Finite variance of the number of stationary points of a Gaussian random field.

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- d a positive integer
- $X \,:\, \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}$ a stationary, Gaussian random field.
- Almost every realization of X is of class $\mathcal{C}^2$.
- $T \subset \mathbb{R}^d$ a bounded rectangle
- $v \in \mathbb{R}^d$
- $\boxed{N(v) = \#\{t \in T \,:\, X'(t) = v\}} \in L^1(\Omega)$

  $N(0)$: number of stationary points of X in T

Problem: $\boxed{N(v) \in L^2(\Omega) \text{ ?}}$

- $r \,:\, t \longmapsto \mathrm{Cov}(X(t), X(0))$: covariance function of X

Former results

$\boxed{d = 1}$

- Cramér and Leadbetter, [1], 1967: if there exists $\delta > 0$ such that :

$$\int_0^\delta \frac{r^{(4)}(0) - r^{(4)}(t)}{t} dt < +\infty,$$

then $N(0) \in L^2(\Omega)$ (Geman condition).

- Geman, [2], 1972: this condition is also necessary.
- Kratz and León, [4], 2006: the same condition is also necessary and sufficient for $N(v)$ to belong to $L^2(\Omega)$, for any level $v \in \mathbb{R}$.

$\boxed{d \geq 1}$

- Elizarov, [3], 1985: a sufficient condition for $N(0)$ to belong to $L^2(\Omega)$.
- Estrade and León, [6] 2015: if X is an isotropic field of class $\mathcal{C}^3$, then for any $v \in \mathbb{R}^d$, $N(v) \in L^2(\Omega)$.

A sufficient condition

Theorem

Suppose $X : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}$ a stationary, Gaussian random field satisfying assumption **(H)** and that its covariance function r satisfies the following condition:

$$\textbf{(G)} \qquad \exists \delta > 0 \quad / \quad \int_{\|t\|<\delta} \frac{\|r^{(4)}(0) - r^{(4)}(t)\|}{\|t\|^d} dt < +\infty,$$

then for any $v \in \mathbb{R}^d$, $N(v) \in L^2(\Omega)$.

$$\textbf{(H)} \begin{cases} \text{Almost every realization of } X \text{ is of class } \mathcal{C}^2, \\ \\ \forall t \neq 0, \ \operatorname{Cov}\left(X'(0), X'(t), (X''_{i,j}(0))_{1\leq i\leq j\leq d}), (X''_{i,j}(t))_{1\leq i\leq j\leq d})\right) \\ \qquad\qquad\qquad \text{is of full rank.} \end{cases}$$

WLOG, we assume that $r(0) = 1$ and $r''(0) = -I_d$.

Rice Formulas

$$\textbf{First moment} \ : \ \mathbb{E}\left[N(v)\right] = \int_T \mathbb{E}\left[|\det X''(t)| \,/\, X'(t) = v\right] p_{X'(t)}(v)\, dt$$
$$= \int_T \mathbb{E}\left[|\det X''(t)|\right] p_{X'(t)}(v)\, dt$$
$$= |T| \; |\det(X''(0))| \; p_{X'(0)}(v),$$

where $|T|$ is the Lebesgue measure of T.

$$\textbf{Second factorial moment} \ : \ \mathbb{E}[N(v)(N(v)-1)]$$
$$= \int_{T\times T} \mathbb{E}[|\det X''(s) \det X''(t)| \,/\, X'(s) = X'(t) = v]\, p_{X'(s),X'(t)}(v,v)\, ds\, dt$$
$$= \int_{T_0} |T \cap (T-t)|\; \mathbf{F(v,t)}\; p_{X'(0),X'(t)}(v,v)\, dt,$$

where $\boxed{F(v,t) = \mathbb{E}\left[|\det X''(0) \det X''(t)| \,/\, X'(0) = X'(t) = v\right] \ ; \ v,t \in \mathbb{R}^d}$,
$T_0 = \left\{t - t',\, (t,t') \in T^2\right\}$.

$$\mathbb{E}[N(v)(N(v)-1)] = \int_{T_0} |T \cap (T-t)| \, \mathbf{F}(\mathbf{v},\mathbf{t}) \, \underbrace{p_{X'(0),X'(t)}(v,v)}_{\leq c\,\|t\|^{-d}} \, dt$$

$$F(v,t) = \mathbb{E}\left[|\det X''(0) \, \det X''(t)| \,/\, X'(0) = X'(t) = v \right]$$

$$F(t,v) \leq (G(v,t)\,G(v,-t))^{1/2}$$

where $\boxed{G(v,t) := \mathbb{E}\left[\det X''(0)^2 \,/\, X'(0) = X'(t) = v\right].}$

Notation: If $g : \mathbb{R}^d \to \mathbb{R}^{d'}$,

$$g \in L^1(\mathcal{V}_0, \|t\|^{-d} dt) \quad \Leftrightarrow \quad \exists \delta > 0 \ : \ \int_{\|t\|<\delta} \|g(t)\| \, \|t\|^{-d} \, dt < +\infty.$$

Lemma

If X satisfies **(H)**, $\qquad G(v,\cdot) \in L^1(\mathcal{V}_0, \|t\|^{-d} dt) \ \Rightarrow \ N(v) \in L^2(\Omega).$

Gaussian regression

$(X''(0)_{i,j})_{1\leq i,j\leq d} \longrightarrow (X''(0))_{1\leq i\leq j\leq d}$ vector of size $K = \frac{d(d+1)}{2}$

$$X''(0) = A(t)\,X'(0) + B(t)\,X'(t) + Z(t),$$

with

- $A(t)$, $B(t)$ matrices of size $K \times d$ depending on $r^{(3)}(0)$ and $r''(t)$
- $Z(t)$ Gaussian centered vector of size K independent of $(X'(0), X'(t))$, its covariance depends on $r''(t)$, $r^{(4)}(0)$ and $r^{(3)}(t)$.

$$\boxed{X''(0)\,/\,\big(X'(0) = X'(t) = v\big) \quad \sim \quad \mathcal{N}\Big((A(t)+B(t))v,\, \Gamma^Z(t)\Big)}$$

Study of $G(v, t)$ and relationship with $G(0, t)$

$$\boxed{G(v,t) := \mathbb{E}\left[\det X''(0)^2 \,/\, X'(0) = X'(t) = v\right]}$$

$$G(v,t) = \mathbb{E}\left[\det\left(\,(A(t)+B(t))v + Z(t)\,\right)^2\right].$$

$$A(t)+B(t) = O(\|t\|) \text{ and } \Gamma^Z(t) = O(1) \text{ as } t \to 0.$$

$$\text{So } G(v,t) = \mathbb{E}\left[\det Z(t)^2\right] + o(\|t\|).$$

Lemma

Let us assume that X fulfills condition **(H)** and let $\mathcal{V} \subset \mathbb{R}^d$ be a compact set. Then
(i) for any $v \in \mathcal{V}$, $G(v,t) = G(0,t) + o(\|t\|)$ as $t \to 0$;
(ii) there exists a homogeneous polynomial $Q_{(d)}$ of degree d, that does not depend on X, such that $G(0,t) = Q_{(d)}(\Gamma^Z(t))$.

An auxiliary function $t \longmapsto \gamma(t)$

For $t \neq 0$, let us introduce $\gamma(t) := (\gamma(t)_{\mathbf{k},\mathbf{l}})_{\substack{1\le \mathbf{k}\le K \\ 1\le \mathbf{l}\le K}}$ the covariance matrix of

$$\boxed{X''(0) \;/\; (X'(0) = X''(0)t = 0)}.$$

A property of symmetric positive matrices:

$$\det\left(X''(0)^2\right) \le \left\langle X''(0)^2 \frac{t}{\|t\|}, \frac{t}{\|t\|}\right\rangle \det(S) = \|t\|^{-2} \left\langle X''(0)t, X''(0)t\right\rangle \det(S).$$

$$\text{So} \quad 0 = \mathbb{E}\left[\det X''(0)^2 \,/\, X'(0) = X''(0)t = 0\right] = \mathbf{Q}_{(\mathbf{d})}(\gamma(\mathbf{t})).$$

$$G(0,t) = \mathbb{E}\left[\det X''(0)^2 \,/\, X'(0) = X'(t) = 0\right] = \mathbf{Q}_{(\mathbf{d})}(\mathbf{\Gamma}^{\mathbf{Z}}(\mathbf{t}))$$

Conclusion

$$G(v,t) = Q_{(d)}(\Gamma^Z(t)) + o(\|t\|) = Q_{(d)}(\Gamma^Z(t)) - Q_{(d)}(\gamma(t)) + o(\|t\|)$$

We deduce from Taylor expansions that

$$\begin{aligned}
\textbf{(G)} \quad & \Leftrightarrow \quad t \mapsto r^{(4)}(0) - r^{(4)}(t) \in L^1(\mathcal{V}_0, \|t\|^{-d} dt) \\
& \Rightarrow \quad \Gamma^Z(t) - \gamma(t) \in L^1(\mathcal{V}_0, \|t\|^{-d} dt) \\
& \Rightarrow \quad G(v,.) \in L^1(\mathcal{V}_0, \|t\|^{-d} dt) \\
& \Rightarrow \quad N(v) \in L^2(\Omega).
\end{aligned}$$

Open questions

- Is the Geman condition a necessary condition for $N(v)$ to belong to $L^2(\Omega)$ in dimension $d > 1$ too?

- If $\phi : \mathbb{R}^d \to \mathbb{R}^d$, what about the second moment of the random variable
$$N(\phi) := \{t \in T \ : \ X'(t) = \phi(t)\}?$$

Main references

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