Order statistics for the inradius of a Poisson line tessellation

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Plan

Poisson line tessellation

- $\hat{\mathbf{X}}$: stationary and isotropic Poisson line process of intensity π in $\mathbf{R}^2$;
- $\mathfrak{m}$: Poisson line tessellation associated with $\hat{\mathbf{X}}$.

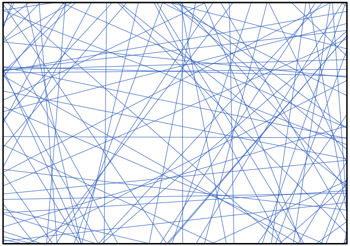

Figure: A realisation of a Poisson line tessellation in a window.

Typical cell

Notation:

- $\mathfrak{m}$: Poisson line tessellation;
- $z(C)$: incenter of C, for each cell $C \in \mathfrak{m}$;
- $a(B)$: area of any Borel subset $B \subset \mathbf{R}^2$.

Definition

The typical cell $\mathcal{C}$ of $\mathfrak{m}$ is a random polygon with distribution given as follows: for each bounded and translation-invariant function $f : \{\text{polygons}\} \to \mathbf{R}$,

$$\mathbb{E}\left[f(\mathcal{C})\right] = \frac{1}{\pi a(B)} \mathbb{E}\left[\sum_{\substack{C \in \mathfrak{m}, \\ z(C) \in B}} f(C)\right],$$

where $a(B) \in (0, \infty)$.

Remarks

- Many results established for the typical cell:
  - mean area: $\mathbb{E}[a(\mathcal{C})] = \frac{1}{\pi}$;
  - mean number of vertices: $\mathbb{E}[n(\mathcal{C})] = 4$;
  - several distributional results, e.g. if $R(C)$ denotes the inradius of any cell $C \in \mathfrak{m}$, we have

$$\mathbb{P}\left(R(\mathcal{C}) \leq v\right) = 1 - e^{-2\pi v}.$$

- Few results for pathological cells (extreme values).

Main problem

Framework:

- Let $\mathfrak{m}$ be a Poisson line tessellation in $\mathbf{R}^2$;
- Let $\mathbf{W}_\rho := \rho^{1/2}B(0, \pi^{-1/2})$, with $\rho > 0$, be a window.
- With each cell $C \in \mathfrak{m}$, we associate:
  - $z(C)$: the incenter of C;
  - $f(C)$: a geometrical characteristic of C, e.g. $f(C)$ is the area, the diameter, the number of vertices of C.

Question: what is the asymptotic behaviour, when $\rho \to \infty$, of

$$M_{f,\rho} = \max_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} f(C)?$$

Remarks

Objective: find $a_{f,\rho}$, $b_{f,\rho}$ such that, for any $t \in \mathbf{R}$, we have

$$\mathbb{P}\left(M_{f,\rho} \leq a_{f,\rho} + b_{f,\rho} t\right) \underset{\rho\to\infty}{\longrightarrow} G(t),$$

where G is non-degenerate. In particular,

- $a_{f,\rho}$: position parameter.
- $b_{f,\rho}$: dispersive parameter.

Remarks:

- many examples provided for numerous geometrical characteristics and random tessellations;
- general results established for random tessellations with a finite range condition, e.g. Poisson-Voronoi and Poisson-Delaunay tessellations.

Framework

Notation:

- $\mathbf{W}_\rho = \rho^{1/2} B(0, \pi^{-1/2})$: window;
- $f(C) := R(C)$: inradius for any cell $C \in \mathfrak{m}$;
- $m_\rho[r]$: r-th smallest value of $R(C)$ over all cells C with incenter $z(C) \in \mathbf{W}_\rho$, $r \geq 1$;
- $M_\rho[r]$: r-th largest value of $R(C)$ over all cells C with incenter $z(C) \in \mathbf{W}_\rho$, $r \geq 1$.

Question: asymptotic behaviours of $m_\rho[r]$ and $M_\rho[r]$?

Asymptotics for the order statistics

Theorem 1

Let $\mathfrak{m}$ be a stationary, isotropic Poisson line tessellation in $\mathbf{R}^2$ and let $r \geq 1$ be fixed. Then

(i) smallest values: for any $t \geq 0$,

$$\mathbb{P}\left(m_\rho[r] \geq (2\pi^2\rho)^{-1} t \right) \underset{\rho\to\infty}{\longrightarrow} e^{-t} \sum_{k=0}^{r-1} \frac{t^k}{k!};$$

(ii) largest values: for any $t \in \mathbf{R}$,

$$\mathbb{P}\left(M_\rho[r] \leq \frac{1}{2\pi}(\log(\rho) + t) \right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}} \sum_{k=0}^{r-1} \frac{(e^{-t})^k}{k!}.$$

Shape of small cells

Theorem 2

Let $\mathfrak{m}$ be a stationary, isotropic Poisson line tessellation in $\mathbf{R}^2$ and let $r \geq 1$ be fixed. Then

$$\mathbb{P}\left(\bigcap_{1 \leq k \leq r} \left\{ n(C_\rho[k]) = 3 \right\} \right) \underset{\rho \to \infty}{\longrightarrow} 1,$$

where $n(C_\rho[k])$ is the number of vertices of the k-th (unique) smallest cell for the inradius.

Remark: in the same spirit, we show that the same fact holds for cells with a small area (Kendall's conjecture).

Remarks and notation

Remarks:

- any cell $C \in \mathfrak{m}$ has a.s. a unique inball $B(C)$;
- exactly three lines are tangent to the border of $B(C)$ and contain three edges of C simultaneously;
- these lines defines a triangle, say $\triangle(C)$.

Notation: for any 3-tuple of lines $H_{1:3} = (H_1, H_2, H_3) \in \hat{\mathbf{X}}^3_{\neq}$, let:

- $\triangle(H_{1:3})$: triangle spanned by $H_{1:3}$;
- $z(H_{1:3})$: incenter of $\triangle(H_{1:3})$.

Sketch of proof of Theorem 1, Part (i)

- Step 1 (intermediate random variable $\overset{\triangle}{m}_\rho[r]$):
  - define $\overset{\triangle}{m}_\rho[r]$ as the r-th smallest value of $R(H_{1:3})$ *over all* 3*-tuples of lines* $H_{1:3} \in \hat{\mathbf{X}}^3_{\neq}$ such that $\triangle(H_{1:3}) \subset \mathbf{W}_\rho$;
  - investigate $\overset{\triangle}{m}_\rho[r]$ by applying a theorem on U-statistics in stochastic geometry.
- Step 2 (deviation between $\overset{\triangle}{m}_\rho[r]$ and $m_\rho[r]$):
  - prove that small cells are triangles with high probability (Theorem 2);
  - deal with boundary effects;
  - deduce that $\overset{\triangle}{m}_\rho[r]$ and $m_\rho[r]$ have asymptotically the same order.

Poisson approximation

For any K-tuple of cells $C_{1:K} = (C_1, \ldots, C_K) \in \mathfrak{m}_{\neq}^{K}$, write $R(C_{1:K}) > v$ to specify that $R(C_i) > v$ for each $i \leq K$, $v > 0$.

Lemma 3

Let v_ρ and $\tau > 0$ be such that, for any $K \geq 1$

$$I^{(K)}(\rho) := \mathbb{E}\left[\sum_{\substack{C_{1:K} \in \mathfrak{m}_{\neq}^{K} \\ z(C_{1:K}) \in \mathbf{W}_\rho}} \mathbf{1}_{R(C_{1:K}) > v_\rho}\right] \underset{\rho \to \infty}{\longrightarrow} \tau^K.$$

Then

$$\mathbb{P}\Big(M_\rho[r] \leq v_\rho\Big) \underset{\rho \to \infty}{\longrightarrow} \sum_{k=0}^{r-1} \frac{\tau^k}{k!} e^{-\tau}.$$

NB: $I^{(K)}(\rho)$ is the so-called mean number of K-tuples of exceedances.

Proof and Remarks for Lemma 3

Proof: thanks to the method of moments and the assumption, we have:

$$\mathbb{P}(M_\rho[r] \leq v_\rho) = \mathbb{P}\left(\sum_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} \mathbf{1}_{R(C) > v_\rho} \leq r - 1 \right) \underset{\rho \to \infty}{\longrightarrow} e^{-\tau} \sum_{k=0}^{r-1} \frac{\tau^k}{k!}.$$

Remarks:

- The previous lemma remains true in any dimension, for any geometrical characteristic and for any random tessellation.
- The method of moments seems necessary to derive Poisson approximation:
  - the largest order statistics are not related to U-statistics;
  - general theorems for random tessellations and Chen-Stein method cannot be applied in our context.

Remarks

- For any fixed $t \in \mathbf{R}$, we apply Lemma 3 (Poisson approximation) to:
  - $v_\rho = v_\rho(t) = \frac{1}{2\pi}(\log(\pi\rho) + t)$;
  - $\tau = \tau(t) = e^{-t}$.

- When $K = 1$, the assumption of Lemma 3 is satisfied. Indeed, denoting by $\mathcal{C}$ the typical cell of $\mathfrak{m}$, we have

$$\mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{W}_\rho}} \mathbf{1}_{R(C)>v_\rho}\right] = \pi\rho\cdot\mathbb{P}\left(R(\mathcal{C}) > v_\rho\right) = e^{-t} = \tau$$

  since $R(\mathcal{C})$ has an exponential distribution with parameter 2π.

- When $K \geq 2$, the main difficulties are:
  1. dependence between cells which share one or two common lines;
  2. dependence between cells which are close to each other (even if they do not share common lines).

Configuration of cells

For any K-tuple of cells $C_{1:K} = (C_1, \ldots, C_K) \in \mathfrak{m}^K{}_{\neq}$, let:

- $L(C_{1:K}) \leq 3K$: number of lines (without repetition) which are tangent to the inballs $B(C_i)$, $1 \leq i \leq K$;
- $n_k(C_{1:K})$: number of connected components of size k of $\bigcup_{i=1}^{K} B\left(z(C_i), R_{\max}(C_{1:K})^3\right)$, where $R_{\max}(C_{1:K}) = \max_{i \leq K} R(C_i)$.

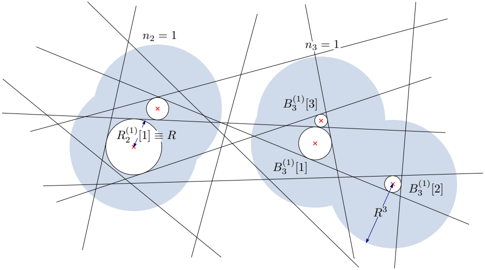

Figure: Example of connected components for $K = 5$ and $(n_1, \ldots, n_5) = (0, 1, 1, 0, 0)$.

Intermediate random variables

Write $I^{(K)}(\rho)$ (mean number of K-tuples of exceedances) as:

$$I^{(K)}(\rho) = \sum \left(I_{S^c}^{(n_{1:K})}(\rho) + I_S^{(n_{1:K})}(\rho) \right),$$

where the sum is taken over all $n_{1:K} = (n_1, \dots, n_K) \in \mathbf{N}^K$ such that $\sum_{k=1}^K k n_k = K$ and where

$$I_{S^c}^{(n_{1:K})}(\rho) := \mathbb{E}\left[\sum_{\substack{C_{1:K} \in \mathfrak{m}_{\neq}^K, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbf{1}_{R(C_{1:K}) > v_\rho} \mathbf{1}_{n_{1:K}(C_{1:K}) = n_{1:K}} \mathbf{1}_{L(C_{1:K}) = 3K} \right],$$

$$I_S^{(n_{1:K})}(\rho) := \mathbb{E}\left[\sum_{\substack{C_{1:K} \in \mathfrak{m}_{\neq}^K, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbf{1}_{R(C_{1:K}) > v_\rho} \mathbf{1}_{n_{1:K}(C_{1:K}) = n_{1:K}} \mathbf{1}_{L(C_{1:K}) < 3K} \right].$$

Intermediate results

Proposition

(i) $I_{S^c}^{(K,0,\dots,0)}(\rho) \underset{\rho\to\infty}{\longrightarrow} \tau^K$;

(ii) $I_{S^c}^{(n_{1:K})}(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$ *for all* $n_{1:K} \neq (K,0,\dots,0)$;

(iii) $I_{S}^{(n_{1:K})}(\rho) \underset{\rho\to\infty}{\longrightarrow} 0$ *for all* $n_{1:K}$.

Heuristic proof:

(i) the inradii of cells behave as though they are independent;

(ii) with high probability, the inradii of neighbouring cells cannot simultaneously exceed v_ρ (uniform bounds);

(iii) the proportion of K-tuples of cells which share at least one line is negligible relative to those that do not.

Perspectives

- rates of convergence for the smallest and largest order statistics;
- intermediate order statistics;
- asymptotics for the inradius in any dimension;
- other geometrical characteristics;
- extremes for various random tessellations (e.g. STIT, Johnson-Mehl, Laguerre).

Thank your for your attention!