

Scaling limit of dynamical percolation on critical Erdős-Rényi random graphs

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Erdős-Rényi random graphs

- ▶ n vertices
- ▶ E_n : set of edges of the complete graph
- ▶ $(U_{e \in E_n})$ i.i.d uniform on $[0, 1]$
- ▶ For $p \in [0, 1]$, state of edge e in $\mathcal{G}(n, p)$: present iff $U_e \leq p$

The birth of a giant connected component (giant = of size $\Theta(n)$):

- ▶ $p = c/n, c < 1 \implies$ no giant component
- ▶ $p = c/n, c > 1 \implies \exists$ a unique giant component
- ▶ **Critical window:** $p = p(n, \lambda) = \frac{1}{n} + \frac{\lambda}{n^{4/3}}, \lambda \in \mathbb{R}$

Background: convergence of rescaled components, λ fixed

$X^{n,\lambda}$:= sequence of **rescaled** connected components listed in decreasing order of size, viewed as **measured metric spaces**:

measure = counting measure on vertices $\times n^{-2/3}$

distance = graph distance $\times n^{-1/3}$

Theorem (Aldous'97 + Addario-Berry, Broutin & Goldschmidt'10)

$(X^{n,\lambda})$ converges in distribution ($n \rightarrow \infty$) to a random sequence of measured metric spaces (which are real graphs) $X^{\infty,\lambda}$.

Topology: Gromov-Hausdorff-Prokhorov

$$d_{GHP}((X, \mu, d), (X', \mu', d')) := \inf_{\phi, \phi'} \{d_H(\phi(X), \phi'(X')) \vee d_{LP}(\phi\#\mu, \phi'\#\mu')\}$$

Background: multiplicative coalescent, λ varying

$s^{n,\lambda}$: = sequence of masses in $X^{n,\lambda}$

When λ increases, connected components i and j , of masses s_i and s_j merge at rate

$$\frac{1}{n^{4/3}} \times s_i n^{2/3} \times s_j n^{2/3} = s_i s_j$$

→ multiplicative coalescent

Theorem (Aldous '97 + Aldous & Limic'98)

Multiplicative coalescent is Feller on ℓ^2 , and the process $(s^{n,\lambda})_{\lambda \in \mathbb{R}}$ converges in distribution ($n \rightarrow +\infty$) to the standard multiplicative coalescent, whose asymptotic sigma-field is trivial.

Dynamical percolation, coalescence and fragmentation on $\mathcal{G}(n, p)$

Dynamical percolation $X^{n,\lambda}(t)$ of intensity γ_n on the graph : take independent Poisson processes on each edge,

- ▶ create the edge at rate $\gamma_n p$ (coalescence)
- ▶ kill the edge at rate $\gamma_n(1-p)$ (fragmentation)

A graph property A_n is γ_n -noise sensitive if

$$\text{Cor}(\mathbb{1}_{A_n}(X^{n,\lambda}(0)), \mathbb{1}_{A_n}(X^{n,\lambda}(t))) \xrightarrow[n \rightarrow \infty]{} 0 \quad \forall t > 0$$

and γ_n -noise stable if

$$\text{Cor}(\mathbb{1}_{A_n}(X^{n,\lambda}(0)), \mathbb{1}_{A_n}(X^{n,\lambda}(t))) \xrightarrow[n \rightarrow \infty]{} 1 \quad \forall t > 0$$

Dynamical percolation, coalescence and fragmentation on measured real graphs

- ▶ **Coalescence** on a measured metric space (X, μ, d) : identify points of a Poisson process of intensity $\frac{1}{2}\mu^2$ on X^2 .
- ▶ **Fragmentation** on a real graph (X, d) : cut points of a Poisson process of intensity the length measure.
- ▶ **Dynamical percolation** on a measured real graph (X, μ, d) : perform **coalescence and fragmentation**, simultaneously and independently.

Main result

Choose intensity $\gamma_n = n^{-1/3}$

Theorem

The dynamical percolation (resp. coalescence, resp. fragmentation) process on $\mathcal{G}(n, n^{-1} + \lambda n^{-4/3})$ in the critical window converges in distribution ($n \rightarrow \infty$) to the dynamical percolation (resp. coalescence, resp. fragmentation) process on the limit $X^{\infty, \lambda}$

Work in progress: Show that dynamical percolation on the limit $X^{\infty, \lambda}$ is mixing

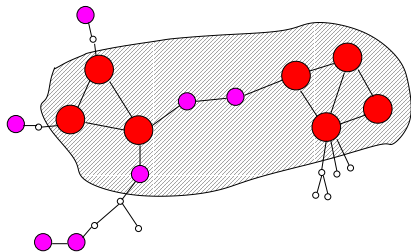
- ▶ Consequence: ε_n -noise sensitivity of properties which are “approximable” in the scaling limit, with $\varepsilon_n \gg n^{-1/3}$
- ▶ **Example:** having a complex component, being planar

Main ideas

- ▶ $1d$ marginals converge \rightarrow only need to show a (kind of) Feller property
 $Coal(X, t) \rightarrow Coal(Y, t)$ when $d_{GHP}(X, Y) \rightarrow 0$
- ▶ Main difficulty for coalescence: total mass is infinite (sizes are in ℓ^2)
- ▶ **Structure** lemma for the multiplicative coalescent

Structure for the multiplicative coalescent

For every $\varepsilon > 0$, there exists $\varepsilon_2 \leq \varepsilon_1 \leq \varepsilon$ such that with probability larger than $1 - \varepsilon$, every component \mathcal{C} of $\text{Coal}(s, t)$ of size at least ε has the following structure



- Large, size $\geq \varepsilon_1$
- Medium, size $\in [\varepsilon_2, \varepsilon_1[$
- Small, size $\leq \varepsilon_2$
- Heart

$\text{Crown} := \mathcal{C} \setminus \text{Heart}$, $\text{size}(\text{Crown}) < \varepsilon_1$

Main steps of AdBrGo'10, λ fixed

- ▶ Define a brownian motion with drift $W_t^\lambda = B_t + \lambda t - \frac{t^2}{2}$,
- ▶ reflect it above past minima: $h_t^{\infty, \lambda} := 2(W_t^\lambda - \min_{0 \leq s \leq t} W_s^\lambda)$,
- ▶ then, the **rescaled height** $h^{n, \lambda}$ of the exploration process converges in distribution to $h^{\infty, \lambda}$ (Marckert & Mokkadem '03 + AdBrGo'10).
- ▶ Additional edges correspond to a Poisson point process of intensity $1/2$ below $h^{\infty, \lambda}$

Control of the width of the crown for $\mathcal{G}(n, p)$

- ▶ In depth-first order, a component of the crown is explored in at most two intervals
- ▶ Thus, the width of the crown is w.h.p less than $w_n(\varepsilon_1)$ with w_n the modulus of continuity of $4h^{n, \lambda+t}$, which goes to zero when ε_1 goes to zero, uniformly in n .

The End

Thanks !!!