The Poisson-Voronoi cell around an isolated nucleus

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GDR GeoSto

1. Historical result

Approximation of a convex body from the inside

- K is a given smooth convex body in the plane $\mathbb{R}^2$

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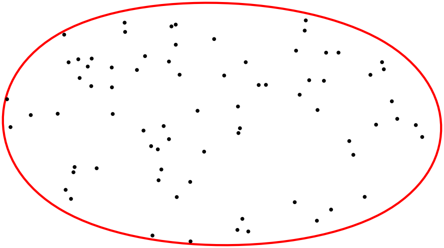

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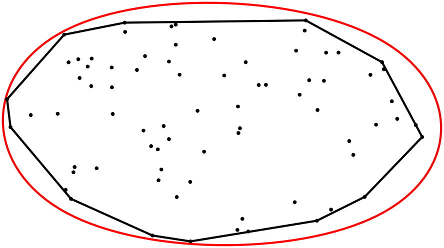

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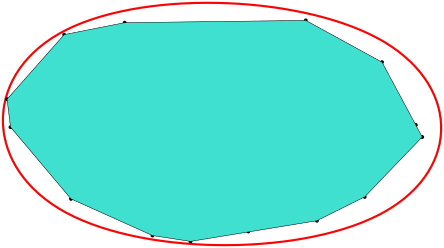

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Question : limit when $\lambda \to \infty$ for

- $\mathcal{A}(K_\lambda)$ = Area of K_λ
- $\mathcal{U}(K_\lambda)$ = Perimeter of K_λ
- $\mathcal{N}(K_\lambda)$ = Number of vertices of K_λ

1. Historical result

Approximation of a convex body from the inside

Theorem [Rényi, Sulanke, 1963-64]
When the intensity $\lambda \to \infty$,

- $\mathcal{A}(K) - \mathbb{E}(\mathcal{A}(K_\lambda)) \sim \lambda^{-\frac{2}{3}} \left(\frac{2}{3}\right)^{\frac{1}{3}} \Gamma(\frac{5}{3}) \int_{\partial K} r_s^{-\frac{1}{3}} \mathrm{d}s$
- $\mathcal{U}(K) - \mathbb{E}(\mathcal{U}(K_\lambda)) \sim \lambda^{-\frac{2}{3}} \frac{3^{\frac{2}{3}}}{2^{\frac{5}{3}}} \Gamma(\frac{5}{3}) \int_{\partial K} r_s^{-\frac{4}{3}} \mathrm{d}s$
- $\mathbb{E}(\mathcal{N}(K_\lambda)) \sim \lambda^{\frac{1}{3}} \left(\frac{2}{3}\right)^{\frac{1}{3}} \Gamma(\frac{5}{3}) \int_{\partial K} r_s^{-\frac{1}{3}} \mathrm{d}s$

where r_s stands for the radius of curvature of ∂K at point s.

2. Approximation of a convex body from the outside

2.1. General issue

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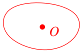

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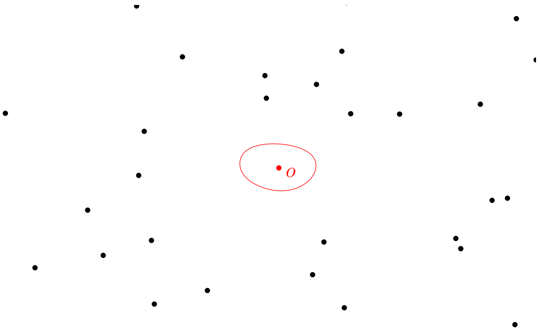

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- Consider a Poisson Point Process Φ with intensity λ with the condition that the Poisson Voronoi tessellation associated with $\Phi \cup \{o\}$ does not intersect K

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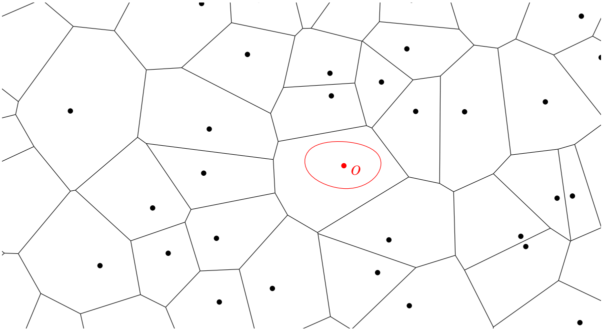

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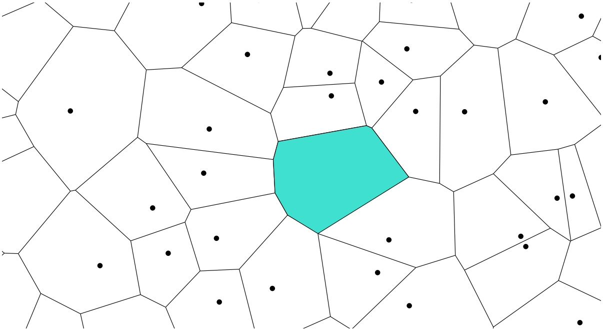

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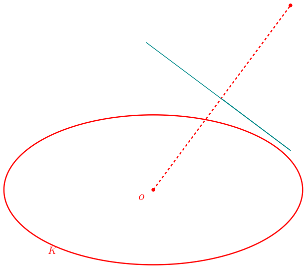
Question : limit of the geometric characteristics of the cell $K_\lambda \supset K$ when $\lambda \to \infty$?

2. Approximation of a convex body from the outside

2.2. Description of the conditioning



Admissible point for the Poisson Point Process Φ

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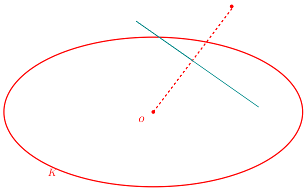

Forbidden point for the Poisson Point Process Φ

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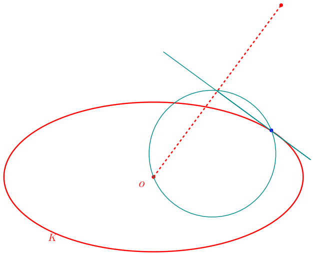

Critical position for a point of the Poisson Point Process Φ

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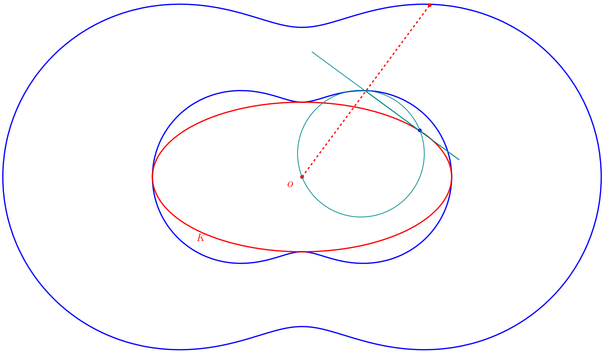

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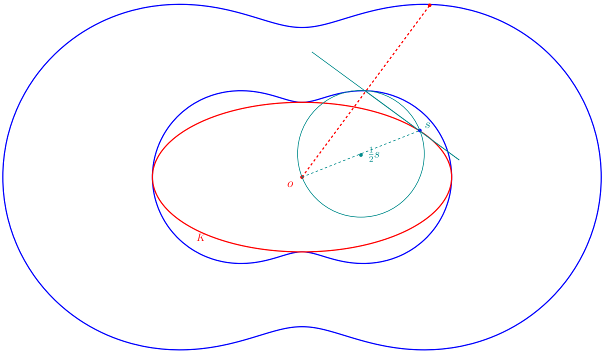


The Poisson point Process Φ is thrown outside a 'security zone' which is twice the Voronoi-flower w.r.t. o defined as :

$$\mathcal{F}_o(K) = \bigcup_{s \in K} B(\tfrac{1}{2}s, \tfrac{1}{2}\|s\|)$$

The boundary of the Voronoi-flower coincides with the set of projections of o onto the tangent lines of ∂K which is the pedal curve of K

3. Main results

Theorem [Calka, Demichel, E.]
When the intensity $\lambda \to \infty$,

- $\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) \sim \lambda^{-\frac{2}{3}} \frac{3^{\frac{2}{3}}}{2^3} \Gamma(\frac{5}{3}) \int_{\partial K} r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} \, \mathrm{d}s$
- $\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K) \sim \lambda^{-\frac{2}{3}} \frac{1}{2 \cdot 3^{\frac{1}{3}}} \Gamma(\frac{5}{3}) \int_{\partial K} (\|s\|^2 - \langle s, n_s \rangle^2) \frac{r_s^{-\frac{2}{3}}}{\langle s, n_s \rangle^{\frac{8}{3}}} \mathrm{d}s$
- $\mathbb{E}(\mathcal{N}(K_\lambda)) \sim \lambda^{\frac{1}{3}} \frac{2}{3^{\frac{1}{3}}} \Gamma(\frac{5}{3}) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} \, \mathrm{d}s$

where r_s and n_s stand respectively for the radius of curvature and normal vector of ∂K at point s.

4. Computation of the exceeding area

- Let $s \in \partial K$. The strategy of the proof relies on the estimation of the distribution of the distance δ_s between s and $\partial K_\lambda \cap s + \mathbb{R}^+ n_s$:

$$\mathbb{P}(\delta_s > h) = \exp\left(-4\lambda \mathcal{A}\big(\mathcal{F}_o(K \cup \{s + hn_s\}) - \mathcal{F}_o(K)\big)\right)$$

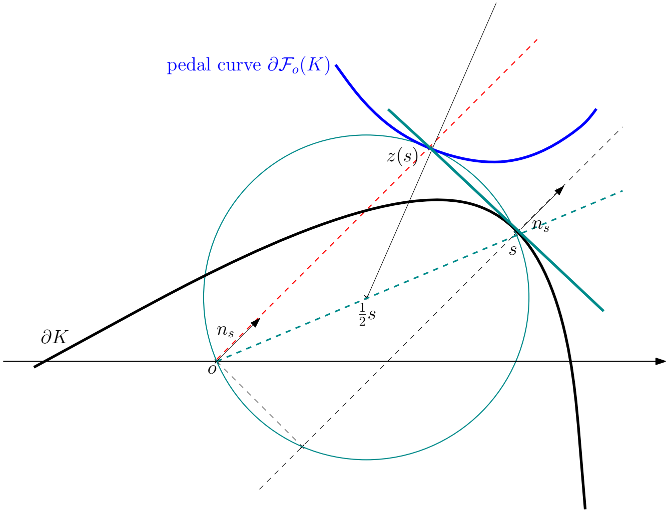
- The asymptotic expected area is obtained by integrating the expectation of this distance along ∂K :

$$\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K) = \int_{\partial K} \int_0^\infty \mathbb{P}(\delta_s > h) \mathrm{d}h \mathrm{d}s$$

- For all $h > 0$:

$$\mathcal{A}\big(\mathcal{F}_o(K \cup \{s + hn_s\}) - \mathcal{F}_o(K)\big) \underset{h \to 0}{=} \frac{4\sqrt{2}}{3} h^{\frac{3}{2}} r_s^{-\frac{1}{2}} \langle s, n_s \rangle + \mathrm{O}(h^2)$$

4. Computation of the exceeding area



$z(s)$ is the projection of o onto the tangent line of ∂K at s and belongs to the circle with diameter $[o, s]$

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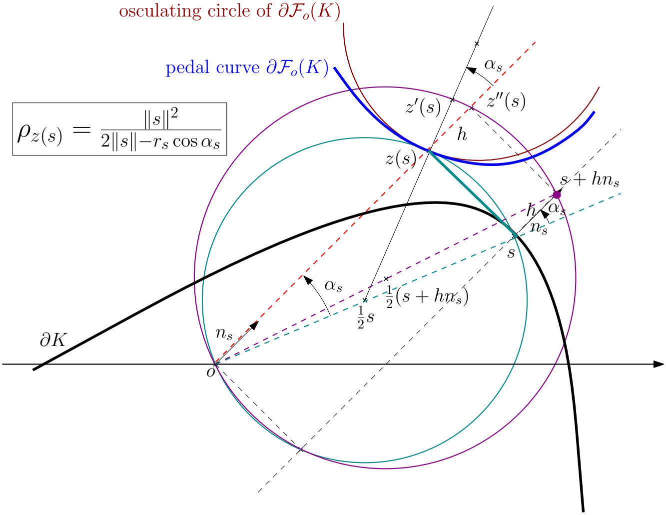
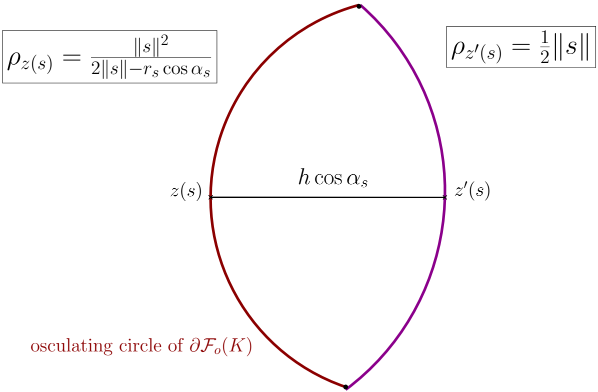


Adding a point $s + hn_s$ in a neighbourhood of s outside K implies an increase of the Voronoi-flower at point $z(s)$

4. Computation of the exceeding area



Idealized picture of the increase of the flower at point $z(s)$

5. Application to two natural problems
5.1. More intrinsic problem

Consider the same problem but with no extra point o : a Poisson Voronoi tessellation is conditioned to have a cell K_λ containing a fixed smooth convex body K.

What happens when the intensity $\lambda \to \infty$?

$\to$ Observe that the germ of K_λ converges to the Steiner point of K

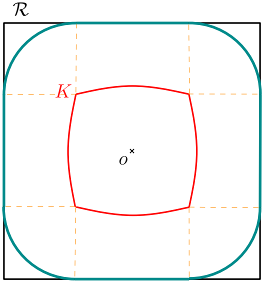
$\to$ Apply the previous results with $o =$ the Steiner point of K

5. Application to two natural problems
5.2. Reverse question

Given a large planar region $\mathcal{R}$ with an origin $o \in \mathcal{R}$, consider a Poisson Point Process outside $\mathcal{R}$ and the associated Poisson Voronoi tessellation.

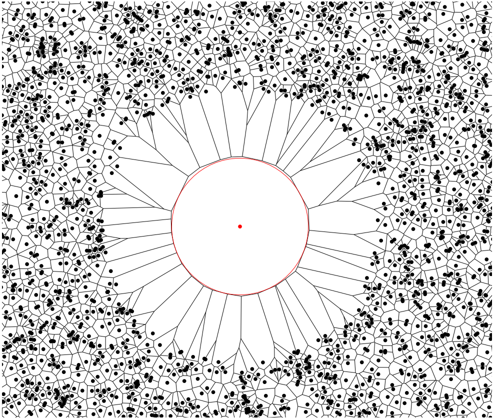
What is the geometry of the cell containing o when the intensity $\lambda \to \infty$?

- $\to$ Consider the maximal flower $\mathcal{F}$ included in $\mathcal{R}$ and the unique convex body K included in $\mathcal{R}$ and whose flower is $\mathcal{F}$
- $\to$ Apply the previous results with K and o



maximal flower $\mathcal{F}$

6. Perspectives



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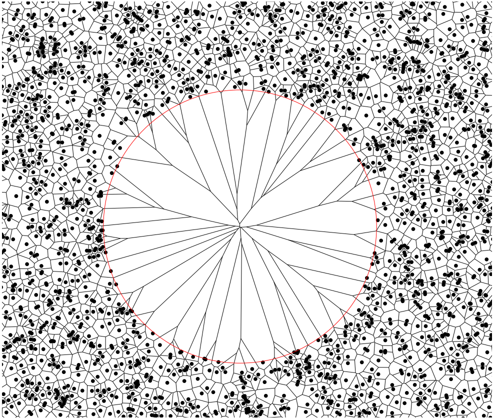

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