4th Stochastic Geometry Days

Aurélien VASSEUR

Asymptotics of some Point Processes Transformations

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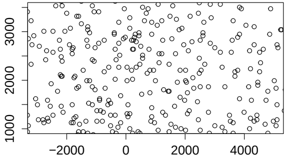
Mobile network in Paris - Motivations



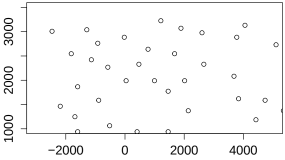


Figure: On the left, positions of all BS in Paris. On the right, locations of BS for one frequency band.

Correlation function - Papangelou intensity

Definition

Correlation function ρ :

$$\mathbb{E}[\sum_{\substack{\alpha\in\widehat{N}_{\mathbb{Y}} \\ \alpha\subset\Phi}} f(\alpha)] = \sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{Y}^k} f\cdot\rho(\{x_1,\ldots,x_k\})\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k)$$

Papangelou intensity c :

$$\mathbb{E}[\sum_{x\in\Phi} f(x,\Phi\setminus\{x\})] = \int_{\mathbb{Y}}\mathbb{E}[c(x,\Phi)f(x,\Phi)]\mu(\mathrm{d}x).$$

Properties

Properties

- Intensity measure : $A \in \mathcal{F}_{\mathbb{Y}} \mapsto \int_A \rho(\{x\})\mu(\mathrm{d}x)$.
- $\rho(\{x\}) = \mathbb{E}[c(x, \Phi)]$.
- If Φ is finite, then :

$$\mathbb{P}(|\Phi| = 1) = \int_{\mathbb{Y}} c(x, \emptyset)\mu(\mathrm{d}x)\ \mathbb{P}(|\Phi| = 0).$$

Poisson point process

Properties

Φ PPP with intensity $M(\mathrm{d}y) = m(y)\mathrm{d}y$.

- Correlation function : $\rho(\alpha) = \prod_{x\in\alpha} m(x)$.
- Papangelou intensity : $c(x,\xi) = m(x)$.

Repulsive point process

Definition

- Point process repulsive if

$$\phi \subset \xi \implies c(x,\xi) \leq c(x,\phi).$$

- Point process weakly repulsive if

$$c(x,\xi) \leq c(x,\emptyset).$$

Determinantal point process

- $Kf(x) = \int_{\mathbb{Y}} K(x,y)f(y)\mu(\mathrm{d}y)$
- $J = (I-K)^{-1}K.$
- $(h_j, j \in \mathbb{N})$ CONB of $L^2(\mathbb{Y}, \mu; \mathbb{C})$ such that :

$$K(x,y) = \sum_{j=1}^{+\infty} \lambda_j h_j(x) h_j(y), \;\; J(x,y) = \sum_{j=1}^{+\infty} \frac{\lambda_j}{1-\lambda_j} h_j(x) h_j(y).$$

Determinantal point process

Definition

Determinantal point process DPP(K, μ) :

$$\rho(\{x_1, \cdots, x_k\}) = \det(K(x_i, x_j),\ 1 \leq i, j \leq k).$$

Proposition

- Papangelou intensity of DPP(K, μ) :

$$c(x_0, \{x_1, \cdots, x_k\}) = \frac{\det(J(x_i, x_j),\ 0 \leq i, j \leq k)}{\det(J(x_i, x_j),\ 1 \leq i, j \leq k)}.$$

- DPP(K, μ) is repulsive.

Ginibre point process

Definition

- Ginibre point process on $\mathcal{B}(0, R)$:

$$K(x, y) = \frac{1}{\pi} e^{-\frac{1}{2}(|x|^2+|y|^2)} e^{x\bar{y}} \mathbf{1}_{\{x \in \mathcal{B}(0,R)\}} \mathbf{1}_{\{y \in \mathcal{B}(0,R)\}}.$$

- β-Ginibre point process on $\mathcal{B}(0, R)$:

$$K_\beta(x, y) = \frac{1}{\pi} e^{-\frac{1}{2\beta}(|x|^2+|y|^2)} e^{\frac{1}{\beta} x\bar{y}} \mathbf{1}_{\{x \in \mathcal{B}(0,R)\}} \mathbf{1}_{\{y \in \mathcal{B}(0,R)\}}.$$

Ginibre point process

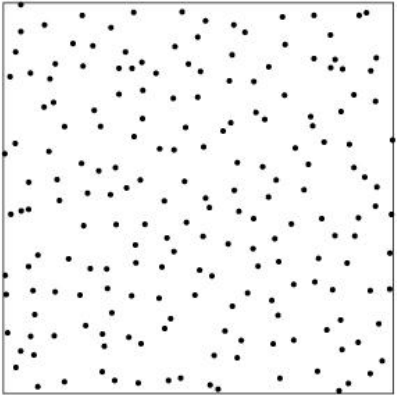


Figure: Ginibre point process.

Kantorovich-Rubinstein distance

Definitions

- Total variation distance :

$$d_{\mathrm{TV}}(\nu_1,\nu_2) := \sup_{\substack{A\in\mathcal{F}_{\mathbb{Y}} \\ \nu_1(A),\nu_2(A)<\infty}} |\nu_1(A)-\nu_2(A)|.$$

- $F : N_{\mathbb{Y}} \to \mathbb{R}$ is 1-Lipschitz ($F \in \mathrm{Lip}_1$) if

$$|F(\phi_1)-F(\phi_2)| \leq \mathrm{d}_{TV}(\phi_1,\phi_2) \text{ for all } \phi_1,\phi_2 \in N_{\mathbb{Y}}.$$

- Kantorovich-Rubinstein distance :

$$d_{\mathrm{KR}}(\mathbb{P}_1,\mathbb{P}_2) = \sup_{F\in\mathrm{Lip}_1}\left|\int_{N_{\mathbb{Y}}} F(\phi)\,\mathbb{P}_1(\mathrm{d}\phi) - \int_{N_{\mathbb{Y}}} F(\phi)\,\mathbb{P}_2(\mathrm{d}\phi)\right|.$$

Stein's method - Theorem

Theorem

- Φ : *finite point process on* $\mathbb{Y}$.
- ζ_M : *PPP with finite control measure* $M(dy) = m(y)\mu(dy)$.

Then, we have :

$$d_{KR}(\mathbb{P}_\Phi, \mathbb{P}_{\zeta_M}) \leq \int_{\mathbb{Y}} \int_{N_{\mathbb{Y}}} |m(y) - c(y, \phi)| \mathbb{P}_\Phi(d\phi)\mu(dy).$$

Superposition of weakly repulsive point processes

- $\Phi_{n,1}, \dots, \Phi_{n,n}$: n independent, finite and **weakly repulsive** point processes on $\mathbb{Y}$.
- $\Phi_n = \sum_{i=1}^{n} \Phi_{n,i}$.
- $R_n := \int_{\mathbb{Y}} |\sum_{i=1}^{n} \rho_{n,i}(x) - m(x)|\mu(\mathrm{d}x)$.

Superposition of weakly repulsive point processes

Proposition

ζ_M : PPP with control measure $M(\mathrm{d}x) = m(x)\mu(\mathrm{d}x)$.

$$d_{\mathrm{KR}}(\mathbb{P}_{\Phi_n}, \mathbb{P}_{\zeta_M}) \leq R_n + \max_{1 \leq i \leq n} \int_{\mathbb{Y}} \rho_{n,i}(x)\mu(\mathrm{d}x).$$

Elements of Proof

$$d_{\mathrm{KR}}(\Phi_n, \pi_M) \leq R_n + \sum_{i=1}^{n} A_{n,i}, \text{ where}$$

$$\begin{aligned} A_{n,i} &= p_{n,i,0} \int_{\mathbb{Y}} |c_{n,i}(x, \varnothing) - \rho_{n,i}(x)| \mu(\mathrm{d}x) \\ &+ \sum_{k \geq 1} \int_{\mathbb{Y}} \mathbb{E}[|c_{n,i}(x, \Phi_{n,i}) - \rho_{n,i}(x)| \mathbf{1}_{\{|\Phi_{n,i}|=k\}}] \mu(\mathrm{d}x). \end{aligned}$$

Lemma

$$|c(x, \varnothing) - \rho(x)| \leq (1 - p_0) c(x, \varnothing).$$

β-Ginibre point processes

Proposition

- Φ_n : β_n-Ginibre process reduced to a compact set Λ.
- ζ : PPP with intensity $1/\pi$ on Λ.

$$d_{\mathrm{KR}}(\mathbb{P}_{\Phi_n}, \mathbb{P}_{\zeta}) \leq C\beta_n.$$

Thinned point processes

- (f_n) : dense sequence in the space of real functions with compact support
- $d^*(\nu_1, \nu_2) = \sum_{n \geq 1} \frac{1}{2^n \|f_n\|_\infty} |\nu_1(f_n) - \nu_2(f_n)|$
- d^*_{KR} : Kantorovich-Rubinstein distance associated to the distance d^*

Thinned point processes

Proposition

For all $n \in \mathbb{N}$, let Φ_n be a finite point process on $\mathbb{Y}$, $p_n : \mathbb{Y} \to [0;1)$ a measurable function and Φ'_n the p_n-thinning of Φ_n. Let γ_M be the Cox process directed by a random finite measure M. Suppose that :

- (p_n) tends to 0 uniformly,
- $d^*_{\mathrm{KR}}(\mathbb{P}_M, \mathbb{P}_{p_n\Phi_n})$ tends to 0.

Then, we have :

$$d^*_{\mathrm{KR}}(\mathbb{P}_{\Phi_n}, \mathbb{P}_{\gamma_M}) \leq 2\mathbb{E}[\sum_{x \in \Phi_n} p_n^2(x)] + d^*_{\mathrm{KR}}(\mathbb{P}_M, \mathbb{P}_{p_n\Phi_n}).$$

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