$ the stabiliser of $\varphi_{v_{i,\mathfrak{t}_i^\perp}}$ in $\mathfrak{p}_{\pi_i}^+$; By Theorem 3.1, it is a maximal reductive stabiliser of $\mathfrak{p}_{\pi_i}^+$. Moreover, $\mathfrak{t}_i$ is a maximal torus of $\mathfrak{r}_i$ and we can write $\mathfrak{r}_i = \mathfrak{t}_i \oplus [\mathfrak{t}_i, \mathfrak{r}_i]$. So, from the inclusion $\mathfrak{t}_i \subset \mathfrak{g}_{\pi_i} \oplus \mathfrak{m}_{\pi_1\cup\pi_2}^+$, the statement follows since $\mathfrak{g}_{\pi_i} \oplus \mathfrak{m}_{\pi_1\cup\pi_2}^+$ is an ideal of $\mathfrak{p}_{\pi_i}^+$. □

Theorem 3.6 (Additivity property)**.** *Assume that $\mathfrak{g}$ is simple of exceptional type and $\mathrm{rk}\,\mathfrak{g} = \mathrm{k}_\mathfrak{g}$. Let $\pi_1, \pi_2 \subset \Pi$ be to subsets which are not connected to each other and assume that $\mathfrak{p}_{\pi_2}^+$ and $\mathfrak{p}_{\pi_2}^+$ are both quasi-reductive. Then $\mathfrak{p}_{\pi_1\cup\pi_2}^+$ is quasi-reductive and $M_*(\mathfrak{p}_{\pi_1}^+) \oplus M_*(\mathfrak{p}_{\pi_2}^+) = M_*(\mathfrak{p}_{\pi_1\cup\pi_2}^+)$. More precisely, there exist a maximal reductive stabiliser $\mathfrak{r}_1$ of $\mathfrak{p}_{\pi_1}^+$ and a maximal reductive stabiliser $\mathfrak{r}_2$ of $\mathfrak{p}_{\pi_2}^+$ such that $\mathfrak{r}_1 \oplus \mathfrak{r}_2$ is a maximal reductive stabiliser of $\mathfrak{p}_{\pi_1\cup\pi_2}^+$.*

Proof. Only need to prove the second part of the statement. Let $i = 1, 2$ and use the notations of Lemma 3.5 and its proof. Denote by $\mathfrak{t}$ the stabiliser of φ_v in $\mathfrak{p}_{\pi_1\cup\pi_2}^+$. We have $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ (cf. [BM, proof of Theorem 2.11]) and $v = v_1 + v_2 - (h + u_{\pi_1\cup\pi_2}^-)$. Denote by $\mathfrak{r}$ the stabiliser of $\varphi_{v_{\mathfrak{t}^\perp}}$ in $\mathfrak{p}_{\pi_1\cup\pi_2}^+$. By Lemma 3.5 and Theorem 3.1, $\mathfrak{r}$ is a maximal reductive stabiliser of $\mathfrak{p}_{\pi_1\cup\pi_2}^+$. We intend to show that $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2$.

Prove first the inclusion $\mathfrak{r}_1 \subseteq \mathfrak{r}$; we similarly prove the inclusion $\mathfrak{r}_2 \subseteq \mathfrak{r}$. From the equality $\left\langle v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{p}_{\pi_1}^+] \right\rangle = 0$, we deduce that $\left\langle v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{p}_{\pi_1\cup\pi_2}^+] \right\rangle = \left\langle v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{g}_{\pi_2}] \right\rangle$. In turn, by Lemma 3.5(ii), $[\mathfrak{r}_1, \mathfrak{g}_{\pi_2}]$ is contained in $\mathfrak{m}_{\pi_1\cup\pi_2}^+$ whence $\left\langle v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{p}_{\pi_1\cup\pi_2}^+] \right\rangle \subset \left\langle v_{1,\mathfrak{t}_1^\perp}, \mathfrak{m}_{\pi_1\cup\pi_2}^+ \right\rangle$. Now, $\left\langle v_{1,\mathfrak{t}_1^\perp}, \mathfrak{m}_{\pi_1\cup\pi_2}^+ \right\rangle = 0$

since $\mathfrak{t}_1^\perp \supset \mathfrak{m}^+_{\pi_1\cup\pi_2}$. To sum up, we have obtained:

$$\left\langle v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{p}^+_{\pi_1\cup\pi_2}] \right\rangle = 0. \tag{2}$$

Let (i,j) be in $\{1,2\}^2$ with $i \neq j$. Since $\mathfrak{t}_j \subset \mathfrak{g}_{\pi_j} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}$, we get $w_i \in \mathfrak{t}_j^\perp$ and $h \in \mathfrak{t}^\perp$. Thus, $v_{i,\mathfrak{t}_i^\perp} = w_{i,\mathfrak{t}_i^\perp} + h + u^-_{\pi_1\cup\pi_2} - u^-_{\pi_1\cup\pi_2,\mathfrak{t}_i}$ and $v_{\mathfrak{t}^\perp} = w_{1,\mathfrak{t}_1^\perp} + w_{2,\mathfrak{t}_2^\perp} + h + u^-_{\pi_1\cup\pi_2,\mathfrak{t}^\perp}$. So, $v_{\mathfrak{t}^\perp} - v_{1,\mathfrak{t}_1^\perp} = v_{2,\mathfrak{t}_2^\perp} - (h + u^-_{\pi_1\cup\pi_2})$ that is $v_{\mathfrak{t}^\perp} - v_{1,\mathfrak{t}_1^\perp} = w_{2,\mathfrak{t}_2^\perp} - u^-_{\pi_1\cup\pi_2,\mathfrak{t}_2}$. In particular, since both $w_{2,\mathfrak{t}_2^\perp} = w_2 - w_{2,\mathfrak{t}_2}$ and $u^-_{\pi_1\cup\pi_2,\mathfrak{t}_2}$ lie in $\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}$, we see that $v_{\mathfrak{t}^\perp} - v_{1,\mathfrak{t}_1^\perp}$ lies in $\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}$. Hence, from the inclusion $\mathfrak{r}_1 \subset \mathfrak{g}_{\pi_1} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}$, we deduce that $[v_{\mathfrak{t}^\perp} - v_{1,\mathfrak{t}_1^\perp}, \mathfrak{r}_1]$ is contained in $\mathfrak{m}^+_{\pi_1\cup\pi_2}$. We deduce:

$$\left\langle v_{\mathfrak{t}^\perp} - v_{1,\mathfrak{t}_1^\perp}, [\mathfrak{r}_1, \mathfrak{p}^+_{\pi_1\cup\pi_2}] \right\rangle = 0. \tag{3}$$

Finally, it follows from (2) and (3) that $\langle v_{\mathfrak{t}^\perp}, [\mathfrak{r}_1, \mathfrak{p}^+_{\pi_1\cup\pi_2}] \rangle = 0$, whence $\mathfrak{r}_1 \subset \mathfrak{r}$.

Turn now to the inclusion $\mathfrak{r} \subseteq \mathfrak{r}_1 + \mathfrak{r}_2$.

Step 1: The fact that $w_{2,\mathfrak{t}_2^\perp} - u^-_{\pi_1\cup\pi_2,\mathfrak{t}_2}$ lies in $\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}$ leads to the equality

$$\left\langle w_{2,\mathfrak{t}_2^\perp} - u^-_{\pi_1\cup\pi_2,\mathfrak{t}_2}, [\mathfrak{g}_{\pi_1} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2}, \mathfrak{p}^+_{\pi_1}] \right\rangle = 0.$$

From this, we deduce that $\mathfrak{r} \cap (\mathfrak{g}_{\pi_1} \oplus \mathfrak{m}^+_{\pi_1\cup\pi_2})$ is contained in $\mathfrak{r}_1$ since $v_{1,\mathfrak{t}_1^\perp} = -(w_{2,\mathfrak{t}_2^\perp} - u^-_{\pi_1\cup\pi_2,\mathfrak{t}_2}) + v_{\mathfrak{t}^\perp}$. Similarly, we show that $\mathfrak{r} \cap (\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_2\cup\pi_2})$ is contained in $\mathfrak{r}_2$.

Step 2: Denote by $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) the sum of all simple factors of $\mathfrak{r}$ which are contained in $\mathfrak{g}_{\pi_1} \oplus \mathfrak{m}^+_{\pi_2\cup\pi_2}$ (resp. $\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_2\cup\pi_2}$), and denote by $\mathfrak{a}_0$ the sum of all simple factors which are neither contained in $\mathfrak{g}_{\pi_1} \oplus \mathfrak{m}^+_{\pi_2\cup\pi_2}$ nor $\mathfrak{g}_{\pi_2} \oplus \mathfrak{m}^+_{\pi_2\cup\pi_2}$. Thus, $\mathfrak{r} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_0$. Furthermore, Step 1 yields $\mathfrak{a}_1 = \mathfrak{r}_1$ and $\mathfrak{a}_2 = \mathfrak{r}_2$. Indeed, for $i = 1, 2$, the inclusions $\mathfrak{r}_i \subset \mathfrak{r}$, shown just above, imply $\mathfrak{r}_i \subset \mathfrak{a}_i$ by Lemma 3.5. Hence $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are in direct sum and $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ is a maximal torus of $\mathfrak{a}_1 \oplus \mathfrak{a}_2$. But, $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ is a maximal torus of $\mathfrak{r}$. This forces $\mathfrak{a}_0 = 0$, whence $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2$. □

4. Reductions in graded Lie algebras and some parabolic subalgebras

In order to compute $M_*(\mathfrak{q})$ one may use approach of Lemma 2.7 and "cut" the nilpotent radical of $\mathfrak{q}$. In practise this is a rather complicated task. Here we present some algorithms for cutting small pieces of the nilpotent radical.

4.1. In this section Q is a linear algebraic group and $\mathfrak{q} = \operatorname{Lie} Q$.

For a linear function α on a Lie algebra $\mathfrak{q}$, let $\hat{\alpha}$ denote the skew-symmetric form on $\mathfrak{q}$ (or any of its subspaces) given by $\hat{\alpha}(\xi, \eta) = \alpha([\xi, \eta])$. Let $\mathfrak{a} \lhd \mathfrak{q}$ be an ideal. Then the Lie algebra $\mathfrak{q}$ acts on $\mathfrak{a}$ and also on $\mathfrak{a}^*$. For $\gamma \in \mathfrak{a}^*$, $\mathfrak{q}_\gamma$ will always refer to the stabiliser in $\mathfrak{q}$ for this action. If γ is extended to a linear function on $\mathfrak{q}$, this is explicitly stated and the extension is denoted by some other symbol. Suppose that $\mathfrak{a} \lhd \mathfrak{q}$ is an Abelian ideal and $\gamma \in \mathfrak{a}^*$. Let $\mathfrak{q}_{\mathbb{C}\gamma}$ be the normaliser of the line $\mathbb{C}\gamma$ in $\mathfrak{q}$. Then $\mathfrak{a} \subset \mathfrak{q}_{\mathbb{C}\gamma}$. Let $\mathfrak{q}(\gamma)$ denote the quotient Lie algebra $\mathfrak{q}_{\mathbb{C}\gamma}/\ker\gamma$, where $\ker\gamma \subset \mathfrak{a}$ is an ideal of $\mathfrak{q}_{\mathbb{C}\gamma}$. (It is not always an ideal in $\mathfrak{q}$.)

Lemma 4.1. *Let $\mathfrak{a} \lhd \mathfrak{q}$ be an Abelian ideal, $\beta \in \mathfrak{q}^*$, $\gamma = \beta|_{\mathfrak{a}}$ and $\check{\beta} = \beta|_{\mathfrak{q}_\gamma}$. Then $(\mathfrak{q}_\gamma)_{\check{\beta}} = \mathfrak{q}_\beta + \mathfrak{a}$.*

Proof. Since $[\mathfrak{a}, \mathfrak{a}] = 0$, we have $\mathfrak{a} \subset \mathfrak{q}_\gamma$. Besides $\check{\beta}([\mathfrak{a}, \mathfrak{q}_\gamma]) = \gamma([\mathfrak{a}, \mathfrak{q}_y]) = 0$. Hence $\mathfrak{a} \subset (\mathfrak{q}_\gamma)_{\check{\beta}}$. Next, $\mathfrak{q}_\beta \subset \mathfrak{q}_\gamma$, because $\mathfrak{a}$ is an ideal, and obviously $\mathfrak{q}_\beta \subset (\mathfrak{q}_\gamma)_{\check{\beta}}$. Now take $\xi \in (\mathfrak{q}_\gamma)_{\check{\beta}}$. Then $\mathrm{ad}^*(\xi)\beta \in (\mathfrak{q}/\mathfrak{q}_\gamma)^*$. On the other hand, $\hat{\beta}$ defines a pairing between $\mathfrak{a}$ and $\mathfrak{q}$, which is non-degenerate on $\mathfrak{q}/\mathfrak{q}_\gamma$ and $\mathfrak{a}/\mathfrak{a}_\beta$. Hence

$$\dim(\mathrm{ad}^*(\mathfrak{a})\beta) = \dim \mathfrak{a} - \dim \mathfrak{a}_\beta = \dim \mathfrak{q} - \dim \mathfrak{q}_\gamma.$$

Since all elements of $\mathrm{ad}^*(\mathfrak{a})\beta$ are zero on $\mathfrak{q}_\gamma$, we obtain $\mathrm{ad}^*(\mathfrak{a})\beta = (\mathfrak{q}/\mathfrak{q}_\gamma)^*$ and there is an element $\eta \in \mathfrak{a}$ such that $\mathrm{ad}^*(\xi)\beta = \mathrm{ad}^*(\eta)\beta$. We have $\xi - \eta \in \mathfrak{q}_\beta$ and hence $\xi \in \mathfrak{q}_\beta + \mathfrak{a}$. □

Lemma 4.2. *Assume that the centre of $\mathfrak{q}$ consists of semisimple elements. Let $A \lhd Q$ be a non-trivial normal Abelian unipotent subgroup and $\mathfrak{a} = \mathrm{Lie}\, A$. Then $\mathfrak{q}$ is quasi reductive if and only if the action of Q/A on $\mathfrak{a}^*$ has an open orbit and, for generic $\gamma \in \mathfrak{a}^*$, $\mathfrak{q}(\gamma)$ is strongly quasi-reductive.*
Moreover, if $\mathfrak{q}$ is quasi-reductive, then $M_*(\mathfrak{q})$ coincides with $M_*(\mathfrak{q}(\gamma))$.

Proof. Suppose first that $\mathfrak{q}$ is quasi-reductive. Then $\mathfrak{q}_\beta$ is reductive for some $\beta \in \mathfrak{q}^*$. By assumptions, the ideal $\mathfrak{a}$ is contained in the nilpotent radical of $\mathfrak{q}$. Hence $\mathfrak{a} \cap \mathfrak{q}_\beta = 0$. Set $\gamma := \beta|_{\mathfrak{a}}$. Since $[\mathfrak{q}, \mathfrak{a}] \subset \mathfrak{a}$ the functions γ and β define the same pairing $\hat{\beta}|_{\mathfrak{q}\times\mathfrak{a}}$ between $\mathfrak{q}$ and $\mathfrak{a}$, which is non-degenerate on $\mathfrak{a}$ since $\ker \hat{\beta} \cap \mathfrak{a} = \mathfrak{a} \cap \mathfrak{q}_\beta = 0$. On the other hand, $\mathfrak{q}_\gamma = \{y \in \mathfrak{q} \mid \hat{\beta}(y, \mathfrak{a}) = 0\}$. Hence $\dim Q\gamma = \dim \mathfrak{a}$ and $Q\gamma$ is a required open orbit. Since A acts trivially on $\mathfrak{a}^*$, $Q\gamma$ coincides with the Q/A-orbit of γ. Let $\check{\beta}$ be the restriction of β to $\mathfrak{q}_\gamma$. According to Lemma 4.1, we have $(\mathfrak{q}_\gamma)_{\check{\beta}} = \mathfrak{q}_\beta \oplus \mathfrak{a}$, in our case the sum is direct. In $\mathfrak{q}_{\mathbb{C}\gamma}$ the stabiliser of (the restriction of) β is almost the same, more precisely, it is equal to $\mathfrak{q}_\beta \oplus \ker \gamma$, since there is an element multiplying γ by a non-zero number and $\xi \in \mathfrak{a}$ such that $\gamma(\xi) \neq 0$ does not stabilise β. Hence, if we consider β is a function on $\mathfrak{q}(\gamma)$, the stabiliser $\mathfrak{q}(\gamma)_\beta$ is equal to $\mathfrak{q}_\beta$ and therefore is reductive. Thus $\mathfrak{q}(\gamma)$ is quasi reductive and its centre necessary consists of semisimple elements. We have proved that each reductive stabiliser in $\mathfrak{q}$ (of a linear function on $\mathfrak{q}$) is also a stabiliser in $\mathfrak{q}(\gamma)$.

Now suppose that $\gamma \in \mathfrak{a}^*$ belongs to an open Q/A-orbit. Then γ is generic. Suppose also that $\mathfrak{q}(\gamma)$ is quasi reductive and its centre consists of semisimple elements. Then there is $\beta \in \mathfrak{q}(\gamma)^*$ such that $\mathfrak{q}(\gamma)_\beta$ is reductive. If β were zero on $\mathfrak{a}/\ker\gamma$, its stabiliser would have contained this quotient, and hence a non-zero nilpotent element. Thereby the restriction of β to $\mathfrak{a}/\ker\gamma$ is non-zero and rescaling β if necessary we may assume that it coincides with γ. Let $\tilde{\beta} \in \mathfrak{q}^*$ be a lifting of β such that $\tilde{\beta}|_{\mathfrak{a}} = \gamma$. Let also $\check{\beta}$ be the restriction of $\tilde{\beta}$ to $\mathfrak{q}_\gamma$. By Lemma 4.1, $(\mathfrak{q}_\gamma)_{\check{\beta}} = \mathfrak{q}_{\tilde{\beta}} + \mathfrak{a}$. Since γ belongs to an open Q-orbit, $\mathfrak{a} \cap \mathfrak{q}_{\tilde{\beta}} = 0$ and, hence, $\mathfrak{q}_{\tilde{\beta}}$ is isomorphic to $\mathfrak{q}(\gamma)_\beta$. In particular, $\mathfrak{q}$ is quasi-reductive. Also each reductive stabiliser in $\mathfrak{q}(\gamma)$ (of a linear function on $\mathfrak{q}(\gamma)$) is also a stabiliser in $\mathfrak{q}$. This proves that maximal reductive stabilisers of two algebras coincide. □

Lemma 4.3. *Suppose that $\mathfrak{q} = \mathfrak{q}(0) \ltimes \mathfrak{a}$ is a semi-direct product of an Abelian ideal $\mathfrak{a}$ consisted of nilpotent elements and a Lie subalgebra $\mathfrak{q}(0)$. Assume further that the centre of $\mathfrak{q}$ consists of semisimple elements.*

Denote by $Q(0)$ the connected subgroup of Q of Lie algebra $\mathfrak{q}(0)$. Then $\mathfrak{q}$ is quasi reductive if and only if the action of $Q(0)$ on $\mathfrak{a}^*$ has an open orbit and for generic $\gamma \in \mathfrak{a}^*$, $\mathfrak{q}(0)_\gamma$ is strongly quasi-reductive.
Moreover, if $\mathfrak{q}$ is quasi reductive, then $M_*(\mathfrak{q})$ coincides with $M_*(\mathfrak{q}(0)_\gamma)$.

Proof. According to Lemma 4.2, all the statements become true if we replace $\mathfrak{q}(0)_\gamma$ by $\mathfrak{q}(\gamma)$. In our case, there is a complementary to $\mathfrak{a}$ subalgebra, namely $\mathfrak{q}(0)$. This implies that $\mathfrak{q}(\gamma) = \mathfrak{q}(0)_\gamma \oplus \mathbb{C}h \oplus \mathbb{C}\xi$, where $\mathfrak{q}(0)_\gamma$ is a subalgebra commuting with ξ, $\mathbb{C}\xi$ is a commutative ideal contained in the nilpotent radical, and h acts on $\mathbb{C}\xi$ via a non-trivial character. Whenever a stabiliser $\mathfrak{q}(\gamma)_\beta$ with $\beta \in \mathfrak{q}(\gamma)^*$ is reductive, β is necessary non-zero on ξ and $\mathfrak{q}(\gamma)_\beta = (\mathfrak{q}(0)_\gamma)_{\check\beta}$ for $\check\beta$ being a restriction of β to $\mathfrak{q}(0)_\gamma$. This completes the proof. □

Definition 4.4. We will say that a Lie algebra $\mathfrak{q}$ is m-graded, if it has a $\mathbb{Z}$-grading with only m non-trivial components: $\mathfrak{q} = \bigoplus\limits_{i=0}^{m-1} \mathfrak{q}(i)$.

Remark 4.5. A semi-direct product structure $\mathfrak{q}(0) \ltimes \mathfrak{a}$ can be also considered as a 2-grading $\mathfrak{q} = \mathfrak{q}(0) \oplus \mathfrak{q}(1)$, where $\mathfrak{q}(1) = \mathfrak{a}$. In the same spirit, if $\mathfrak{q}$ is m-graded, then $\mathfrak{q}(m-1)$ is an Abelian ideal.

Lemma 4.6. *Assume that the centre of $\mathfrak{q}$ consists of semisimple elements. Suppose that $\mathfrak{q}$ is 3-graded $\mathfrak{q} = \mathfrak{q}(0) \oplus \mathfrak{q}(1) \oplus \mathfrak{q}(2)$ and for generic $\alpha \in \mathfrak{q}(2)^* \subset \mathfrak{q}^*$ the skew-symmetric form $\hat\alpha$ is non-degenerate on $\mathfrak{q}(1)$. Set $\mathfrak{a} = \mathfrak{q}(2)$. Then $\mathfrak{q}$ is quasi reductive if and only if the action of $Q(0)$ on $\mathfrak{a}^*$ has an open orbit and for generic $\gamma \in \mathfrak{a}^*$, $\mathfrak{q}(0)_\gamma$ is strongly quasi reductive.*
Moreover, if $\mathfrak{q}$ is quasi reductive, then $M_*(\mathfrak{q})$ coincides with $M_*(\mathfrak{q}(0)_\gamma)$.

Proof. First of all, if $\mathfrak{q}$ is quasi reductive, then by Lemma 4.2, Q has an open orbit in $\mathfrak{a}^*$. Since the normal subalgebra $\mathfrak{q}(1) \oplus \mathfrak{q}(2)$ acts on $\mathfrak{a}^*$ trivially, that open orbit is also an open orbit of $Q(0)$. For the rest of the proof assume that $\overline{Q(0)\gamma} = \mathfrak{a}^*$. If $\mathfrak{q}_\beta$ is reductive for some $\beta \in \mathfrak{q}^*$, then $\beta|_\mathfrak{a}$ lies in $Q(0)\gamma$. Replacing β by a conjugate one, we may (and will) assume that the restriction of β to $\mathfrak{a}$ equals γ. By our assumptions, $\hat\beta$ is non-degenerate on $\mathfrak{q}(1)$. Again replacing β by a Q-conjugate linear function, one may assume that β is zero on $\mathfrak{q}(1)$. Summing up, if we are interested in the existence of reductive stabilisers and their possible types it suffices to consider $\beta \in \mathfrak{q}^*$ such that $\beta|_{\mathfrak{q}(1)} = 0$ and $\beta|_\mathfrak{a} = \gamma$. In this case β can be considered as a function on $\hat{\mathfrak{q}} = \mathfrak{q}(0) \ltimes \mathfrak{q}(2)$ and $\mathfrak{q}_\beta = \hat{\mathfrak{q}}_\beta$. Now all the claims follow from Lemma 4.3 applied to $\hat{\mathfrak{q}}$. □

Lemma 4.6 provides an alternative proof of Theorem 3.4. Let $\mathfrak{g}$ be a simple Lie algebra and $e \in \mathfrak{g}$ a minimal nilpotent element. It can be included into an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$. Then the h-grading of $\mathfrak{g}$ looks as follows

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2),$$

where $\mathfrak{g}(2) = \mathbb{C}e$ and $\mathfrak{g}(-2) = \mathbb{C}f$. Set $\mathfrak{p}(e) := \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. It is a maximal parabolic subalgebra in $\mathfrak{g}$ and it is 3-graded. The Levi part, $\mathfrak{g}(0)$, is generated by all simple roots orthogonal to the highest root, the semisimple element h, and, in type A only, another semisimple element.

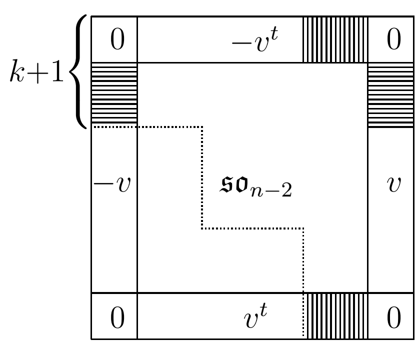
Proposition 4.7. *Let $\mathfrak{p} \subset \mathfrak{p}(e)$ be a parabolic subalgebra of $\mathfrak{g}$. Then $M_*(\mathfrak{p}) = M_*(\mathfrak{g}(0)_e \cap \mathfrak{p})$ and the conjugacy class of $M_*(\mathfrak{p})$ has a representative embedded into the intersection on the right hand side.*

Proof. Set $\hat{\mathfrak{p}} = \mathfrak{p} \cap \mathfrak{g}(0)_e$. Then $\mathfrak{p} = \hat{\mathfrak{p}} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and this decomposition is a 3-grading.

Using the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$, one can consider f as a linear function on $\mathfrak{g}$ and $\mathfrak{p}(e)$. Since the pairing $\langle \mathfrak{g}(-1), \mathfrak{g}(1)\rangle$ is non-degenerate, the skew-symmetric form $\hat{f}$ is non-degenerate on $\mathfrak{g}(1)$. Thus conditions of Lemma 4.6 are satisfied. To conclude, note that $\mathfrak{g}(0)_e = \mathfrak{g}(0)_f$. $\square$

Remark 4.8. The intersection $\mathfrak{p} \cap \mathfrak{g}(0)_e$ is a parabolic subalgebra of $\mathfrak{g}(0)_e$. If $\mathfrak{g}$ is an exceptional Lie algebra, then $\mathfrak{g}(0)_e$ is simple and is generated by all simple roots orthogonal to the highest one. For extended Dynkin diagrams see Table 3 in Section 6.

4.2. In our reductions the following situation will appear quite often. Let P be a standard proper parabolic subgroup of $Spin_n$ (or SO_n) such that the Levi part of $\mathfrak{p}$ contains the first k simple roots $\alpha_1, \ldots, \ldots, \alpha_k$ of $\mathfrak{so}_n$ and does not contain α_{k+1}. Let Pv be an orbit of the maximal dimension in the defining representation $\mathbb{C}^n$. It is assumed that v is not an isotropic vector, i.e., $(SO_n)_v = SO_{n-1}$. The orbit Pv is an open subset of a complex sphere. In order to understand the stabiliser $\mathfrak{p}_v$, we choose a complementary to $\mathfrak{p}$ subalgebra $\mathfrak{so}_{n-1} \subset \mathfrak{so}_n$ embedded as shown in Picture 3. As a vector space, we have $\mathfrak{so}_{n-1} = \mathfrak{so}_{n-2} \oplus \mathbb{C}^{n-2}$. The big matrix is skew-symmetric with respect to the anti-diagonal, v stands for a vector in $\mathbb{C}^{n-2}$, and v^t is v transposed with respect to the anti-diagonal. The parabolic $\mathfrak{p}$ is schematically shown by a dotted line, that is, it lies above the dotted line.



Pic. 3

Lemma 4.9. *Let $\mathfrak{p} \subset \mathfrak{so}_n$ be a parabolic subalgebra as above and $\mathbb{C}^n$ a defining representation of $\mathfrak{so}_n$ with an invariant bilinear form chosen as $(x_j, x_{n-j}) = 1$, $(x_j, x_t) = 0$ for $t \neq n-j$. Take $v = x_1 + x_n$. Assume further that $k \geqslant 2$ is even. Then $M_*(\mathfrak{p}_v) = M_*(\mathfrak{p}_0)$, where $\mathfrak{p}_0 \subset \mathfrak{p}_v$ is a point wise stabiliser of the plane $\mathbb{C}x_1 \oplus \mathbb{C}x_n$ and a parabolic subalgebra of $\mathfrak{so}_{n-2}$.*

Proof. According to our choice, v is not isotropic and Pv is an open subset of the sphere containing v. Hence for $(\mathfrak{so}_n)_v = \mathfrak{so}_{n-1}$ we have $(\mathfrak{so}_n)_v + \mathfrak{p}_v = \mathfrak{so}_n$. The stabiliser $\mathfrak{p}_v$ decomposes as $\mathfrak{p}_v = \mathfrak{p}_0 \oplus V$, where $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{so}_{n-2}$ is a parabolic in $\mathfrak{so}_{n-2}$, and V comes, partly, from the intersection of $\mathfrak{gl}_{k+1}$-part of the Levi with "v-part" of the $\mathfrak{so}_{n-1}$. In Picture 3, the embedding $V \subset \mathfrak{so}_n$ is shown by four coated segments. Here $\dim V = k$ and $[\mathfrak{p}_0, V] \subset V$.

Let $\mathfrak{n}_0$ be the nilpotent radical of $\mathfrak{p}_0$ and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ a Levi decomposition. Then $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{so}_{n-2} = \mathfrak{n} \cap \mathfrak{p}_v$. On one hand, $[\mathfrak{n}_0, V] \subset \mathfrak{n}$, on the other hand, $[\mathfrak{n}_0, V] \subset V$, since it lies in the complementary to $\mathfrak{so}_{n-2}$ subspace. Hence $[\mathfrak{n}_0, V] = 0$ for the simple reason that $V \cap \mathfrak{n} = 0$. Observe that $[V, V] \cong \Lambda^2 V$ lies in the centre of $\mathfrak{n}_0$. For $k \geqslant 2$, the subspace $[V, V] = \Lambda^2 \mathbb{C}^k$ is non-zero, and in that case it coincides with the centre of $\mathfrak{n}_0$.

In the Levi part of P the passage from P to P_v results in replacing GL_{k+1} by $GL_k \ltimes V$. The subspace V is acted upon only by the reductive part of P_0, more precisely, by GL_k.

Set $\mathfrak{a} = [V, V]$. It is an Abelian ideal in $\mathfrak{p}_v$. Note that GL_k, and hence P_0 and P_v, acts on $\mathfrak{a}^*$ with an open orbit. Suppose that a stabiliser $(\mathfrak{p}_v)_\beta$ (with $\beta \in \mathfrak{p}_v^*$) is reductive. Then the restriction $\gamma = \beta|_{\mathfrak{a}}$ lies in the open P_v-orbit and, since k is even, the form $\hat{\beta}$ is non-degenerate on V. Replacing β by an element of $\exp(V)\beta$ we may assume that β is zero on V. Then it can be also considered as a function on $\mathfrak{p}_0$. Since $[\mathfrak{p}_0, V] \subset V$, all elements in the orbit $P_0\beta$ are zero on V and hence $(\mathfrak{p}_v)_\beta = (\mathfrak{p}_0)_\beta$.

Now suppose that $\beta \in \mathfrak{p}_0^*$ is of reductive type. Then again $\gamma = \beta|_{\mathfrak{a}}$ lies in the open P_0-orbit. We also consider β as a linear function on $\mathfrak{p}_v$ such that $\beta(V) = 0$. Since the form $\hat{\beta}$ is non-degenerate on V, one gets $(\mathfrak{p}_0)_\beta = (\mathfrak{p}_v)_\beta$. Thus the maximal reductive stabiliser of P_v is the same as of P_0. $\square$

4.3. Suppose that $\mathfrak{g}$ is a Lie algebra of type F_4 or E_8. We use Vinberg-Onishchik numbering of simple root (see Table 2 in Section 6). Let $H \subset G$ be a maximal parabolic subgroup with the Lie algebra $\mathfrak{h} = \mathfrak{p}_\pi^+$ such that $\pi = \{\alpha_2, \alpha_3, \alpha_4\}$ in the F_4 case and in the E_8 case π contains to all simple roots except α_7. Both these parabolics are 3-graded, more precisely, they are

$$(\mathbb{C}^\times \times Spin_7) \ltimes \exp(\mathbb{C}^8 \oplus \mathbb{C}^7) \text{ and } (\mathbb{C}^\times \times Spin_{14}) \ltimes \exp(\mathbb{C}^{64} \oplus \mathbb{C}^{14}).$$

In both cases $H(0)$ acts on $\mathfrak{h}(1)$ via a half-spin representation and on $\mathfrak{h}(2)$ via the defining representation. The reductive part, $H(0)$, has an open orbit in $\mathfrak{h}(2)^*$ and $\mathfrak{h}(1)$ remains irreducible after the restriction to a generic stabiliser $H(0)_*(\mathfrak{h}(2)^*)$. Hence for a generic point $\alpha \in \mathfrak{h}(2)^*$ the skew-symmetric form $\hat{\alpha}$ is non-degenerate on $\mathfrak{h}(1)$. Here generic means that α lies in the open $H(0)$-orbit. (Another way to see that $\hat{\alpha}$ is non-degenerate, is to notice that the above grading is related to a nilpotent element of height 2.) By Lemma 4.6, $M_*(\mathfrak{h}) = \mathfrak{h}(0)_\alpha$ is either $\mathfrak{so}_6$ or $\mathfrak{so}_{13}$, depending on $\mathfrak{g}$.

Proposition 4.10. *Let $H \subset G$ be as above, $P \subset G$ a parabolic subgroup, which is contained in H, and $\alpha \in \mathfrak{h}(2)^*$ generic. Then $M_*(\mathfrak{p})$ is equal to the maximal reductive stabiliser of $\mathfrak{p} \cap \mathfrak{h}(0)_\alpha = (\mathfrak{p} \cap \mathfrak{h}(0))_\alpha$.*

Proof. Clearly $P \cap H(0)$ is a parabolic in $H(0)$. Let $B \subset (P \cap H(0))$ be a Borel subgroup. As is well known, B acts on $\mathfrak{h}(2)^*$ with an open orbit. Let us choose α such that $B\alpha$ is that open orbit. The nilpotent radical $\mathfrak{h}(1) \oplus \mathfrak{h}(2)$ of $\mathfrak{h}$ is contained in $\mathfrak{p}$. Thus $\mathfrak{p}$ is 3-graded: $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{h}(0)) \oplus \mathfrak{h}(1) \oplus \mathfrak{h}(2)$. Since $\hat{\alpha}$ is still non-degenerate on $\mathfrak{h}(1)$, the claim follows from Lemma 4.6. $\square$

Combining Proposition 4.10 and Lemma 4.9 we get the following.

Corollary 4.11. *Let P be a parabolic subgroup of G such that $P \subset H$ and*

- *if $\mathfrak{g}$ is of type F_4, then $\mathfrak{p} = \mathfrak{p}_\pi^+$ with $\pi = \{\alpha_3, \alpha_4\}$;*
- *if $\mathfrak{g}$ is of type E_8, then the Levi part of $\mathfrak{p}$ contains α_1 and is of type A_k with even k.*

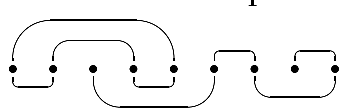
Let $\mathfrak{so}_m \subset \mathfrak{h}(0)$ be the standard Levi subalgebra of $\mathfrak{g}$ corresponding to α_2, α_3 in F_4 and $\alpha_2, \alpha_3, \ldots, \alpha_6, \alpha_8$ ($m = 12$) in E_8. Then $M_*(\mathfrak{p})$ is equal to the maximal reductive stabiliser of $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{so}_m$, where $\mathfrak{p}_0$ is a parabolic subalgebra of $\mathfrak{so}_m$.

5. Classical Lie algebras

5.1. The $\mathfrak{gl}_n$ case. Recall that a seaweed subalgbera $\mathfrak{q}$ of a reductive Lie algebra $\mathfrak{g}$ is an intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ of two parabolics such that $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{g}$. In case $\mathfrak{g} = \mathfrak{gl}_n$ a parabolic subalgebra is defined up to conjugation by a flag of $\mathbb{C}^n$ or by a composition of n. Fixing a maximal torus in $\mathfrak{gl}_n$ and a root system, one may say that a seaweed is given by two compositions of n. Our goal is to describe $M_*(\mathfrak{q})$ in terms of these compositions.

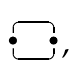
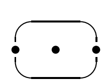
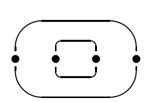
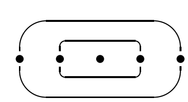
Remark 5.1. If $\mathfrak{q} \subset \mathfrak{gl}_n$ is a seaweed, then $\mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{sl}_n$ is seaweed in $\mathfrak{sl}_n$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}_0$. Since also the centre of GL_n acts on $\mathfrak{q}^*$ trivially, we conclude that $M_*(\mathfrak{q}_0) = M_*(\mathfrak{q}) \cap \mathfrak{sl}_n$.

Let $(\bar{a}|\bar{b}) = (a_1, a_2, \ldots, a_m | b_1, b_2, \ldots b_l)$ be two compositions of n and $\mathfrak{q} = \mathfrak{q}(\bar{a}|\bar{b})$ a corresponding seaweed in $\mathfrak{gl}_n$. Following [DK00], we associate to this object a graph with n vertices and several edges constructed by the following principle: take first a_1 vertices and connect vertex 1 with a_1, 2 with $a_1{-}1$ and so on; repeat it for vertices $a_1{+}1, a_1{+}2, \ldots, a_1{+}a_2$, namely connecting $a_1{+}1$ with $a_1{+}a_2$; do the same for all intervals $(a_1 + \ldots + a_k + 1, a_1 + \ldots + a_k + a_{k+1})$; finally, repeat the procedure using the composition $\bar{b}$ (and the same set of vertices). Let $\Gamma(\bar{a}|\bar{b}) = \Gamma(\mathfrak{q})$ denote the obtained graph. Each vertex has valencity 1 or 2, hence connected components of $\Gamma(\bar{a}|\bar{b})$ are simple cycles or segments. Below is an example of such a graph of a seaweed in $\mathfrak{gl}_9$.

$\Gamma(5,2,2|2,4,3)=$ 

Definition 5.2. Let Y be a cycle, X either a segment or a cycle in $\Gamma(\bar{a}|\bar{b})$ and $x_1 > \ldots > x_r$, $y_1 > \ldots > y_t$ the vertices of X, Y, respectively. We say that X *lies inside* Y if $y_{2i} < x_1 < y_{2i-1}$ for some i. (This means that in the 2-dimensional picture of $\Gamma(\bar{a}|\bar{b})$ X lies inside Y.) We say that X is *maximal* if it does not lie inside any other cycle.

Let us consider the simplest example when the seaweed is just $\mathfrak{gl}_n$. Then the corresponding graph $\Gamma(n|n)$ consist of $[n/2]$ cycles and for n odd there is also a single vertex in the middle. Only one cycle is maximal.

$\Gamma(2|2)=$ , $\Gamma(3|3)=$ , $\Gamma(4|4)=$ , $\Gamma(5|5)=$ .

To each cycle we attach a number, its *dimension*, which is equal to the sum 2#(cycles lying inside)+ #(segments lying inside)+2. We will see later that the second summand is either 1 or 0. According to this formula, the maximal cycle arising in the $\mathfrak{gl}_n$ example has dimension n. By convention, segments are of dimension one and if a segment is maximal, it is considered as a maximal cycle of dimension 1.

To each maximal cycle $X \subset \Gamma(\bar a|\bar b)$ of dimension r we associate a subgroup $GL_r \subset GL_n$, embedded in the following way. Let $x_1 > \ldots > x_t$ be the vertices of X. If X is a segment, the corresponding GL_1 is a diagonal torus with the same $c \in \mathbb{C}^\times$ on places x_i and 1's on all other places. If X is not a segment, then necessary t is even and $x_{2i-1} - x_{2i} = r{-}1$ for all i (see Lemma 5.4(iii) below). Our GL_r is the diagonal in the product of $t/2$ copies of GL_r corresponding to columns and rows intervals $[x_{2i}, x_{2i-1}]$.

Theorem 5.3. *The product of all GL_r over all maximal cycles in $\Gamma(\bar a, \bar b)$ is a maximal reductive stabiliser for the corresponding seaweed.*

The proof is based on a reduction procedure introduced by Panyushev [Pa01, Prop. 4.1]. First, notice that if $a_1 = b_1$, then $\mathfrak q$ is a direct sum of $\mathfrak{gl}_{a_1}$ and a seaweed $\mathfrak q'' \subset \mathfrak{gl}_{n-a_1}$; the graph $\Gamma(\bar a, \bar b)$ is a disjoint union of $\Gamma(a_1|a_1)$ and $\Gamma(\mathfrak q'')$.

Suppose that $a_1 \neq b_1$. Interchanging $\bar a$ and $\bar b$, one may assume that $a_1 < b_1$. We define a new seaweed $\mathfrak q'' \subset \mathfrak{gl}_{n''}$ and a new tuple of numbers according to the following rule:

if $2a_1 \leqslant b_1$, then $n'' = n - a_1$ and $\mathfrak q''$ corresponds to $(a_2, \ldots, a_m | b_1 - 2a_1, a_1, b_2, \ldots, b_l)$;
if $2a_1 > b_1$, then $n'' = n - b_1 + a_1$ and $\mathfrak q''$ corresponds to $(2a_1 - b_1, a_2, \ldots, a_m | a_1, b_2, \ldots, b_l)$.

Lemma 5.4. (i) *In case $a_1 \neq b_1$, the passage from $\Gamma(\mathfrak q)$ to $\Gamma(\mathfrak q'')$ preserves segments and cycles as well as inclusion relation among then and therefore dimensions.*
(ii) *If two connected components X_1, X_2 of $\Gamma(\mathfrak q)$ lie inside a cycle Y, then one of them lies inside the other.*
(iii) *For each cycle $X \subset \Gamma(\mathfrak q)$ of dimension r with vertices $x_1 > \ldots > x_t$ all differences $x_{2i-1} - x_{2i}$ are equal to $r{-}1$.*

Proof. For convenience, we draw $\Gamma(\bar a, \bar b)$ in a 3-dimensional space, putting all vertices on a line, preserving the order and choosing a separate plane for each edge.

For the proof of part (i) we may assume that $a_1 < b_1$. Consider first the case where $2a_1 \leqslant b_1$. Here we contract $\bar b$-edges connecting 1 with b_1, 2 with $b_1{-}1$, and so on finishing with the $\bar b$-edge $(a_1, b_1 - a_1 + 1)$. At the same time also the vertices are identified in each pair. More precisely, the pair $(i, b_1 - i + 1)$ is now a single vertex $b_1 - a_1 + i$ for $1 \geqslant i \geqslant a_1$. Other vertices are renumbered $j \to j{-}a_1$. Topologically speaking, the transformation was just a contraction. Thus all cycles and segments remain cycles and segments, no new connected components appear. We still have a graph with no self-intersections and this new graph corresponds now to the seaweed $\mathfrak q''$.

Now suppose a connected component $X \subset \Gamma(\mathfrak q)$ is lying inside a cycle Y with vertices $y_1 > \ldots > y_t$. Let $x_1 > \ldots > x_r$ be the vertices of X. By definition, there is i such that $y_{2i-1} > x_1 > y_{2i}$. We need to show that this kind of an inequality remains after the modification. Note that because of the $\bar{b}$-edges, if X has a vertex in the interval $[b_1{-}a_1{+}1, b_1]$, then it also have a vertex in $[1, a_1]$. (The same holds for Y.) Therefore necessary $x_1 > a_1$ and there is nothing to prove.

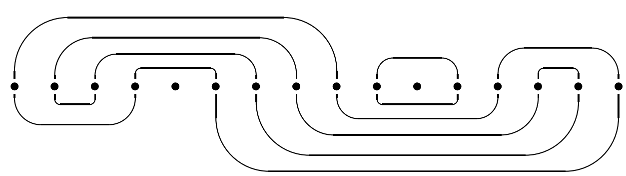
Consider now the second case, where $2a_1 > b_1$. Here $\Gamma(\mathfrak{q}'')$ is obtained from $\Gamma(\mathfrak{q})$ in three steps. First we contract $\bar{b}$-edges connecting 1 with b_1, 2 with $b_1{-}1$, and so on finishing with the $\bar{b}$-edge $(b_1{-}a_1, a_1+1)$. At the same time also the vertices are identified in each pair. More precisely, the pair (i, b_1-i+1) is now a single vertex i for $1 \geqslant i \geqslant b_1{-}a_1$. Next we apply the central symmetry to $[1, a_1]$ and renumber the vertices $i \to a_1{-}i{+}1$ for $1 \geqslant i \geqslant a_1$; $j \to j{-}b_1{+}a_1$ for $j > b_1$. Finally the first a_1 $\bar{a}$-edges are turned into $\bar{b}$-edges and the first $(2a_1{-}b_1)$ of the $\bar{b}$-edges into $\bar{a}$-edges. One can easily see that the passage preserves connected components.

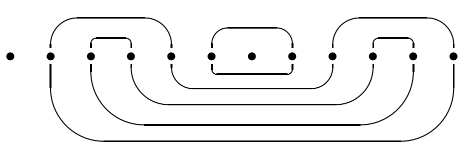
It remains to treat connected components X and Y. Let X'', $Y'' \subset \Gamma(\mathfrak{q}'')$ be the modified X and Y. If $x_1 > a_1$, then $y''_{2i-1} > x''_1 > y''_{2i}$ for the same i as before the modification. The other possibility is that $x_1 \in [b_1{-}a_1{+}1, a_1]$. Nevertheless, in that case $x''_1 = x_1{-}b_1{+}a_1$ and again $y''_{2i-1} > x''_1 > y''_{2i}$ for the same i as before the modification.

In order to prove parts (ii) and (iii), we argue by induction on n. For $n = 1$ there is nothing to prove. If $a_1 \neq b_1$, we can pass to $\Gamma(\mathfrak{q}'')$. By part (i) the passage preserves connected components and inclusions among them. Moreover, $\mathfrak{q}'' \subset \mathfrak{gl}_{n''}$ with $n'' < n$. Thus it remains to show that this passage preserves the differences between vertices in a cycle Y. In case $2a_1 \leqslant b_1$, we only have to notice that the number $\#(Y \cap [1, a_1]) = \#(Y \cap [b_1{-}a_1{+}1, b_1])$ is even. In case $2a_1 > b_1$, an observation that the set $Y \cap [b_1{-}a_1{+}1, a_1]$ is invariant under the central symmetry does the job.

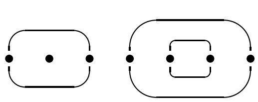
If $a_1 = b_1$, then $\Gamma(\mathfrak{q})$ is a disjoint union of $\Gamma(a_1|a_1)$ and $\Gamma(\mathfrak{q}'')$ with $\mathfrak{q}'' \subset \mathfrak{gl}_{n-a_1}$. Clearly both statements hold for $\Gamma(a_1|a_1)$ and by induction they hold for $\Gamma(\mathfrak{q}'')$. This finishes the proof of the lemma. $\square$

Example 5.5. We illustrate reductions of Lemma 5.4 by a seaweed $\mathfrak{q}(9, 3, 4|4, 1, 11)$ in $\mathfrak{gl}_{16}$.

$\Gamma(9, 3, 4|4, 1, 11)=$  ,

$\Gamma(\mathfrak{q}'')=$  .

We are reduced to a seaweed (parabolic) $\mathfrak{p}(4, 3, 4|11) \in \mathfrak{gl}_{11}$ and

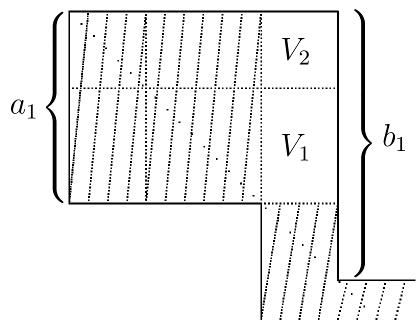
$\Gamma(\mathfrak{p}'')=$  .

One can conclude that $M_*(\mathfrak{q}) = \mathbb{C} \oplus M_*(\mathfrak{p}) = \mathbb{C} \oplus \mathfrak{gl}_3 \oplus \mathfrak{gl}_4$.

Proof of Theorem 5.3. (cf. [Pa05, proof of Theorem 4.3].) We argue by induction on n. If $n = 1$, there is nothing to prove. Assume that $n > 1$ and suppose first that $a_1 = b_1$. Then MRS of $\mathfrak{q}$ is equal to the direct product of GL_{a_1} and MRS of $\mathfrak{q}''$. By induction, the theorem holds for $\mathfrak{q}'' \subset \mathfrak{gl}_{n-a_1}$. Suppose now that $a_1 \neq b_1$. Then

$$\mathfrak{q} = \widehat{\mathfrak{q}} \ltimes V,$$

where $\widehat{\mathfrak{q}}$ is a seaweed subalgebra of $\mathfrak{gl}_n$ defined by $(\bar{a}|a_1, b_1{-}a_1, b_2, \ldots, b_l)$ and V is an Abelian ideal. Moreover $\widehat{Q}$ acts on V^* with an open orbit. More precisely, $V = V_1 \oplus V_2$ with $V_1 \cong (\mathbb{C}^{a_1})^* \otimes \mathbb{C}^c$, V_2 being isomorphic to a space of $a_1{\times}a_1$-matrices in case $2a_1 \leqslant b_1$; and $V_1 \cong (\mathbb{C}^c)^* \otimes \mathbb{C}^d$, V_2 being isomorphic to a space of $d{\times}d$-matrices for $d = b_1{-}a_1$ in case $2a_1 > b_1$; in both cases $c = |2a_1{-}b_1|$. We illustrate this decomposition in Picture 4 for $2a_1 > b_1$. In both cases there is a point $\gamma \in V^*$ in the $\widehat{Q}$-open orbit



The subalgebra $\widehat{\mathfrak{q}}$ is shaded.

Pic. 4

such that $\gamma(V_1) = 0$. One can easily see that $\widehat{Q}_\gamma \cong Q''$ for this γ. By Lemma 4.3, MRS of Q is equal to the MRS of Q'' and its embedding into Q can be read from the embedding of $\widehat{Q}_\gamma$. By Lemma 5.4(i), the graphs $\Gamma(\mathfrak{q})$ and $\Gamma(\mathfrak{q}'')$ have the same maximal cyles. Therefore our descriptions of MRS as an abstract group is justified and it remains only to specify the embedding into GL_n.

Assume that γ is given as the identity matrix in V_2^*. We detail the first case, $2a_1 \leqslant b_1$. Let $H \cong GL_{a_1}$ be a subgroup of GL_n corresponding to a diagonal square at rows $[(b_1{-}a_1{+}1), b_1]$. Then $\mathfrak{q}''$ has a subalgebra isomorphic to $\mathfrak{h} \cap \mathfrak{q} = \mathfrak{h} \cap \widehat{\mathfrak{q}}$; and $\widehat{Q}_\gamma$ has a subgroup isomorphic to $H \cap Q$ embedded diagonally into $GL_{a_1} {\times} H$. Let $X \subset \Gamma(\mathfrak{q})$ be a maximal cycle of dimension 1 and GL_1 the corresponding subgroup of the MRS of $\mathfrak{q}$. As a subgroup of Q'' it is defined by the properties that diagonal entries on places x_i'' are all equal and entries on other diagonal places are 1's. By means of $\widehat{Q}_\gamma$ this embedding is extended to GL_{a_1}. If $x_i \in [1, a_1]$ is a vertex of X, then also $a_1{-}x_1{+}1$ and $b_1{-}a_1{+}x_i$ are. From the description of $\widehat{Q}_\gamma$, we get that diagonal entries on places x_i and $b_1{-}a_1{+}x_i$ are equal. If $i \in [1, a_1]$ is not a vertex of X, then neither is $a_1{-}i+1$ or $b_1{-}a_1{+}i$, and the diagonal entry on place i is equal to 1.

Now let $Y \subset \Gamma(\mathfrak{q})$ be a maximal cycle of dimension $r > 1$ and $H_r \cong GL_r$ the corresponding subgroup of the MRS of Q''. Recall that because of the $\bar{b}$-edges the number $\#(Y \cap [a_1{+}1, b_1{-}a_1])$ is even. Hence each interval $[y_{2i}, y_{2i-1}]$ with $y_{2i-1} \leqslant a_1$ after going over $\bar{a}$-edges and $\bar{b}$-edges is shifted by $b_1 - a_1$ to an interval $[y_{2j}, y_{2j-1}]$. Hence H_r projects isomorphically on a GL_r subgroup of GL_{a_1} corresponding to columns and rows interval $[y_{2i}, y_{2i-1}]$.

In case $2a_1 > b_1$ the proof goes with evident changes. For example, here $H \cong GL_d$ corresponds to rows and columns $[a_1+1, b_1]$. One also has to notice that for a maximal cycle Y of dimension $r > 1$, the number $\#(Y \cap [b_1 - a_1 + 1, a_1])$ is even. $\square$

5.2. The $\mathfrak{sp}_n$ and $\mathfrak{so}_n$ cases. In this subsection, E is the vector space $\mathbb{C}^n$ endowed with a non-degenerate bilinear form B which is either symmetric or alternating. Set $\ell = \left[\frac{n}{2}\right]$ and assume that $\ell \geqslant 1$. We have $B(v, w) = \varepsilon B(w, v)$ for all $v, w \in E$ where $\varepsilon \in \{1, -1\}$. The Lie subalgebra of $\mathfrak{gl}_n(E)$ preserving B is denoted by $\mathfrak{g}^\varepsilon$. Thus $\mathfrak{g}^{+1}$ is $\mathfrak{so}(E) \simeq \mathfrak{so}_n$ and $\mathfrak{g}^{-1}$ is $\mathfrak{sp}(E) \simeq \mathfrak{sp}_{2\ell}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a set of simple roots for $\mathfrak{g}^\varepsilon$. We use Vinberg-Onishchik numbering of simple roots, which in the classical case coincides with the Bourbaki numbering.

The stabiliser of a flag of isotropic subspaces of E in $\mathfrak{g}^\varepsilon$ is a parabolic subalgebra of $\mathfrak{g}^\varepsilon$ and any parabolic subalgebra of $\mathfrak{g}^\varepsilon$ is obtained in this way. A composition $\bar{a} = (a_1, \ldots, a_t)$ of an integer $r \in \{1, \ldots, \ell\}$ determines a standard (with respect to Π) flag $\mathcal{V}(\bar{a}) = \{\{0\} = V_0(\bar{a}) \subsetneq \cdots \subsetneq V_t(\bar{a})\}$ of isotropic subspaces of E. Note that $\dim V_t(\bar{a}) = r$ and $\dim V_i(\bar{a}) - \dim V_{i-1}(\bar{a}) = a_i$ for $i = 1, \ldots, t$. Let $\mathfrak{p}_n^\varepsilon(\bar{a})$ denote the stabiliser of $\mathcal{V}(\bar{a})$ in $\mathfrak{g}^\varepsilon$, and $\mathfrak{p}_r(\bar{a})$ will stand for the stabiliser in $\mathfrak{gl}(V_t(\bar{a}))$ of the flag $\mathcal{V}(\bar{a})$. The Levi part of $\mathfrak{p}_n^\varepsilon(\bar{a})$ is $\mathfrak{gl}_{a_1} \oplus \ldots \oplus \mathfrak{gl}_{a_t} \oplus \mathfrak{g}^\varepsilon_{n-2r}$, where $\mathfrak{g}^\varepsilon_{n-2r}$ is either $\mathfrak{sp}_{n-2r}$ or $\mathfrak{so}_{2n-r}$, depending on ε.

As has been noticed, any parabolic subalgebras of $\mathfrak{g}^{-1} \simeq \mathfrak{sp}_{2\ell}$ is (strongly) quasi-reductive, see [Pa05]. Using a reduction of Panyushev [Pa03, Proof of Theorem 5.2], we describe in the following theorem the maximal reductive stabilisers for parabolics $\mathfrak{p} \subset \mathfrak{sp}_{2l}$.

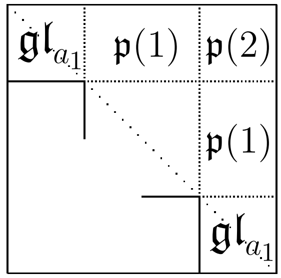
Theorem 5.6. *Let $\bar{a} = (a_1, \ldots, a_t)$ be a composition of r, with $1 \leqslant r \leqslant \ell$. Then*

$$M_*(\mathfrak{p}_n^{-1}(\bar{a})) = \mathfrak{so}_{a_1} \oplus \cdots \oplus \mathfrak{so}_{a_t} \oplus \mathfrak{sp}_{n-2r}\ .$$

The embedding of $M_*(\mathfrak{p}_n^{-1}(\bar{a}))$ into a Levi of $\mathfrak{p}_n^{-1}(\bar{a})$ is obvious.

Proof. We argue by induction on $n = 2\ell$. Clearly the theorem is true for $n = 2$. Let $n \geqslant 4$ and assume that the theorem is true for any standard parabolic subalgebra of $\mathfrak{sp}_{n'}$ with $n' < n$.

For $\gamma \in \Delta$ and $\alpha \in \Pi$, $[\gamma : \alpha]$ will stand for the component of γ in α written in the basis Π. We define a $\mathbb{Z}$-grading on $\mathfrak{g}^{-1}$ by letting $\mathfrak{g}^{-1}(i)$, for $i \neq 0$, be the sum of all root spaces $\mathfrak{g}^{-1}_\gamma$ with $[\gamma : \alpha_{a_1}] = i$, and $\mathfrak{g}^{-1}(0)$ be the sum of $\mathfrak{h}$ and all root spaces $\mathfrak{g}^{-1}_\gamma$ with $[\gamma : \alpha_{a_1}] = 0$. Restricting this grading to $\mathfrak{p} = \mathfrak{p}_n^{-1}(\bar{a})$, we obtain a 3-term grading on $\mathfrak{p}$, $\mathfrak{p} = \mathfrak{p}(0) \oplus \mathfrak{p}(1) \oplus \mathfrak{p}(2)$, where $\mathfrak{p}(0) = \mathfrak{p} \cap \mathfrak{g}^{-1}(0)$ and $\mathfrak{p}(i) = \mathfrak{g}^{-1}(i)$ for $i = 1, 2$, see Picture 5. It follows from the construction that $\mathfrak{p}(0)$ is isomorphic to $\mathfrak{gl}_{a_1} \oplus \mathfrak{p}^{-1}_{n-2a_1}(\bar{b})$, where $\bar{b} = (a_2, \ldots, a_t)$, and $\mathfrak{p}(2) \cong S^2\mathbb{C}^{a_1}$. For a non-degenerate (as a matrix) $\xi \in \mathfrak{p}(2)^*$, we have $\mathrm{ad}^*(\mathfrak{p}(0))\xi = \mathfrak{p}(2)$ and $\hat{\xi}$ is non-degenerate on $\mathfrak{p}(1)$. Therefore Lemma 4.6 applies and $M_*(\mathfrak{p}) = M_*(\mathfrak{p}(0)_\xi) = M_*(\mathfrak{p}^{-1}_{n-2a_1}(\bar{b})) \oplus \mathfrak{so}_{a_1}$. By our induction applied to the parabolic subalgebra $\mathfrak{p}^{-1}_{n-2a_1}(\bar{b})$ of $\mathfrak{sp}_{n-2a_1}$, we obtain the expected result. $\square$



Pic. 5. Reduction for parabolics in $\mathfrak{sp}_{2l}$.

We now turn to the $\mathfrak{so}_n$ case. From now on, B is assumed symmetric.

Definition 5.7. We will say that a composition $\bar{a} = (a_1, \ldots, a_t)$ of r, with $1 \leqslant r \leqslant \ell$, satisfies the property $(*)$ if it does not contain pairs (a_i, a_{i+1}) with a_i odd and a_{i+1} even.

If $\bar{a} = (a_1, \ldots, a_t)$ is a composition of r, with $1 \leqslant r \leqslant \ell$, we set $\bar{a}' := (a_1, \ldots, a_{t-1})$ if r is odd and equal to $n/2$, and $\bar{a}' := \bar{a}$ otherwise.

The characterization of quasi-reductive parabolic subalgebras in $\mathfrak{so}_n \simeq \mathfrak{g}^{+1}$ in term of flags has been established in [DKT]. Recall here this result, which is also stated in [BM, Theorem 1.7]:

Theorem 5.8 ([DKT])**.** *Let $\bar{a} = (a_1, \ldots, a_t)$ be a composition of r, with $1 \leqslant r \leqslant \ell$. Then, $\mathfrak{p}_n^{+1}(\bar{a})$ is quasi-reductive if and only if $\bar{a}'$ has property $(*)$.*

Remark 5.9. Explicit description of $M_*(\mathfrak{p})$ also proves that $\mathfrak{p}$ is quasi-reductive if it satisfies condition $(*)$.

To each composition $\bar{a} = (a_1, \ldots, a_t)$ of r, with $1 \leqslant r \leqslant \ell$, such that $\bar{a}'$ has property $(*)$, and each $s \in \{1, \ldots, t\}$, we assign a subalgebra $\mathfrak{r}_s(\bar{a})$:

$$\mathfrak{r}_s(\bar{a}) = \Big(\bigoplus_{\substack{i \in \{1,\ldots,s\}, \\ a_i \text{ even}}} \mathfrak{sp}_{a_i} \Big) \oplus \Big(\bigoplus_{\substack{i \in \{1,\ldots,s\}, \\ a_{i-1},\ a_i \text{ odd}}} \mathfrak{sp}_{a_{i-1}-1} \oplus \mathfrak{sp}_{a_i - 1} \Big).$$

By convention, $a_0 := 0$ and a_0 is even. Moreover, $r_0(\bar{a}) := 0$ and $\mathfrak{sp}_0 := 0$, $\mathfrak{so}_0 := 0$.

Theorem 5.10. *Let $\bar{a} = (a_1, \ldots, a_t)$ be a composition of r, with $1 \leqslant r \leqslant \ell$, such that $\bar{a}'$ has property $(*)$. Then, setting $\mathfrak{p} = \mathfrak{p}_n^{+1}(\bar{a})$, $M_*(\mathfrak{p})$ is given by the following formulas, depending on the different cases:*

(1) *r is even: $M_*(\mathfrak{p}) = \mathfrak{r}_t(\bar{a}) \oplus \mathfrak{so}_{n-2r}$.*
(2) *$r < \ell$ is odd: $M_*(\mathfrak{p}) = \mathfrak{r}_{t-1}(\bar{a}) \oplus \mathfrak{sp}_{a_t - 1} \oplus \mathfrak{so}_{n-2r-1}$.*
(3) *$r = \ell$ is odd and $a_t = 1$: $M_*(\mathfrak{p}) = \mathfrak{r}_{t-1}(\bar{a}) \oplus \mathbb{C}$.*
(4) *$r = \ell$ is odd and $a_t > 1$ is odd: $M_*(\mathfrak{p}) = \mathfrak{r}_{t-1}(\bar{a}) \oplus \mathfrak{sp}_{a_t - 3}$.*
(5) *$r = \frac{n}{2}$ is odd and a_t is even: $M_*(\mathfrak{p}) = \mathfrak{r}_{t-2}(\bar{a}) \oplus \mathfrak{sp}_{a_{t-1}-1} \oplus \mathfrak{sp}_{a_t - 2}$.*

The above cases are the only possibilities since $\bar{a}'$ has property $(*)$. Note that the index of $\mathfrak{p}$ is described in [DKT] for each of these cases. More generally, a formula for the index of any seaweed subalgebra has been obtained in [J06].

Proof. We argue by induction on n. By small rank isomorphisms, e.g. $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, the statement is known for $n \leqslant 6$. Let $n > 6$ and assume that the theorem is true for any standard parabolic subalgebra of $\mathfrak{so}_{n'}$ with $n' < n$. Set $\mathfrak{p} = \mathfrak{p}_n^{+1}(\bar{a})$ as in the theorem and set $r_i := a_1 + \cdots + a_i$ for all $i \in \{1, \ldots, t\}$; thus $r_t = r$.

Step 1: Assume first that there is $k \in \{1, \ldots, t\}$ such that r_k is even. Define a $\mathbb{Z}$-grading on $\mathfrak{g}$ as in the proof of Theorem 5.6 with respect to α_{r_k}. So, $\mathfrak{p} = \mathfrak{p}(0) \oplus \mathfrak{p}(1) \oplus \mathfrak{p}(2)$ where $\mathfrak{p}(i) = \mathfrak{g}^{\varepsilon}(i)$ for $i = 1, 2$. We have $\mathfrak{p}(0) = \mathfrak{p}' \oplus \mathfrak{p}''$ where $\mathfrak{p}'$ and $\mathfrak{p}''$ are parabolic subalgebras of $\mathfrak{gl}_{r_k}$ and $\mathfrak{so}_{n-2r_k}$ respectively. Namely, $\mathfrak{p}' = \mathfrak{p}_{r_k}(\bar{a}^{(k)})$ and $\mathfrak{p}'' = \mathfrak{p}_{n-2r_k}^{+1}(\bar{b})$ with $\bar{a}^{(k)} := (a_1, \ldots, a_k)$ and $\bar{b} := (a_{k+1}, \ldots, a_t)$. Moreover, $\mathfrak{p}(2) \cong \Lambda^2\mathbb{C}^{r_k}$ and $\mathfrak{p}' \oplus \mathfrak{p}(2) \cong \mathfrak{p}_{2r_k}^{+1}(\bar{a}^{(k)})$ (cf. [Pa01, proof of Theorem 6.1]), finally $[\mathfrak{p}'', \mathfrak{p}(2)] = 0$. Let ξ be a generic element of $\mathfrak{p}(2)^*$. Then $\mathrm{ad}^*(\mathfrak{p}(0))\xi = \mathfrak{p}(2)$ and the form $\hat{\xi}$ is non-degenerate on $\mathfrak{p}(1)$. By Lemma 4.6, $M_*(\mathfrak{p})$ is equal to the M_* of the stabiliser $\mathfrak{p}(0)_\xi$. In view of Lemma 4.3, this also can be expressed as $M_*(\mathfrak{p}) = M_*(\mathfrak{p}(0) \oplus \mathfrak{p}(2))$. To summarize,

$$M_*(\mathfrak{p}) = M_*(\mathfrak{p}_{2r_k}^{+1}(\bar{a}^{(k)}) \oplus M_*(\mathfrak{p}_{n-2r_k}^{+1}(\bar{b})) \ .$$

Since $\bar{a}'$ satisfies the property $(*)$, the inductive step does not work only in the following three cases:

(a) $t = 1$ and $a_1 = n/2$;

(b) $t = 1$ and a_1 is odd;

(c) $t = 2$, a_1 is odd, and $r = n/2$.

These different cases will be discussed in Step 2.

Step 2: Define a 3-term $\mathbb{Z}$-grading on $\mathfrak{p}$ as above with respect to α_r. We have $\mathfrak{p}(0) \cong \mathfrak{p}_r(\bar{a}) \oplus \mathfrak{so}_{n-2r}$. Whenever $r > 1$, $\mathfrak{p}(1)$ is isomorphic to $\mathbb{C}^r \otimes \mathbb{C}^{n-2r}$ as a $\mathfrak{p}(0)$-module, and $\mathfrak{p}(2) \cong \Lambda^2\mathbb{C}^r$. Otherwise, $\mathfrak{p}(1) \cong \mathbb{C}^{n-2}$ is Abelian, and $\mathfrak{p}(2) = 0$. Note that there is an open $P(0)$-orbit in $\mathfrak{p}(2)^*$. From now on, we fix an element $\xi \in \mathfrak{p}(2)^*$ in this open orbit.

Case (a): Here $\mathfrak{p}(1) = 0$ and by Lemma 4.3, $M_*(\mathfrak{p}) = M_*(\mathfrak{p}(0)_\xi)$. If a_1 is even, this is just $\mathfrak{sp}_{a_1}$. If $a_1 = 2s + 1$ is odd, then $\mathfrak{p}(0)_\xi = (\mathbb{C} \oplus \mathfrak{sp}_{2s}) \ltimes \mathbb{C}^{2s}$. This algebra is 2-graded and, by Lemma 4.3, we are reduced to computing MRS of $\mathfrak{q} = \mathfrak{sp}_{2s-2} \oplus \mathbb{C} \ltimes (\mathbb{C}^{2s-2} \oplus \mathbb{C})$, where the second summand is a semidirect product of a one-dimensional reductive torus and a Heisenberg Lie algebra. Since the second summand is a Lie algebra of index zero, we conclude that $M_*(\mathfrak{q}) = \mathfrak{sp}_{2s-2}$ and $M_*(\mathfrak{p}) = \mathfrak{sp}_{a_1-3}$.

Case (b): We may safely assume that $a_1 < n/2$, i.e., either $1 \leqslant r < \ell$ or $r = \ell$ and $n = 2\ell + 1$. If $a_1 = 1$, then $\mathfrak{p} = \mathfrak{p}(0) \oplus \mathfrak{p}(1)$ and $\mathfrak{p}(1) \cong \mathbb{C}^{n-2}$ is an Abelian ideal of $\mathfrak{p}$. Let v be a non-isotropic vector of $(\mathbb{C}^{n-2})^*$. Then it belongs to an open $P(0)$-orbit and $\mathfrak{p}(0)_v \cong (\mathfrak{so}_{n-2})_v \cong \mathfrak{so}_{n-3}$. Hence, Lemma 4.3 yields $M_*(\mathfrak{p}) \cong \mathfrak{so}_{n-3}$.

Assume now that $r > 1$. Apply first Lemma 4.2 to the Abelian nilpotent ideal $\mathfrak{p}(2)$. We have

$$\mathfrak{p}_{\mathbb{C}\xi} = (\mathfrak{gl}_r(\xi) \oplus \mathfrak{so}_{n-2r}) \ltimes ((\mathbb{C}^r \otimes \mathbb{C}^{n-2r}) \oplus \Lambda^2\mathbb{C}^r),$$

where $\mathfrak{gl}_r(\xi) \subset \mathfrak{gl}_r$ is the nomaliser of $\mathbb{C}\xi$. Thus, $\mathfrak{p}(\xi) \cong (\mathfrak{gl}_r(\xi) \oplus \mathfrak{so}_{n-2r}) \ltimes ((\mathbb{C}^r \otimes \mathbb{C}^{n-2r}) \oplus \mathbb{C})$. There is a one-dimensional $\mathfrak{gl}_r(\xi)$-invariant subspace $\mathbb{C}w \subset \mathbb{C}^r$ such that $\mathbb{C}w \otimes \mathbb{C}^{n-2r} \subset \mathfrak{p}(1)$ is the kernel of $\hat{\xi}$. Let $W \subset \mathbb{C}^r$ be a complement of $\mathbb{C}w$ and $\mathfrak{a} = \mathbb{C}w \otimes \mathbb{C}^{n-2r} \oplus \mathbb{C}$ an Abelian ideal of $\mathfrak{p}(\xi)$. Note that $P(\xi)$ acts on $\mathfrak{a}^*$ with an open orbit. Assertion of Lemma 4.2, reduces computation of the MRS to a 3-graded Lie algebra

$$\mathfrak{q} = (\mathfrak{so}_{n-2r-1} \oplus ((\mathfrak{sp}_{r-1} \oplus \mathbb{C}) \ltimes \mathbb{C}^{r-1})) \oplus W \otimes \mathbb{C}^{2n-r} \oplus \mathbb{C},$$

with the last $\mathbb{C}$ being $\mathfrak{q}(2)$. We can change the first and the second grading components, making $\mathfrak{q}(0) = \mathfrak{sp}_{r-1} \oplus \mathfrak{so}_{n-2r-1}$ reductive and putting $\mathfrak{q}(1) = \mathbb{C}^{r-1} \oplus W \otimes \mathbb{C}^{2n-r}$. Here $W \otimes \mathbb{C}^{2n-r}$ decomposes as a sum of two $\mathfrak{q}(0)$-stable subspaces $W_1 \oplus W_2$ with $\dim W_1 = r-1$ and $\dim W_2 = (r-1)(2n-r-1)$. Moreover $[x, W_1] = \mathbb{C}$ for each non-zero $x \in \mathbb{C}^{r-1}$ and $[\mathbb{C}^{r-1}, W_2] = 0$. Since our old form $\hat{\xi}$ is non-degenerate on W_2, we conclude that $\hat{\eta}$ is non-degenerate on $\mathfrak{q}(1)$ for a non-zero $\eta \in \mathfrak{q}(2)^*$. By Lemma 4.6, $M_*(\mathfrak{q}) = M_*(\mathfrak{q}(0)_\eta)$. The finall result is that $M_*(\mathfrak{p}) = \mathfrak{sp}_{a_1-1} \oplus \mathfrak{so}_{n-2r-1}$.

Case (c): Here $\mathfrak{p}(1) = 0$ and $\mathfrak{p}$ has only two graded components, with $\mathfrak{p}(0) = \mathfrak{p}_r(\bar{a})$ being a parabolic in $\mathfrak{gl}_r$ and $\mathfrak{p}(2) = \Lambda^2\mathbb{C}^r$. As above, $P(0)\xi$ is an open orbit in $\mathfrak{p}(2)^*$. By virtue of Lemma 4.3, $M_*(\mathfrak{p}) = M_*(\mathfrak{q})$ for $\mathfrak{q} = \mathfrak{p}_r(\bar{a})_\xi$. Description of $\mathfrak{q}$ depends on the parity of a_2. In order to calculate this stabiliser one may consider the intersection of $\mathfrak{p}(0)$ with a complementary (in $\mathfrak{gl}_r$) subalgebra $(\mathfrak{gl}_r)_\xi$.

If a_2 is odd, then $\mathfrak{q}$ is isomorphic to $(\mathfrak{sp}_{a_1-1} \oplus \mathfrak{sp}_{a_2-1} \oplus \mathbb{C}) \ltimes ((\mathbb{C}^{a_1-1} \oplus \mathbb{C}^{a_2-1}) \oplus \mathbb{C})$. Here $(\mathbb{C}^{a_1-1} \oplus \mathbb{C}^{a_2-1}) \oplus \mathbb{C}$ is the nilpotent radical, which is also a Heisenberg Lie algebra with the centre $\mathbb{C}$. According to Lemma 4.6, $M_*(q) = \mathfrak{sp}_{a_1-1} \oplus \mathfrak{sp}_{a_2-1}$. Therefore, we have obtained that $M_*(\mathfrak{p}) = \mathfrak{sp}_{a_1-1} \oplus \mathfrak{sp}_{a_2-1}$.

If a_2 is even, then $\mathfrak{q} = \mathfrak{q}(0) \oplus \mathfrak{q}(1) \oplus \mathfrak{q}(2)$ with $\mathfrak{q}(0) = \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{sp}_{a_1-1} \oplus \mathfrak{sp}_{a_2-2}$ being reductive, $\mathfrak{q}(1) = \mathbb{C}^{a_1-1} \oplus V_1 \oplus V_2$, and $\mathfrak{q}(2) = \mathbb{C}_I \oplus \mathbb{C}_{II}$. The non-evident commutator relations are $[\mathbb{C}^{a_1-1}, \mathbb{C}^{a_1-1}] = \mathbb{C}_I$, $[V_1, V_1] = \mathbb{C}_I$, $[V_1, V_2] = \mathbb{C}_{II}$, and $[V_2, V_2] = 0$. Note also that $V_1 \cong V_2 \cong \mathbb{C}^{a_2-2}$. It is not difficult to see that conditions of Lemma 4.6 are satisfied and therefore $M_*(\mathfrak{p}) = M_*(\mathfrak{q}) = \mathfrak{sp}_{a_1-1} \oplus \mathfrak{sp}_{a_2-2}$. □

6. Exceptional Lie algebras

In this section $\mathfrak{g}$ is a simple exceptional Lie algebra and Π a set of simple roots of $\mathfrak{g}$ as in Section 3. For each quasi-reductive standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ we explicitly describe the Lie algebra $M_*(\mathfrak{p})$ of its maximal reductive stabiliser. We use the Vinberg-Onishchik numberings of simple roots. For convenience of the reader, it is presented in Table 2.

In the exceptional case, there is a unique simple root, say $\widetilde{\alpha}$, which is not orthogonal to the highest positive root (see e.g. Table 3). Set $\widetilde{\Pi} = \Pi \setminus \{\widetilde{\alpha}\}$. This is in fact the same $\widetilde{\Pi}$ as in Section 3. A subset $\pi \subset \Pi$ defines a standard parabolic subalgebra $\mathfrak{p} := \mathfrak{p}(\pi) := \mathfrak{p}_\pi^+$ in $\mathfrak{g}$, and all parabolic subalgebras arise in this way. Let $P \subset G$ denote the corresponding (connected) parabolic subgroup. If $\pi \subset \widetilde{\Pi}$, then for the description of $M_*(\mathfrak{p})$ the highest root reduction (Thm 3.4, Prop. 4.7) can be applied. It

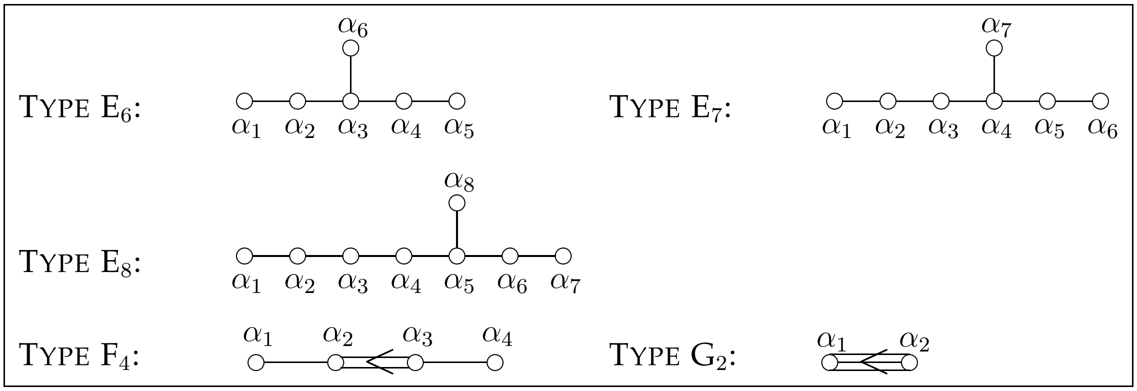


TABLE 2. Vinberg-Onishchik numbering of simple roots

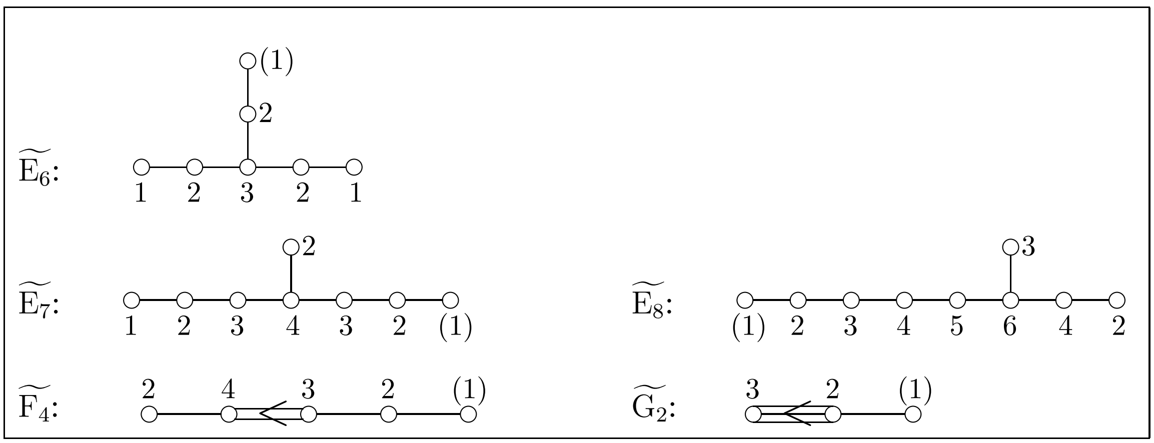


TABLE 3. Extended Dynkin diagrams with the highest root coefficients

reduces parabolics in E_8 to parabolics in E_7, in E_7 to D_6, in E_6 to A_5, in F_4 to C_3, and, finally, in G_2 to A_1. Therefore we will assume that $\widetilde{\alpha} \in \pi$. Outside the E_6 type, the additivity property holds and therefore we will consider only connected subsets π.

Unfortunately, some explicit computations are needed, especially in type E_6, where additivity does not work. Here reductions are done by cutting ideals in the nilpotent radical of $\mathfrak{p}$. The Lie algebra structure of $\mathfrak{p}$ can be read from the root system of $\mathfrak{g}$. For example, $\mathfrak{p}$ always has an m-grading where $\mathfrak{p}(0)$ is the Levi part, lowest weight vectors of $\mathfrak{p}(1)$ (w.r.t. $P(0)$) correspond to simple roots in $\Pi \setminus \pi$, and $\mathfrak{p}(m{-}1)$ is an irreducible $P(0)$-module with the highest weight vector being the highest root vector of $\mathfrak{g}$. If $\pi = \Pi \setminus \alpha_i$, then m is the coefficient of α_i in the decomposition of the highest root (these coefficients can be found in Table 3).

Explanations concerning tables: we let ϖ_r denote the fundamental weights and the $R(\varpi_r)$ corresponding irreducible representations; embedding $M_*(\mathfrak{p}) \subset \mathfrak{p}$ is described in terms of the restriction to $M_*(\mathfrak{p})$ of the defining representation of the Levi (usually $R(\varpi_1)$); id stands for the 1-dimensional trivial representation. If $\mathfrak{g}$ is of type E_6, then MRS of $\mathfrak{p}$ is not always semisimple. We give the index

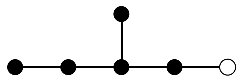
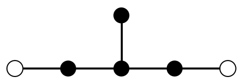
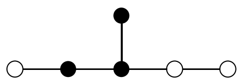
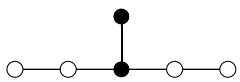
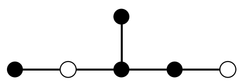
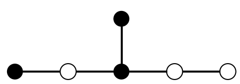
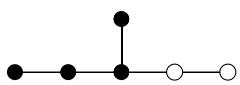
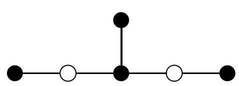
π		$\operatorname{ind}\mathfrak{p}_\pi^+$	$M_*(\mathfrak{p}_\pi^+)$	Embedding
(1)		2	G_2	$R(\varpi_1)+3\mathrm{id}$
(2)		4	$\mathrm{B}_3\oplus\mathbb{C}$	$R(\varpi_3);\check{h}_1-\check{h}_5$
(3)		2	$\mathrm{A}_1\oplus\mathbb{C}$	$R(\varpi_1)+2\mathrm{id};\check{h}_1-\check{h}_5$
(4)		3	$\mathrm{A}_1\oplus 2\mathbb{C}$	$R(\varpi_1)+\mathrm{id};\check{h}_1-\check{h}_5,h_2-h_4$
(5)		1	A_1	$2\mathrm{id}\otimes(R(\varpi_1)+2\mathrm{id})$
(6)		2	$\mathrm{A}_1\oplus\mathbb{C}$	$2\mathrm{id}\otimes(R(\varpi_1)+\mathrm{id});h_2-h_4$
(7)		0	0	trivial
(8)		3	$2\mathrm{A}_1\oplus\mathbb{C}$	$R(\varpi_1)\otimes(R(\varpi_1')+\mathrm{id})\otimes R(\varpi_1);\check{h}_2-\check{h}_4$

TABLE 4. $M_*(\mathfrak{p}_\pi^+)$ in E_6

of $\mathfrak{p}$ and indicate generators of the centre using the Chevalier basis (e_i,h_i,f_i) and elements $\check{h}_i$ of the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ such that $\alpha_i(\check{h}_j)=\delta_{i,j}$. A method for computing the index of a parabolic is given in [J07].

6.1. E_6. According to [BM, Subsection 8.2], $\mathfrak{p}$ is not quasi-reductive if either $\{\alpha_6\}$ is a connected component of π or $\pi=\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$ up to the diagram automorphism.

Now we describe (up to the diagram automorphism) all subsets π leading to quasi-reductive parabolic subalgebras, separating them by cardinality. By our assumptions, $\alpha_6\in\pi$. Since α_6 cannot be a connected component of π, it must contain α_3. Therefore for $|\pi|=2$, there is just one possibility, $\pi=\{\alpha_3,\alpha_6\}$. This is item (4) in Table 4.

Let $|\pi|=3$. To $\{\alpha_3,\alpha_6\}$ we can add any other root. This leads to items (3) and (6). If $|\pi|=4$, then again nothing is forbidden and we get items (7), (2), (8), and (5). The last case is $|\pi|=5$. If $\alpha_2\notin\pi$, the parabolic is not quasi-reductive. Thus the only possibility is item (1).

All cases are treated separately.

Case (1): We have $P=(\mathbb{C}^\times\times Spin_{10})\ltimes\mathbb{C}^{16}$. This parabolic is 2-graded. The reductive part, $P(0)$, acts on the dual space of the Abelian nilpotent radical, $\mathfrak{p}(1)=\mathbb{C}^{16}$, with an open orbit and the corresponding stabiliser is $P''=(\mathbb{C}^\times\times Spin_7)\ltimes\mathbb{C}^8$, see e.g. [El72]. Due to Lemma 4.3, we are reduced to computing $M_*(\mathfrak{p}'')$. Here again $\mathbb{C}^\times\times Spin_7$ acts on $(\mathbb{C}^8)^*$ with an open orbit and the corresponding stabiliser is of type G_2.

The next three parabolics are treated in a unified way. More precisely, parabolics (3) and (4) are contained in the second one. We start with the largest and then apply some reductions. In case (2), the Levi part of P is $L = P(0) = \mathbb{C}^\times\times\mathbb{C}^\times\times Spin_8$, the nilpotent radical of $\mathfrak{p}$ is two-step nilpotent and can be decomposed as $\mathfrak{p}(1)\oplus\mathfrak{p}(2)$, where L acts on $\mathfrak{p}(1)$ via $R(\varpi_1) + R(\varpi_4)$ and on $\mathfrak{p}(2)$ via $R(\varpi_3)$. All representations of the group $Spin_8$ are self dual. The group $P(0)$ and its Borel subgroup have open orbits in $\mathfrak{p}(2)^*$. When restricted to a generic stabiliser $L_*(\mathfrak{p}(2)) = \mathbb{C}^\times\times Spin_7$, both representations $R(\varpi_1), R(\varpi_4)$ stay irreducible. Hence $\hat{\alpha}$ is non-degenerate on $\mathfrak{p}(1)$ for generic $\alpha \in \mathfrak{p}(2)^*$. By virtue of Lemma 4.6, MRS of $\mathfrak{p}$ is equal to $\mathbb{C}^\times\times Spin_7$. Central part of $M_*(\mathfrak{p})$ coincides with the kernel of the highest root of E_6 in the centre of $\mathfrak{p}(0)$.

The parabolics $\mathfrak{p}(\pi_3)$, $\mathfrak{p}(\pi_4)$ in lines (3), (4), respectively, both are semi-direct sums of the parabolics $\mathfrak{l}\cap\mathfrak{p}(\pi_i)$ and the nilpotent radical $\mathfrak{p}(1)\oplus\mathfrak{p}(2)$ of $\mathfrak{p}(\pi_2)$. Therefore, for them MRS is equal to the maximal reductive stabiliser of the intersection $P(\pi_i) \cap L_*(\mathfrak{p}(2))$, where $L_*(\mathfrak{p}(2))$ is chosen to be transversal to $P(\pi_i)$.

Case (3): Here the Levi of P is $\mathbb{C}^\times\times\mathbb{C}^\times\times GL_4$ and $Q := P\cap L_*(\mathfrak{p}(2))$ equals $(\mathbb{C}^\times\times GL_3)\ltimes\exp(\mathbb{C}^3\oplus\Lambda^2\mathbb{C}^3)$, where the first $\mathbb{C}^\times$ lies in the centre of Q. We can disregard the semisimple central elements and assume that the reductive part of Q is just GL_3. One readily sees that $\mathfrak{q} = \mathrm{Lie}\,Q$ is 3-graded with $\mathfrak{q}(2) = \Lambda^2\mathbb{C}^3$, $\mathfrak{q}(1) = \mathbb{C}^3$.

Take a non-zero $\alpha \in \mathfrak{q}(2)^*$, considered just as a linear function on $\mathfrak{q}(2)$, not on $\mathfrak{q}$. Then $\mathfrak{q}(\alpha) = \mathfrak{q}_{\mathbb{C}\alpha}/\ker\alpha$ is a Lie algebra $((\mathfrak{gl}_2\oplus\mathbb{C})\ltimes\mathbb{C}^2)\ltimes(\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C})$. We apply Lemma 4.2. Let V_0 be the first $\mathbb{C}^2$ (a subset of $\mathfrak{q}(0)$); $V_1 = \mathbb{C}$ a subset of $\mathfrak{q}(1)$, $V_1' = \mathbb{C}^2$ the second part of $\mathfrak{q}(1)$, and, finally, $V_2 = \mathfrak{q}(2)/\ker\alpha$. Then non-zero commutators in the nilpotent radical of $\mathfrak{q}(\alpha)$ are $[V_0, V_1'] = V_1$ and $[V_1', V_1'] = V_2$. The centre of $\mathfrak{gl}_2$ acts on V_1 by a non-trivial character. Thus $\mathfrak{q}(\alpha) = (\mathfrak{gl}_2\oplus\mathbb{C}) \oplus (V_0\oplus V_1') \oplus (V_1\oplus V_2)$ is 3-graded and conditions of Lemma 4.6 are satisfied. Hence $M_*(\mathfrak{q}(\alpha)) = \mathfrak{sl}_2 = M_*(\mathfrak{q})$. In order to get $M_*(\mathfrak{p})$ we have to add the same central element as in case (2).

Case (4): The Levi subgroup of P is $(\mathbb{C}^\times)^3\times GL_3$. Hence for the description of $P \cap L_*(\mathfrak{p}(2))$ we can use Lemma 4.9 with $k = 2$. This gives a reduction to a parabolic subalgebra $\mathfrak{p}_1 \subset \mathfrak{so}_6$, which in this situation is equal to $(\mathbb{C}\oplus\mathfrak{gl}_2) \ltimes (2\mathbb{C}^2\oplus\Lambda^2\mathbb{C}^2)$. For $\mathfrak{p}_1$ the maximal reductive stabiliser is $\mathbb{C}^\times\times SL_2$. Remembering the central element in $L_*(\mathfrak{p}(2))$ we get that $M_*(\mathfrak{p})$ is of type $A_1 \oplus 2\mathbb{C}$.

Parabolics $\mathfrak{p}(\pi_5)$ and $\mathfrak{p}(\pi_6)$ are contained in $\mathfrak{p}(\pi_1)$ and the Abelian nilpotent radical of $\mathfrak{p}(\pi_1)$ is also an ideal of both of them. By Lemma 4.3, $M_*(\mathfrak{p})$ in both this cases is equal to $M_*(\mathfrak{p}\cap\mathfrak{p}'')$ with $\mathfrak{p}'' \subset \mathfrak{p}(\pi_1)$ transversal to $\mathfrak{p}$. Recall that $\mathfrak{p}'' = (\mathbb{C}\oplus\mathfrak{so}_7) \ltimes \mathbb{C}^8$. More precisely, $\mathfrak{so}_7$ is embedded (via the spin-representation) into a regular subalgebra $\mathfrak{so}_8 \subset \mathfrak{so}_{10} \subset \mathfrak{p}(\pi_1)$ and $\mathbb{C}^8$ is an $\mathfrak{so}_8$-invariant subspace in the complement of $\mathfrak{so}_8$ in $\mathfrak{so}_{10}$.

Case (5): First we compute the intersection $\mathfrak{q} := \mathfrak{p}(\pi_5)\cap\mathfrak{p}''$. Outside of $\mathfrak{so}_7$ it equals $\mathbb{C}\oplus\mathbb{C}$ with the first $\mathbb{C}$ consisting of semisimple elements and the second of nilpotent. To compute the intersection inside $\mathfrak{so}_7$ we pass to a different realisation of the embedding $Spin_7 \subset SO_8$, namely to $SO_7 \subset SO_8$. Then the second object, $\mathfrak{so}_8 \cap \mathfrak{p}(\pi_5)$, is a parabolic $\mathfrak{gl}_4 \ltimes \Lambda^2\mathbb{C}^4$. Thus $\mathfrak{q}$ is a direct sum of two subalgebras, the first of which is of index zero and the second, $\mathfrak{gl}_3 \ltimes (\mathbb{C}^3 \oplus \Lambda^2\mathbb{C}^3)$, is the same as the one we came across in case (3). Hence, the maximal reductive stabiliser is $\mathfrak{sl}_2$.

Case (6): We proceed as in case (5) and compute $\mathfrak{p}(\pi_6) \cap \mathfrak{p}''$. The only difference here is that the parabolic $\mathfrak{p}(\pi_6)$ is slightly smaller than $\mathfrak{p}(\pi_5)$. Namely, the $\mathfrak{gl}_4$ is replaced by its maximal parabolic subalgebra with the reductive part $\mathfrak{gl}_3 \oplus \mathbb{C}$. This means that $\mathfrak{p}(\pi_6) \cap \mathfrak{p}''$ is a direct sum of an index zero subalgebra and a Lie algebra $\mathfrak{p}_1$ arising in case (4). Therefore $M_*(\mathfrak{p}(\pi_6))$ is of type $A_1 \oplus \mathbb{C}$ and is a subalgebra of $M_*(\mathfrak{p}(\pi_4))$.

Case (7): The parabolic is of index zero. Hence $M_*(\mathfrak{p}) = 0$ and there is nothing to describe.

Case (8): In the last line $\pi = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$ and it is a union of two disconected subsets $\{\alpha_1, \alpha_5\}$ and $\{\alpha_3, \alpha_6\}$. Let $\mathfrak{p}_1 \subset \mathfrak{g}$ be the standard parabolic subalgebra corresponding to the first subset, and $\mathfrak{p}_2$ to the second. Since $\mathrm{rk}\,\mathfrak{g} \neq \mathrm{k}_{\mathfrak{g}}$, we cannot apply Theorem 3.6. However, we intend to show that the additivity property works for the derived algebras of $M_*(\mathfrak{p}_1)$ and of $M_*(\mathfrak{p}_2)$.

By the proofs of [BM, Corollary 2.10, Theorem 3.6, and Lemma 5.5], there is $(a', a'', a) \in (\mathbb{C}^\times)^3$ such that φ_{v_1} (resp. φ_{v_2}, φ_v) is regular and has reductive type for $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2, \mathfrak{p}$) with $v_1 := a'x_{\alpha_1} + a''x_{\alpha_5} + u_\varnothing^-$, $v_2 := ax_{\alpha_3+\alpha_6} + u_\varnothing^-$ and $v := v_1 + v_2 - u_\varnothing^-$. Let $\mathfrak{t}$ (resp. $\mathfrak{r}$) be the stabiliser of φ_v (resp. $\varphi_{v_{\mathfrak{t}^\perp}}$) in $\mathfrak{p}$ and $\mathfrak{t}_i$ (resp. $\mathfrak{r}_i$) the stabilisers of φ_{v_i} (resp. $\varphi_{v_{i,\mathfrak{t}_i^\perp}}$) in $\mathfrak{p}_i$, for $i = 1, 2$. Let $\mathfrak{g}$ stand for the derived algebra of the Levi $\mathfrak{l}$ of $\mathfrak{p}$ and $\mathfrak{m}$ for the nilpotent radical of $\mathfrak{p}$. The same applies to $\mathfrak{p}_i$ with adding an index i. Let also $\mathfrak{s}_i$ denote the image of $\mathfrak{t}_i$ under the projection map from $\mathfrak{p}_i$ to $\mathfrak{g}_i \oplus \mathfrak{m}_i$, and by $\mathfrak{k}_i$ the intersection of $\mathfrak{t}_i$ with the centre of $\mathfrak{l}_i$.

Let $i = 1, 2$. We have, $\mathfrak{t}_i = \mathfrak{k}_i \oplus \mathfrak{s}_i$ and $\mathfrak{t} = \mathfrak{k}_1 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2$. Note that $\dim \mathfrak{k}_1 = 1$, $\dim \mathfrak{k}_2 = 2$ and $\dim \mathfrak{s}_i = 1$ for $i = 1, 2$. Moreover, $\mathfrak{s}_i$ is contained in $\mathfrak{g}_i \oplus \mathfrak{m}$. All this comes from [BM] (mostly from Lemma 2.9).

By the $\mathfrak{sl}_6$ case and the highest root reduction (Thm 3.4, Prop. 4.7), one knows that $\mathfrak{r}_1$ has type $\mathrm{A}_1 \oplus \mathbb{C}$ and, by the case (4) of E_6, $\mathfrak{r}_2$ has type $\mathrm{A}_1 \oplus 2\mathbb{C}$. The centre of $\mathfrak{r}_i$ is $\mathfrak{k}_i$ and we can write $\mathfrak{r}_i = \mathfrak{k}_i \oplus \mathfrak{a}_i$, where $\mathfrak{a}_i$ is a reductive Lie subalgebra complementary to $\mathfrak{k}_i$ in $\mathfrak{r}_i$ with $\mathfrak{s}_i$ as a maximal torus. Further, $\mathfrak{r} = \mathfrak{k}_1 \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a reductive Lie subalgebra complementary to $\mathfrak{k}_1$ in $\mathfrak{r}$ with $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ as a maximal torus. We have $\mathfrak{a}_i = \mathfrak{s}_i \oplus [\mathfrak{s}_i, \mathfrak{a}_i]$ and $\mathfrak{a}_i \subseteq \mathfrak{g}_i \oplus \mathfrak{m}$. Also, $\mathfrak{a} = \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{a}]$ and $\mathfrak{a} \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{m}$. Our goal is to show that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$.

Prove first the inclusion $\mathfrak{a}_i \subseteq \mathfrak{a}$. We argue as in the proof of Theorem 3.6 (inclusion $\mathfrak{r}_i \subseteq \mathfrak{r}$). The same line of arguments gives $\left\langle v_{i,\mathfrak{t}_i^\perp}, [\mathfrak{a}_i, \mathfrak{p}_i] \right\rangle = 0$ since $\mathfrak{a}_i$ is contained in $\mathfrak{g}_i \oplus \mathfrak{m}$. Next, setting $w_1 = a'x_{\alpha_1} + a''x_{\alpha_5}$ and $w_2 = ax_{\alpha_3+\alpha_6}$, we also show that $\left\langle v_{\mathfrak{t}^\perp} - v_{i,\mathfrak{t}_i^\perp}, [\mathfrak{a}_i, \mathfrak{p}_i] \right\rangle = \{0\}$. Indeed, it suffices to observe that $w_i \in \mathfrak{t}_j^\perp$ for $j \neq i$. Therefore, we get $\mathfrak{a}_i \subset \mathfrak{r}$; so $\mathfrak{a}_i \subset \mathfrak{a}$.

Prove now the inclusion $\mathfrak{a} \subseteq \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Again, we argue as in the proof of Theorem 3.6 (inclusion $\mathfrak{r} \subseteq \mathfrak{r}_1 \oplus \mathfrak{r}_2$). Here, Step 1 gives us $\mathfrak{r} \cap (\mathfrak{g}_i \oplus \mathfrak{m}) \subseteq \mathfrak{r}_i$; so $\mathfrak{r} \cap (\mathfrak{g}_i \oplus \mathfrak{m}) \subseteq \mathfrak{a}_i$. We resume now Step 2 with $\mathfrak{a}$, $\mathfrak{a}_1$, and $\mathfrak{a}_2$ instead of $\mathfrak{r}$, $\mathfrak{r}_1$, and $\mathfrak{r}_2$, respectively. We obtain here that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$.

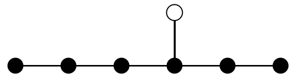
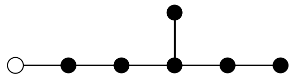
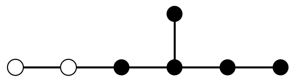
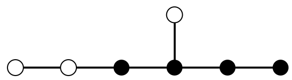
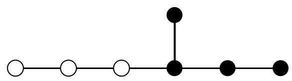
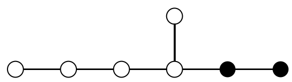
π		$\operatorname{ind}\mathfrak{p}_\pi^+$	$M_*(\mathfrak{p}_\pi^+)$	Embedding
(1)		3	C_3	$R(\varpi_1)+\mathrm{id}$
(2)		4	F_4	$R(\varpi_1)+\mathrm{id}$
(3)		4	B_4	$R(\varpi_1)+\mathrm{id}$
(4)		2	C_2	$R(\varpi_1)+\mathrm{id}$
(5)		2	C_2	$R(\varpi_1)+\mathrm{id}$
(6)		1	A_1	$R(\varpi_1)+\mathrm{id}$

TABLE 5. $M_*(\mathfrak{p}_\pi^+)$ in E_7

Since $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have both type A_1, we deduce that $\mathfrak{a}$ has type $2\mathrm{A}_1$. By the index considerations, $M_*(\mathfrak{p}) = \mathfrak{sl}_2{\oplus}\mathfrak{sl}_2{\oplus}\mathbb{C}$. We also obtain that the first A_1 is a diagonal in $\mathfrak{sl}_2{\oplus}\mathfrak{sl}_2 \subset \mathfrak{sl}_5 \subset \mathrm{E}_6$ and the second is embedded into A_2-subalgebra corresponding to $\{\alpha_3, \alpha_6\}$ in the same way as in case (4).

Now we briefly described an alternative approach to the same parabolic $\mathfrak{p}$, which makes use of the 5-grading:

$$\mathfrak{p} = \mathfrak{gl}_2{\oplus}\mathfrak{gl}_2{\oplus}\mathfrak{sl}_3 \ltimes \left((\mathbb{C}^2_{\mathrm{I}}{\otimes}\mathbb{C}^3{\oplus}\mathbb{C}^3{\otimes}\mathbb{C}^2_{\mathrm{II}}) \oplus \mathbb{C}_{\mathrm{I}}{\otimes}\mathbb{C}^2_{\mathrm{II}}{\otimes}(\mathbb{C}^3)^* \oplus (\mathbb{C}^2_{\mathrm{I}}{\oplus}\mathbb{C}^2_{\mathrm{II}}) \oplus \mathbb{C}^3\right),$$

where subscripts I, II indicate non-trivial actions if the first and the second $\mathfrak{sl}_2$-factors. Note also that $[\mathfrak{p}(i), \mathfrak{p}(j)] = \mathfrak{p}(i{+}j)$ whenever $i + j \leqslant 4$ and the kernel of each commutator is trivial. Lemma 4.2 applied to $\mathfrak{a} = \mathfrak{p}(3){\oplus}\mathfrak{p}(4)$ and $\alpha \in \mathfrak{a}^*$, which is zero on $\mathfrak{p}(3)$ and non-zero on $\mathfrak{p}(4)$, reduces calculation of $M_*(\mathfrak{p})$ to

$$\mathfrak{p}(\alpha) = \mathfrak{gl}_2{\oplus}\mathfrak{gl}_2{\oplus}(\mathfrak{gl}_2{\oplus}\mathbb{C}^2) \ltimes V_1 \oplus V_2 \oplus \mathbb{C},$$

where $[V_1, V_1] \subset V_2$, $[V_1, V_2] = 0$, $[V_2, V_2] = \mathbb{C}$, and $V_1 \cong \mathbb{C}^2_{\mathrm{I}} \otimes \mathbb{C}^2{\oplus}\mathbb{C}^2{\otimes}\mathbb{C}^2_{\mathrm{II}}$, $V_2 \cong \mathbb{C}^2_{\mathrm{I}}{\otimes}\mathbb{C}^2_{\mathrm{II}} \oplus \mathbb{C}^2_{\mathrm{I}}{\otimes}\mathbb{C}^2_{\mathrm{II}}{\otimes}\mathbb{C}^2$. Note that the nilpotent part of $\mathfrak{p}(0)_\alpha$ acts on V_2 non-trivially. The central element h of $M_*(\mathfrak{p})$ is defined by the conditions that it commutes with the semisimple part of $M_*(\mathfrak{p})$, stabilises α, and also that $\varpi_2(h) = -\varpi_4(h)$. Together they amount to $h = \overset{\vee}{h}_2 - \overset{\vee}{h}_4$, up to a scalar.

6.2. E_7. Recall that if π has as a connected component one of the following sets

$$\{\alpha_6\}, \{\alpha_6, \alpha_5, \alpha_4\}, \{\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\},$$

then $\mathfrak{p}$ is not quasi-reductive, cf [BM, Table 6]. Quasi-reductive parabolics corresponding to connected subsets of roots π are listed in Table 5 and for each of them we find a suitable reduction.

The four last items are dealt with in a unified way. Take first a maximal parabolic subgroup P with the Levi $\mathbb{C}^\times \times Spin_{10} \times SL_2$, so all simple roots except α_2 are contained in it. Then $\mathfrak{p}$ is 3-graded with $\mathfrak{p}(1) = \mathbb{C}^{16} \otimes \mathbb{C}^2$ and $\mathfrak{p}(2) = \mathbb{C}^{10}$. There is an open $P(0)$-orbit in $\mathfrak{p}(2)^*$ with the stabiliser $Spin_9 \times SL_2$, $p(1)$ remains irreducible under restriction to this stabiliser, hence the skew-symmetric form $\hat{\alpha}$ is non-degenerate for generic $\alpha \in \mathfrak{p}(2)^*$ and Lemma 4.6 yields that MRS equals $Spin_9 \times SL_2$.

If now $P(\pi_i)$ is one of the parabolics (3)–(6) in the table, then $P(\pi_i) \subset P$ and $M_*(\mathfrak{p}(\pi_i))$ is equal to a maximal reductive stabiliser of the intersection $\mathfrak{p}(\pi_i) \cap \mathfrak{so}_9$ for the complementary to $\mathfrak{p}(\pi_i)$ subalgebra $\mathfrak{so}_9$. This readily shows that $M_*(\mathfrak{p}(\pi_3)) = \mathfrak{so}_9$. In the remaining three cases the Levi part of $P(\pi_i)$ is either GL_3 or GL_5. Thus Lemma 4.9 provides a reduction to a parabolic $\mathfrak{p}_1 \subset \mathfrak{so}_8$. Here the answer follows from Theorem 5.10.

Items (1) and (2) are treated separately in a different way.

Case (1): Here $\mathfrak{p}$ is 3-graded:

$$\mathfrak{p} = \mathfrak{gl}_7 \oplus \Lambda^4\mathbb{C}^7 \oplus \mathbb{C}^7.$$

Unfortunately, a skew-symmetric form $\hat{\alpha}$ is degenerate on $\mathfrak{p}(1)$ for all $\alpha \in \mathfrak{p}(2)^*$ by a simple reason that $\mathfrak{p}(1)$ is odd dimensional. We apply Lemma 4.2 to $\mathfrak{p}(2)$ and a non-zero $\alpha \in \mathfrak{p}(2)^*$. As a result we get

$$\mathfrak{q} = \mathfrak{p}(\alpha) = (\mathfrak{gl}_6 \oplus \mathbb{C} \oplus \mathbb{C}^6) \oplus (\Lambda^3\mathbb{C}^6 \oplus \Lambda^4\mathbb{C}^6) \oplus \mathbb{C},$$

where $\Lambda^4\mathbb{C}^6$ is the kernel of $\hat{\alpha}$ and the first $\mathbb{C}$ consists of semisimple elements. This is again a 3-grading with $\mathfrak{q}(0) = \mathfrak{gl}_6 \oplus \mathbb{C}$, $\mathfrak{q}(1) = \mathbb{C}^6 \oplus \Lambda^3\mathbb{C}^6$, and $\mathfrak{q}(2) = \Lambda^4\mathbb{C}^6 \oplus \mathbb{C}$. The commutators are $[\Lambda^3\mathbb{C}^6, \Lambda^3\mathbb{C}^6] = \mathbb{C}$ (the form $\hat{\alpha}$ is non-degenerate on $\Lambda^3\mathbb{C}^6$); and $[\mathbb{C}^6, \Lambda^3\mathbb{C}^6] = \Lambda^4\mathbb{C}^6$. Let us check that for generic $\beta \in \mathfrak{q}(2)^*$ the skew-symmetric form $\hat{\beta}$ is non-degenerate on $\mathfrak{q}(1)$. Take $\beta = \gamma + \alpha$ with generic $\gamma \in (\Lambda^4\mathbb{C}^6)^*$. The stabiliser of β in $\mathfrak{q}(0)$ is $\mathfrak{sp}_6$. As a representation of Sp_6, the subspace $\mathfrak{q}(1)$ decomposes as $\mathbb{C}^6 \oplus V(\varpi_1) \oplus V(\varpi_3)$, where $V(\varpi_j)$ is a space of the irreducible representation with the highest weight ϖ_j. Note that $\hat{\alpha}$ is non-degenerate on $V(\varpi_3)$ and $\hat{\gamma}$ defines a non-degenerate pairing between $\mathbb{C}^6$ and $V(\varpi_1)$. Since $\hat{\gamma}(\mathbb{C}^6, V(\varpi_3)) = 0$, we conclude that $\hat{\beta}$ is indeed non-degenerate on $\mathfrak{q}(1)$ and the maximal reductive stabiliser is Sp_6 by Lemma 4.6.

Case (2): Here $\mathfrak{p}$ is 2-graded with $\mathfrak{p}(0) = \mathrm{E}_6 \oplus \mathbb{C}$ being reductive and $\mathfrak{p}(1) = V(\varpi_1) = \mathbb{C}^{27}$ an Abelian ideal. The reductive past acts on $\mathfrak{p}(1)$ with an open orbit and a generic stabiliser F_4. This is a maximal reductive stabiliser.

6.3. $\mathbf{E}_8$. Recall that if π has as a connected component one of the following sets

$$\{\alpha_1\}, \{\alpha_3, \alpha_2, \alpha_1\}, \{\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\},$$
$$\{\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_8\},$$

then $\mathfrak{p}$ is not quasi-reductive, cf [BM, Table 6]. We list all connected subsets π corresponding to quasi-reductive parabolics in Table 6. By the assumptions, $\alpha_1 \in \pi$. Since π is connected, either $\alpha_7 \in \pi$, then π is the A_7-line and $\mathfrak{p}(\pi)$ is not quasi-reductive, or $\alpha_7 \notin \pi$. As one readily sees, no subset in Table 6 contains α_7.

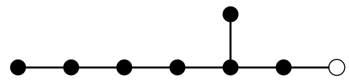
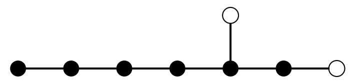
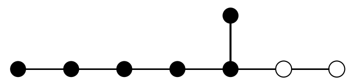
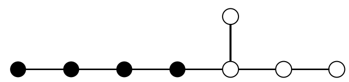
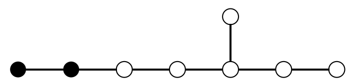
π		$\operatorname{ind} \mathfrak{p}_\pi^+$	$M_*(\mathfrak{p}_\pi^+)$	Embedding
(1)		6	B_6	$R(\varpi_1) + \mathrm{id}$
(2)		3	C_3	$R(\varpi_1) + \mathrm{id}$
(3)		3	C_3	$R(\varpi_1) + \mathrm{id}$
(4)		2	C_2	$R(\varpi_1) + \mathrm{id}$
(5)		1	A_1	$R(\varpi_1) + \mathrm{id}$

TABLE 6. $M_*(\mathfrak{p}_\pi^+)$ in E_8

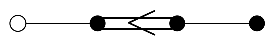
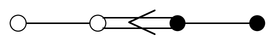
π		$\operatorname{ind} \mathfrak{p}_\pi^+$	$M_*(\mathfrak{p}_\pi^+)$	Embedding
(1)		3	$D_3(\simeq A_3)$	$R(\varpi_1) + \mathrm{id}$
(2)		1	A_1	$R(\varpi_1) + \mathrm{id}$

TABLE 7. $M_*(\mathfrak{p}_\pi^+)$ in F_4

The item in the line (1) was considered in section 4.3 and $M_*(\mathfrak{p}(\pi_1)) = \mathfrak{so}_{12}$. For all other lines Corollary 4.11 provides a reduction to a parabolic in $\mathfrak{so}_{12}$.

6.4. $\mathbf{F}_4$. Recall that if one of the connected components of π is $\alpha_4 = \alpha_\iota$, then $\mathfrak{p}(\pi)$ is not quasi-reductive, cf [BM, Table 6]. By our assumption, $\alpha_4 \in \pi$. Since it cannot be a connected component, $\alpha_3 \in \pi$. Because also π is assumed to be connected, we need to consider only two parabolics.

The item in the first line was treated in section 4.3 and here $M_*(\mathfrak{p}) = \mathfrak{so}_6$. For line (2) Corollary 4.11 provides a reduction to a parabolic $\mathfrak{p}_0$ in $\mathfrak{so}_5$.

6.5. $\mathbf{G}_2$. The only parabolic that could be of interest corresponds to $\pi = \{\alpha_2\}$, but it is not quasi-reductive.

7. CONCLUSION

As was already noticed, if $\mathfrak{q} \subset \mathfrak{gl}_n$ is a seaweed, then MRS is connected. In general, MRS are not always connected. Take for example the parabolic in line (1) of Table 7. A maximal reductive stabiliser is equal to $\mathbb{Z}/2\mathbb{Z} \times SL_4$ and the component group acts on $M_*(\mathfrak{p})$ non-trivially.

In this paper we have studied maximal reductive stabilisers on the Lie algebra level. Given any particular quasi-reductive parabolic subalgebra it is not difficult to compute the component group of its MRS, for instance, along the lines indicated in the Introduction. Also our explicit calculations presented in Section 6 allow to get MRS on the group level for all considered Lie algebras. Nevertheless, an intriguing question remains, does the "additivity" property hold on the group level? We believe it does and would like to find a general proof, not a case-by-case verification.

Contemplating results of Section 5, one may notice that "additivity" works for all parabolics in type C. If $\mathfrak{p}(\pi) \subset \mathfrak{sl}_n$ is a parabolic and $\pi = \pi_1 \cup \pi_2$, where π_1, π_2 are not connected to each other and both are invariant under the diagram automorphism, then "additivity" holds for the derived algebras of $M_*(\mathfrak{p})$, $M_*(\mathfrak{p}(\pi_1)), M_*(\mathfrak{p}(\pi_2))$. A more careful analysis shows that it holds in type B and in type D fails only for D_{2m+1} and either $r = 2m{+}1$, a_t even (case (5) of Theorem 5.10) or $r = n-4$, a_t odd. As is well known, a Weyl involution is inner in type D_{2m} and outer in type D_{2m+1}. In the exceptional case, the "additivity" does not work only in E_6 and here holds in the same sense as for type A. All these observations indicate a relation with the existence of outer automorphisms or, more precisely, with the condition that a Weyl involution is inner. It would be nice to clarify this dependence.

Two further questions in the same spirit were suggested to us by Michel Duflo. What is the dimension of the centre of a maximal reductive stabilser and what is the number of its simple factors? Analysing Section 6 and Theorem 5.10, one might conclude that these questions are also related to the existence of outer automorphisms.

One of the most interesting direction for further investigation is: what happens for real Lie algebras? What kind of unitary representations come out as results of Duflo's construction for $\gamma \in \mathfrak{q}^*_{\rm red}$ of nilpotent type or other linear forms of reductive type?

Stabilisers $\mathfrak{q}_\gamma$ play an important rôle in representation theory, in all characteristics. They are of particular interest if γ is of nilpotent type, i.e., vanishes on $\mathfrak{q}_\gamma$ (for relevant results in finite characteristic see e.g. [Pr98], [PS99]). Studying the nilpotent cone in $\mathfrak{q}^*$ and linear forms of nilpotent type on $\mathfrak{q}$ for strongly quasi-reductive Lie algebras looks like a very promising project.

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ANNE MOREAU, LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UNIVERSITÉ DE POITIERS, FRANCE
E-mail address: anne.moreau@math.univ-poitiers.fr

OKSANA YAKIMOVA, EMMY-NOETHER-ZENTRUM, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN-NÜRNBERG, GERMANY
E-mail address: yakimova@mccme.ru