Conformal Measure Ensembles *

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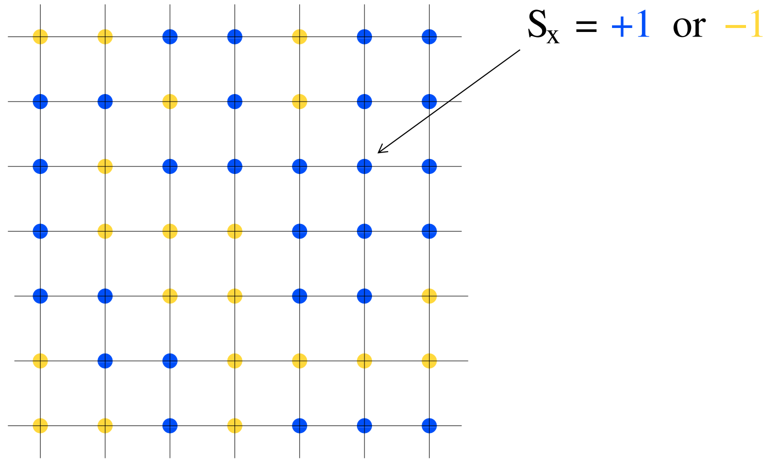
*Based on joint work with Christophe Garban and Charles M. Newman, and with Demeter Kiss and René Conijn.

Ising Model on $\mathbb{Z}^2$



Probability $\propto$ exp($\beta \sum_{\{x,y\}} S_x S_y$)

Spins: $S_x, S_y = \pm 1$
Edges: $e = \{x, y\}$ ($||x - y|| = 1$)

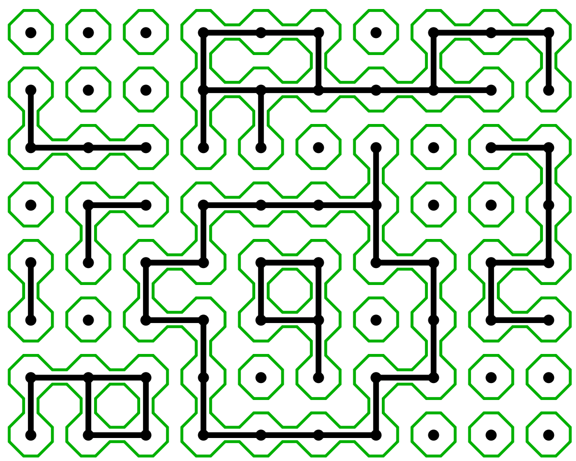
Continuum scaling limit: replace $\mathbb{Z}^2$ by $a\,\mathbb{Z}^2$ and let $a \to 0$.

FK Representation

Let $p = 1 - e^{-2\beta}$ and choose random subset of edges with probab.

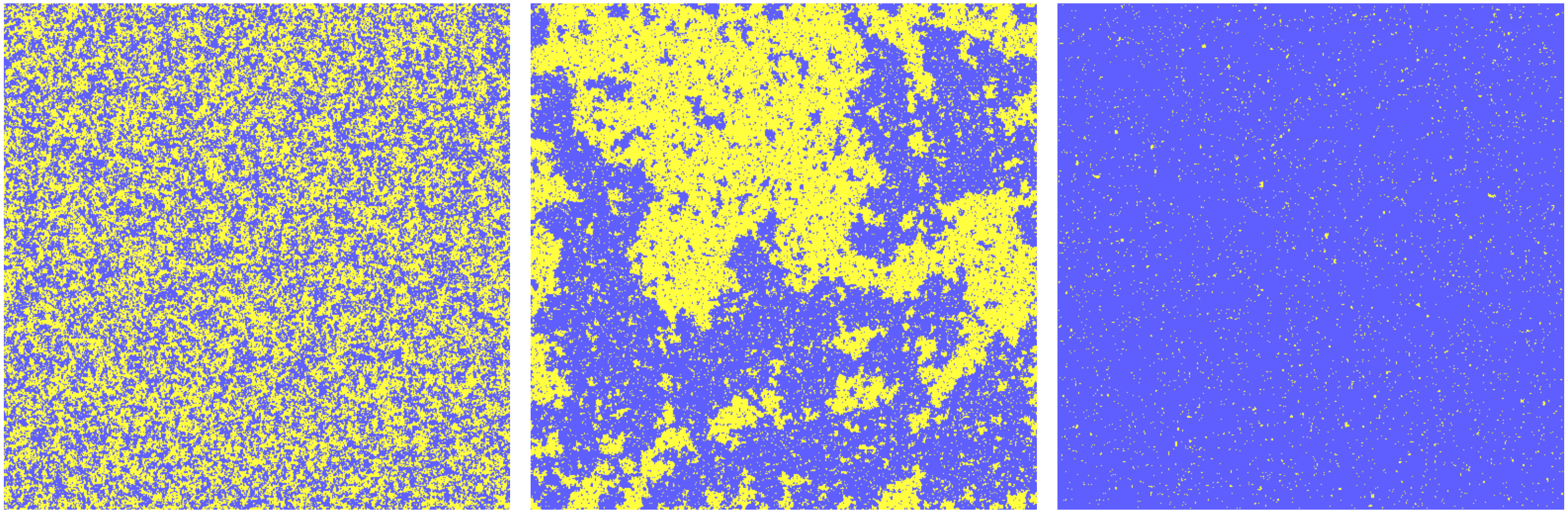
$$p^{\#\text{ chosen edges}}(1-p)^{\#\text{ empty edges}}2^{\#\text{ clusters}}.$$

Then assign to each cluster spin ± 1 with equal probability.



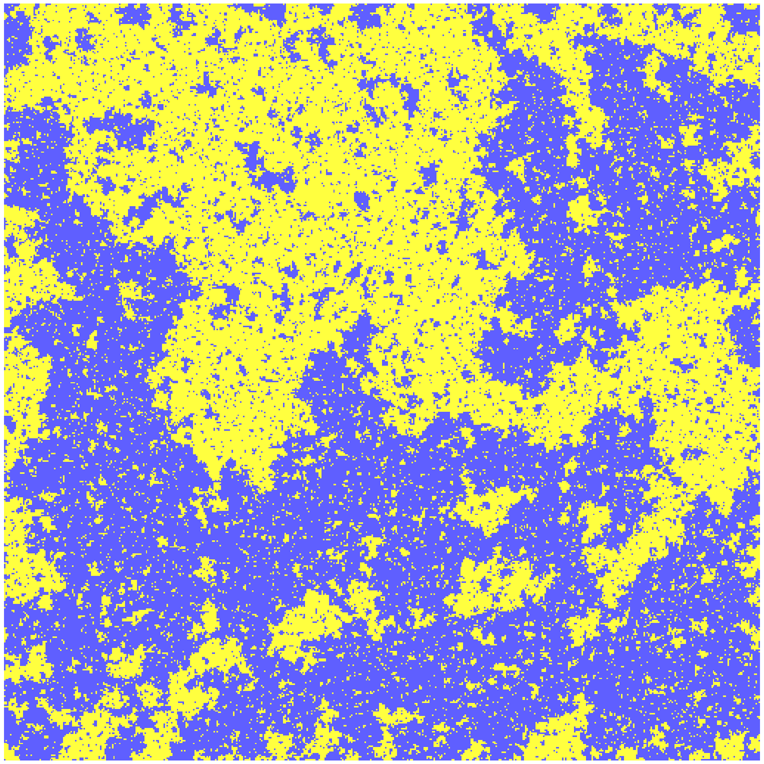
$$E(S_x S_y) = P(x, y \text{ belong to same cluster})$$

Three Regimes



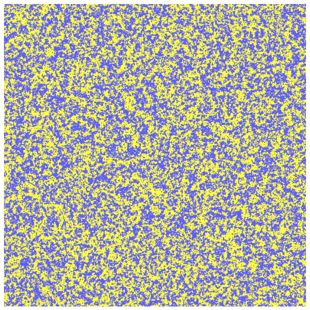
Evidence of a phase transition.

The Critical Point: $\beta = \beta_c = \frac{1}{2}\log(1+\sqrt{2})$

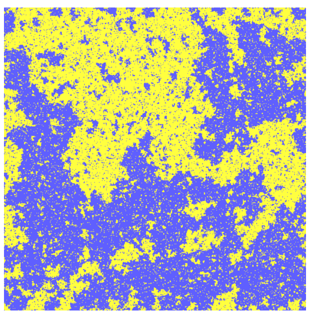


From now on $\beta = \beta_c$, unless otherwise stated.

Scaling Limits for the Magnetization



High temperature: $\frac{1}{\sqrt{1/a^2}} \sum_{x \in \text{square}} S_x \overset{a \to 0}{\longrightarrow} M \sim$ Normal dist.



Critical temperature: classical CLT does not hold.

Critical Magnetization Field

F.C., C. Garban, C.M. Newman

Ann. Probab. (2015) and Ann. Inst. H. Poincaré (to appear)

$$\Phi^a := a^{15/8} \sum_{x \in \mathbb{Z}^2} S_x \delta(z - ax)$$

Critical scaling limit: $\beta = \beta_c$, $a \to 0$

$\Phi^a \to$ random generalized function Φ^0: massless field (power-law decay of correlations)

The limiting magnetization field is not Gaussian:

$$\log \mathbb{P}(\Phi^0([0,1]^2) > x) \overset{x \to \infty}{\sim} -c\, x^{16}$$

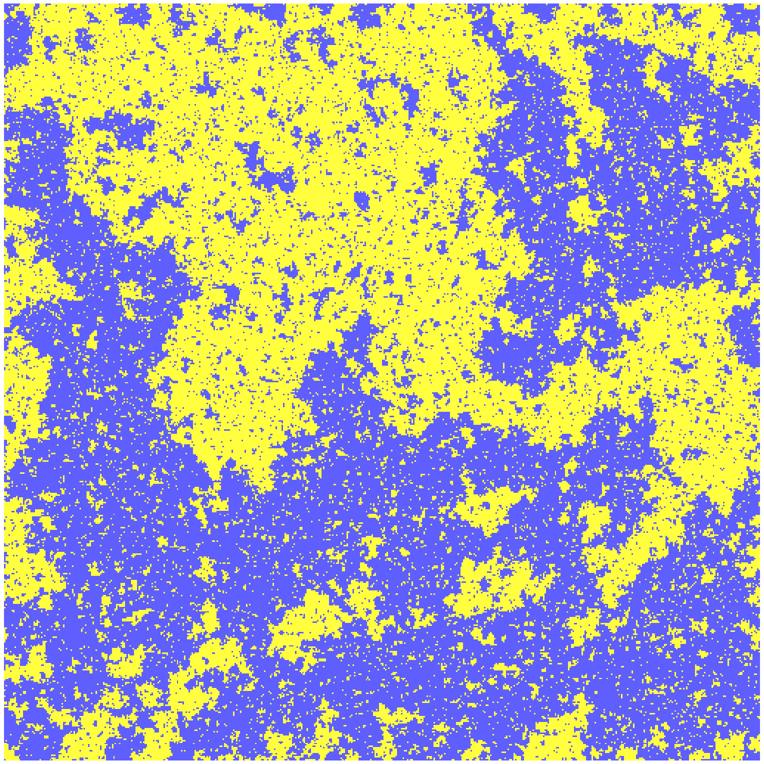
$f(z)$: test function of bounded support on $\mathbb{R}^2$
C_i's are FK clusters

$$\begin{aligned}
\langle \Phi^a, f \rangle &= \int_{\mathbb{R}^2} f(z)\Phi^a(z)dz \\
&= \int_{\mathbb{R}^2} f(z)[a^{15/8} \sum_{x \in \mathbb{Z}^2} S_x \delta(z - ax)]dz \\
&\stackrel{dist.}{=} \sum_i \sigma_i \int_{\mathbb{R}^2} f(z)[a^{15/8} \sum_{x \in C_i} \delta(z - ax)]dz \\
&\stackrel{dist.}{=} \sum_i \sigma_i \mu_i^a(f),
\end{aligned}$$

with σ_i's i.i.d. symmetric (± 1)-valued random variables and $\mu_i^a = a^{15/8} \sum_{z \in C_i} \delta(z - ax)$.

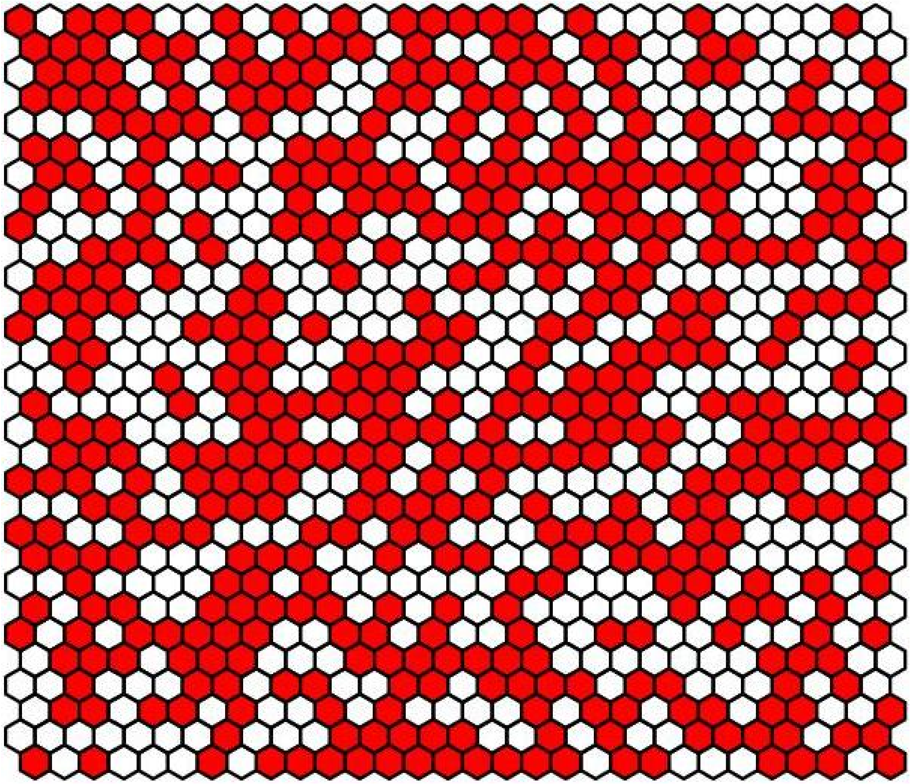
Scaling limit: $a \to 0$, $\langle \Phi^0, f \rangle \stackrel{dist.}{=} \sum_j \sigma_j \mu_j^0(f)$ **???**

Collection of Clusters, their Boundaries and Areas



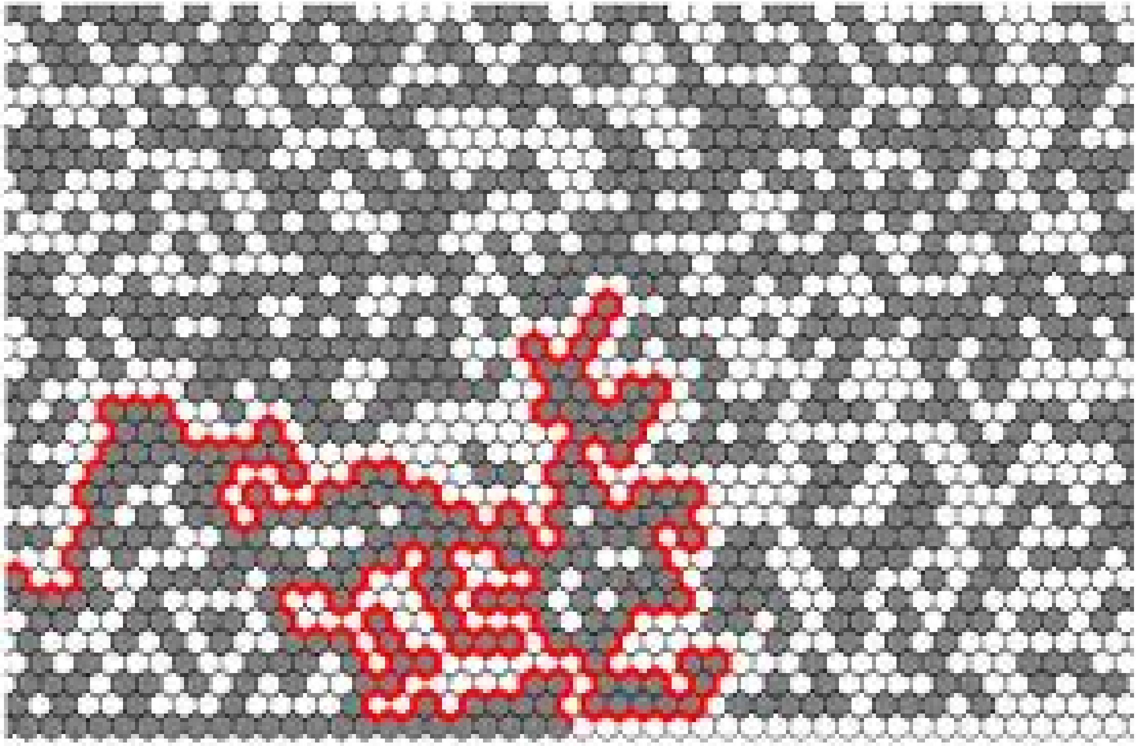
$$(C_i^a, \gamma_i^a, \mu_i^a)_i \xrightarrow{?} (C_j^0, \gamma_j^0, \mu_j^0)_j \text{ with } C_j^0 = \text{supp}(\mu_j^0)$$

Percolation: A Random Coloring Model



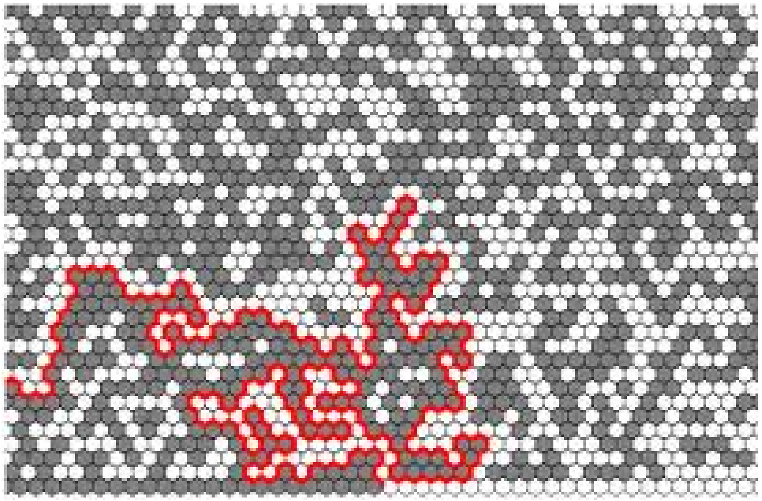
Critical point: equal density of red and white

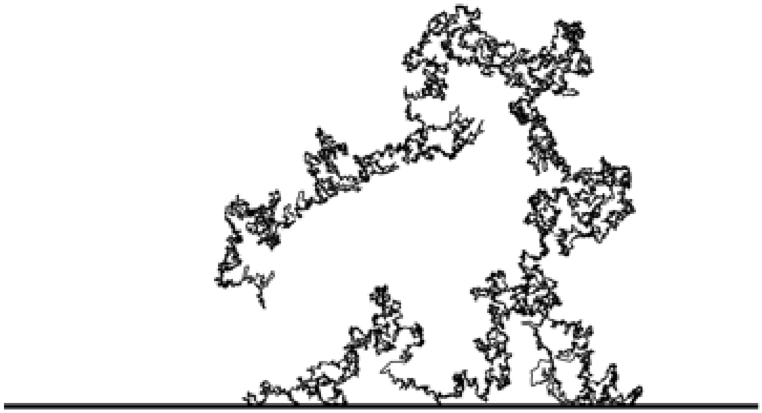
Exploration Path I



Can one describe the exploration path in the scaling limit?

Percolation Exploration Path





Exp. path $\to$ SLE$_6$ (O. Schramm; S. Smirnov; F.C.-C.M. Newman)

Conformal Loop and Measure Ensembles: Percolation

F.C. and C.M. Newman (2006): Collection of percolation interfaces $(\gamma_i^a)_i \stackrel{a \to 0}{\longrightarrow}$ CLE$_6$ (Conformal Loop Ensemble).

F.C., R. Conijn and D. Kiss (2015): Let

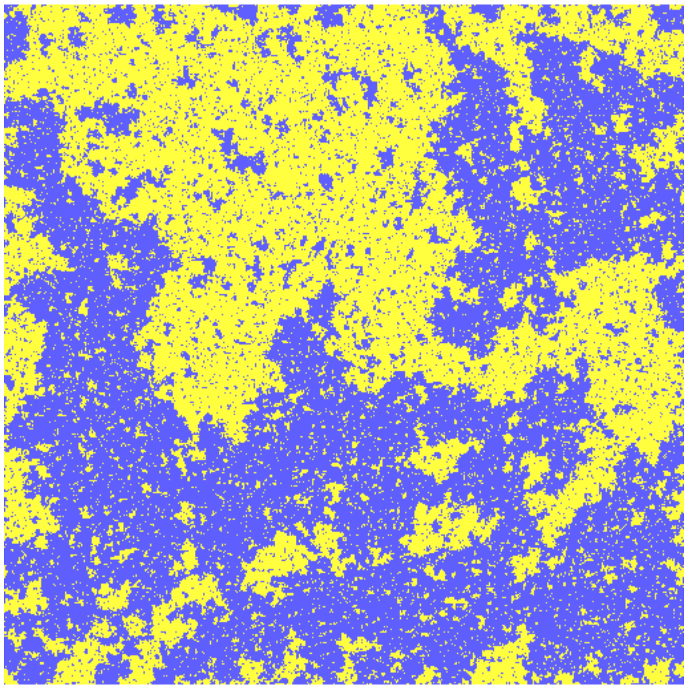
$$\mu_i^a = a^{91/48} \sum_{z \in C_i} \delta(z - ax),$$

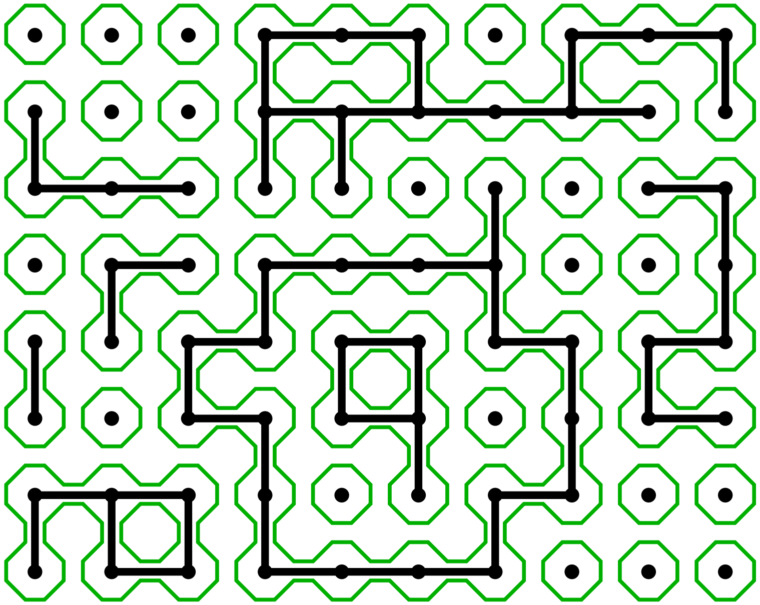
then $(C_i^a, \gamma_i^a, \mu_i^a)_i \stackrel{a \to 0}{\longrightarrow} (C_j^0, \gamma_j^0, \mu_j^0)_j$ with $C_j^0 = \text{supp}(\mu_j^0)$.

If φ is conformal map and $z' = \varphi(z)$,

$$\text{“}\mu_j^0(dz') \stackrel{\text{dist.}}{=} |\varphi'(z)|^{91/48} \mu_j^0(dz)\text{”}$$

Ising Interfaces
(S. Smirnov et al.)





Left: Interfaces between spin clusters $\rightarrow \mathrm{SLE}_3$
Right: FK interfaces $\rightarrow \mathrm{SLE}_{16/3}$

Conformal Loop and Measure Ensembles: Ising

Assumption: Collection of FK interfaces $(\gamma_i^a)_i \xrightarrow{a\to 0} \mathrm{CLE}_{16/3}$.

F.C., R. Conijn and D. Kiss (2015): Let

$$\mu_i^a = a^{15/8} \sum_{z \in C_i} \delta(z - ax),$$

then $(C_i^a, \gamma_i^a, \mu_i^a)_i \xrightarrow{a\to 0} (C_j^0, \gamma_j^0, \mu_j^0)_j$ with $C_j^0 = \mathrm{supp}(\mu_j^0)$.

If $z' = \alpha z + \beta$,

$$\text{“}\mu_j^0(dz') \overset{\text{dist.}}{=} \alpha^{15/8} \mu_j^0(dz)\text{”}$$

Geometric Representation of the Magnetization Field

F.C., R. Conijn and D. Kiss (2015): Let

$$\Phi^0_\varepsilon = \sum_{C_j:\mathrm{diam}(C_j)>\varepsilon} \sigma_j \mu^0_j.$$

Then $\langle \Phi^0_\varepsilon, f \rangle$ is in L^2 and

$$\langle \Phi^0_\varepsilon, f \rangle \stackrel{\varepsilon \to 0}{\longrightarrow} \langle \Phi^0, f \rangle \text{ in } \mathrm{L}^2.$$