Sublinearity of semi-infinite branches for geometric random trees

David Coupier - Université Lille 1

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1. Two models of geometric random trees ($d = 2$)
2. What about their semi-infinite branches?
3. Sketch of the proof

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Model 1

The Radial Poisson Tree $\mathcal{T}_\rho$

$\mathcal{N}$ is a homogeneous Poisson Point Process in $\mathbb{R}^2$ with intensity 1.
$\rho > 0$ is a parameter.
Assume $\mathcal{N} \cap B(O, \rho) = \emptyset$.

$X \in \mathcal{N}$: $\mathrm{Cyl}(X, \rho) := ([O; X] \oplus B(O, \rho)) \cap B(O, |X|)$

- If $\mathrm{Cyl}(X, \rho) \cap \mathcal{N} = \emptyset$ then $A(X) := O$.
- Otherwise

$$A(X) := \operatorname{argmax} \{|Y|, Y \in \mathrm{Cyl}(X, \rho) \cap \mathcal{N}\}.$$

$A(X)$ is the ancestor of X. It is a.s. unique.

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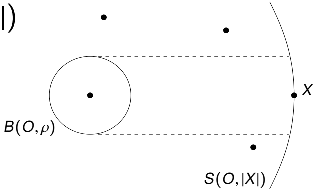

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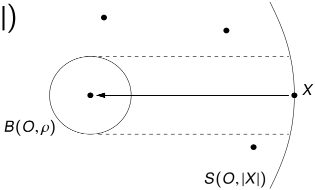

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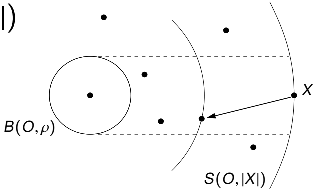

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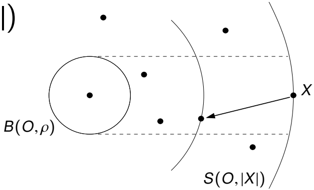

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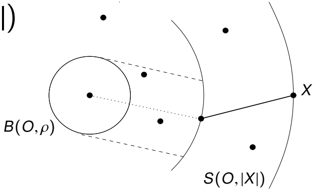

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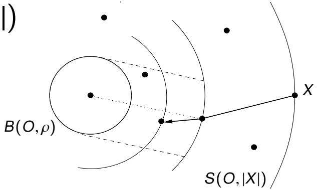

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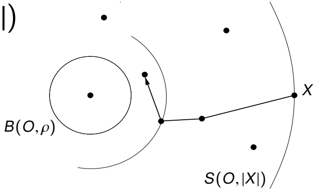

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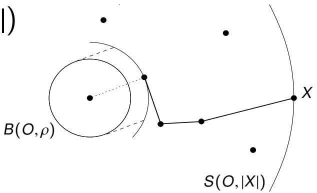

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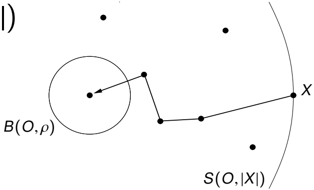

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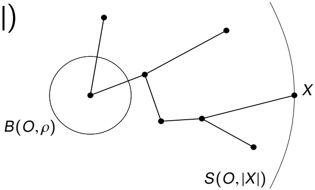

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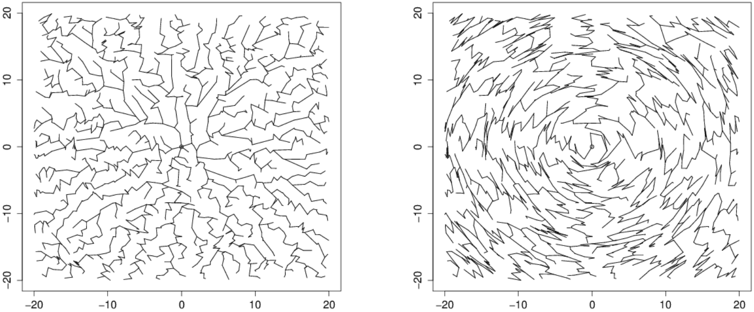
Two simulations of the RPT $\mathcal{T}_\rho$



Built on the same PPP $\mathcal{N}$, the RPT with $\rho = 1$ (to the left) and $\rho = 3$ (to the right).

Remarks about the RPT $\mathcal{T}_\rho$

- Closely related (in some sense) to a directed forest introduced by Ferrari, Landim & Thorisson in '04.
- Its graph structure is local.
- $\forall \rho > 0$, its branches do not cross.
- $\forall \rho > 0$ and $\forall X \in \mathcal{N} \cup \{O\}$, $\{Y \in \mathcal{N}, A(Y) = X\}$ is a.s. finite.

  $\Rightarrow \exists$ at least one semi-infinite branch in $\mathcal{T}_\rho$.

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Model 2

The Euclidean FPP Tree $\mathcal{T}_\alpha$

$\mathcal{N}$ is a homogeneous PPP in $\mathbb{R}^2$ with intensity 1.
$\alpha > 0$ is a parameter.
X^O is the closest point of $\mathcal{N}$ to O.

Any $X \in \mathcal{N}$ is linked to X^O by its geodesic γ_X:

$$\gamma_X := \operatorname{argmin} \left\{ \sum_{0 \leq i \leq n-1} |X_i - X_{i+1}|^\alpha \, , \begin{array}{c} X_1 = X^O, X_2, \ldots, X_{n-1}, X_n = X \\ \text{are points of } \mathcal{N} \text{ and } n \geq 2 \end{array} \right\} .$$

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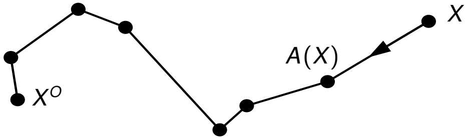

Remarks about the Euclidean FPP Tree $\mathcal{T}_\alpha$

- $\forall \alpha > 0$ and $\forall X \in \mathcal{N}$, existence and uniqueness a.s. of γ_X.

  $\Rightarrow \mathcal{T}_\alpha$ is a tree rooted at X^O.

- Introduced by Howard & Newman in '97 and '01.
- Its graph structure is global.
- $\forall \alpha \geq 2$, its branches do not cross.
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Plan

Straight trees

Howard & Newman in '01 have developed an efficient method describing the semi-infinite branches of straight trees...

Let $\mathcal{T}$ be a geometric random tree in $\mathbb{R}^2$.

- For any vertex X, $\mathcal{T}_X^{\mathrm{out}}$ is the subtree of $\mathcal{T}$ rooted at X.
- $X \in \mathbb{R}^2$ and $\varepsilon > 0$, $C(X, \varepsilon) := \{Y \in \mathbb{R}^2, \mathrm{ang}(X, Y) \leq \varepsilon\}$.

Definition

$\mathcal{T}$ is straight if $\exists f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\ell \to \infty} f(\ell) = 0$ such that a.s. for all but finitely many vertices X, $\mathcal{T}_X^{\mathrm{out}} \subset C(X, f(|X|))$.

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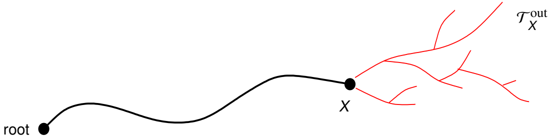

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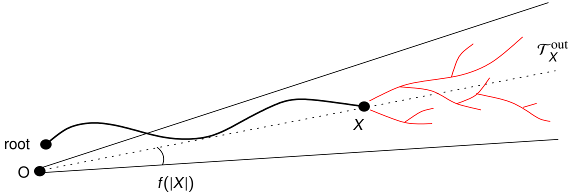

Result of Howard & Newman '01

$(X_n)_{n\in\mathbb{N}}$ has an asymptotic direction $\theta \in [0;2\pi)$ if $\lim_{n\to\infty} \frac{X_n}{|X_n|} = e^{\imath\theta}$.

Theorem (Howard & Newman '01)

Let $\mathcal{T}$ be a straight geometric random tree in $\mathbb{R}^2$ built on a PPP $\mathcal{N}$. Then:

(1) *a.s. every semi-infinite branch of $\mathcal{T}$ has an asymptotic direction;*

(2) *a.s. for every $\theta \in [0;2\pi)$, there is at least one semi-infinite branch of $\mathcal{T}$ with asymptotic direction θ;*

- (C. '14) $\forall \rho > 0$, $\mathcal{T}_\rho$ is straight with $f(\ell) = \ell^{-\frac{1}{2}+\varepsilon}$ and $0 < \varepsilon < \frac{1}{2}$.
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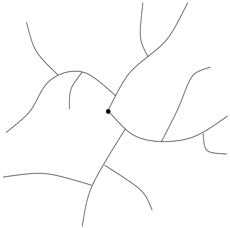
Sublinearity of semi-infinite branches of $\mathcal{T} = \mathcal{T}_\rho$ or $\mathcal{T}_\alpha$

χ_r : number of semi-infinite branches of $\mathcal{T}$ crossing $S(O,r)$.

$\chi_r(\theta, c)$: number of semi-infinite branches of $\mathcal{T}$ crossing the arc of $S(O,r)$ centered at $re^{\imath\theta}$ and with length c.



Rmk: • $\chi_r \overset{\text{a.s.}}{\to} \infty$.

• The mean number of edges of $\mathcal{T}$ crossing $S(O,r)$ is of order r.

Theorem (C. '14)

For $\mathcal{T} = \mathcal{T}_\rho$ with $\rho > 0$ or $\mathcal{T} = \mathcal{T}_\alpha$ with $\alpha \geq 2$:

$$\lim_{r\to\infty} \mathbb{E}\frac{\chi_r}{r} = 0 \quad \textit{and} \quad \chi_r(\theta, c) \longrightarrow 0 \;\; \textit{in } L^1 \textit{ but not a.s.}$$

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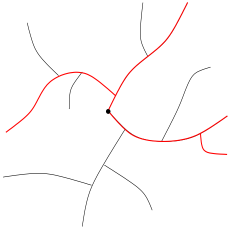

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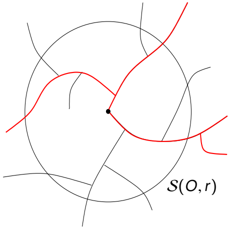

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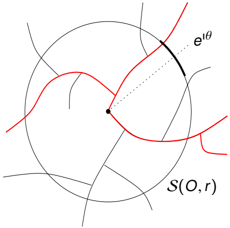

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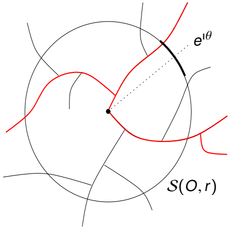

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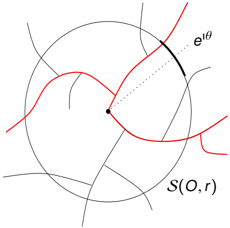

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Plan

1. Two models of geometric random trees ($d = 2$)
2. What about their semi-infinite branches?
3. Sketch of the proof

A robust proof

Our proof is based on a general method which *should* apply to straight trees:

- whose graph structure is local as
  1. the RPT $\mathcal{T}_\rho$, for $\rho > 0$;
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For any local function F,

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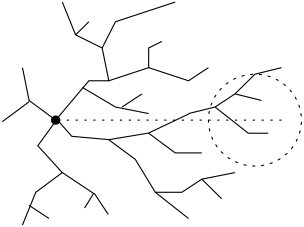

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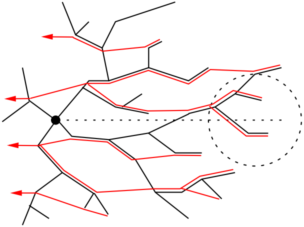

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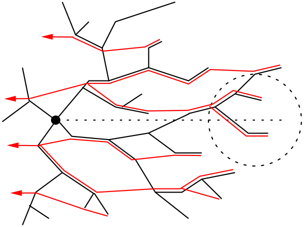

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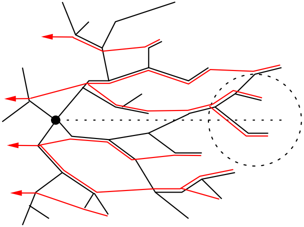

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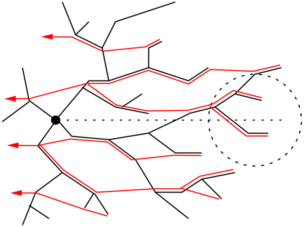

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