
LECTURES SERIES ON MOTIVIC INTEGRATION

—
APPLICATIONS TO JET SCHEMES OF THE
NILPOTENT CONE OF A REDUCTIVE LIE ALGEBRA

by

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Abstract. — This note corresponds to my lecture series at ETH Zürich on spring 2008 about the following result of Mustață [27]

“The m -order jet scheme of a locally complete intersection X is irreducible if and only if X has rational singularities.”

The main technic used to prove this result is motivic integration, as developed by Kontsevich, Denef and Loeser and Batyrev. The course starts with an introduction to Kontsevich’s theory of motivic integration. Then, we give a description of the proof of Mustață’s result. At last, we present applications of motivic integration theory and Mustață’s result to Lie theory. More precisely, we will detail the following two applications:

* *Jet schemes of the nilpotent cone of a reductive Lie algebra*, following David Eisenbud and Edward Frenkel [27](Appendix).

* *Nilpotent bicone of a reductive Lie algebra*, joint with Jean-Yves Charbonnel [7].

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INTRODUCTION

Motivic integration was introduced by M. Kontsevich in a lecture at Orsay in 1995 [17]. It is an integration theory on the *arc space* $J_\infty(X)$ of a smooth complex variety X . The points of this space corresponds to “formal arcs” on X and can be thought as an infinitesimal curve centered at a point of X . Likewise, the points of the *m-order jet scheme* $J_m(X)$ of X can be viewed as a *m-order infinitesimal curve* centered at a point of X .

The starting point of these lecture series is the following result of Mircea Mustața [27], conjectured by David Eisenbud and Edward Frenkel, who intended to apply it to the case where X is the nilpotent cone of a reductive Lie algebra:

Theorem 1 (Mustața, 2001). — *Let X be a complex algebraic variety. If X is locally a complete intersection variety, then $J_m(X)$ is irreducible for any $m \geq 1$ if and only if X has rational singularities, where $J_m(X)$ is the m -order jet scheme of X .*

Close relationships between the geometry of arc spaces and the singularities of X were conjectured by Nash in 1995. Thus, Mustața’s result is a nice illustration of Nash’s predictions. The main technic used in the proof of Theorem 1 is motivic integration as developed by Kontsevich, Denef and Loeser, and Batyrev. The basic idea is that the *Hodge realization* of the *motivic measure* gives information we need about the jet schemes $J_m(X)$. In the appendix of [27], David Eisenbud and Edward Frenkel apply Theorem 1

to the nilpotent cone of a reductive Lie algebra to extend Kostant's results in the setting of jet schemes. In addition, Jean-Yves Charbonnel and myself have recently studied the *nilpotent bicone* using Mustață's theorem and the main ideas of its proof. For all of these various reasons, the presentation of the Mustață's result is a great opportunity to get onto various aspects during these courses:

- * Part 1 is an introduction to the motivic integration theory.

- * Part 2 concerns Theorem 1 and its proof. We tempt to explain how motivic integration appears in the proof.

- * In Part 3, we present various applications of Mustață's theorem to reductive Lie algebras. We start by a review of known results concerning the nilpotent cone. Then, we detail two applications. The first one concerns jet schemes of the nilpotent cone (following the appendix of [27]). The second one concerns the nilpotent bicone: in [7], we use Mustață's theorem (and technics from motivic integration as developed in its proof) to prove that the nilpotent bicone is a complete intersection. This in particular answers positively a conjecture of H. Kraft and N. Wallach [19]. The study of the nilpotent bicone was originally the reason why Michel Duflo brought [27] to our attention few years ago.

PART I MOTIVIC INTEGRATION

We present in this part a short introduction to motivic integration theory. The theory for smooth varieties is due to M. Kontsevich [17] and was generalized later by J. Denef and F. Loeser [10]. For the sake of the simplicity we deal from Subsection with smooth varieties. But most of results can be extended to singular varieties according to [10]. We refer to [10], [25], [9], [5] for more explanations about this part.

1. Introduction

1.1. Short historic. — Motivic integration was initially introduced by M. Kontsevich [17] to prove the following result, conjectured by Batyrev:

Theorem 1.1 (Kontsevich). — *Birationally equivalent Calabi-Yau varieties have the same Hodge numbers.*

Here, by *Calabi-Yau* variety, we mean a normal projective variety X of dimension n whose canonical divisor K_X is equivalent to 0. Using p -adic integration, Batyrev proved that birationally equivalent Calabi-Yau varieties have the same Betti numbers $h^i := \dim H^i(-, \mathbb{C})$ [2]. This led Kontsevich to invent motivic integration to prove Theorem 1.1. So we should not be so surprised to observe some similarities between the construction of motivic integrals and p -adic integration.

Approach of the Kontsevich's proof. — Let us give the main idea of the proof to motivate the future construction of motivic integrals. The main idea is to assign to any variety X a *volume* $[X]$ with value in a suitable ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ which encodes information we need about the varieties.

$$X \longrightarrow [X] \in \widehat{\mathcal{M}}_{\mathbb{C}} \rightsquigarrow \text{"encodes the Hodge numbers"}$$

The definition of the ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ is quite subtle and we explain it later. It is now enough to prove that birationally equivalent Calabi-Yau varieties have the same *volume*. This is achieved via a suitable *motivic integral* and using the *transformation rule* of motivic integrals. More explicitly, if $\gamma : Y \rightarrow X$ is a proper birational map, the class $[X]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$ can be expressed as a motivic integral which only depends on Y and the discrepancy divisor

$$W_{\gamma} := K_Y - \gamma^* K_X$$

of γ . To conclude the proof, let X_1 and X_2 be two birationally equivalent Calabi-Yau varieties. The birational map can be resolved by an *Hironaka hut*:

$$\begin{array}{ccc} & Y & \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & \text{-----} & X_2 \end{array}$$

By the Calabi-Yau assumption, the canonical divisor K_{X_i} is equivalent to zero, for $i = 1, 2$, and the divisors W_{γ_1} and W_{γ_2} are numerically equivalent. By the transformation rules, $[X_1]$ is an expression depending only on Y and $W_{\gamma_1} = W_{\gamma_2}$. The same goes for X_2 , whence $[X_1] = [X_2]$, as expected. \square

To use a volume associated to a variety to study its properties is the standard idea of motivic integration. Indeed, in certain "favourable" situations —

that we will precise later — the integral so obtained can be explicitly computed while we get information from the volume associated to the variety. For instance, to prove Theorem 1, we will apply this principle to the jet schemes $J_m(X)$ (see Part 2).

1.2. Setting of the theory. — We suppose that the ground field is \mathbb{C} (we can replace \mathbb{C} by any field k algebraically closed and of characteristic 0). We fix a complex algebraic variety X (= separated and reduced scheme, of finite type over \mathbb{C}) of dimension d .

Since we attend to define a notion of "integration", we need to introduce the different blocks of the theory (as in any theory of integration):

- (1) A domain of integration: *the space of formal arcs* $J_\infty(X)$ over X .
- (2) An algebra of measurable sets of $J_\infty(X)$: *algebra containing the "cylinders"*.
- (3) The value ring of the measure: $\widehat{\mathcal{M}}_{\mathbb{C}}$, a ring constructed from the Grothendieck ring of varieties.
- (4) A measure defined from the algebra (2) with values in the ring (3).
- (5) An interesting class of measurable/integrable functions: *order functions associated to a divisor*.
- (6) A change of variables formula: *the transformation rule*.

The important point is the construction of the value ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ of the measure. Contrary to classical integration theory, the values of the measure will not lie in \mathbb{R} or \mathbb{C} . Instead they lie in a huge ring, constructed from the Grothendieck ring of varieties by a process of localization and completion. This ingenious construction is the feature key of the theory. In order to explain and motivate this construction, we start with the domain of integration.

We give in the following table the corresponding different blocks of the theory in the Lebesgues integration theory and the p -adic integration theory:

	Lebesgues	p -adic	Motivic
Space	\mathbb{R}^N	$(\mathbb{Q}_p)^N$	$J_\infty(\mathbb{A}^N)$
Values of measure	$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$	$\mathbb{Z} \subseteq \mathbb{Z}[p^{-1}] =: \mathbb{Z}_p \subseteq \mathbb{Q}_p$	$K_0(\text{Var}_{\mathbb{C}}) \subseteq \mathcal{M}_{\mathbb{C}} \subseteq \widehat{\mathcal{M}}_{\mathbb{C}}$
Cubes around point a	$\{x \in \mathbb{R}, \ x - a\ \leq 1/m\}$	$\{x \in \mathbb{Z}_p, \ x - a\ \leq 1/m\}$	$\{\gamma \in J_\infty(\mathbb{A}^N), \ x - a\ \leq 1/m\}$
Volume of cube	$(2/m)^N$	q^{-mN}	$\mathbb{L}^{-(m+1)N}$
Transformation rule	\mathbb{Q} -analytic isomorphism/Jacobian	Diffeomorphism/Jacobian	Birational map/Discrepancy

In this table, the norm $\|\cdot\|$ on $J_\infty(\mathbb{A}^N)$ is defined as follows:

$$\|\gamma\| \leq \frac{1}{m} \iff \forall i, \gamma_i \in (t^m)\mathbb{C}[[t]]$$

if $\gamma = (\gamma_1, \dots, \gamma_N)$ is a N -tuple of power series in $J_\infty(\mathbb{A}^N)$. We refer to [24] for a quick review concerning p -adic integration.

2. The space of arcs $J_\infty(X)$

2.1. Definitions. — Arc spaces were first introduced by Nash [28] who conjectured interesting relationships between the geometry of the arc spaces and the singularities of X . Recent works of Mustařă support his predictions by showing that the arc spaces contain information about singularities (see Theorem 1).

Let us give some definitions. An m -order jet of X over $x \in X$ is an m -order infinitesimal curve in X , that is a morphism

$$\mathcal{O}_{X,x} \longrightarrow \mathbb{C}[t]/t^{m+1} .$$

The subset $J_m(X)$ of m -order jets has a natural structure of scheme and the \mathbb{C} -valued points of $J_m(X)$ are in natural bijection with the $\mathbb{C}[t]/t^{m+1}$ -valued point of X . In particular, there are canonical isomorphisms $J_0(X) \simeq X$, $J_1(X) \simeq TX$, where TX is the total tangent space of X .

For instance, suppose that X is contained in an affine space $X \subseteq \mathbb{A}^N$, $X = \text{Spec}(R)$, where $R = \mathbb{C}[x_1, \dots, x_N]/(f_1, \dots, f_r)$.

[In fact, by [27](Proposition 1.1), the construction of jet schemes is compatible with open immersions. As a consequence, in order to describe $J_m(X)$, we can restrict ourselves to the affine case]

In this case, X is an affine variety defined by the equations $f_i(\mathbf{x}) = 0$, $i = 1, \dots, r$, where $\mathbf{x} = (x_1, \dots, x_N)$. Then $J_m(X)$ is defined by the equations

$$f_i(\mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_m t^m) \equiv 0 \pmod{t^{m+1}} ,$$

where $\mathbf{x}_i \in \mathbb{A}^N$. Therefore, we have an immersion

$$J_m(X) \hookrightarrow \mathbb{A}^{(m+1)N}$$

and $J_m(X)$ is defined by $(m+1)r$ equations. In particular, the dimension of $J_m(X)$ is at least $(N-r)(m+1)$.

By truncation, we have canonical morphisms

$$\pi_{m,n} : J_m(X) \longrightarrow J_n(X) ,$$

for $m \geq n$. In general, the morphisms $\pi_{m,n}$ are not necessarily surjective. When X is smooth, $\pi_{m,n}$ is a locally trivial fibration with fiber $\mathbb{A}^{d(m-n)}$ (see Example 2.1 and Proposition 2.2). These morphisms induce a projective system $(J_m(X), \pi_{l,m})$. The *space of the arcs* is by definition the projective limit of the system $(J_m(X), \pi_{l,m})$:

$$J_\infty(X) := \text{proj} \lim_m J_m(X)$$

The subset $J_\infty(X)$ has a natural structure of scheme. The \mathbb{C} -valued points of $J_\infty(X)$ are in natural bijection with the $\mathbb{C}[[t]]$ -valued point of X . Thus, an

arc on X is roughly an "infinitesimal curve" on it. Again by truncation, there are canonical morphisms:

$$\pi_{\infty, m} : J_{\infty}(X) \longrightarrow J_m(X) ,$$

for $m \geq 0$.

Suppose that $X \subseteq \mathbb{A}^N$ is defined by r equations $f_i(\mathbf{x}) = 0$, for $i = 1, \dots, r$, then, $J_{\infty}(X)$ is the set of formal series $\gamma = \mathbf{x}_0 + \mathbf{x}_1 t + \mathbf{x}_2 t^2 + \dots +$, with coefficients \mathbf{x}_i in \mathbb{A}^N , satisfying

$$f_i(\mathbf{x}_0 + \mathbf{x}_1 t + \mathbf{x}_2 t^2 + \dots) = 0 ,$$

where $\mathbf{x}_i \in \mathbb{A}^N$. For any arc γ on X , i.e. an element $\gamma \in J_{\infty}(X)$, we call $\pi_{\infty, 0}(\gamma)$ the *origin* of γ .

Example 2.1. — Let $Y = \text{Spec}\mathbb{C}[x_1, \dots, x_N] = \mathbb{A}^N$ be an affine space. We can view the elements x_i as elements of $\mathbb{C}[J_n(Y)]$ and $\mathbb{C}[J_{\infty}(Y)]$. Define a function $x_i^{(\alpha)}$, $0 \leq \alpha \leq m$, on $J_m(Y)$, for $m \geq 0$ by the formula

$$x_i^{(\alpha)}(\mathbf{y}(t)) := x_i(\partial_t^{\alpha}(\mathbf{y}(t))|_{t=0}) ,$$

for $\mathbf{y}(t) = \mathbf{y}_0 + \mathbf{y}_1 t + \dots + \mathbf{y}_m t^m \in J_m(Y)$. Then we see that

$$J_m(Y) \simeq \text{Spec}\mathbb{C}[x_1^{(0)}, \dots, x_N^{(0)}, \dots, x_1^{(m)}, \dots, x_N^{(m)}] \simeq \mathbb{A}^{N(m+1)} .$$

Example 2.1 is almost a "general" situation for smooth varieties. More precisely, the smooth case is described by the following proposition:

Proposition 2.2 ([5](**Proposition 2.3**)). — Let Y be a smooth scheme of dimension N . Then $J_m(Y)$ is locally an \mathbb{A}^{Nm} -bundle over Y . In particular $J_m(Y)$ is smooth of dimension $N(m+1)$. In the same way, $J_{m+1}(Y)$ is an \mathbb{A}^N -bundle over $J_m(Y)$.

We will give additional properties about jet schemes next part in the proof of Theorem 1.

2.2. The cylinder sets and stable sets. — We introduce now a particularly interesting class of subsets in $J_{\infty}(X)$.

Definition 2.3. — A subset $C \subset J_{\infty}(X)$ of the space of arcs is called cylinder set if $C = \pi_{\infty, m}^{-1}(B_m)$, for $m \geq 0$, and B_m a constructible subset of $J_m(X)$.

As a *constructible subset* is a finite, disjoint union of locally closed subvarieties, it is quite clear the the collection of all the cylinder sets forms an algebra of sets, that we denote by $\tilde{\mathbf{B}}_X$

[this will not be exactly the "right" algebra, that we will call then \mathbf{B}_X .]

For instance $J_\infty(X) = \pi_{\infty,0}^{-1}(X)$ is a cylinder set, as well as any finite unions and complement (and hence finite intersection) of cylinders sets.

We would like that cylinder sets are *measurable* for the motivic measure that we are going to define. With this definition, it will be not easy to define a measure in such way that this measure is independent of m . The *stable sets* turns out to be a more judicious choice of subsets class.

Recall that $f : S \rightarrow B$ is a *piecewise trivial fibration* with fiber constant fiber F if we can write $B = \bigsqcup B_i$ as a finite union of locally closed subsets B_i such that over each B_i , we have $f^{-1}(B_i) \cong B_i \times F$ and f is given by the first projection onto B_i .

Definition 2.4. — *A subset $A \subseteq J_\infty(X)$ is called stable if*

- (1) for $m \gg 0$, $A_m := \pi_{\infty,m}(A)$ is a constructible subset of $J_m(X)$,
- (2) $A = \pi_{\infty,m}^{-1}(A_m)$, for $m \gg 0$, and,
- (3) $\pi_{n+1,n} : A_{n+1} \rightarrow A_n$ is a piecewise trivial fibration over $\pi_{\infty,n}(A)$ with fiber \mathbb{A}^d , for $n \geq m$.

The stable sets are cylinders, but the converse is not true in general. By Proposition 2.2, if X is smooth, then cylinders are stable. Then in smooth case, the condition (3) is superfluous and the two notions coincide. But in general way, (3) is absolutely crucial. In fact, the main technical part to define motivic measure on singular spaces — what did Denef and Loeser in [10] — is to show that the class of stable sets can be enlarged to an algebra of *measurable subsets* which contains cylinders. In particular $J_\infty(X)$ will be measurable.

To avoid these technical part, we assume until the end of this part that Y is a smooth variety over \mathbb{C} of dimension N . In most of cases we can even suppose that $Y = \mathbb{A}^N$. For instance, in Part 3, we will consider the case where Y is a complex finite dimensional reductive Lie algebra \mathfrak{g} . We will reserve the term X for subvarieties, eventually singular, of Y . In Part 3 we will consider the case where X is the nilpotent cone or the *principal cone* of \mathfrak{g} .

2.3. The order functions F_X associated to a closed subvariety. —

We consider an interesting class of functions defined on $J_\infty(Y)$ with values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$: the order functional associated to a divisor (and more generally to any closed variety). We would like to define an algebra \mathbf{B}_Y on $J_\infty(Y)$ with respect to which these functionals are *measurable*.

For $X \subseteq Y$ a subvariety of Y defined by an ideal I_X , we associate the function

$$F_X : J_\infty(Y) \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

which maps an arc $\gamma \in J_\infty(Y)$ to the order of vanishing of γ along Y , i.e. the biggest integer e such that the ideal $\gamma(\mathcal{I}_X)$ of $\mathbb{C}[[t]]$ is contained in the ideal (t^e) . In the same way, we can define a function, that we denote by the same symbol F_X , from $J_m(Y)$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. The function F_X is called *order function* (with respect, or associated to, X).

Example 2.5. — Suppose that $Y = \mathbb{A}^N$ and $X \subseteq Y$ is defined by equations $f_i(\mathbf{z}) = 0$, for $i = 1, \dots, r$. Let $\gamma = \mathbf{z}_0 + \mathbf{z}_1 t + \mathbf{z}_2 t^2 + \dots \in J_\infty(Y)$. Then $F_X(\gamma)$ is the biggest integer e such that

$$f_i(\gamma) = f_i(\mathbf{z}_0 + \mathbf{z}_1 t + \mathbf{z}_2 t^2 + \dots) \equiv 0 \pmod{t^e},$$

for any $i = 1, \dots, r$.

Clear that, for $\gamma \in J_\infty(Y)$ and $m \geq 0$, we have the following interpretation in terms of jets:

- 1) $F_X(\gamma) \neq 0 \iff \pi_{\infty,0}(\gamma) \in X$,
- 2) $F_X(\gamma) \geq m + 1 \iff \pi_{\infty,m}(\gamma) \in J_m(X)$,
- 3) $F_X(\gamma) = \infty \iff \gamma \in J_\infty(X)$.

The following proposition is consequently straightforward, since

$$F_X^{-1}(s) = \pi_{\infty,s-1}^{-1}(J_{s-1}(X)) \setminus \pi_{\infty,s}^{-1}(J_s(X)).$$

Proposition 2.6. — The subset $F_X^{-1}(s)$ is a cylinder.

A very interesting case is when X is an effective divisor D of Y . Let $x \in Y$ and g a local defining equation for D on a neighborhood U of y . Then the function

$$F_D : J_\infty(Y) \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

associated to D maps $\gamma \in J_\infty(Y)$ to $F_D(\gamma)$ the order of vanishing of the formal series $g(\gamma(z))$ at $z = 0$. If we write $D = \sum_{i=1}^r a_i D_i$ as a linear combination of prime divisors then g decomposes as a product $g = \prod_{i=1}^r g_i^{a_i}$ of defining equations for D_i , hence $F_D = \sum_{i=1}^r a_i F_{D_i}$.

If X is a closed subvarieties X in Y , then $F_X^{-1}(s)$ is a cylinder, by Proposition 2.6. It is worth to notice that $F_X^{-1}(\infty)$ is not a cylinder.

[Indeed, suppose the contrary, so there is a constructible subset $B_m \subseteq J_m(Y)$ such that $F_X^{-1}(\infty) = \pi_{\infty,m}^{-1}(B_m)$. Each arc $\gamma \in F_X^{-1}(\infty)$ is an N -tuple of power series whose at least one is identically zero. But each $\gamma \in \pi_{\infty,m}^{-1}(B_m)$ is an N -tuple of power series whose terms of degree higher than m may take any complex value, whence a contradiction.]

Nevertheless, we observe that

$$F_X^{-1}(\infty) = \bigcap_{k \in \mathbb{Z}_{\geq 0}} \pi_{\infty, m}^{-1}(\pi_{\infty, m}(F_X^{-1}(\infty))) ,$$

since a power series is identically zero if and only if its truncation to degree m is the zero polynomial for any $m \geq 0$. Then, it is easy to see that $\pi_{\infty, m}(F_X^{-1}(\infty))$ is a constructible set, for any m . Therefore, we deduce

Proposition 2.7 ([9](**Proposition 2.6**)). — $F_X^{-1}(\infty)$ is a countable intersection of cylinders.

We will see next section how construct an algebra in such a way that F_X is measurable. In particular the algebra $\tilde{\mathbf{B}}_Y$ of cylinders is certainly not sufficient.

3. The value ring of the motivic measure

We define in this section the value ring of the motivic measure.

3.1. We denote by $K_0(\text{Var}_{\mathbb{C}})$ the Grothendieck ring of the category of varieties $\text{Var}_{\mathbb{C}}$ over \mathbb{C} . Recall that it is the ring formally generated by the isomorphism classes $[S]$ of all the finite type complex varieties S with the relation

$$[S] = [S \setminus S'] + [S'] ,$$

for any closed subvariety S' in S . The product structure is given by

$$[S] \cdot [S'] = [S \times S'] .$$

We denote by \mathbb{L} the class of the affine line \mathbb{A}^1 in $K_0(\text{Var}_{\mathbb{C}})$ and by 1 the class of any point. Using a stratification of S by smooth varieties, we show that $K_0(\text{Var}_{\mathbb{C}})$ is generated by smooth varieties.

Examples: 1) $[\mathbb{C}^*] = [\mathbb{C} \setminus \{0\}] = [\mathbb{C}] - [\{0\}] = \mathbb{L} - 1$.

2) $[\mathbb{P}^N] = \mathbb{L}^N + \mathbb{L}^{N-1} + \dots + \mathbb{L} + 1$.

3) *Piecewise trivial fibration*: suppose that $f : S \rightarrow B$ is a piecewise trivial fibration with constant fiber F . Then $[S] = [B] \cdot [F]$. Also, if $f : S \rightarrow B$ is a locally trivial fibration with fiber F , then $[S] = [B] \cdot [F]$.

[Indeed, this means that we can write $B = \bigsqcup B_i$ as a finite union of locally closed subsets B_i such that over each B_i , we have $f^{-1}(B_i) \cong B_i \times F$ and f is given by the first projection onto B_i . Then $S = f^{-1}(B) = \bigsqcup f^{-1}(B_i) \cong \bigsqcup B_i \times F$, whence the equality.]

The map $S' \mapsto [S']$ from the set of closed subvarieties of S extends uniquely in a mapping $W \mapsto [W]$ from the set of constructible sets of S to $K_0(\text{Var}_{\mathbb{C}})$. This map satisfies $[W \cup W'] = [W] + [W'] - [W \cap W']$.

An *additive invariant* λ from the category $\text{Var}_{\mathbb{C}}$ with values in a ring R is a map satisfying:

$$\begin{aligned} \lambda(S) &= \lambda(S'), \text{ if } S \cong S' \\ \lambda(S) &= \lambda(S') + \lambda(S \setminus S'), \text{ for } S' \text{ closed in } S, \\ \lambda(S \times S') &= \lambda(S) \cdot \lambda(S'), \text{ for any } S, S'. \end{aligned}$$

An additive invariant naturally extends to the set of constructible sets. The map $S \mapsto [S]$ is clearly an additive invariant with values in $K_0(\text{Var}_{\mathbb{C}})$ and satisfies the following universal property: *for any additive invariant $\lambda : \text{Var}_{\mathbb{C}} \rightarrow R$ with values in a ring R , there is a unique map $\bar{\lambda}$ such that $\bar{\lambda}([S]) = \lambda(S)$, for any S in $K_0(\text{Var}_{\mathbb{C}})$:*

$$\begin{array}{ccc} \text{Var}_{\mathbb{C}} & \xrightarrow{\lambda} & R \\ \downarrow [-]! & \nearrow \bar{\lambda} & \\ K_0(\text{Var}_{\mathbb{C}}) & & \end{array}$$

Example 3.1 (Hodge Polynomials). — *There is an unique additive invariant $h : \text{Var}_{\mathbb{C}} \rightarrow \mathbb{Z}[u, v]$, which assigns to a smooth variety S over \mathbb{C} its Hodge polynomial*

$$h(S, u, v) := \sum_{p, q} (-1)^{p+q} H^{p, q}(S) u^p v^q,$$

where $H^{p, q}(S) = \dim H^q(S, \Omega_S^p)$ is the (p, q) -Hodge number of S . The map h factors through $K_0(\text{Var}_{\mathbb{C}})$ according the universal property.

3.2. We denote by $\mathcal{M}_{\mathbb{C}}$ the localization at \mathbb{L} of the Grothendieck ring $K_0(\text{Var}_{\mathbb{C}})$

$$\mathcal{M}_{\mathbb{C}} := K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}].$$

Recall that $\tilde{\mathbf{B}}_Y$ is the algebra generated by cylinders (see Definition 2.3). First we define a “naive” measure $\tilde{\mu}_Y$ on $\tilde{\mathbf{B}}_Y$ with values in $\mathcal{M}_{\mathbb{C}}$ by setting:

$$(1) \quad \tilde{\mu}_Y(\pi_{\infty, m}^{-1}(B_m)) := [B_m] \cdot \mathbb{L}^{-N(m+1)},$$

for any constructible subset B_m in $J_m(Y)$. As Y is smooth, $J_{\infty}(Y)$ is a cylinder and we have

$$\tilde{\mu}_Y(J_{\infty}(Y)) = [Y] \cdot \mathbb{L}^{-N}.$$

Notice that Relation (1) makes sense because the right hand is independent on the choice B_m and on the level m . This comes from the fact that cylinders are stable sets (see Definition 2.4) in the case where Y is smooth.

[If $B = \pi_{\infty, m}^{-1}(B_m) = \pi_{\infty, m+1}^{-1}(\pi_{m+1, m}^{-1}(B_m))$, then $\tilde{\mu}_Y(B) = [\pi_{m+1, m}^{-1}(B_m)] \cdot \mathbb{L}^{-N(m+2)} = [\mathbb{A}^N] \cdot [B_m] \cdot \mathbb{L}^{-N(m+2)} = [B_m] \cdot \mathbb{L}^{-N(m+1)}$, according to condition (3) of Definition 2.4 and above example (3). On the other hand, it is easy to check that the relation $\pi_{\infty, m}^{-1}(B_m) = \pi_{\infty, m}^{-1}(C_m)$ implies the relation $[B_m] = [C_m]$.]

Remark 3.2. — For singular spaces, we can define $\tilde{\mu}_Y$ only for stable sets by setting $\tilde{\mu}_Y(A) := [\pi_{\infty, m}(A)] \cdot \mathbb{L}^{-N(m+1)}$, for any stable set $A = \pi_{\infty, m}^{-1}(A_m)$. In this case, condition (3) of Definition 2.4 is required.

Let X be a closed subvariety in Y . By Proposition 2.6, the set $F_X^{-1}(s)$ is $\tilde{\mu}_Y$ -measurable for any $s \in \mathbb{Z}_{\geq 0}$. However, F_X is not $\tilde{\mu}_Y$ -measurable because $F_X^{-1}(\infty)$ is not a cylinder. To process, we extend $\tilde{\mu}_Y$ to a measure μ_Y . The following discussion intends to motivate the definition of μ_Y . First of all, as $[-]$ is additive, it is straightforward to show that

$$\tilde{\mu}_Y\left(\bigsqcup_{i=1}^l C_i\right) = \sum_{i=1}^l \tilde{\mu}_Y(C_i),$$

for cylinders C_1, \dots, C_l . Now, the set $J_{\infty}(Y) \setminus F_X^{-1}(\infty)$ is a countable disjoint union of cylinder sets, by Proposition 2.7:

$$\begin{aligned} & J_0(Y) \setminus \pi_{\infty, 0}^{-1}(\pi_{\infty, 0}(F_X^{-1}(\infty))) \\ \sqcup & \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \pi_{\infty, k+1}^{-1}(J_{k+1}(Y) \setminus \pi_{\infty, k+1}(F_X^{-1}(\infty))) \setminus \pi_{\infty, k}^{-1}(J_k(Y) \setminus \pi_{\infty, k}(F_X^{-1}(\infty))) . \end{aligned}$$

Our goal is to define a measure μ_Y defined on the collection of countable disjoint unions of cylinder sets so that the set $J_{\infty}(Y) \setminus F_X^{-1}(\infty)$, and hence its complement $F_X^{-1}(\infty)$ too, is μ_Y -measurable. We would like to define μ_Y so that:

$$(2) \quad \mu_Y\left(\bigsqcup_{i \in \mathbb{N}} C_i\right) := \sum_{i \in \mathbb{N}} \mu_Y(C_i) = \sum_{i \in \mathbb{N}} \tilde{\mu}_Y(C_i), \text{ for cylinders } C_i, \text{ with } i \in \mathbb{N}.$$

The problem is that the sum $\sum_{i \in \mathbb{N}} \tilde{\mu}_Y(C_i)$ is not in $\mathcal{M}_{\mathbb{C}}$ yet. Furthermore, it is not clear that the relation (2) is independent on the choice of the C_i . Kontsevich solved both of these problems by completing the ring $\mathcal{M}_{\mathbb{C}}$.

3.3. We can process by analogy with p -adic integration (see [24]): $K_0(\text{Var}_{\mathbb{C}})$ is the analogue of \mathbb{Z} and $\mathcal{M}_{\mathbb{C}}$ is the analogue of $\mathbb{Z}[p^{-1}]$. In \mathbb{R} , p^{-i} has limit 0 when i go to $-\infty$. Here we expect that \mathbb{L}^{-i} has "limit" 0 in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Let $F^m(\mathcal{M}_{\mathbb{C}})$ be the subgroup of $\mathcal{M}_{\mathbb{C}}$ generated by the elements of the form $[S]\mathbb{L}^{-i}$, with $\dim S - i \leq -m$. We have $F^{m+1}(\mathcal{M}_{\mathbb{C}}) \subset F^m(\mathcal{M}_{\mathbb{C}})$, $F^m(\mathcal{M}_{\mathbb{C}}) \cdot F^n(\mathcal{M}_{\mathbb{C}}) \subset F^{m+n}(\mathcal{M}_{\mathbb{C}})$ and $\mathbb{L}^{-m} \in F^m(\mathcal{M}_{\mathbb{C}})$. We denote by $\widehat{\mathcal{M}}_{\mathbb{C}}$ the completion with respect to this filtration.

[We can explicit this completion; there is a natural notion of dimension on $K_0(\text{Var}_{\mathbb{C}})$. We say that $\tau \in K_0(\text{Var}_{\mathbb{C}})$ has *dimension* d if we can write $\tau = \sum a_i[S_i]$, where a_i is in \mathbb{Z} and S_i is a variety of dimension at most d for any i and if in addition not all S_i has dimension at most $d - 1$. We decide that the dimension of the empty set is $-\infty$. It is easy to check that the dimension map

$$\dim : K_0(\text{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

satisfies $\dim(\tau \cdot \tau') \leq \dim \tau + \dim \tau'$ and $\dim(\tau + \tau') \leq \max\{\dim \tau, \dim \tau'\}$ with equality if and only if $\dim \tau \neq \dim \tau'$. The dimension map extends to $\mathcal{M}_{\mathbb{C}}$ by setting $\dim(\mathbb{L}^{-1}) := -1$. There is a filtration induced by the dimension. Then $F^m(\mathcal{M}_{\mathbb{C}})$ is the subgroup of elements τ in $\mathcal{M}_{\mathbb{C}}$ such that $\dim \tau$ is at most $-m$. Then, the convergence in $\widehat{\mathcal{M}}_{\mathbb{C}}$ is quite easy. A sequence of elements $\tau_i \in \widehat{\mathcal{M}}_{\mathbb{C}}$ if and only if the dimensions $\dim \tau_i$ tend to $-\infty$ when i goes to ∞ . Then, a sum $\sum_{i=1}^{\infty} \tau_i$ converges if and only if the sequence of summands converges to zero.]

Remark 3.3. — *It is not already known whether the morphism $\mathcal{M}_{\mathbb{C}} \rightarrow \widehat{\mathcal{M}}_{\mathbb{C}}$ is injective. But, at the end we plan to consider the compound $h \circ \mu_Y$ and we can show that h factors through the image of $\mathcal{M}_{\mathbb{C}}$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, so it is not too serious.*

We dispose now of the values ring for our feature measure.

4. The motivic measure and the motivic integral

4.1. Motivic measure. — Denote by φ the natural completion map $\varphi : \mathcal{M}_{\mathbb{C}} \longrightarrow \widehat{\mathcal{M}}_{\mathbb{C}}$. Then we still denote by $\tilde{\mu}_Y$ the compound map $\varphi \circ \tilde{\mu}_Y$ with is so in values in $\widehat{\mathcal{M}}_{\mathbb{C}}$. For a sequence of cylinders $\{C_i\}$, there is now a meaning to ask whether or not the sequence $\tilde{\mu}_Y(C_i)$ converges to 0 when i go to $-\infty$. We pass over the details concerning the construction of the definitive motivic measure. So we admit Proposition-Definition 4.1:

Definition-Proposition 4.1. — *There exists an algebra \mathbf{B}_Y of subsets in $J_{\infty}(Y)$, which contains cylinder subsets, and an unique map $\mu_Y : \mathbf{B}_Y \longrightarrow \widehat{\mathcal{M}}_{\mathbb{C}}$ satisfying the following conditions:*

- 1) *If $A = \pi_{\infty, m}^{-1}(A_m)$, with $m \in \mathbb{Z}_{\geq 0}$, is a cylinder set, then $\mu_Y(A) = [A_m] \cdot \mathbb{L}^{-(m+1)N}$.*
- 2) *If $A \in \mathbf{B}_Y$ is contained in $J_{\infty}(Z)$ where Z is a closed subvariety of Y with $\dim Z < \dim Y$, then $\mu_Y(A) = 0$.*

3) Let $\{A_i\}_{i \in I}$ be a sequence in \mathbf{B}_Y . Assume that the A_i are pairwise distinct and that the union $A := \bigsqcup_{i \in I} A_i$ is in \mathbf{B}_Y . Then $\sum_{i \in I} \mu_Y(A_i)$ converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$ to $\mu_Y(A)$.

4) If A and B are in \mathbf{B}_Y , $A \subseteq B$, and if $\mu_Y(B)$ belongs to the closure $F^m(\widehat{\mathcal{M}}_{\mathbb{C}})$ of $F^m(\mathcal{M}_{\mathbb{C}})$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, then $\mu_Y(A) \in F^m(\widehat{\mathcal{M}}_{\mathbb{C}})$.

The measure μ_Y is called the motivic measure on Y .

Replacing ‘‘cylinder subsets’’ by ‘‘stable subsets’’ in Proposition-Definition 4.1, all the statements still hold if Y is a singular variety (see [10](Definition-Proposition 3.2)). Actually, in the case where Y is not necessary smooth, the algebra \mathbf{B}_Y is the algebra of *semi-algebraic subsets* of $J_{\infty}(Y)$. The main difficulty in this case is to construct an algebra \mathbf{B}_Y which contains cylinders and define the measure on cylinders as limit of measure of stable sets. In the case where Y is smooth, the algebra \mathbf{B}_Y is the collection of countable disjoint unions of cylinders sets $\bigsqcup_{i \in I} C_i$ for which $\tilde{\mu}_Y(C_i)$ converges to 0 when i go to $-\infty$, as well as the complements of such sets. Then the measure μ_Y maps $\bigsqcup_{i \in I} C_i$ to $\sum_{i \in I} \tilde{\mu}_Y(C_i)$. The difficulty is to show that this definition is independent on the choice of the C_i (see [10]).

Corollary 4.2. — *Let Z be a closed subvariety Z of Y . We suppose that Z is strictly contained in Y . Then $\mu_Y(F_Z^{-1}(\infty)) = 0$*

Proof. — Let us remark that $\nu \in F_Z^{-1}(\infty)$ if and only if $\nu \in J_{\infty}(Z)$. But the measure of $J_{\infty}(Z)$ is zero by Definition-Proposition 4.1, (2), since Z is strictly contained in Y . \square

4.2. Motivic integral. — For A in \mathbf{B}_Y and $\alpha : A \rightarrow \mathbb{Z} \cup \{\infty\}$ a function such that $\alpha^{-1}(s) \in \mathbf{B}_Y$ for any $s \in \mathbb{Z} \cup \{\infty\}$ and $\mu_Y(\alpha^{-1}(\infty)) = 0$, we can set

$$(3) \quad \int_A \mathbb{L}^{-\alpha} d\mu_Y := \sum_{s \in \mathbb{Z}} \mu_Y(A \cap \alpha^{-1}(s)) \mathbb{L}^{-s}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$, whenever the right hand side converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$. In this case, we say that $\mathbb{L}^{-\alpha}$ is *integrable* on A .

An important example is the case where $\alpha = F_Z$ is an order functional of a closed subvariety Z of Y . We suppose that Z is strictly contained in Y . By Corollary 4.2, $\mu_Y(F_Z^{-1}(\infty)) = 0$. Then, the formula (3) applied to F_Z gives:

$$\int_{J_{\infty}(Y)} \mathbb{L}^{-F_Z} d\mu_Y := \sum_{s \in \mathbb{Z}} \mu_Y(F_Z^{-1}(s)) \mathbb{L}^{-s} .$$

Notice that the sum of the right hand side does converge since the virtual dimension of the summands approaches negative infinity. Let us give some examples:

Example 4.3. — $Z = \emptyset$, then $F_Z \equiv 0$ and $\int_{J_\infty(Y)} \mathbb{L}^{-F_Z} d\mu_Y = \mu_Y(F_Z^{-1}(0)) = [Y]\mathbb{L}^{-N}$. Then we see now how the volume $[Y]$ can be expressed as a motivic integral.

Example 4.4. — $Z = D$ is a smooth divisor of Y . Then

$$F_D^{-1}(s) = \pi_{\infty, s-1}^{-1}(J_{s-1}(D)) \setminus \pi_{\infty, s}^{-1}(J_s(D)) .$$

Moreover $J_s(D)$ is locally a $\mathbb{A}^{(N-1)s}$ -bundle over D , since D is smooth of dimension $N-1$. Therefore the measure of $F_D^{-1}(s)$ is $[D](\mathbb{L}-1)\mathbb{L}^{-(s+N)}$ and we get:

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu_Y = \sum_{s \in \mathbb{Z}} \mu_Y(F_D^{-1}(s)) = ([Y-D] + \frac{[D]}{\mathbb{P}^1})\mathbb{L}^{-N} .$$

[Explanation: we have

$$\begin{aligned} \int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu_Y &= \sum_{s \in \mathbb{Z}} \mu_Y(F_D^{-1}(s))\mathbb{L}^{-s} \\ &= [Y-D]\mathbb{L}^{-N} + \sum_{s \in \mathbb{Z}_{\geq 1}} ([D](\mathbb{L}-1)\mathbb{L}^{-(s+N)})\mathbb{L}^{-s} \\ &= [Y-D]\mathbb{L}^{-N} + [D](\mathbb{L}-1)\mathbb{L}^{-N} \frac{1}{\mathbb{L}^2(1-\mathbb{L}^{-2})} \\ &= ([Y-D] + \frac{[D]}{\mathbb{L}+1})\mathbb{L}^{-N} = ([Y-D] + \frac{[D]}{\mathbb{P}^1})\mathbb{L}^{-N} . \end{aligned}$$

]

Example 4.5. — $Z = D = \sum_{i=1}^t a_i D_i$ is an effective divisor of Y . We detail this example, since it is one of the key of the theory. Moreover, we will use it Part 2 and Part 3 for the nilpotent bicone. For $J \subseteq \{1, \dots, t\}$ any subset, define:

$$D_J := \begin{cases} \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \quad \text{and} \quad D_J^0 := D_J \setminus \bigcup_{i \in \{1, \dots, t\} \setminus J} D_i .$$

These subvarieties stratify Y and define a partition of the space of arcs into cylinders sets:

$$Y = \bigsqcup_{J \subseteq \{1, \dots, t\}} D_J^0 \quad \text{and} \quad J_\infty(Y) = \bigsqcup_{J \subseteq \{1, \dots, t\}} \pi_{\infty, 0}^{-1}(D_J^0) .$$

For any $s \in \mathbb{Z}_{\geq 0}$ and $J \in \{1, \dots, r\}$, define

$$M_{J,s} := \{(m_1, \dots, m_t) \in \mathbb{Z}_{\geq 0} \mid \sum_{i=1}^t a_i m_i = s \text{ with } (m_i > 0 \iff j \in J)\} .$$

We suppose that $D = \sum_{i=1}^t a_i D_i$ is an effective divisor with simple normal crossing (SNC) (see Appendix A, Definition A.1) and such that all the D_i are smooth.

Lemma 4.6. — Let J be in $\{1, \dots, t\}$, let s be in $\mathbb{Z}_{\geq 0}$, and let $\mathbf{m} := (m_1, \dots, m_t)$ be in $M_{J,s}$. For $q \geq \max_i \{m_i\}$, the subset

$$S_{\mathbf{m}}^q := \pi_{\infty,q}(\{ \nu \in J_{\infty}(Y) \mid F_{D_i}(\nu) = m_i, \forall i \in J \})$$

is a locally trivial fibration over D_J^0 with fiber $(\mathbb{C} \setminus \{0\})^{|J|} \times \mathbb{C}^{nq - \sum_i m_i}$.

Proof. — We follow the proof of [9](Proposition 2.5). Since the divisor $D = \sum_{i=1}^t a_i D_i$ on Y has only simple normal crossing, for any y in Y , there exists a neighborhood U of y in Y with global coordinates z_1, \dots, z_N on U for which a local defining equation for D is given by

$$(4) \quad g = z_1^{a_1} \cdots z_{j_y}^{a_{j_y}} ,$$

for some $j_y \leq N$. We cover $Y = \bigcup U$ by finitely many charts on which D has a local equation of the form (4), and we lift to cover $J_q(Y) = \bigcup \pi_{q,0}^{-1}(U)$. Hence $S_{\mathbf{m}}^q$ is covered by the open subsets

$$U_{\mathbf{m}}^q := \bigcap_{i \in J} F_{D_i}^{-1}(m_i) \cap \pi_{q,0}^{-1}(U) .$$

As m_i is at least 1 for $i \in J$, the subset $U_{\mathbf{m}}^q$ is contained in $\pi_{q,0}^{-1}(D_J^0 \cap U)$. If J is not contained in $\{1, \dots, j_y\}$ then $D_J^0 \cap U$ is empty, and so $U_{\mathbf{m}}^q$. Thus we can suppose that J is contained in $\{1, \dots, j_y\}$.

The key observation is that when we regard each arc $\nu \in \pi_{q,0}^{-1}(U)$ as an N -tuple $(f_1(z), \dots, f_N(z))$ of polynomials of degree at most q with zero constant term, each condition $F_{D_i}(\nu) = m_i$ is equivalent to a condition on the truncation of $f_i(z)$ to degree m_i . Indeed, since D_i is cut out by $z_i = 0$ on U , it follows that $F_{D_i}(\nu) = \{\text{order of } f_i(z) \text{ at } z = 0\}$. Thus $\nu \in F_{D_i}^{-1}(m_i)$ if and only if the truncation of $f_i(\nu)$ to degree m_i is of the form $c_{m_i} z^{m_i}$ where c_{m_i} is different from 0. Then we obtain $N - |J|$ polynomials of degree q with zero constant term, and, for each $j \in J$, a polynomial of the form

$$f_j(\nu) = 0 + \cdots + 0 + c_{m_j} z^{m_j} + c_{m_j+1} z^{m_j+1} + \cdots + c_{m_q} z^{m_q} ,$$

for c_{m_j} in $\mathbb{C} \setminus \{0\}$ and for c_k in \mathbb{C} , for any k strictly bigger than j . So, the space of all such N -tuples is isomorphic to

$$\mathbb{C}^{q(N-|J|)} \times (\mathbb{C} \setminus \{0\})^{|J|} \times \mathbb{C}^{q|J| - \sum_{i \in J} m_i} .$$

As a consequence, $U_{\mathbf{m}}^q$ is isomorphic to

$$\pi_{m,0}^{-1}(U \cap D_J^0) \times \mathbb{C}^{q(N-|J|)} \times (\mathbb{C} \setminus \{0\})^{|J|} \times \mathbb{C}^{q|J| - \sum_{i \in J} m_i} ,$$

whence the lemma. \square

We deduce from Lemma 4.6, the relation:

$$\begin{aligned} \mu_Y(\pi_{\infty,q}^{-1}(S_{\mathbf{m}}^q)) &= [\{ \nu \in J_{\infty}(Y) \mid F_{D_i}(\nu) = m_i, \forall i \in J \}] \mathbb{L}^{-N(q+1)} \\ &= [D_J^0] (\mathbb{L} - 1)^{|J|} \mathbb{L}^{-N - \sum_{i \in J} m_i} . \end{aligned}$$

Notice now that $\pi_{\infty,q}^{-1}(S_{\mathbf{m}}^q)$ is equal to $\left(\bigcap_{i=1, \dots, t} F_{D_i}^{-1}(m_i) \right)$.

Proposition 4.7. — We suppose that $D = \sum_{i=1}^t a_i D_i$ is an effective divisor with SNC and such that all the D_i are smooth. Then

$$\int_{J_{\infty}(Y)} \mathbb{L}^{-F_D} d\mu_Y = \sum_{J \subseteq \{1, \dots, s\}} [D_J^0] \left(\prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1} \right) \mathbb{L}^{-N} = \sum_{J \subseteq \{1, \dots, s\}} \left(\prod_{j \in J} \frac{[D_J^0]}{\mathbb{P}^{a_j}} \right) \cdot \mathbb{L}^{-N} .$$

Proof. — We observe that

$$\gamma \in \pi_{\infty,0}^{-1}(D_J^0) \cap F_D^{-1}(s) \iff (F_{D_i}(\gamma), \dots, F_{D_r}(\gamma)) \in M_{J,r} .$$

As a result we produce a finite partition:

$$(5) \quad F_D^{-1}(s) = \bigsqcup_{J \subseteq \{1, \dots, t\}} \bigsqcup_{(m_1, \dots, m_t) \in M_{J,s}} \left(\bigcap_{i=1, \dots, t} F_{D_i}^{-1}(m_i) \right) .$$

Then we use partition (5) and Lemma 4.6 to compute the motivic integral...

[Details of the computations:

$$\begin{aligned}
\int_{J_\infty(Y)} \mathbb{L}^{-FD} d\mu_Y &= \sum_{s \in \mathbb{Z}_{\geq 0}} \mu_Y(F_D^{-1}(s)) \cdot \mathbb{L}^{-s} \\
&= \sum_{s \in \mathbb{Z}_{\geq 0}} \sum_{J \subseteq \{1, \dots, t\}} \sum_{(m_1, \dots, m_t) \in M_{J,s}} \mu_Y \left(\bigcap_{i=1, \dots, t} F_D^{-1}(m_i) \right) \cdot \mathbb{L}^{-\sum_{i \in J} a_i m_i} \\
&= \sum_{s \in \mathbb{Z}_{\geq 0}} \sum_{J \subseteq \{1, \dots, t\}} \sum_{(m_1, \dots, m_t) \in M_{J,s}} [D_J^0] (\mathbb{L} - 1)^{|J|} \mathbb{L}^{-N} \cdot \prod_{i \in J} \mathbb{L}^{-(a_i+1)m_i} \\
&= \sum_{J \subseteq \{1, \dots, r\}} [D_J^0] \cdot \prod_{i \in J} \left((\mathbb{L} - 1) \cdot \sum_{m_j > 0} \mathbb{L}^{-(a_i+1)m_i} \right) \cdot \mathbb{L}^{-N} \\
&= \sum_{J \subseteq \{1, \dots, r\}} [D_J^0] \cdot \prod_{i \in J} \left((\mathbb{L} - 1) \cdot \left(\frac{1}{1 - \mathbb{L}^{-(a_i+1)}} - 1 \right) \right) \cdot \mathbb{L}^{-N} \\
&= \sum_{J \subseteq \{1, \dots, r\}} [D_J^0] \cdot \left(\prod_{i \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i+1} - 1} \right) \cdot \mathbb{L}^{-N} \\
&= \sum_{J \subseteq \{1, \dots, r\}} \left(\prod_{i \in J} \frac{[D_J^0]}{\mathbb{P}^{a_i}} \right) \cdot \mathbb{L}^{-N}
\end{aligned}$$

]

□

5. The transformation rule for the integral

The power of the theory resides in the existence of a formula describing how the motivic integral transforms under birational morphism (this is the key of the Kontsevich's proof of Theorem 1.1).

Let $\gamma : Y' \rightarrow Y$ be a birational morphism between smooth varieties. Then, for m in $\mathbb{N} \cup \{\infty\}$, there is a canonical morphism $\gamma_m : J_m(Y') \rightarrow J_m(Y)$ induced by γ making the following diagram commutative:

$$\begin{array}{ccc}
J_\infty(Y') & \xrightarrow{\gamma_\infty} & J_\infty(Y) \\
\downarrow \pi_{\infty,m}^{Y'} & & \downarrow \pi_{\infty,m} \\
J_m(Y') & \xrightarrow{\gamma_m} & J_m(Y) \\
\downarrow \pi_{m,0}^{Y'} & & \downarrow \pi_{m,0} \\
Y' & \xrightarrow{\gamma} & Y
\end{array}$$

Recall that the *discrepancy divisor* $W_\gamma = K_{Y'} - \gamma^*K_Y$ of γ is the divisor of the Jacobian determinant of f , where K_Y and $K_{Y'}$ are the canonical divisors of Y and Y' respectively, (see Appendix A, Definitions A.3 and A.4). Then the following theorem can be view as a change of variables formula for the motivic integral.

Theorem 5.1. — *If F is an integrable functional over $J_\infty(Y)$ with respect to μ_Y , then*

$$\int_{J_\infty(Y)} \mathbb{L}^{-F} d\mu_Y = \int_{J_\infty(Y')} \mathbb{L}^{-(F \circ \gamma_\infty + F_{W_\gamma})} d\mu_{Y'} .$$

For example, if D is an effective divisor on Y , then

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu_Y = \int_{J_\infty(Y')} \mathbb{L}^{-F_{\gamma^{-1}(D) + W_\gamma}} d\mu_{Y'} ,$$

since $F_D \circ \gamma_\infty = F_{\gamma^{-1}(D)}$. We pass over the proof. We refer for instance to [5] for a proof.

Example 5.2. — *In order to check the coherence of the formula, consider the case where $Y' = \text{Bl}_Z(Y)$ is a blowing-up of Y along a smooth subvariety Z of codimension c in Y , and $F = 0$. Then by Appendix A (Example A.5), we have $W_p = (c-1)E$, where E is the exceptional divisor and p the morphism of the blowing-up. Then using Theorem 5.1, we have to compute*

$$\int_{J_\infty(Y')} \mathbb{L}^{-F_{W_p}} d\mu_{Y'} = \int_{J_\infty(Y')} \mathbb{L}^{-F_{(c-1)E}} d\mu_{Y'}$$

By Proposition 4.7 applied to $D = (c - 1)E$, we get $\int_{J_\infty(Y')} \mathbb{L}^{-F_{W_p}} d\mu_{Y'} = [Y' - E] + \frac{[E]}{\mathbb{P}^{c-1}} \cdot \mathbb{L}^{-N}$. Since E is locally isomorphic to $Z \times \mathbb{P}^{c-1}$ and since $Y' \setminus E \simeq Y \setminus Z$, we get $\int_{J_\infty(Y')} \mathbb{L}^{-F_{W_p}} d\mu_{Y'} = ([Y - Z] + [Z]) \cdot \mathbb{L}^{-N} = [Y] \cdot \mathbb{L}^{-N}$. On the other hand $\int_{J_\infty(Y)} \mathbb{L}^{-F} d\mu_Y = \mu_Y(J_\infty(Y)) = [Y] \cdot \mathbb{L}^{-N}$!

The philosophy in most applications is to encode information in a motivic integral, then using the *transformation rule of variables change* (see next section), the computation of this integral can be reduced to the computation of the integral of \mathbb{L}^{-F_D} , where D is a SNC divisor. To process, we construct in general a sequence of blowing-up and we apply Theorem A.2 (see Appendix A). Then, by Example 4.5, this integral can be explicitly computed. This procedure will be illustrated at least twice in the following (Part 2, Section 8, and Part 3, Section 11).

PART II

JET SCHEMES OF LOCALLY COMPLETE INTERSECTIONS

6. Introduction

In this part, X is a complex variety of dimension d . We quick recall Appendix B what means for X to have rational singularities (see Definition B.1). Now, we remind the result of Mustață:

Theorem 6.1 (Mustață). — *If X is locally a complete intersection, then $J_m(X)$ is irreducible for all $m \geq 1$ if and only if X has rational singularities.*

The study of singularities via the space of the arcs was initiated by Nash in [28]. He suggested that the study of the images $\pi_{\infty, m}(J_\infty(X)) \subseteq J_m(X)$ should give information about the fibers over the singular points in the desingularization of X . Theorem 6.1 supports his predictions. It says that rational singularities of X can be detected by the irreducibility of the jet schemes for local intersections. Let us give some examples .

Example 6.2. — *If X is smooth, connected, then by Proposition 2.2, $J_m(X)$ is an affine bundle over X with fiber \mathbb{A}^{mn} . So, in this case, $J_m(X)$ is smooth, connected and irreducible, and has dimension $(m + 1)n$.*

Example 6.3. — $X \subseteq \mathbb{A}^N$, $X = \text{Spec}(R)$, where $R = \mathbb{C}[x_1, \dots, x_N]/(f_1, \dots, f_r)$. Then remember that $J_m(X)$ is defined by the equations

$$f_i(\mathbf{z}_0 + \mathbf{z}_1 t + \dots + \mathbf{z}_m t^m) \equiv 0 \pmod{t^{m+1}},$$

where $\mathbf{z}_i \in \mathbb{A}^N$. Therefore, $J_m(X)$ is defined by $(m+1)r$ equations. In particular, $J_m(X)$ has dimension at least $(N-r)(m+1)$. In the case where $J_m(X)$ has exactly dimension $(N-r)(m+1)$, then it is a complete intersection. Moreover, we have an explicit description of $J_m(X)$. To process we generalize Example 2.1. Namely, $J_m(X)$ is defined in $\mathbb{A}^{d(m+1)}$ by the equations $f_\alpha^{(j)} = D^j(f_\alpha)$, where D is the derivation mapping $x_i^{(\alpha)}$ to $x_i^{(\alpha+1)}$ for $\alpha \leq m+1$. Then one can sometimes directly test the irreducibility condition with this description (see next Example 6.4).

Example 6.4. — Let $X_c \subset \mathbb{A}^{rs}$ be the determinantal variety of $r \times s$ matrices of rank at most c , that is, X_c is defined by I_{c+1} , the ideal of $(c+1)$ -minors of a $r \times s$ matrix (x_{ij}) . So an m -jet of X_c is a truncated arc

$$\gamma(t) = \mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_m t^m,$$

where $\mathbf{x} = (x_{ij})$ is in \mathbb{A}^{rs} , satisfying the conditions $\Delta(\gamma) \equiv 0 \pmod{t^{m+1}}$, for any Δ in I_{c+1} . Determinantal varieties have always rational singularities. We know that the determinantal variety of singular square matrices is an hypersurface. So, by Theorem 6.1, all of its jet schemes are irreducible. Though determinantal varieties always have rational singularities, they are rarely complete intersections, so Theorem 6.1 is not applicable to more general determinantal varieties. In fact, jet schemes of determinantal varieties are not always irreducible (see [32]).

The previous example give a lot of examples of Gorenstein varieties with rational singularities whose jet schemes are not irreducible. In particular, they illustrate that Mustață's result cannot be weakened: we may not replace the local complete intersection hypothesis with a Gorenstein hypothesis. Mustață also gave an example of a toric variety to illustrate this fact (see [27](Example 4.6)).

7. First properties of jet schemes

Let X be a lci of pure dimension d . We denote by X_{reg} the smooth part of X and by X_{sing} the complementary of X_{reg} in X .

Proposition 7.1. — i) The scheme $J_m(X)$ is pure dimensional if and only if $\dim J_m(X) \leq d(m+1)$.

ii) The scheme $J_m(X)$ is irreducible if and only if $\dim \pi_{m,0}^{-1}(X_{\text{sing}}) < d(m+1)$

Proof. — i) It is enough to embed X into \mathbb{A}^N , then we have already seen that $J_m(X)$ is the nullvariety in $\mathbb{A}^{N(m+1)}$ of $r(m+1)$ equations, if r is the number of equations defining locally X .

ii) We have the following decomposition:

$$(6) \quad J_m(X) = \pi_{m,0}^{-1}(X_{\text{sing}}) \cup \overline{\pi_{m,0}^{-1}(X_{\text{reg}})}.$$

In general $\pi_{m,0}^{-1}(X_{\text{reg}})$ is an irreducible component of $J_m(X)$ of dimension $d(m+1)$ (see Example 6.2). So the “only if” part holds without the hypothesis of lci (as in (i) by the way). Suppose now that $\dim \pi_{m,0}^{-1}(X_{\text{sing}}) < d(m+1)$. Then this implies that $\dim J_m(X) \leq d(m+1)$, so $J_m(X)$ is pure dimensional (by (i) since X is a lci) and the decomposition (6) shows that $J_m(X)$ is irreducible. \square

As the “only if” part of both (i) and (ii) of Proposition 7.1 holds without hypothesis of lci, (ii) gives a sufficient condition for $J_m(X)$ to be reducible. Concretely, if we want to show $J_m(X)$ is reducible, it suffices to show $\dim \pi_{m,0}^{-1}(X_{\text{sing}}) \geq \dim \pi_{m,0}^{-1}(X_{\text{reg}})$. Let us give some examples:

Example 7.2. — *Then if X is an integral curve, then for any $m \geq 1$, $J_m(X)$ is irreducible if and only if X is non singular. Here X don't need to be supposed a lci.*

[Indeed, if x is a singular point, then by [27](Lemma 4.1) $\dim \pi_{m,0}^{-1}(x) \geq m+1$. Therefore $\pi_{m,0}^{-1}(x)$ gives an irreducible component of $J_m(X)$.]

Example 7.3. — *Come back to Example 6.4: Cornelia Yuen use this sufficient condition to prove the following: “let X be the variety of $r \times s$ size matrices of rank at most one. Assume $r > s \geq 3$. Then $J_m(X)$ has precisely $\lfloor \frac{m+1}{2} \rfloor + 1$ irreducible components.*

Remark 7.4. — *There are examples when $J_m(X)$ is reducible even though $\dim \pi_{m,0}^{-1}(X_{\text{sing}}) < \dim \pi_{m,0}^{-1}(X_{\text{reg}})$. For example (still following [32]), if X is the variety of 3×3 size matrices of rank at most one. Then $\dim \pi_{1,0}^{-1}(X_{\text{sing}}) = 9 < 10 = \dim \pi_{1,0}^{-1}(X_{\text{reg}})$. However, the jet scheme $J_1(X)$ has two irreducible components and is therefore reducible. But here, X is not a lci!*

According to Proposition 7.1, to prove that $J_m(X)$ is irreducible it is enough to prove that:

- 1) : $\dim J_m(X) = \dim X(m+1)$,
- 2) : $J_m(X)$ has only one irreducible component of maximal dimension.

8. Irreducibility of jet schemes via motivic integration

We suppose that X is a lci. We use here motivic integration to give a necessary and sufficient condition for X to have all the jet schemes of the expected dimension and precisely one component of maximal dimension.

8.1. The following construction is very important in order to use arguments from motivic integration. We will take this construction again latter for the study of the nilpotent bicone.

We fix an embedding $X \hookrightarrow Y$, where Y is a smooth variety. Let r be the codimension of X in Y and denote by N the dimension of Y . Consider the blowing-up

$$p : B := Bl_X(Y) \rightarrow Y$$

of Y along X , and let $F = p^{-1}(X)$ be the exceptional divisor. We review Appendix A some results about vanishing theorems for integral divisors. By Theorem A.2, there is a morphism $\tilde{p} : \tilde{Y} \rightarrow B$ which is proper, an isomorphism over the complementary of a proper closed subset of F , and such that \tilde{Y} is smooth and $\tilde{p}^{-1}(F)$ is a divisor with only simple normal crossings (SNC), since the exceptional locus of \tilde{p} is contained in $\tilde{p}^{-1}(F)$. Let γ be the compound map $p \circ \tilde{p}$

$$\gamma : \tilde{Y} \xrightarrow{\tilde{p}} B \xrightarrow{p} Y$$

We can write $\gamma^{-1}(X) = \tilde{p}^{-1}(F) = \sum_{i=1}^t a_i E_i$, where E_2, \dots, E_t are the exceptional divisors of \tilde{p} , and E_1 is the proper transform of F . Recall that the discrepancy W_γ of γ is defined by the formula (see Appendix A, Definition A.4)

$$K_{\tilde{Y}} = \gamma^{-1}(K_Y) + W_\gamma .$$

We write $W_\gamma = \sum_{i=1}^t b_i E_i$. The following proposition, whose we pass over the proof, gives a criterion for X to have rational singularities.

Proposition 8.1 ([27](Theorem 2.1)). — (i) *With the above notations, X has rational singularities if and only if $b_i \geq ra_i$ for any $i \geq 2$.*

(ii) *We can assume $a_1 = 1$ and $b_1 = r - 1$.*

[The key step in the proof of Propostion 8.1 is to notice that since X is a lci, X has rational singularities if and only if X has canonical singularities (see Appendix B for a little bit more explanations).]

With these notations, we intend to prove the following results:

Theorem 8.2. — *The following two statements are equivalent:*

i) *For any $i \geq 2$, we have $b_i \geq ra_i$.*

ii) For any $i \geq 1$, $\dim J_m(X) = (m + 1) \dim X$, and $J_m(X)$ has only one irreducible component of maximal dimension.

Then Theorem 8.2 give Theorem 6.1, by Propositions 8.1 and 7.1.

8.2. Proof of Theorem 8.2. — Recall that the Hodge polynomials $h : \text{Var}_{\mathbb{C}} \rightarrow \mathbb{Z}[u, v]$ assigns to a smooth variety S over \mathbb{C} its Hodge polynomial

$$h(S) := \sum_{1 \leq p, q \leq \dim S} (-1)^{p+q} H^{p,q}(S) u^p v^q ,$$

where $H^{p,q}(S) = \dim H^q(S, \Omega_S^p)$ is the (p, q) -Hodge number of S (see Subsection 3.1). We have already noticed that the map h factors through $K_0(\text{Var}_{\mathbb{C}})$ according the universal property. So we have a morphism h from $K_0(\text{Var}_{\mathbb{C}})$ to $\mathbb{Z}[u, v]$ such that $h(S)$ is equal to $h([S])$ for any smooth variety S . In particular, $h(\mathbb{L}) = uv$. Setting $h(\mathbb{L}^{-1}) = (uv)^{-1}$, h extends to a morphism from $\mathcal{M}_{\mathbb{C}}$ to $\mathbb{Z}[u, v][u^{-1}, v^{-1}]$. Then by continuity, h extends in an unique way to a morphism from $\widehat{\mathcal{M}}_{\mathbb{C}}$ to the ring $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]]$ of Laurent power series in two variables u^{-1} and v^{-1} . Thus, there is a meaning to consider the compound $h \circ \mu_Y$. The “new” measure

$$h \circ \mu_Y : \mathbf{B}_Y \rightarrow \mathbb{Z}[u, v][[u^{-1}, v^{-1}]] , \pi_{\infty, m}^{-1}(S) \mapsto h([S])(uv)^{-(m+1)N}$$

gives the Hodge realizations of motivic integrals.

$$\begin{array}{ccc}
 \text{Var}_{\mathbb{C}} & \xrightarrow{h} & \mathbb{Z}[u, v][[u^{-1}, v^{-1}]] \\
 \downarrow [-] & \nearrow & \nearrow \\
 K_0(\text{Var}_{\mathbb{C}}) & & \\
 \downarrow & \nearrow & \nearrow \\
 \mathcal{M}_{\mathbb{C}} & & \\
 \downarrow & \nearrow & \nearrow \\
 \mathbf{B}_Y \xrightarrow{\mu_Y} \widehat{\mathcal{M}}_{\mathbb{C}} & &
 \end{array}$$

The most important thing for us is that $h([S])$ is a polynomial of degree $c(uv)^{\dim S}$, where c is the number of irreducible components of S of maximal dimension. In order to prove Theorem 8.2, we will chose a suitable function $f : \mathbb{N} \rightarrow \mathbb{N}$ (which we extend by $f(\infty) = \infty$) and we will integrate $F := f \circ F_X$ on Y_{∞} with respect to the measure $h \circ \mu_Y$. This is allowed since

$\mu_Y(F^{-1}(\infty)) = \mu_Y(F_X^{-1}(\infty)) = 0$. Then the transformation rule gives:

$$\mathbf{I} := h\left(\int_{J_\infty(Y)} \mathbb{L}^{-F} d\mu_Y\right) = h\left(\int_{J_\infty(\tilde{Y})} \mathbb{L}^{-(F \circ \gamma_\infty + F_{W_\gamma})} d\mu_{Y'}\right),$$

in the sense that one member exists if and only if the other one does, and in this case they are equal. We have

$$F \circ \gamma_\infty = f \circ F_{\gamma^{-1}(X)},$$

and since $\gamma^{-1}(X) \cup W_\gamma$ has SNC, the right-side member can be explicitly computed by Proposition 4.7, while for a suitable choice of f , the left-hand side contains the information we need about the dimension of $J_m(X)$ and the number of its irreducible components of maximal dimension.

So, let us fix a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any $m \geq 0$,

$$(7) \quad f(m+1) > f(m) + \dim J_m(X) + C(m+1),$$

where C is a constant “well-chosen”. We extend f by setting $f(\infty) = \infty$.

1) What is information about $J_m(X)$ contained the integral \mathbf{I} ?

For any $m' = f(m)$, we have

$$\mu_Y(F^{-1}(m')) = \mu_Y(F_X^{-1}(m)) = \pi_{\infty, m-1}^{-1}(J_{m-1}(X)) \setminus \pi_{\infty, m}^{-1}(J_m(X)).$$

Then $\mathbf{I} = h\left(\sum_{m' \geq 0} \mu_Y(F^{-1}(m')) \mathbb{L}^{m'}\right) = \mathbf{S}_1 - \mathbf{S}_2$, with

$$\begin{aligned} \mathbf{S}_1 &= \sum_{m \geq 0} h([J_{m-1}(X)])(uv)^{-mN - f(m)}, \\ \mathbf{S}_2 &= \sum_{m \geq 0} h([J_m(X)])(uv)^{-(m+1)N - f(m)}, \end{aligned}$$

where by convention $J_{-1}(X) = Y$. For $m \geq 1$, denote by $\boxed{c_m}$ the number of irreducible components of maximal dimension of $J_m(X)$. Remember that the highest degree term of $h([J_m(X)])$ is $c_m (uv)^{\dim J_m(X)}$. Then, we claim:

(*) Every monomial which appears in the m -th term of \mathbf{S}_1 has degree bounded by $2P_1(m)$ and below by $2P_2(m)$, where

$$\begin{aligned} P_1(m) &= \dim J_{m-1}(X) - mN - f(m), \\ P_2(m) &= -mN - f(m), \end{aligned}$$

for any $m \geq 0$. Moreover we have exactly one term of degree $2P_1(m)$ whose coefficient is c_{m-1} .

(*) Every monomial which appears in the m -th term of \mathbf{S}_2 has degree bounded by $2Q_1(m)$ and below by $2Q_2(m)$, where

$$\begin{aligned} Q_1(m) &= \dim J_m(X) - (m+1)N - f(m) , \\ Q_2(m) &= -(m+1)N - f(m) , \end{aligned}$$

for any $m \geq 0$. Moreover we have exactly one term of degree $2Q_1(m)$ whose coefficient is c_m .

Remark 8.3. — *Using additionally the “good” choice of f , we can deduce from the previous discussion that F is indeed integrable.*

Lemma 8.4 ([27](Lemma 1.2)). — *For any proper subscheme $Z \subseteq Y$, $(\dim J_m(Z) - N(m+1)) \rightarrow -\infty$.*

By Lemma 8.4, the inequality $\dim J_m(X) \leq \dim J_{m-1}(X) + N$ is strict for infinitely many m . Studying more carefully the sums \mathbf{S}_1 and \mathbf{S}_2 and their highest degree terms, we can furthermore state:

Proposition 8.5. — (i) $P_1(m+1) < \min(P_1(m), P_2(m))$ and $Q_1(m) \leq P_1(m)$

(ii) *In \mathbf{S}_1 , the term $c_{m-1}(uv)^{P_1(m)}$ appears precisely once for any $m \geq 0$. In \mathbf{S}_2 , the term $c_m(uv)^{P_1(m)}$ appears precisely at most once. It appears if and only if $m \geq 1$ and $\dim J_m(X) = \dim J_{m-1} + N$.*

By Lemma 8.4 and Proposition 8.5, if it happens that the term $c_m(uv)^{P_1(m)}$ appears in \mathbf{S}_2 , it is at most for finitely many m .

2) Explicit computation of the integral I. As foretold, we apply the transformation rule to F . Hence, we get:

$$\begin{aligned} \mathbf{I} &= h\left(\int_{J_\infty(Y)} \mathbb{L}^{-F} d\mu_Y\right) = h\left(\int_{J_\infty(\tilde{Y})} \mathbb{L}^{-(F \circ \gamma_\infty + F W_\gamma)} d\mu_{Y'}\right) \\ &= h\left(\int_{J_\infty(\tilde{Y})} \mathbb{L}^{-F(f \cdot \gamma^{-1}(X) + W_\gamma)} d\mu_{Y'}\right) . \end{aligned}$$

Up this form, \mathbf{I} can be explicitly computed since $f \cdot \gamma^{-1}(X) + W_\gamma$ has SNC. We apply the proof of Proposition 4.7 to the effective divisor with SNC

$$f \cdot \gamma^{-1}(X) + W_\gamma = \sum_{i=1}^t (fa_i + b_i)E_i .$$

Precisely, using the second equality in this proof, we have

$$h\left(\int_{J_\infty(Y)} \mathbb{L}^{-(F \circ \gamma_\infty + F_W)} d\mu_{Y'}\right) = \sum_{J \subseteq \{1, \dots, t\}} \mathbf{S}_J,$$

where

$$\mathbf{S}_J = \sum_{\alpha_i \geq 1, i \in J} h([E_J^0])(uv - 1)^{|J|} \cdot (uv)^{-N - \sum_{i \in J} \alpha_i (b_i + 1) - f(\sum_{i \in J} a_i \alpha_i)}.$$

Similarly as 1), we can compute the exact bounds for the degree of \mathbf{S}_J and discuss about the terms of fixed degrees. Namely, each monomial in the term \mathbf{S}_J corresponding to $(\alpha_i)_{i \in J}$ has degree bounded above by $2R_1(\alpha_i, i \in J)$ and below by $2R_2(\alpha_i, i \in J)$, where

$$R_1(\alpha_i, i \in J) = - \sum_{i \in J} \alpha_i (b_i + 1) - f\left(\sum_{i \in J} a_i \alpha_i\right)$$

and $R_2(\alpha_i, i \in J) = R_1(\alpha_i, i \in J) - N$. This comes the fact that E_J^0 has dimension $N - |J|$. And, arguing as in 1), we obtain:

Proposition 8.6. — *The only monomial of the form $(uv)^{P_1(m)}$ which appears in the term corresponding to J and $(\alpha_i)_{i \in J}$ is for $m = \sum_{i \in J} a_i \alpha_i$.*

3) The end of the proof. We leave the part “ii) \Rightarrow i)” of the proof out. For the implication “i) \Rightarrow ii)”, suppose that $b_i \geq ra_i$, for all $i \geq 2$ and suppose that $\dim J_m(X) > (m + 1) \dim X$. We expect a contradiction. Studying more carefully the sum \mathbf{S}_J , we see that $(uv)^{P_1(m+1)}$ does not appear in the sum \mathbf{S}_J , for every J . By Proposition 8.5 this implies that $\dim J_{m+1}(X) = \dim J_m(X) + N$. In particular $\dim J_{m+1}(X) > (m + 2) \dim X > 0$. Continuing in this way, we get by that $\dim J_{p+1}(X) = \dim J_p(X) + N$, for any $p \geq m$, whence the expected contradiction by Lemma 8.4. Therefore we must have $\dim J_m(X) = (m + 1) \dim X$, for any $m \geq 1$. It remains to prove $c_m = 1$, for any $m \geq 1$. The coefficient of $(uv)^{P_1(m+1)}$ in \mathbf{I} is c_m for all $m \geq 1$, by Proposition 8.5. Then we see that for all $m \geq 0$ the term $(uv)^{P_1(m+1)}$ appears in \mathbf{S}_J if and only if $J = \{1\}$ and in this case it has coefficient 1, since $E_{\{1\}}^0$ is irreducible. Therefore $c_m = 1$, for every $m \geq 1$.

PART III

APPLICATIONS TO REDUCTIVE LIE ALGEBRAS

In this part \mathfrak{g} is supposed to be a complex semisimple Lie algebra of finite dimension. Most of results can be easily extended to reductive Lie algebras. This just allows to kill the center, that is simplifies sometimes the definitions. We denote by G the adjoint group of \mathfrak{g} and by $\langle \cdot, \cdot \rangle$ the Killing form of \mathfrak{g} .

9. The nilpotent cone

Many results of this section are proved in [18]. We refer also to [8](Chapter 3) for a review of some of these results.

Recall that an element $x \in \mathfrak{g}$ is called *nilpotent* if $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent, where ad is the adjoint representation of \mathfrak{g} . This evidently agrees with the classical notion of nilpotency when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. We denote by \mathfrak{N} the set of all the nilpotent elements of \mathfrak{g} . Clearly, \mathfrak{N} is a G -invariant closed cone of \mathfrak{g} . Here, by *cone*, we means that \mathfrak{N} is invariants by the maps $x \mapsto ax$, for any $a \in \mathbb{C}^*$. The set \mathfrak{N} is so called the *nilpotent cone* of \mathfrak{g} . We study in this section some of algebraic and geometric properties of the nilpotent cone.

9.1. Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of polynomial functions on \mathfrak{g} . The adjoint action of G on \mathfrak{g} induces a G -action on $\mathbb{C}[\mathfrak{g}]$ and we denote by $\mathbb{C}[\mathfrak{g}]^G$ the set of G -invariant polynomial functions on \mathfrak{g} .

We fix a Borel subalgebra \mathfrak{b} in \mathfrak{g} with nilradical \mathfrak{u} . Let B the connected subgroup of G whose Lie algebra is \mathfrak{b} . Recall that an element $x \in \mathfrak{g}$ is *regular* if its centralizer has minimal dimension, that is the rank of \mathfrak{g} . The following result is due to Richardson:

Lemma 9.1 (Richardson). — *The nilradical \mathfrak{u} consists in a single B -orbit, namely $B.e$, where e is any regular nilpotent element contained in \mathfrak{b} .*

Example 9.2. — *Suppose that $\mathfrak{g} = \mathfrak{sl}_n$, the classical Lie algebra of n -size square matrices of trace zero. Let \mathfrak{b} the Borel subalgebra of upper triangular matrices of trace zero. Then B is the subgroup of GL_n of upper triangular matrices and \mathfrak{u} is the subspace of strictly upper triangular matrices. Lemma 9.1 says that \mathfrak{u} is the closure of the conjugation class under B of the n -size full Jordan block,*

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

which is a well-known result.

We deduce from Lemma 9.1 that for any G -invariant polynomial p on \mathfrak{g} , the restriction to \mathfrak{u} of p is constant, by continuity.

We denote by $\mathbb{C}[\mathfrak{g}]_+^G$ the augmentation ideal of $\mathbb{C}[\mathfrak{g}]^G$, that is the set of G -invariant polynomials without constant terms. The following result is due to Kostant. It generalizes a classical result of linear algebra: an element $x \in \mathfrak{sl}_n(\mathbb{C})$ is nilpotent if and only if $\text{Tr}(x^{i+1}) = 0$, for any $i = 1, \dots, n-1$

Proposition 9.3 (Kostant). — *An element $x \in \mathfrak{g}$ is nilpotent if and only if for any $p \in \mathbb{C}[\mathfrak{g}]_+^G$, we have $p(x) = 0$.*

Proof. — Let x be nilpotent. Then x belongs to \mathfrak{u} for some Borel algebra \mathfrak{b} . By Lemma 9.1, the restriction to \mathfrak{u} of any $p \in \mathbb{C}[\mathfrak{g}]_+^G$ is constant. Therefore, $p(x) = p(0) = 0$.

Conversely, suppose that $p(x) = 0$, for any $p \in \mathbb{C}[\mathfrak{g}]_+^G$. We observe that the coefficients of the characteristic polynomial $\det(\lambda \cdot \text{Id} - \text{ad}x)$ except the first one belongs to $\mathbb{C}[\mathfrak{g}]_+^G$. Hence, they vanish and $\det(\lambda \cdot \text{Id} - \text{ad}x) = \lambda^{\dim \mathfrak{g}}$. It follows that $\text{ad}x$ is nilpotent, so x is nilpotent. \square

By Proposition 9.3, the nilpotent cone is the subscheme of \mathfrak{g} corresponding to the ideal $\mathbb{C}[\mathfrak{g}]_+^G$, that is

$$\mathfrak{N} = \text{Spec} \mathbb{C}[\mathfrak{g}] / \mathbb{C}[\mathfrak{g}]_+^G .$$

By Chevalley's Theorem, the subalgebra $\mathbb{C}[\mathfrak{g}]^G$ is polynomial in rank of \mathfrak{g} variables. In particular, Proposition 9.3 implies that the dimension of \mathfrak{N} is at least $\text{rk} \mathfrak{g}$, the rank of \mathfrak{g} . We will see next section that \mathfrak{N} has exactly codimension $\text{rk} \mathfrak{g}$. In other words, we will see that \mathfrak{N} is a complete intersection.

9.2. Let \mathcal{B} be the set of all the Borel subalgebras in \mathfrak{g} and let \mathfrak{b} be the dimension of Borel subalgebras of \mathfrak{g} . By definition \mathcal{B} is the closed subvariety of the Grassmannian of \mathfrak{b} -dimensional subspaces in \mathfrak{g} consisting in all solvable Lie subalgebras. We set

$$\tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

and we denote by $\gamma : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ the first projection. Notice that the map γ is proper since \mathcal{B} is a projective variety. Now we set:

$$\tilde{\mathfrak{N}} := \gamma^{-1}(\mathfrak{N}) = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\} .$$

Fix a Borel subalgebra \mathfrak{b} in \mathcal{B} . If V is a finite dimensional vector space over \mathbb{C} and $\rho : B \rightarrow GL(V)$ is a morphism of algebraic groups then (V, ρ) is

called a B -module. The *contracted product* $G \times_B V$ is defined as the quotient of $G \times V$ under the right action of B given by

$$(g, v)b := (gb, \rho(b^{-1})v) ,$$

for all $g \in G$, $v \in V$ and $b \in B$. As \mathfrak{u} is a B -module, we can define $G \times_B \mathfrak{u}$. The group G acts on $G \times_B \mathfrak{u}$ by the left action of G on G , while G on $\tilde{\mathfrak{N}}$ by the diagonal action. Then we claim.

Lemma 9.4. — *The variety $G \times_B \mathfrak{u}$ is an irreducible smooth variety and we get a G -equivariant isomorphism $\tilde{\mathfrak{N}} \simeq G \times_B \mathfrak{u}$.*

Proof. — The fiber of the second projection $\pi : \tilde{\mathfrak{N}} \rightarrow \mathcal{B}$ over any Borel subalgebra in \mathcal{B} is isomorphic to the nilradical of this Borel subalgebra. Therefore $\tilde{\mathfrak{N}}$ is a vector bundle over \mathcal{B} with fiber \mathfrak{u} . In particular $\tilde{\mathfrak{N}}$ is a smooth variety.

On the other hand the map $\tau : G \times \mathfrak{u} \rightarrow \tilde{\mathfrak{N}}$, $(g, x) \mapsto (g(x), g(\mathfrak{b}))$ factors through $G \times_B \mathfrak{u}$:

$$\begin{array}{ccc} G \times \mathfrak{u} & \xrightarrow{\tau} & \tilde{\mathfrak{N}} \\ \downarrow & \nearrow \bar{\tau} & \\ G \times_B \mathfrak{u} & & \end{array}$$

The maps τ and $\bar{\tau}$ are surjective, since any nilpotent element is conjugated to an element in \mathfrak{u} . Moreover, if $(g(x), g(\mathfrak{b})) = (g'(x'), g'(\mathfrak{b}))$, for some $x, x' \in \mathfrak{u}$ and $g, g' \in G$, then $g^{-1}g'$ stabilizes B . So $g' = gb$, for $b \in B$, since the normalizer of B is equal to B . It follows that $(g(x), g(\mathfrak{b}))$ and $(g'(x'), g'(\mathfrak{b}))$ are equal in $G \times_B \mathfrak{u}$. Hence $\bar{\tau}$ is an isomorphism. At last, it is clear that $\bar{\tau}$ is G -equivariant. \square

The nilpotent cone \mathfrak{N} is always singular, at least at the origin. By Lemma 9.4, the morphism $\gamma : \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ gives a proper morphism from a smooth variety to the nilpotent cone. Moreover, we have the following theorem:

Theorem 9.5. — *The map $\gamma : \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ is a birational map. Thus γ is a resolution of singularities for \mathfrak{N} .*

Proof. — The map γ is proper and is surjective since any nilpotent element is conjugated to an element in the nilradical \mathfrak{u} of \mathfrak{b} . Moreover, γ is an isomorphism over the Zariski open set of \mathfrak{N} formed by regular nilpotent element. This comes from the fact that any regular nilpotent element is contained in an unique Borel subalgebra. Thus γ is a resolution of singularities for \mathfrak{N} , since $\tilde{\mathfrak{N}}$ is smooth. \square

The map $\gamma : \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ is called the *Springer's resolution*.

Example 9.6. — Let $\mathfrak{g} = \mathfrak{sl}_2$. Then \mathfrak{N} is isomorphic to the cone in \mathbb{C}^3 ,

$$\mathfrak{N} = \left\{ x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid \det x = -a^2 - bc = 0 \right\},$$

and $\tilde{\mathfrak{N}}$ is a line bundle over $\mathcal{B} \simeq \mathbb{P}^1$ with fiber $\mathfrak{u} \simeq \mathbb{C}$. In this case, \mathfrak{N} has an unique singularity at the origin.

Corollary 9.7. — The nilpotent cone is irreducible and has dimension $\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}$.

Proof. — By Theorem 9.5, as γ is surjective, the nilpotent cone is irreducible since $\tilde{\mathfrak{N}}$ is irreducible. Moreover, as γ is surjective, the dimension of \mathfrak{N} is smaller than

$$\dim \tilde{\mathfrak{N}} = 2 \dim \mathfrak{u} = \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}.$$

The other inequality is already known. \square

9.3. We establish in this subsection that \mathfrak{N} has rational singularities using the Springer's resolution. We start by proving that \mathfrak{N} is normal. We admit the following result:

Proposition 9.8. — The number of nilpotent G -orbits in \mathfrak{g} is finite.

For \mathfrak{sl}_n this result is well-known; it follows from the decomposition of nilpotent matrices into Jordan blocks which gives a bijection between nilpotent G -orbits and partitions of n .

Since \mathfrak{N} is irreducible and G -stable, we deduce from Proposition 9.8, that there is an unique dense open orbit \mathcal{O} whose dimension is $\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}$. This implies that the centralizer of any element $x \in \mathcal{O}$ has dimension $\operatorname{rk} \mathfrak{g}$, that is x is regular. Hence, we get:

Corollary 9.9. — The regular nilpotent orbit is dense in \mathfrak{N} .

Let r be the rank of \mathfrak{g} . Recall that by Chevalley's Theorem, $\mathbb{C}[\mathfrak{g}]^G$ is a polynomial algebra in r variables. We denote by p_1, \dots, p_r homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ of degree d_1, \dots, d_r respectively. The following result is proven in [18]:

Lemma 9.10. — Let x be in \mathfrak{g} . The differentials at x of p_1, \dots, p_r are linearly independent if and only if x is regular.

Recall a Serre's criterium for normality (we present here a simpler version of the Serre-Cohen-Macaulay criterium): Let X be an algebraic variety satisfying the following two conditions:

- (1) X is a complete intersection,
- (2) X is regular in codimension 1, that is to say, the singular locus of X has codimension in X at least 2,

Then, X is normal.

By Lemma 9.10, Corollary 9.9 and Proposition 9.3, \mathfrak{N} is regular in codimension 1.

[Indeed, as the dimension of any G -orbit in \mathfrak{g} is even, an element $x \in \mathfrak{N}$ belongs to the singular locus of the scheme \mathfrak{N} if and only if x belongs to $\mathfrak{N} \setminus \mathcal{O}_{\text{reg}} = \overline{\mathcal{O}_{\text{subreg}}}$ by Lemma 9.10. But $\mathcal{O}_{\text{subreg}}$ has dimension at most $\dim \mathfrak{N} - 2$ (in fact has exactly this dimension...)]

Therefore, by Serre's criterium and Corollary 9.7, we can now state:

Proposition 9.11. — *The nilpotent cone is normal.*

The definition of *rational singularities* is recalled in Appendix B, Definition B.1. In order to apply Theorem 6.1 to the nilpotent cone, it remains to prove that \mathfrak{N} has rational singularities. By Theorem 9.11, the nilpotent cone \mathfrak{N} is normal and by Theorem 9.5, γ is a resolution of singularities for \mathfrak{N} . Hence, by the following result due to Hesselink,

Proposition 9.12 (Hesselink [14]). — *We have $\gamma^*(\mathcal{O}_{\tilde{\mathfrak{N}}}) = \mathcal{O}_{\mathfrak{N}}$ and $R^p\gamma_*(\mathcal{O}_{\tilde{\mathfrak{N}}}) = 0$, for any $p \geq 1$.*

we deduce:

Theorem 9.13. — *The nilpotent cone \mathfrak{N} is a complete intersection in \mathfrak{g} of codimension r and it has rational singularities.*

10. Jet schemes of the nilpotent cone

We present in this section a result due to D. Eisenbud and E. Frenkel (see Appendix of [27]). In [18], B. Kostant proves:

Theorem 10.1 (Kostant). — *The algebra $\mathbb{C}[\mathfrak{g}]$ is free over $\mathbb{C}[\mathfrak{g}]^G$.*

[More precisely, we have $\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G \otimes H$, where H is any graded subspace such that $\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]_+^G + H$.]

This result has important applications. In particular, it implies that the universal enveloping algebra $U(\mathfrak{g})$ is free over its center.

David Eisenbud and Edward Frenkel originally conjectured Theorem 6.1 in order to extend Theorem 10.1 in the setting of jet schemes. Denote simply by \mathfrak{g}_n the n -order jet scheme $J_n(\mathfrak{g})$ of \mathfrak{g} . Then $\mathfrak{g}_n = \mathfrak{g}[t]/(t^{n+1})$ and $\mathfrak{g}_\infty = \mathfrak{g}[[t]]$. Let G_n be the n -order jet scheme of G and let G_∞ be the infinite jet scheme of G . Then the Lie algebra of G_n is \mathfrak{g}_n and the Lie algebra of G_∞ is \mathfrak{g}_∞ . Denote by $\mathbb{C}[\mathfrak{g}_n]^{G_n}$ (resp. $\mathbb{C}[\mathfrak{g}_\infty]^{G_\infty}$) the subalgebra of G_n -invariants (resp. G_∞ -invariants) of $\mathbb{C}[\mathfrak{g}_n]$ (resp. $\mathbb{C}[\mathfrak{g}_\infty]$) under the adjoint action. The analogue of the Kostant freeness theorem is:

Theorem 10.2 (Eisenbud-Frenkel). — (i) $\mathbb{C}[\mathfrak{g}_n]$ is free over $\mathbb{C}[\mathfrak{g}_n]^{G_n}$.
(ii) $\mathbb{C}[\mathfrak{g}_\infty]$ is free over $\mathbb{C}[\mathfrak{g}_\infty]^{G_\infty}$.

Let N be the dimension of \mathfrak{g} and let x_1, \dots, x_N be a basis of \mathfrak{g}^* , which we take as a set of generators of $\mathbb{C}[\mathfrak{g}]$. Fix p_1, \dots, p_r homogenous generators of $\mathbb{C}[\mathfrak{g}]^G$ as before of degree d_1, \dots, d_r respectively. Recall that $p_i^{(m)}$, for $m \geq 1$ and $i = 1, \dots, r$ are defined for instance Example 6.3 by the relation:

$$(8) \quad p_i^{(m)}(\mathbf{y}(t)) = p_i(\partial_t^m \mathbf{y}(t))|_{t=0}, \quad \mathbf{y}(t) \in \mathfrak{g}_n, n \geq m, \text{ or } \mathfrak{g}_\infty.$$

We have: $\deg p_i^{(m)} = d_i + m$. Since the set $\{p_1, \dots, p_r\}$ is algebraically independent, it follows from relation (8) that the set $\{p_1^{(m)}, \dots, p_r^{(m)}\}_{m \geq 0}$ is also algebraically independent. Again using relation (8), the elements $p_i^{(m)}$ are G_n -invariant for $n \geq m$ and G_∞ -invariant, because the elements p_i are G -invariant. We refer to [27](Appendix) for the proof of the following proposition:

Proposition 10.3 ([4]). — The ring $\mathbb{C}[\mathfrak{g}_n]^{G_n}$ (resp. $\mathbb{C}[\mathfrak{g}_\infty]^{G_\infty}$) is generated by the algebraically independent elements $p_1^{(m)}, \dots, p_r^{(m)}$, $0 \leq m \leq n$ (resp. $m \geq 0$). Thus, $\mathbb{C}[\mathfrak{g}_n]^{G_n} = \mathbb{C}[p_1^{(m)}, \dots, p_r^{(m)}]_{0 \leq m \leq n}$ and $\mathbb{C}[\mathfrak{g}_\infty]^{G_\infty} = \mathbb{C}[p_1^{(m)}, \dots, p_r^{(m)}]_{m \geq 0}$.

Let $\mathbb{C}[\mathfrak{g}_n]_+^{G_n}$ be the augmentation ideal of the graded ring $\mathbb{C}[\mathfrak{g}_n]^{G_n}$. By Proposition 10.3, the ideal $\mathbb{C}[\mathfrak{g}_n]_+^{G_n}$ is equal to $(p_1^{(m)}, \dots, p_r^{(m)})_{0 \leq m \leq n}$. Hence we obtain that the n -order jet scheme \mathfrak{N}_n of the nilpotent cone \mathfrak{N} is $\text{Spec} \mathbb{C}[\mathfrak{g}_n] / \mathbb{C}[\mathfrak{g}_n]_+^{G_n}$. Likewise, $\mathfrak{N}_\infty = \text{Spec} \mathbb{C}[\mathfrak{g}_\infty] / \mathbb{C}[\mathfrak{g}_\infty]_+^{G_\infty}$.

By Proposition 9.7, the nilpotent cone is a complete intersection, which is irreducible and reduced. Moreover, by Theorem 9.13, it has rational singularities. Therefore, we obtain by Theorem 6.1:

Theorem 10.4. — \mathfrak{N}_n is irreducible, reduced and a complete intersection.

In Theorem 10.4, we apply also [27](Proposition 1.5) for the reduced property.

Corollary 10.5. — The natural map $\mathbb{C}[\mathfrak{N}_n] \longrightarrow \mathbb{C}[\mathfrak{N}_{n+1}]$ is an embedding.

Proof. — Let Y be the open dense G -orbit of regular elements in \mathfrak{N} . By Theorem 10.4, \mathfrak{N}_n is irreducible. Hence Y_n is dense in \mathfrak{N}_n . Since Y is smooth, the map $Y_{n+1} \rightarrow Y_n$ is surjective. Therefore, the map $\mathfrak{N}_{n+1} \rightarrow \mathfrak{N}_n$ is dominant. \square

We can now prove Theorem 10.2:

Proof of Theorem 10.2. — Let us admit (i), which actually doesn't use Corollary 10.5.

ii) Set $I_n := \mathbb{C}[\mathfrak{g}_n]^{G_n}$, for $n \in \mathbb{N} \cup \{\infty\}$. We use the embedding of Corollary 10.5 to construct “step by step” a basis of $\mathbb{C}[\mathfrak{g}_\infty]$ over I_∞ .

By (i), we can choose a graded basis S_n of $\mathbb{C}[\mathfrak{g}_n]$ over $\mathbb{C}[\mathfrak{g}_n]^{G_n}$. Then the image S'_n of S_n in $\mathbb{C}[\mathfrak{N}_n]$ is a \mathbb{C} -basis of $\mathbb{C}[\mathfrak{N}_n]$. By Corollary 10.5, the image of S'_n in $\mathbb{C}[\mathfrak{N}_{n+1}]$ can be extended to a \mathbb{C} -basis of $\mathbb{C}[\mathfrak{N}_{n+1}]$. Hence the image of S_n in $\mathbb{C}[\mathfrak{g}_{n+1}]$ can be extended to a I_{n+1} -basis of $\mathbb{C}[\mathfrak{g}_{n+1}]$.

Let S be the union of all the set S_n , $n \geq 0$. Then we claim that S is the basis of $\mathbb{C}[\mathfrak{g}_\infty]$ over $\mathbb{C}[\mathfrak{g}_\infty]^{G_\infty}$. \square

11. Dimension of the nilpotent bicone via motivic integration

11.1. Introduction. — Fix p_1, \dots, p_r homogenous generators of $\mathbb{C}[\mathfrak{g}]^G$ as before of degree d_1, \dots, d_r respectively. The 2-order polarizations $p_{i,m,n}$ of p_i are defined by the following relation:

$$p_i(ax + by) = \sum_{m+n=d_i} a^m b^n p_{i,m,n}(x, y) ,$$

for any (a, b) in \mathbb{C}^2 and any (x, y) in $\mathfrak{g} \times \mathfrak{g}$.

Definition 11.1. — The nilpotent bicone \mathcal{N} of \mathfrak{g} is by definition the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of the 2-order polarizations $p_{i,m,n}$, $1 \leq m + n \leq d_i$.

As the nilpotent cone \mathfrak{N} is the nullvariety in \mathfrak{g} of the p_i by Theorem 9.3, we deduce that the geometric locus of the nilpotent bicone is the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ whose subspace generated by x and y is contained in the nilpotent cone. In particular \mathcal{N} is a G -invariant closed bicone of $\mathfrak{g} \times \mathfrak{g}$.

By a classical result, the cardinality of the set of polarizations $\{p_{i,m,n}, 1 \leq m + n \leq d_i\}$ is equal to $b + r$, where b is as before the dimension of Borel subalgebras of \mathfrak{g} . As a result, any irreducible component of \mathcal{N} has dimension at least $3(b - r)$, since $\mathfrak{g} \times \mathfrak{g}$ has dimension $2(2b - r)$.

Example 11.2. — Let \mathfrak{g} be the classical Lie algebra \mathfrak{sl}_3 of 3-size square matrices of trace zero. The rank of \mathfrak{g} is 2 and we can choose for p_1 and p_2 the polynomials $x \mapsto \text{Tr}(x^2)$ and $x \mapsto \text{Tr}(x^3)$. Then the 2-order polarizations of p_1 and p_2 are the following polynomials:

$$(x, y) \longmapsto \text{Tr}(x^2), \text{Tr}(xy), \text{Tr}(y^2), \text{Tr}(x^3), \text{Tr}(x^2y), \text{Tr}(xy^2), \text{Tr}(y^3) .$$

In particular, the nilpotent bicone is defined by $b + r = 5 + 2 = 7$ equations. So, any irreducible component of \mathcal{N} has dimension at least $16 - 7 = 9$. Moreover, we can describe the union of the irreducible components of \mathcal{N} whose images by the first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} is equal to the nilpotent

cone. Indeed, let X be this union and let e be the regular nilpotent element $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the fiber X_e at e in X is an union of two irreducible components of dimension 3: \mathfrak{u} and

$$\left\{ \begin{pmatrix} a & b & c \\ d & -2a & e \\ 0 & -d & a \end{pmatrix} \mid 2a^3 - ad(e-b) + cd^2 = 0, 3a^2 - d(e-b) = 0 \right\}.$$

In a joint work with Jean-Yves Charbonnel [7], we prove the following result:

Theorem 11.3. — *The nilpotent bicone \mathcal{N} is a complete intersection (non-reduced) of dimension $3(b-r)$. Moreover the images of any irreducible components of \mathcal{N} by the first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} is equal to the nilpotent cone.*

The *nullcone* of $\mathfrak{g} \times \mathfrak{g}$ is the nullvariety of the augmentation ideal of $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G$. In [19], N. Wallach and H. Kraft conjecture:

Conjecture 11.4 (Kraft-Wallach). — *The nullcone of $\mathfrak{g} \times \mathfrak{g}$ is an irreducible component of $\mathcal{N}_{\mathfrak{g}}$.*

As a by-product of Theorem 11.3, we answer their conjecture affirmatively.

Other motivation: Recall that the *commuting variety* $\mathcal{C}_{\mathfrak{g}}$ of \mathfrak{g} is the set of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. The commuting variety has been studied for many years. According to a result of R.W. Richardson [30], $\mathcal{C}_{\mathfrak{g}}$ is irreducible. In addition, $\mathcal{C}_{\mathfrak{g}}$ is the nullvariety of the ideal generated by the elements $(x, y) \mapsto \langle v, [x, y] \rangle$, where v runs through \mathfrak{g} . An old unsolved question is to know whether this ideal is prime [23]. In other words, we want to know if this ideal is the ideal of definition of $\mathcal{C}_{\mathfrak{g}}$, since $\mathcal{C}_{\mathfrak{g}}$ is irreducible. It is not easy to see but it is known that the properties of the nilpotent bicone are very important for the understanding of the commuting variety. The study of the commuting variety and of its ideal of definition is a main motivation for our work.

The main difficulty encountered with the nilpotent bicone is that the subscheme $\mathcal{N}_{\mathfrak{g}}$ is not reduced. In order to deal with this problem, we introduce an auxiliary reduced variety, that we call *the principal bicone* (see Subsection 11.2). The nilpotent bicone and the principal bicone are closely related to the jet schemes of the nilpotent cone and the principal cone respectively. We apply Mustața's result to the nilpotent cone and the principal cone and we use arguments from motivic integration to study the nilpotent bicone and the principal bicone (see Subsections 11.3 and 11.4).

11.2. The principal cone and the principal bicone. — Let (e, h, f) be a *principal* \mathfrak{sl}_2 -triple of \mathfrak{g} . This means that (e, h, f) satisfies the *\mathfrak{sl}_2 -triple relations* and that e and f are regular nilpotent elements of \mathfrak{g} . Hence h is a regular semisimple element. By a theorem of Kostant, the G -orbit of h doesn't depend on the principal \mathfrak{sl}_2 -triple choosen.

Definition 11.5. — *The principal cone \mathfrak{X} is the G -invariant closed cone generated by h .*

Notice that the principal cone contains \mathfrak{N} . The following theorem is the analogue of Theorem 9.13 for the principal cone:

Theorem 11.6. — *\mathfrak{X} is a normal complete intersection of codimension $r - 1$. Moreover, \mathfrak{X} has rational singularities.*

Denote by q_1, \dots, q_{r-1} generators of the ideal of definition of \mathfrak{X} . We define the *principal bicone* with respect to the principal cone as the nilpotent bicone was defined with respect to the nilpotent cone. Namely:

Definition 11.7. — *The principal bicone \mathcal{X} is the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of the 2-order polarizations $q_{i,m,n}$ of the q_i , for $1 \leq i \leq r - 1$.*

We can take for p_1 the Casimir element $x \mapsto \langle x, x \rangle$. Then, the nilpotent bicone is the nullvariety in \mathcal{X} of $p_{1,0,1}$, $p_{1,1,1}$ and $p_{1,1,0}$, the 2-order polarizations of p_1 . Denote by π_1 and π_2 the first and second projections of from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} . We intend to prove:

Theorem 11.8. — (i) *The images of any irreducible component of maximal dimension of \mathcal{N} by π_1 and π_2 are equal to \mathfrak{N} .*

(ii) *The images of any irreducible component of maximal dimension of \mathcal{X} by π_1 and π_2 are equal to \mathfrak{X} .*

We admit that Theorem 11.8 implies the following Theorem:

Theorem 11.9. — (i) *The principal bicone is a reduced complete intersection of dimension $3(b - r + 1)$.*

(ii) *The nilpotent bicone is a complete intersection (not reduced) of dimension $3(b - r)$.*

We prove Theorem 11.8 next subsections using arguments from motivic integration, following the main ideas of [27].

11.3. The principal bicone and the nilpotent bicone in terms of jet schemes. — As defined, the nilpotent bicone and the principal bicone can be identified to subsets of $J_\infty(\mathfrak{N})$ and $J_\infty(\mathfrak{X})$ respectively. More precisely, (x, y) belongs to \mathcal{N} (resp \mathcal{X}) if and only if the arc $t \rightarrow x + ty$ belongs to $J_\infty(\mathfrak{N})$ (resp $J_\infty(\mathfrak{X})$). With this “naive” identification, the motivic measure $\mu_{\mathfrak{g}}$ of the subsets obtained in this way is equal to zero according to Definition-Proposition 4.1, (2). Then we can’t expect to obtain information from arguments of motivic integration by this way. So we need to make a subtler construction...

Let us explain roughly this construction. We introduce more general notations. Let V be a finite dimensional vector space. As V is a vector space, there is a canonical injection $\iota_{m,m+1}$ from $J_m(V)$ into $J_{m+1}(V)$. The first projection from $V \times V$ to V is still denoted by π_1 . Let K be a connected closed subgroup of $\mathrm{GL}(V)$ and let X be an irreducible closed cone in V . We suppose that X is a complete intersection in V with rational singularities and that X is a finite union of K -orbits. Let N be the dimension of V and let r be the codimension of X in V . Let T be a closed bicone of $X \times V$ satisfying the two following conditions:

- 1) T is K -invariant under the diagonal action of K in $V \times V$,
- 2) for any (x, y) in T , y is a tangent vector of X at x ,
- 3) The image of T by π_1 is X .

We will apply this in the following to $V = \mathfrak{g}$, $X = \mathfrak{N}$ or $X = \mathfrak{X}$ and $T = \mathrm{T}\mathfrak{N} \simeq J_1(\mathfrak{N})$ or $T = \mathrm{T}\mathfrak{X} \simeq J_1(\mathfrak{X})$.

We define by induction on m a subset C_m of $J_m(V)$. We explain the construction of C_m in the case where $X = \mathfrak{N}$ and $T = \mathrm{T}\mathfrak{N}$ by the following picture:

$$\begin{array}{ccccc}
\mathbf{C}_1 := T\mathfrak{N} \subset \mathfrak{g}_1 & \xrightarrow[1]{\iota_{12}} & C'_2 = \iota_{12}(C_1) \subset \mathfrak{g}_2 & \xrightarrow{\iota_{23}} & C'_3 = \iota_{23}(C_2) \subset \mathfrak{g}_3 \dashrightarrow \dots \\
& & \uparrow & \nearrow 4 & \uparrow \\
& & \mathbf{C}_2 := \bigcap_{p \geq 2} \overline{\pi_{\infty,2}(C_{\infty,2,p})} & & \mathbf{C}_3 := \bigcap_{p \geq 3} \overline{\pi_{\infty,3}(C_{\infty,3,p})} \\
& & \uparrow 3 & & \uparrow 6 \\
& & \pi_{\infty,2} & & \pi_{\infty,3} \\
& & \downarrow 2 & & \downarrow 5 \\
C_{\infty,2,p} = \pi_{\infty,2}^{-1}(C'_2) \cap F_X^{-1}(\geq p), p \geq 2 & & & & C_{\infty,3,p} = \pi_{\infty,3}^{-1}(C'_3) \cap F_X^{-1}(\geq p), p \geq 3
\end{array}$$

Notice that for $m = 1$, C_1 is irreducible and has dimension $2 \dim X$ by Theorem 6.1. By induction, the subsets C_m are closed subsets of $J_m(V)$, for any positive integer m . For m bigger than 1, we set

$$D_m := \{ (x, y) \in X \times V \mid t \mapsto x + ty \in C_m \} .$$

Proposition 11.10. — *For m big enough we have*

$$D_m := \{ (x, y) \in T \mid x + ty \in X, \forall t \in \mathbb{C} \} .$$

In other words, for m big enough \mathcal{N} and \mathcal{X} are identified to a subset C_m , for a good choices of V , X and T . Now, by Proposition 11.10, in order to Theorem 11.8, it is enough to prove:

Theorem 11.11. — *For m big enough, the image by $\pi_{m,0}$ of any irreducible component of maximal dimension of C_m is equal to X .*

11.4. Proof of Theorem 11.11 via motivic integration. — Here we process almost as in Section 8. We denote by Z the union of K -orbits in X which are not of maximal dimension. Let Bl_X be the blowing up of V whose center is X and let

$$p : Bl_X \longrightarrow V$$

be the canonical morphism from Bl_X to V . By the theorem of embedded desingularization of Hironaka (see Theorem A.2), there exists a desingularization (\tilde{Y}, \tilde{p}) of Bl_X such that $(p \circ \tilde{p})^{-1}(X)$ is a SNC divisor. Let us denote by γ the morphism $p \circ \tilde{p}$. Denoting by E_1, \dots, E_t the irreducible components of $\gamma^{-1}(X)$, we can assume that the following conditions are fulfilled:

- E_1 is the only prime divisor dominating X ,
- the divisor $\gamma^{-1}(X)$ is equal to $\sum_{i=1}^t a_i E_i$,

- c) the discrepancy W_γ of γ is equal to $\sum_{i=1}^t b_i E_i$,
- d) a_1 is equal to 1 and $b_1 + 1$ is equal to r ,
- e) $\gamma^{-1}(Z)$ is contained in the union of E_2, \dots, E_t ,

since K has finitely many orbits in X . By condition (b), E_1, \dots, E_t are K -invariant. Moreover, by condition (e), Z is the image by γ of the union of E_2, \dots, E_t . So there exist nonnegative integers c_2, \dots, c_t , such that $\gamma^{-1}(Z)$ is the divisor equal to $\sum_{i=2}^t c_i E_i$.

Proof of Proposition 11.11. — We warn that we voluntarily present here a very simplified version of this proof. It contains some inaccuracies. We refer to [7] for more explanations and more precisions.

Let C'_m be the union of irreducible components of C_m whose image by $\pi_{m,0}$ is equal to X . Denote by d_m (resp. d'_m) and c_m (resp. c'_m) the dimension of C_m (resp. C'_m) and the number of irreducible components of maximal dimension of C_m (resp. C'_m). Then, it is enough to prove: $c_m = c'_m$ and $d_m = d'_m$.

Recall that the Hodge polynomial h was defined and extended to $\widehat{\mathcal{M}}_{\mathbb{C}}$ in Section 8. We consider the motivic integral

$$\mathbf{I}_{m,k} := h \left(\int_{\pi_{\infty,m}^{-1}(C_m)} \mathbb{L}^{-kF_Z(\nu)} d\mu_V(\nu) \right), \quad k \geq 0.$$

The end of the proof consists in the following steps:

1) We claim that the highest degree term of $\mathbf{I}_{m,k}$ doesn't depend on k for m big enough.

2) Then, with $k = 0$ in $\mathbf{I}_{m,k}$, we obtain the polynomial $h \circ \mu_V(\pi_{\infty,m}^{-1}(C_m))$ whose highest degree $c_m(uv)^{d_m}$ term contains the information we need.

3) On the other hand, by the transformation rule for motivic integrals, we have:

$$\mathbf{I}_{m,k} = h \left(\int_{\gamma_{\infty}^{-1}(\pi_{\infty,m}^{-1}(C_m))} \mathbb{L}^{-F_{W_\gamma}(\nu) - k \sum_{i=2}^t c_i F_{E_i}(\nu)} d\mu_{\widehat{Y}}(\nu) \right).$$

Since X is a lci and has rational singularities, we can use the criterium of Proposition 8.1. Moreover, as $W_\gamma + k \sum_{i=2}^t c_i E_i$ is a SNC divisor, we can explicitly compute the integral $\mathbf{I}_{m,k}$. Then, we are able to prove that, for k and m big enough, the highest degree term of $\mathbf{I}_{m,k}$ only depend on the irreducible components of C_m whose image by $\pi_{m,0}$ is X , that is C'_m .

4) Arguing in the same way with C'_m instead of C_m in Steps 1), 2), 3) we deduce the two equalities: $c_m = c'_m$ and $d_m = d'_m$.

□

Appendix A Vanishing Theorems

We review in this appendix some basics about vanishing theorems for integral divisors. We intend through the results of this appendix to explain how process to use the transformation rule for motivic integrals (Section 5 (Theorem 5.1)). We refer to [21] for more details about the following results.

Let X be a non-singular variety of dimension n .

Definition A.1 (Simple Normal Crossing (SNC))

An effective divisor $D = \sum_i D_i$ on X has simple normal crossings (and D is a simple normal crossing or SNC divisor) if D is reduced, each component D_i is smooth, and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 z_2 \dots z_k = 0 ,$$

for $k \leq n$. A divisor $E = \sum_i a_i D_i$ has simple normal crossings support if the underlying reduced divisor $D = \sum_i D_i$ has simple normal crossings.

Confronted with an arbitrary divisor, the first step in many situations is to perform some blowing-up to bring it into normal crossing divisor. Hironaka's theorem on resolution of singularities guarantees that this is possible:

Theorem A.2 (Hironaka's embedded resolutions of singularities)

Let X be an irreducible complex variety and let $D \subseteq X$ be an effective divisor on X .

i) There is a proper birational morphism

$$\mu : X' \longrightarrow X ,$$

where X' is non-singular and μ has divisorial exceptional locus $\text{except}(\mu)$, such that

$$\mu^* D + \text{except}(\mu)$$

is a divisor with SNC support, where μ^* is the inverse image of μ .

ii) We can construct X' via a sequence of blowing-up along smooth centers supported in the singular loci of D and X . In particular, one can assume that μ is an isomorphism over

$$X \setminus (\text{Sing}(X) \cup \text{Sing}(D)) .$$

The variety X' is called *an embedded of the pair* (X, D) . Recall that the *exceptional locus* of μ is the set of points at which μ fails to be biregular. When X is itself smooth — which is the case in many situations — this is the same as the locus of points that lie on a positive dimensional fiber of μ .

Recall now some notions which come into the transformation rule for motivic integral. Let X be a complex algebraic variety and denote by \mathcal{O}_X the structure sheaf of X . Recall that any divisor D on X determines a line bundles $\mathcal{O}_X(D)$. Conversely, if X is additionally reduced or projective, then any line bundle arises from a divisor.

Definition A.3 (Canonical divisor). — Suppose that X is a non-singular algebraic variety of dimension n . Denote by ω_X the line bundle Ω_X^n , which is the n th exterior power of the cotangent bundle Ω_X on X . Then, the canonical divisor on X , that we denote by K_X , is the divisor on X defined by

$$\mathcal{O}_X(K_X) = \omega_X .$$

Definition A.4 (Discrepancy). — Let $\gamma : X' \rightarrow X$ be a proper birational map between algebraic varieties. Then the discrepancy divisor W_γ of γ is defined by the relation

$$W_\gamma := K_{X'} - \gamma^*(K_X) .$$

Example A.5. — Let Y be a smooth variety and let X be a smooth variety in Y of codimension r in Y . Denote by $p : Y' = \text{Bl}_X(Y) \rightarrow Y$ the blowing-up of Y along X . The morphism p is a proper birational map and by [13](II, Exercise 8.5, (b)), we have $W_p = (r-1)D$, where D is the exceptional divisor of the blowing-up, that is $p^{-1}(X)$.

Appendix B

Rational singularities

We recall in this appendix some definitions about varieties and existing relationships between them.

Let X be a complex algebraic variety of dimension d . We said that X has a *resolution of singularities* if we can find a non-singular variety X' and a proper birational map from X' to X , which is an isomorphism over the non-singular points of X' . (The condition that the map is proper is needed to exclude

trivial solutions, such as taking X to be the subvariety of non-singular points of X .)

Definition B.1 (Rational singularities). — We say that X has rational singularities if X is normal and if there exists a resolution $\gamma : X' \rightarrow X$ such that the higher right derived functors of γ_* applied to $\mathcal{O}_{X'}$ are trivial. That is, $R^i\gamma_*\mathcal{O}_{X'} = 0$, for $i > 0$. If there is one such resolution, then it follows that all resolutions share this property, since any two resolutions of singularities can be dominated by another.

We refer to [12] for the definition of *canonical singularities*:

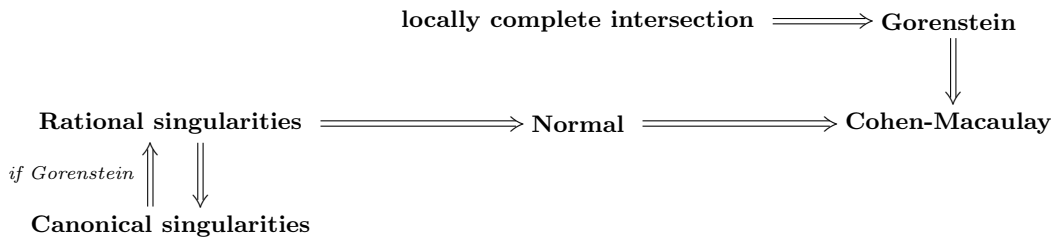
Theorem B.2 ([12](Theorem 1)). — If Y is a scheme of finite type over a field with characteristic zero with rational singularities, then Y has canonical singularities.

Definition B.3 (Gorenstein). — (i) A local Cohen-Macaulay ring R is called Gorenstein if there is a maximal R -regular sequence in the maximal ideal generating an irreducible ideal.

(ii) A variety X is called Gorenstein if the local ring $\mathcal{O}_{X,x}$ of regular function at x is Gorenstein.

For example, every locally complete intersection is Gorenstein and every smooth variety is Gorenstein. If X has rational singularities. Then it is in particular normal and Cohen-Macaulay). But Gorenstein varieties don't need to have rational singularities.

If X is a locally complete intersection (lci), hence Gorenstein, by a result of Elkik, X has rational singularities if and only if it has canonical singularities. As a conclusion, we present in the following diagram the relations between all of these notions.



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