Traffic in a random Poissonian network

(limit theorems with improper Poisson line processes).

4[th] Stochastic Geometry Days
LMA, Université de Poitiers

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27[th] August 2015

Probability for spatial networks.

This is "Generic Applied Probability": it has appeared as WSK (2014).
Spatial networks involved in transportation / communication.

Consider generic model for traffic in an idealized network.

Questions:

Is there a limiting distribution of traffic in the centre under a suitable re-scaling?

How to simulate it?

JFCK: "Rise and Fall of the City of Mahagonny" `en.wikipedia.org/wiki/Rise_and_Fall_of_the_City_of_Mahagonny`.

City Calculations Impropriety Simulation Conclusion References

Warwick Statistics

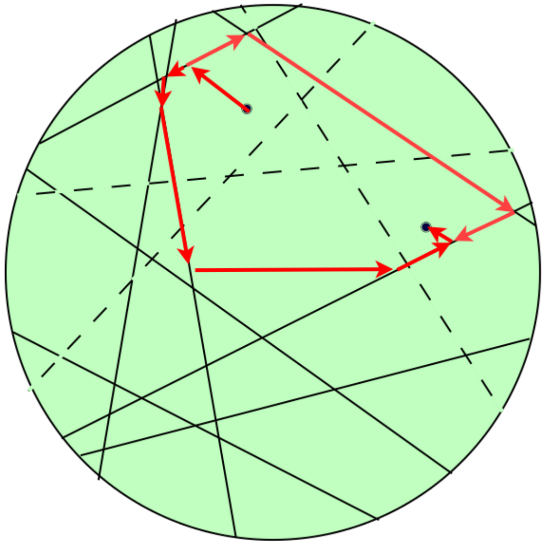
The image presents a case of congestion in a major capital city.
How might we get a handle on volume of traffic?
We need a useful model for a network of road connections in a city, and a good model for traffic in the network.

1. Is there a limiting mean for traffic through the centre?
2. Is there a limiting distribution?
3. or might there be a "congestion effect"?

The Poissonian City (I)

Design an idealized city!

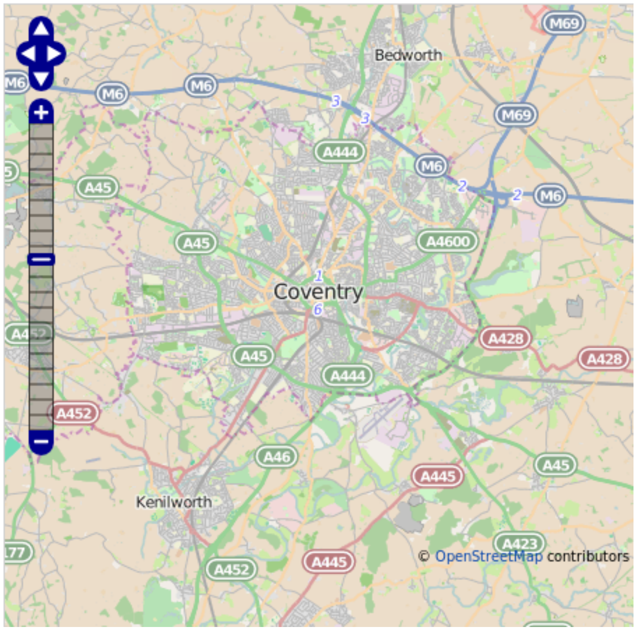
Complete the 2 routes using off-road sections.



This model is found in WSK (2011). Aldous and WSK (2008) discussed these semi-cell perimeters ("near-geodesics"), and showed they compare well to true geodesics in the network. WSK (2015) also Kahn (2015) apply these ideas to the construction of scale-invariant random spatial networks (SIRSN).

1. Generic source / destination pair **x**/**y** is indicated.
2. Defined *via* parametrization of lines.
3. Follows Aldous and WSK (2008): construction of near-geodesics.
4. We need to get on to the network!
5. So progress "cross-country".
6. Two possible directions!
7. Note; two possible near-geodesics.
8. Hence: two possible routes.

Real city plan



Dark humour from the city: "Coventry is a nice place to visit, although some say our best town planning was done by the Luftwaffe."
Best not use this. Instead, report the following factoid: "If everyone in Coventry put their car on the road network, there would be no room left."

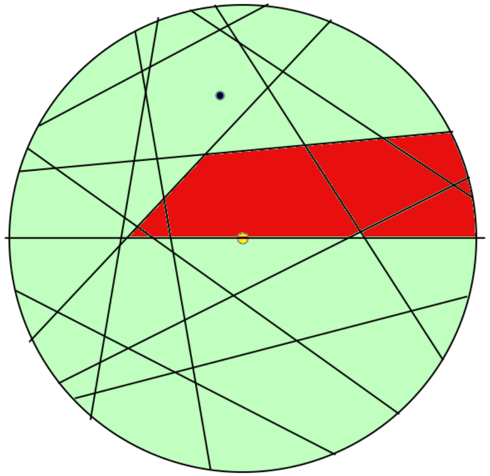
Comparison of Coventry city with Poissonian city:
curved not straight;
however, not too curved.

Note: My current PhD student Rodolpho Gameros Leal will be considering the issue of curved roads . . .

The Poissonian City (II)

Aldous idea: Every point-pair x/y contributes infinitesimal amount $\mathrm{d}\,x\,\mathrm{d}\,y$ split equally between the two near-geodesics.

Compute 4-volume of random polytope!



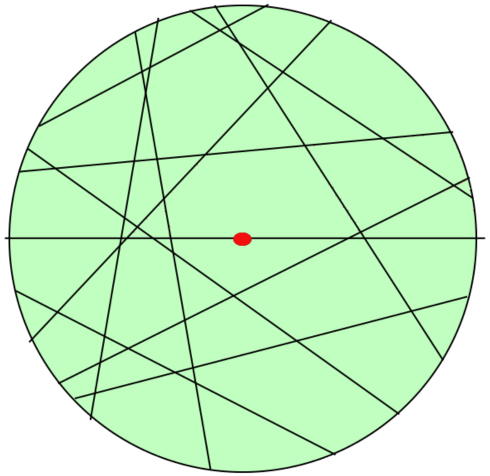
1. Slivnyak theorem: condition on road through centre.
2. Each pair contributes $\frac{1}{2}\,\mathrm{d}\,x\,\mathrm{d}\,y$, namely half the infinitesimal amount, or nothing.
3. Image shows a 2-dimensional slice.

Measure congestion?

To repeat: we are only dealing with a *mathematical* idealization!
In particular, we do not model interactions (and these will capture most if not all the congestion behaviour).
However it may be an interesting caricature.

Questions:

1. What is the mean traffic flow through the centre?
2. Is there a limiting scaled distribution?
3. Is the limiting distribution accessible to computation?



1. Mean is easy.
2. Here's an interesting probabilistic limit theorem!
3. This will be a real challenge.

Mean traffic computation

- Mean traffic flow through centre can be obtained by direct arguments from stochastic geometry:

  for $\rho = \sqrt{r^2 + s^2 - 2rs\cos\theta}$,

$$\mathbb{E}\left[T_n\right] = \int_0^{\pi}\int_0^{n}\int_0^{n} \exp\left(-\tfrac{1}{2}\left(r+s-\rho\right)\right) r \,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\theta .$$

  Asymptotically for large city radius ($n \to \infty$),

$$\mathbb{E}\left[T_n\right] = 2n^3 + O\left(n^2\sqrt{n}\right) .$$

There is interesting stochastic geometry going on here.

Consider the invariant measure μ of lines separating A from BC.

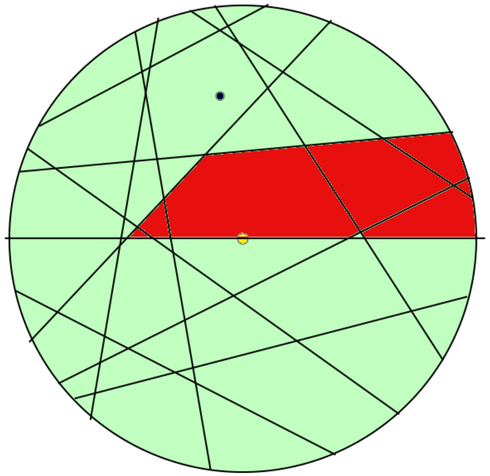
A counting argument shows this is given by
$\frac{1}{2}(\mu([AB]) + \mu([AC]) - \mu[BC]))$.

Hence an answer to our first question: there is indeed a simple asymptotic for the mean. This provides us with a guess as to the correct scaling.

Don't worry too much about factors of 2!

Existence of limiting distribution?

Is there a limiting distribution for T_n/n^3 as $n \to \infty$?



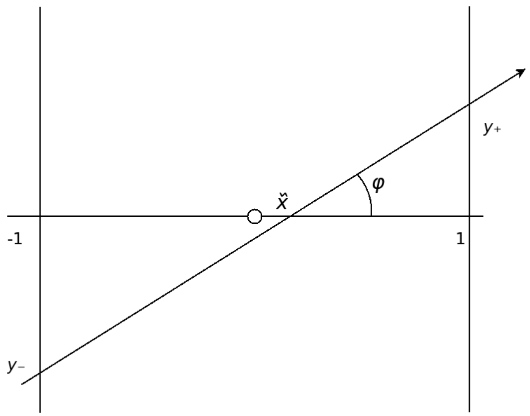
The 4-dimensional random geometry is hard to visualize!

Here we indicate a 2-dimensional section obtained by fixing one of the two points.

In the impending improper limit, this idea suggested the key to a useful representation.

Improper line processes to the rescue (I)

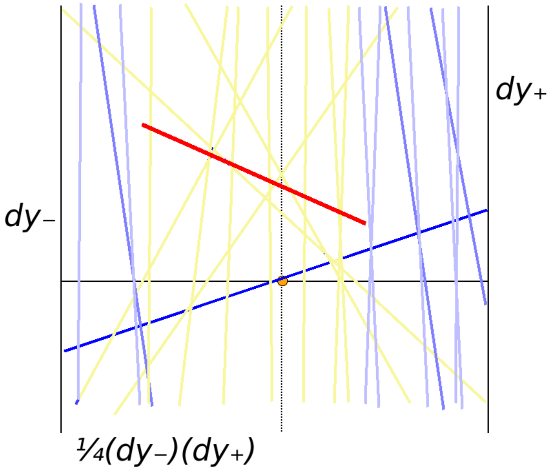
- Consider *stretching* the city.
- Vertically, rescale by $1/\sqrt{n}$. Horizontally, rescale by $1/n$. (Affine transformation.)
- New coordinates y_- and y_+.
- New (improper) intensity $\frac{1}{4}(y_- + y_+)$,



There is actually no such thing as a *fully* "improper" line or point process: see WSK (2000). For consider a stationary random countable dense set entirely on its own, taking account of no extra structure. All meaningful probabilities are zero or one!
Extra structure is vital for proper formulation of problems. (Probabilists learn, in their arduous measure-theoretic apprenticeship, to avoid speaking of improper structures without adding extra explicative structure.)
The density here is *improper* because it provides an infinite tangle of nearly vertical lines. The factor $\frac{1}{4}$ comes from the interval $[-1, 1]$.

Improper line processes to the rescue (II)

Infinitely many nearly vertical lines running near any point.

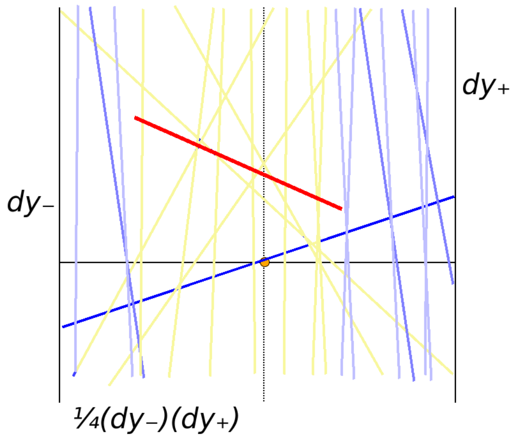


Centre sees no traffic when separated off by lines.
Coupling argument: limiting distribution non-degenerate.

Coupling argument is eased, because we can realize the scaling so that monotonically more lines as we scale further. However, care is needed because we integrate over larger and larger regions. Fortunately it is relatively easy to use an L^1 / dominated convergence argument.

There is a limiting distribution, but how to access it?

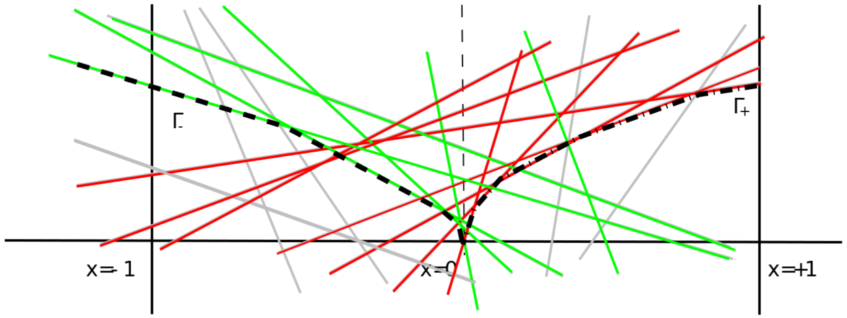
- Can't find analytic expression for the limiting distribution $\lim_{n\to\infty} \mathcal{L}\left(T_n/n^3\right)$.
- We now can recognize T_n as 4-dimensional volume of random polytope (use $\frac{1}{4}\,\mathrm{d}\,y_-\ \mathrm{d}\,y_+$).
- Can we produce achievable approximating simulations?



We want to get some notion of how good are the obvious truncated approximations.
In a mathematical version of "fake it till you make it", one might say "If you can't calculate it, can you simulate it?".

Seminal curves

WSK (2014):



- Focus on upper half-plane (quadrants $Q_\pm$) and subset of source/destinations in $Q_- \times Q_+$ not separated from $\mathbf{o}$.
- Identify two sub-families of improper line process Π_∞:

$$\Pi_{\infty,\pm} \quad = \quad \{\ell \in \Pi_\infty : \text{slope}(\ell) = \pm, \ell \text{ intercepts } \mp x\text{-axis}\}\,.$$

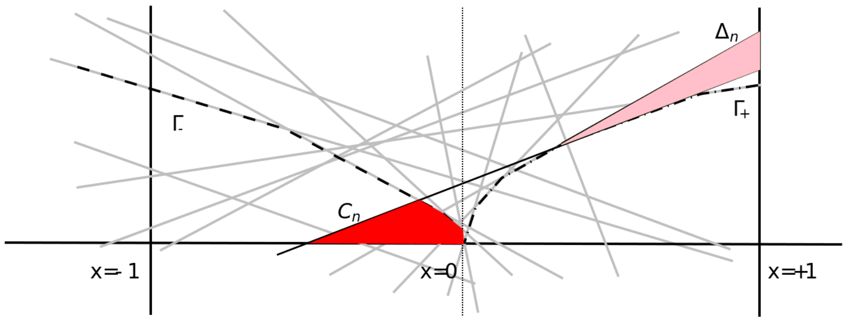
- Define *seminal curves* $\Gamma_\pm$ as the concave lower envelopes of the unions of lines in $\Pi_{\infty,\pm}$:

The region we focus on here is 4-dimensional, hence hard to visualize. Seminal curves help us get a handle on it, as we can use them to provide convenient representations / factorizations of the 4-dimensional volume being studied.

The easy way of getting flow: if x is below Γ_+ and y below Γ_- (or, of course, *vice versa*, or using up-down reflection symmetry).

A series expression (I)

Using areas of regions indicated in the following diagram



we obtain a series expression for the relevant volume . . .

$$\left(\int_0^1 \Gamma_-(-s)\,\mathrm{d}\,s\right) \times \left(\int_0^1 \Gamma_+(s)\,\mathrm{d}\,s\right)$$

$$+ \sum_{n=0}^{\infty} \text{Leb}_2(C_n^+)\,\text{Leb}_2(\Delta_n^+) + \sum_{n=0}^{\infty} \text{Leb}_2(C_n^-)\,\text{Leb}_2(\Delta_n^-)$$

Now we can get a grip on this 4-volume.

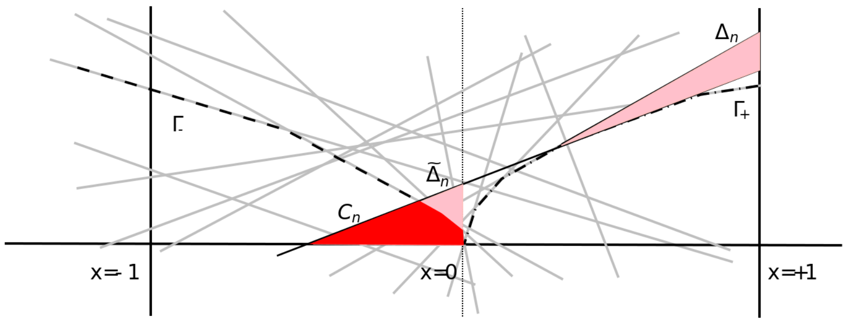
$\Delta_n^\pm$: triangle

$C_n^\pm$: more complicated region.

The only extra way of getting flow is if for some n we have x in $\Delta_n^\pm$ and y in $C_n^\pm$.

A series expression (II)

...and a tail upper bound



$$\sum_{n=N}^{\infty} \text{Leb}_2(C_n^+)\,\text{Leb}_2(\Delta_n^+) \quad \leq \quad \sum_{n=N}^{\infty} \left(\text{Leb}_2(\widetilde{\Delta}_n^+) \times \text{Leb}_2(\Delta_n^+)\right) .$$

We can get a convenient tail bound to control simulation error. We replace the more complicated regions $C_n^\pm$ by larger triangles $\widetilde{\Delta}_n^\pm$.

Hence we obtain an upper bound that looks as if it *might* be controllable.

Recursive representation of $\Gamma_\pm$

Theorem

Consider times of slope-change of $\Gamma = \Gamma_+$ in reversed time: $1 = S_0 > S_1 > S_2 > \ldots > 0$.
Successive tangents $\ell_0, \ell_1, \ell_2, \ldots, \ell_n$: slopes $\Gamma'(s)$ for $S_n \geq s > S_{n+1}$.
Let $Y_n = \Gamma(S_n) - S_n\Gamma'(S_n)$ be intercept of ℓ_n on y axis,

$$\frac{1}{S_{n+1}} = \frac{1}{S_n} + \frac{4}{Y_n^2} E_{n+1} ,$$

$$\Gamma'(S_{n+1}) = \Gamma'(S_n) + \frac{Y_n}{S_{n+1}} \sqrt{U_{n+1}} .$$

for $E_n \sim \text{Exponential}(1)$, $U_n \sim \text{Uniform}[0,1]$, independent of each other and $\Gamma(S_0) = \Gamma(1)$, $\Gamma'(S_0) = \Gamma'(1)$.
($\Gamma(1), \Gamma'(1)$ joint distribution is computable.)

Missing out a lot of details!

Key idea: “everything is a perpetuity” — or at least a probabilistic recurrence. Calculations of distributions use (a) calculations concerning $\frac{1}{4}\,\mathrm{d}\,y_-\ \mathrm{d}\,y_+$ under suitable re-parametrizations; (b) independence for Poisson counts in disjoint subsets of line-space.

Simulation Error Estimation

Theorem

The L^1 error of the 4-volume approximation

$$\int_{Q_+}\int_{Q_-} \mathbb{I}_{[\mathbf{o}\in\partial C(\mathbf{x},\mathbf{y})]}\,\mathrm{d}\,\mathbf{x}\,\mathrm{d}\,\mathbf{y} \approx \left(\int_0^1 \Gamma_-(-s)\,\mathrm{d}\,s\right)\times\left(\int_0^1 \Gamma_+(s)\,\mathrm{d}\,s\right) + \sum_{n=0}^{N} \mathrm{Leb}_2(C_n^+)\,\mathrm{Leb}_2(\Delta_n^+) + \sum_{n=0}^{N} \mathrm{Leb}_2(C_n^-)\,\mathrm{Leb}_2(\Delta_n^-)$$

is bounded above by

$$\frac{20}{7}\times 3^{-N} + \frac{20}{27}\times 6^{-N}.$$

We now have the desired control. Remaining issue: what is the actual computational complexity of $C_n^{\pm}$?
My PhD student Rodolpho Gameros Leal has some partial results.

Conclusion

Traffic at the centre of the Poissonian city:

- There is a limiting distribution under scaling.
- There is a theoretical basis for developing an effective simulation algorithm.
- Actual implementation?!
- Analytical representation?
- Perfect simulation?
- Interactions?

Some things we now know, and some things we known we don't yet know.
The things we don't yet know we don't know are reserved for the Questions slot at the end.
Note: not yet found time to implement this algorithm.
Interactions: Nash equilibria, also called Wardrop equilibria in traffic analysis (contemporary with Nash!)

Questions?

Not all traffic networks experience congestion.
This road is one of a network of ”Green Roads“ in Northern England, around 2000 years old.

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