The Black–Scholes semigroup of mathematical finance is chaotic

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Abstract

To nonmathematicians, the three standard everyday examples of "chaos" are the weather, marriage, and the stock market. Weather prediction led to the theory of chaos, which was thought to be solely a nonlinear phenomenon. But in recent years it was discovered that chaos could occur in infinite dimensional linear situations. There are nice "toy" examples of this, but these examples do not arise from a scientific context. Of the three examples mentioned above, the first two are clearly nonlinear. But in developing a theory of stock options in the 1970s, Black, Merton and Scholes derived an important linear parabolic PDE that greatly advanced mathematical finance and earned a Nobel Prize in Economics. As was typical in a Markov process context, the governing semigroup was contractive on a sup norm space, and hence nonchaotic as it could not have a dense orbit. But the correct "initial value" (from their derivation) for their stock option equation was an unbounded function. So the "right" space or spaces for the problem had to be found. We found candidates for these spaces and proved that the corresponding BMS semigroup is chaotic on these spaces. The spaces are

$$Y^{s,\tau} = \{u \in C_0(0,\infty) : \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0, \quad \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0\},$$

with norm $\|u\|_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right| < \infty$. Our main theorem is that the Black-Scholes semigroup is strongly continuous and chaotic on $Y^{s,\tau}$ for $s > 1,\ \tau \geq 0,$ with $s\sigma > \sqrt{2}$, where σ is the volatility. The proof uses the Godefroy-Shapiro hypercyclicity criterion. The lecture will attempt to explain all this and more.