Continuum Random Cluster Model

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Notations

Let Ω be the space of locally finite subsets of $\mathbb{R}^2 \times \mathbb{R}^+$ embedded with the classical σ-algebra $\mathcal{F}$.
For a configuration $\omega \in \Omega$ and a bounded subset Λ of $\mathbb{R}^2$, ω_Λ denotes the subset $\omega \cap (\Lambda \times \mathbb{R}^+)$.
For a positive real number z and a probability measure Q on $\mathbb{R}^+$, $\pi^{z,Q}$ denotes the law of a Poisson point process of intensity measure $z\lambda^{(2)} \otimes Q$, and $\pi_\Lambda^{z,Q}$ is it restriction to the set $\Lambda \times \mathbb{R}^+$.

Local number of connected components

Let Λ be a bounded subset of $\mathbb{R}^2$ and ω a configuration. We define

$$N_{cc}^{\Lambda}(\omega) = \lim_{\Delta \to \mathbb{R}^2} N_{cc}(\omega_{\Delta}) - N_{cc}(\omega_{\Delta \setminus \Lambda}),$$

where

- N_{cc} is number of connected components, defined only for finite configurations.
- The limit is taken along any increasing sequence converging to $\mathbb{R}^2$.

Definition 1

A probability measure P is a CRCM(z, Q, q) if, for every bounded Λ and every measurable bounded function f we have

$$\int f dP = \int \int f(\omega'_\Lambda + \omega_{\Lambda^c}) \frac{q^{N^\Lambda_{cc}(\omega'_\Lambda + \omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \pi^{z,Q}_\Lambda (d\omega') P(d\omega).$$

Theorem 2

- *If Q has bounded support then for every $z > 0$ and $q > 0$, there exists a stationary CRCM(z, Q, q).*
- *If $\int R^2 Q(dR)$ is finite then for every $z > 0$ and $q \geq 1$, there exists a stationary CRCM(z, Q, q).*

Theorem 3

If $\int R^2 Q(dR) = \infty$ then there exists a $z_c > 0$ such that for every z positive and every q positive integer, we have the existence of a stationary CRCM(z, Q, q) different from $\pi^{z,Q}$.

Theorem 2 : Sketch of the proof

Step 1 : Finding a good "candidate".
For $\Lambda_n =]-n,n]^2$ we define

$$P_n(d\omega) = \frac{q^{N_{cc}(\omega)}}{Z_n} \pi_{\Lambda_n}^{z,Q}(d\omega).$$

From this sequence we define a stationary sequence $(\bar{P}_n)$.

Definition 4

A sequence of measure (ν_n) converge to ν for the local convergence topology if for all local bounded function f we have

$$\int f \, d\nu_n \underset{n\to\infty}{\longrightarrow} \int f \, d\nu.$$

(A function is local if there exists a bounded Δ such that $f(\omega) = f(\omega_\Delta)$ for every ω)

Definition 5

For a stationary probability measure ν, the *specific entropy* *is*

$$\mathcal{I}^z(\nu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n}(\nu|\pi^{z,Q}), \text{ where}$$

$$\mathcal{I}_{\Lambda_n}(\nu|\pi^{z,Q}) = \begin{cases} \int f \ \ln(f) \ d\pi^{z,Q}_{\Lambda_n} & \text{if} \quad \nu_{\Lambda_n} \ll \pi^{z,Q}_{\Lambda_n}, \ f = \frac{d\nu_{\Lambda_n}}{d\pi^{z,Q}_{\Lambda_n}}, \\ +\infty & \text{else} \end{cases}.$$

Theorem 6 (Georgii)

For any $C > 0$, the set

$$\{\nu \text{ stationary probability measure}, \ \mathcal{I}^z(\nu) \leq C\}$$

is compact for the local convergence topology.

Using Theorem 6, we have the existence of a cluster point $\bar{P}$ of the sequence $(\bar{P}_n)$.

Step 2 : Proving the DLR equations

$$\int f \ d\bar{P} = \int \int f(\omega'_\Lambda + \omega_{\Lambda^c}) \frac{q^{N^\Lambda_{cc}(\omega'_\Lambda + \omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \pi^{z,Q}_\Lambda(d\omega')\bar{P}(d\omega).$$

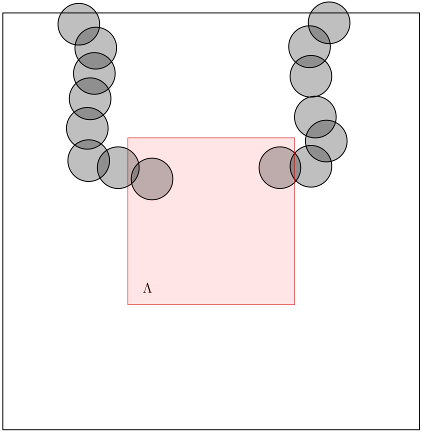
Idea : Since each $\bar{P}_n$ satisfies those equations, we get the result by passing through the limit.
Problem : We can take f local, but $q^{N^\Lambda_{cc}(\omega'_\Lambda + \cdot)}$ and Z_Λ are not local functions.

Idea :

$$\int f \mathbb{1}_{H_k} \, d\bar{P} = \int \int \mathbb{1}_{H_k} f(\omega'_\Lambda + \omega_{\Lambda^c}) \frac{q^{N_{cc}^\Lambda(\omega'_\Lambda + \omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \pi_\Lambda^{z,Q}(d\omega') \bar{P}(d\omega),$$

where (H_k) is a good sequence of "localizing" events.

- N_{cc}^Λ is local on each H_k.
- $\bar{P}(H_k) \to 1$.
- Sometimes we also need $\sup_n \bar{P}_n(H_k) \to 1$.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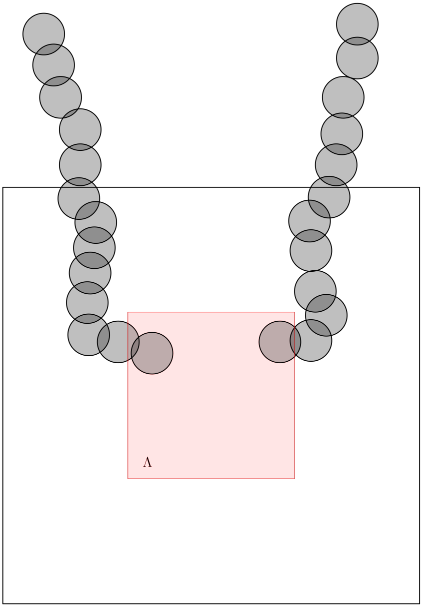
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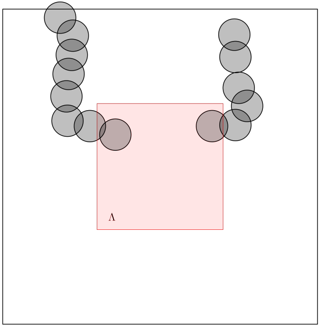
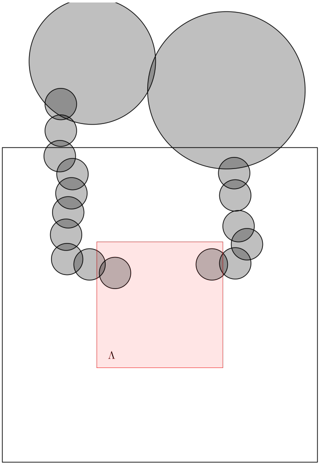
Proposition 7

With $\bar{P}$-probability one, we have at most one infinite connected component.





We need a control for the radii. We obtain this control by stochastic comparison, using a result from the paper of Georgii,Küneth.

Theorem 3 : Sketch of the proof

Method :

$$\text{Widom-Rowlinson model} \xrightarrow[\text{the colors}]{\text{forgetting}} \text{Continuum Random Cluster Model}$$

Let $\tilde{\Omega}$ be the space of colored configurations, ie the space of locally finite subsets of $\mathbb{R}^2 \times \mathbb{R}^+ \times \{1, \dots, q\}$, embedded with the classical σ-algebra $\tilde{\mathcal{F}}$.
$\tilde{\pi}^{z,Q,q}$ denotes the law of the "colored" Poisson point process of intensity measure $z\lambda^{(2)} \otimes Q \otimes \mathcal{U}_q$.
$\mathcal{A}$ denotes the event of authorized configurations, where balls of different color do not overlap.

Definition 8

A probability measure ν on $\tilde{\Omega}$ is a WR(z, Q, q) if

- $\nu(\mathcal{A})=1$.
- *For every measurable bounded function f and every bounded subset Λ of $\mathbb{R}^2$, we have*

$$\int f \, d\nu = \int \int f(\tilde{\omega}'_\Lambda + \tilde{\omega}_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\tilde{\omega}'_\Lambda + \tilde{\omega}_{\Lambda^c})}{\tilde{Z}_\Lambda(\tilde{\omega}_{\Lambda^c})} \tilde{\pi}_\Lambda^{z,Q,q}(d\tilde{\omega}')\nu(d\tilde{\omega})$$

Proposition 9

If $\int R^2 Q(dR) = \infty$ then there exists a $z_c > 0$ such that for every $z < z_c$, there exists a stationary WR(z, Q, q) for which there is at least two phases (two colors).

Step 1 (as before) : $\nu_n(d\tilde{\omega}) = \frac{\mathbb{1}_{\mathcal{A}}(\tilde{\omega})}{\tilde{Z}_n} \tilde{\pi}^{z,Q,q}_{\Lambda_n}(d\tilde{\omega})$.

With this sequence we construct a sequence $(\bar{\nu}_n)$. With the specific entropy we can prove that this sequence has a cluster point $\bar{\nu}$ for the local convergence topology.

Step 2 : $\bar{\nu}(\mathcal{A}) = 1$.

Step 3 : We want to prove

$$\int f \; d\bar{\nu} = \int \int f(\tilde{\omega}'_\Lambda + \tilde{\omega}_{\Lambda^c}) \frac{\mathbb{1}_{\mathcal{A}}(\tilde{\omega}'_\Lambda + \tilde{\omega}_{\Lambda^c})}{\tilde{Z}_\Lambda(\tilde{\omega}_{\Lambda^c})} \tilde{\pi}_\Lambda^{z,Q,q}(d\tilde{\omega}')\bar{\nu}(d\tilde{\omega})$$

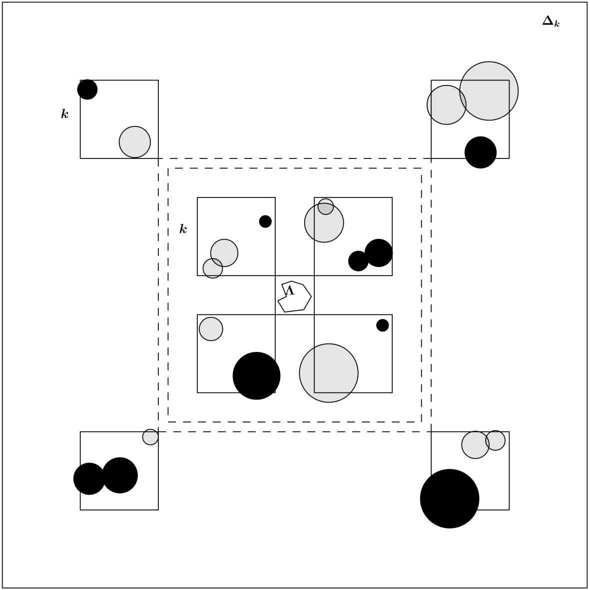
knowing that this equation is realize by each $\bar{\nu}_n$.

Problem : $\mathbb{1}_{\mathcal{A}}$ is not a local function.

Idea : Same as before, using localizing events.

Remark : since $\bar{\nu}(\mathcal{A}) = 1$, we have $\tilde{\omega}_{\Lambda^c} \in \mathcal{A}$ and so to determine the value $\mathbb{1}_{\mathcal{A}}(\tilde{\omega}'_\Lambda + \tilde{\omega}_{\Lambda^c})$ we have to look at

- balls of of $\tilde{\omega}'_\Lambda$.
- balls of of $\tilde{\omega}_{\Lambda^c}$ which intersect balls of $\tilde{\omega}'_\Lambda$.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Problem : In order to have $\bar{\nu}(''\textit{shield}'') \underset{k\to\infty}{\longrightarrow} 1$, we need

$$\bar{\nu}(\{\tilde{\omega} \text{ having at least two colors}\}) = 1,$$

but it is (probably) not true.
Idea : If this event has a not zero probability, then by conditioning we will get this property.
Method : We prove that $\bar{\nu}$ is different from every stationary "monochromatic" probability measures, by comparing there specific entropy.

- $\mathcal{I}^z(\text{"monochromatic measures"}) \geq \frac{q-1}{q} z$.
- For z smaller than some positive z_c we have

$$\mathcal{I}^z(\bar{\nu}) < \frac{q-1}{q} z.$$

- Dereudre, Houdebert : *Infinite Volume Continuum Random Cluster Model*
- *Georgii : Gibbs Measures and Phase Transitions*
- *Georgii, Küneth : Stochastic comparison of point random fields*