Local digital algorithms for estimation of intrinsic volumes

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Digital stereology

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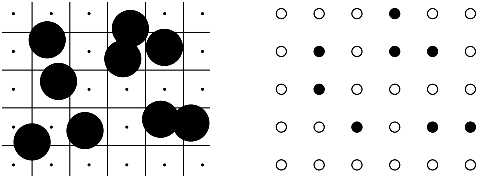
The aim of *digital stereology* is to compute geometric characteristics of X based on an (ideal) digital image.

In this talk: want to estimate the *intrinsic volumes* $V_0(X), \dots, V_d(X)$.

- $V_d(X)$ is volume.
- $2V_{d-1}(X)$ is surface area.
- $c_d V_{d-2}(X)$ is integrated mean curvature.
- $V_0(X)$ is Euler characteristic.

Black-and-white images

A *black-and-white image* is modeled as follows:



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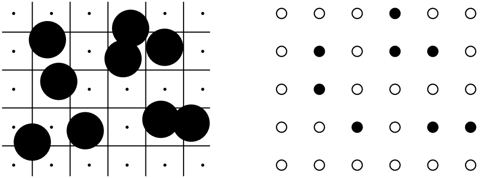

Mathematically, the information we have is

$$X \cap \mathbb{L},$$

where $\mathbb{L}$ is the lattice formed by the pixel midpoints.

Convergence of estimators

Let $\hat{V}_k$ be an estimator for V_k.

It is unlikely that $\hat{V}_k(X \cap \mathbb{L}) = V_k(X)$.

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We assume that the lattice is randomly translated with respect to X, i.e. $\mathbb{L} = c + \mathbb{Z}^d$ where $c \in [0,1]^d$ is uniform random.

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Change of resolution corresponds to scaling $\mathbb{L}$ by some $a > 0$. The image becomes:

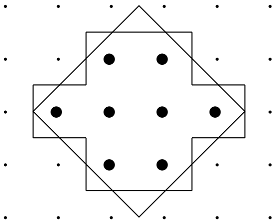
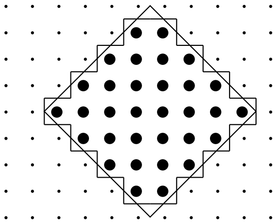
$$X \cap a\mathbb{L}.$$

We will require

$$\lim_{a \to 0} \mathbb{E}\hat{V}_k(X \cap a\mathbb{L}) = V_k(X).$$

Example

Approximate X by the union of black pixels and compute intrinsic volumes of approximation.

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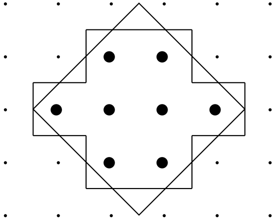

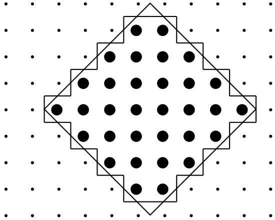

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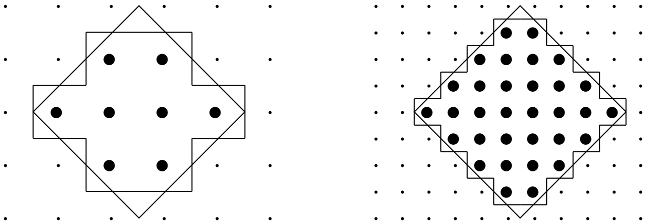

Good approximation of volume: $\mathbb{E}\hat{V}_d(X \cap \mathbb{L}) = V_d(X)$.

Bad approximation of boundary length:

$$\mathbb{E}\hat{V}_{d-1}(X \cap a\mathbb{L}) \approx \sqrt{2} V_{d-1}(X).$$

Local algorithms

A $n \times \cdots \times n$ configuration is a way of coloring an $n \times \cdots \times n$ pixel block in black and white.

For instance, the possible 2×2 configurations in 2D are:

$$\begin{matrix} \circ & \circ \\ \circ & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ \\ \bullet & \circ \end{matrix} \qquad \begin{matrix} \circ & \circ \\ \bullet & \bullet \end{matrix} \qquad \begin{matrix} \bullet & \circ \\ \circ & \bullet \end{matrix} \qquad \begin{matrix} \circ & \bullet \\ \bullet & \bullet \end{matrix} \qquad \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$$

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Index the possible configurations by $l = 1, \ldots, 2^{n^d}$.

Assign to each configuration a weight $w_l \in \mathbb{R}$.

Let $N_l(X \cap a\mathbb{L})$ be the number of times configuration l occurs in the image.

A local algorithm estimates $V_k(X)$ by

$$\hat{V}_k(X) = a^k \sum_{l=1}^{2^{n^d}} w_l N_l(X \cap a\mathbb{L}).$$

Convergence results

Is there a local algorithm for V_k such that

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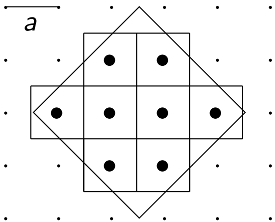
For volume estimation, the answer is yes: The estimator

$$\hat{V}_d(X \cap a\mathbb{L}) = a^d \cdot \#\text{black lattice points}$$

is unbiased.



Convergence (continued)

Theorem (S. 2014)

There is no local algorithm for V_k with $k < d$ such that

$$\lim_{a \to 0} \mathbb{E} V_k(X \cap a\mathbb{L}) = V_k(X)$$

whenever X is a compact convex polytope with interior points.

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In fact:

- When $k < d - 1$, the worst case asymptotic bias is 100%. When $k = d - 1$, $d = 3$ it is $\approx 4\%$ (Ziegel, Kiderlen 2010).
- There is a natural measure on the set of compact convex polytopes such that $\hat{V}_k$ is asymptotically biased on a set of positive measure.

What if we assume that X has smooth boundary?

Theorem (Pavlidis, 1982; Stelldinger, Latecki, Siqueira, 2007)

There exists a local algorithm for the Euler characteristic in 2D and 3D such that for any smooth d-dimensional manifold X with boundary

$$\hat{V}_0(X \cap a\mathbb{L}) = V_0(X)$$

whenever a is sufficiently small.

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Theorem (S. 2015)

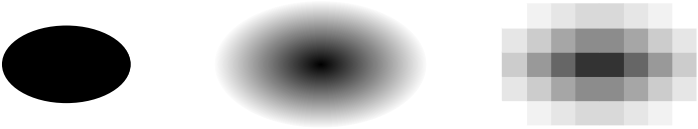
If $k = d - 1$ or $k = d - 2$ and $k \neq 0$, then there is no local algorithm for V_k such that

$$\lim_{a \to 0} \mathbb{E} V_k(X \cap a\mathbb{L}) = V_k(X)$$

whenever X is a smooth d-dimensional manifold with boundary.

Grey-scale images

In practice, one typically observes a blurred grey-scale image, rather than a sharp image.

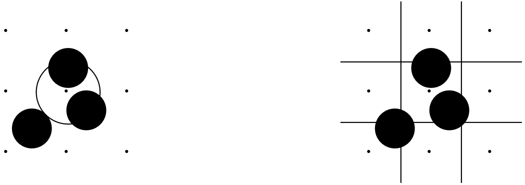


Solutions:

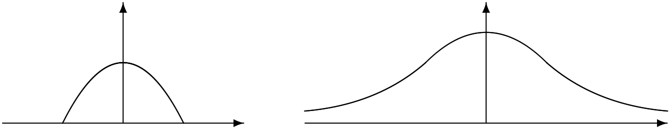
- Thresholding
- Use grey-values directly

Examples of grey-scale images

At each lattice point we measure the light coming from a neighbourhood:



The measured intensity may depend on the distance to the lattice point:



Model for grey-scale images

The light intensity we can measure at $x \in \mathbb{R}^d$ is

$$\theta^X(x) = \mathbb{1}_X * \rho(x) \in [0, 1].$$

Here $\rho : \mathbb{R}^d \to [0, \infty)$ is the *point spread function* satisfying

$$\int_{\mathbb{R}^d} \rho(x) dx = 1.$$

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Introduce a parameter $b > 0$ that controls the blurring, replacing ρ by

$$\rho_b(x) = b^{-d} \rho(b^{-1} x).$$

Local algorithms for grey-scale images

We consider only $1 \times \cdots \times 1$ configurations (single points).

A configuration consists of a grey-value in $[0, 1]$.

A weight is now a function $f : [0, 1] \to \mathbb{R}$.

Definition

A local algorithm in the grey-scale setting estimates $V_k(X)$ by:

$$\hat{V}_k^f(X) = a^d b^{k-d} \sum_{z \in a\mathbb{L}} f(\theta_b^X(z)).$$

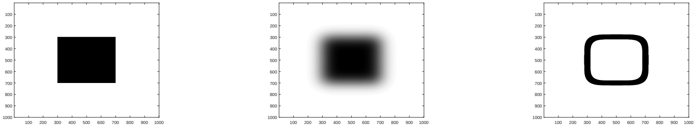
Example - a surface area estimator

Take $f = \mathbb{1}_{[t,1-t]}$. Then

$$\hat{V}^f_{d-1}(X) = a^d b^{-1} \sum_{z \in a\mathbb{L}} \mathbb{1}_{\{\theta^X_b(z) \in [t,1-t]\}}$$

is the number of grey-values in $[t, 1-t]$.

The picture below shows a square, a blurred image of the square, and the pixels with grey-values between $1/3$ and $2/3$.



Convergence

We can compute the mean estimator:

$$\begin{aligned}
\mathbb{E}\hat{V}_k^f(X) &= a^d b^{k-d} \int_{[0,1]^d} \sum_{z \in a(\mathbb{Z}^d + c)} f(\theta_b^X(z))dc \\
&= a^d b^{k-d} \sum_{z \in \mathbb{Z}^d} \int_{[0,1]^d} f(\theta_b^X(a(z+c)))dc \\
&= a^d b^{k-d} \int_{\mathbb{R}^d} f(\theta_b^X(az))dz \\
&= b^{k-d} \int_{\mathbb{R}^d} f \circ \theta_b^X(z)dz.
\end{aligned}$$

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Independent of a (resolution)!

We say that $\hat{V}_k^f(X)$ is asymptotically unbiased if

$$\lim_{b \to 0} \mathbb{E}\hat{V}_k^f(X) = V_k(X).$$

Convergence for surface area estimators

Assume:

(i) $\rho(x) = \rho(|x|)$ + mild technical conditions.

(ii) $X \subseteq \mathbb{R}^d$ is a gentle set (e.g. a finite union of full-dimensional convex sets or a smooth manifold with boundary).

(iii) f is continuous at all but finitely many points. The support of f is contained in $[t, 1-t] \subseteq (0,1)$.

Theorem (S. 2014)

Under conditions (i)–(iii),

$$\lim_{b \to 0} \mathbb{E} \hat{V}^f_{d-1}(X) = c_1(f, \rho) V_{d-1}(X).$$

If $f = \mathbb{1}_{[t,1-t]}$, *then* $c_1(f, \rho) > 0$.

Convergence - estimators for integrated mean curvature

Assume:

(i) $\rho(x) = \rho(|x|)$, ρ is continuous with compact support + mild technical conditions.

(ii) $X \subseteq \mathbb{R}^d$ is a smooth manifold with boundary.

(iii) f is C^1 at all but finitely many points. The support of f is contained in $[t, 1-t] \subseteq (0,1)$. Moreover, $f(t) = -f(1-t)$.

Theorem (S. 2014)

Under conditions (i)–(iii),

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For instance, $f = \mathbb{1}_{[t,1/2]} - \mathbb{1}_{[1/2,1-t]}$.

We do not know whether it is possible to continue for $k < d-2$.

A variance bound

Assume that $\mathbb{L}$ is also randomly rotated.

Theorem (S. 2014)

Suppose X is a smooth manifold, and f and ρ are smooth. For a, b small,

$$\mathrm{Var}(\hat{V}_{d-1}^{f}(X)) \leq b^{-1} a^{d} C_1(X) C_2(f, \rho).$$

Simple tests (Thanks to Sune Darkner!)

Consider $\hat{V}^f_{d-1}$ with $f = \mathbb{1}_{[t,1-t]}$ and $\rho(x) = (2\pi)^{-1}\exp(-|x|^2/2)$.

Square, sidelength 400:

t	a	b	V_{d-1}	$\hat{V}_{d-1}$	Error
1/3	1	50	800	747	6.6%
1/3	0.1	50	800	744	7.0%
1/3	1	25	800	785	1.9%
1/4	1	25	800	777	2.9%

Disk, radius 200:

β	a	b	V_{d-1}	$\hat{V}_{d-1}$	Error
1/3	1	50	628	617	1.8%
1/3	0.1	50	628	617	1.8%
1/3	1	25	628	625	0.48%
1/4	1	25	628	625	0.48%

Conclusions

Summary:

- Local algorithms are in many cases biased.
- Get asymptotically unbiased estimators by using grey-values.
- Fast and simple algorithms.
- Variance is relatively well-behaved.

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- Local algorithms are in many cases biased.
- Get asymptotically unbiased estimators by using grey-values.
- Fast and simple algorithms.
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Open questions:

- Performance in finite resolution.
- Applicability in practical situations.
- Best choice of algorithm.

Thank you for the attention!