Cells With Many Facets in a Hyperplane Mosaic

Gilles Bonnet,
joint work with Matthias Reitzner and Pierre Calka

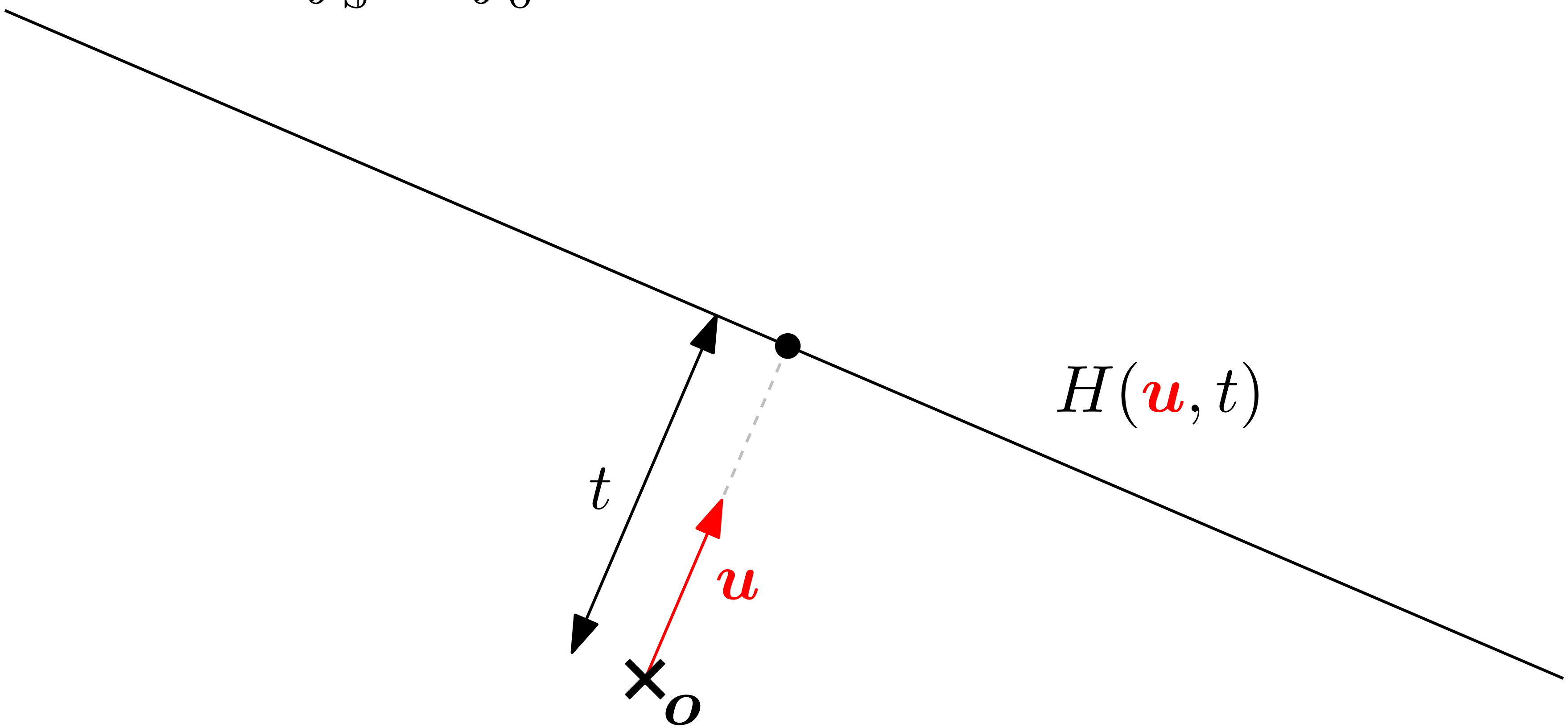
4th Stochastic Geometry Days
Poitiers, Friday 28th August 2015

Stationary Poisson Hyperplane Mosaic in $\mathbb{R}^d$

η Poisson Hyperplane Process of **intensity measure** Θ

φ **directional distribution** (even measure on $\mathbb{S}^{d-1}$)

$$\Theta(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}\{H(\boldsymbol{u}, t) \in \cdot\} \, \mathrm{d}t \, \varphi(\mathrm{d}\boldsymbol{u})$$



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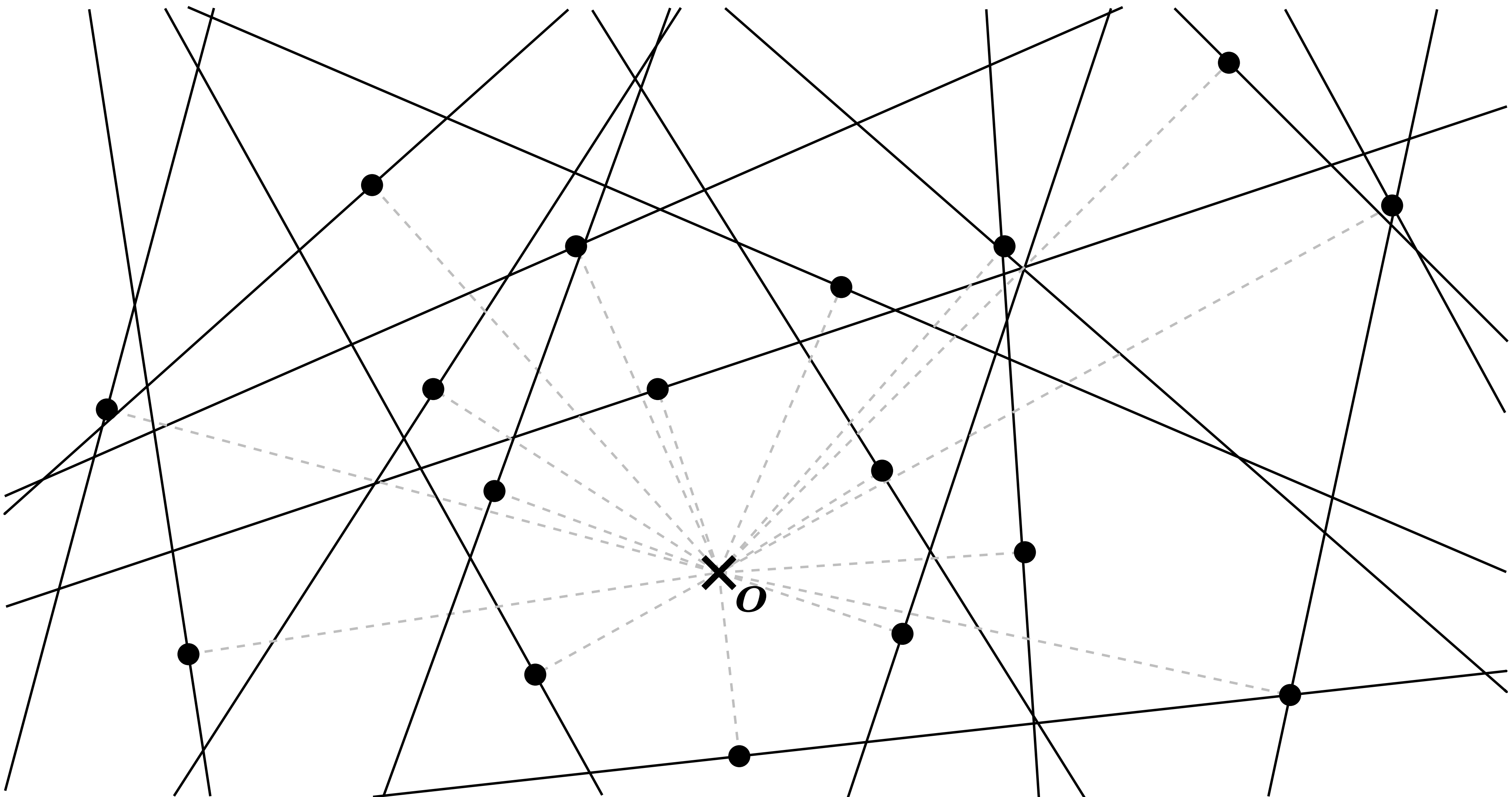

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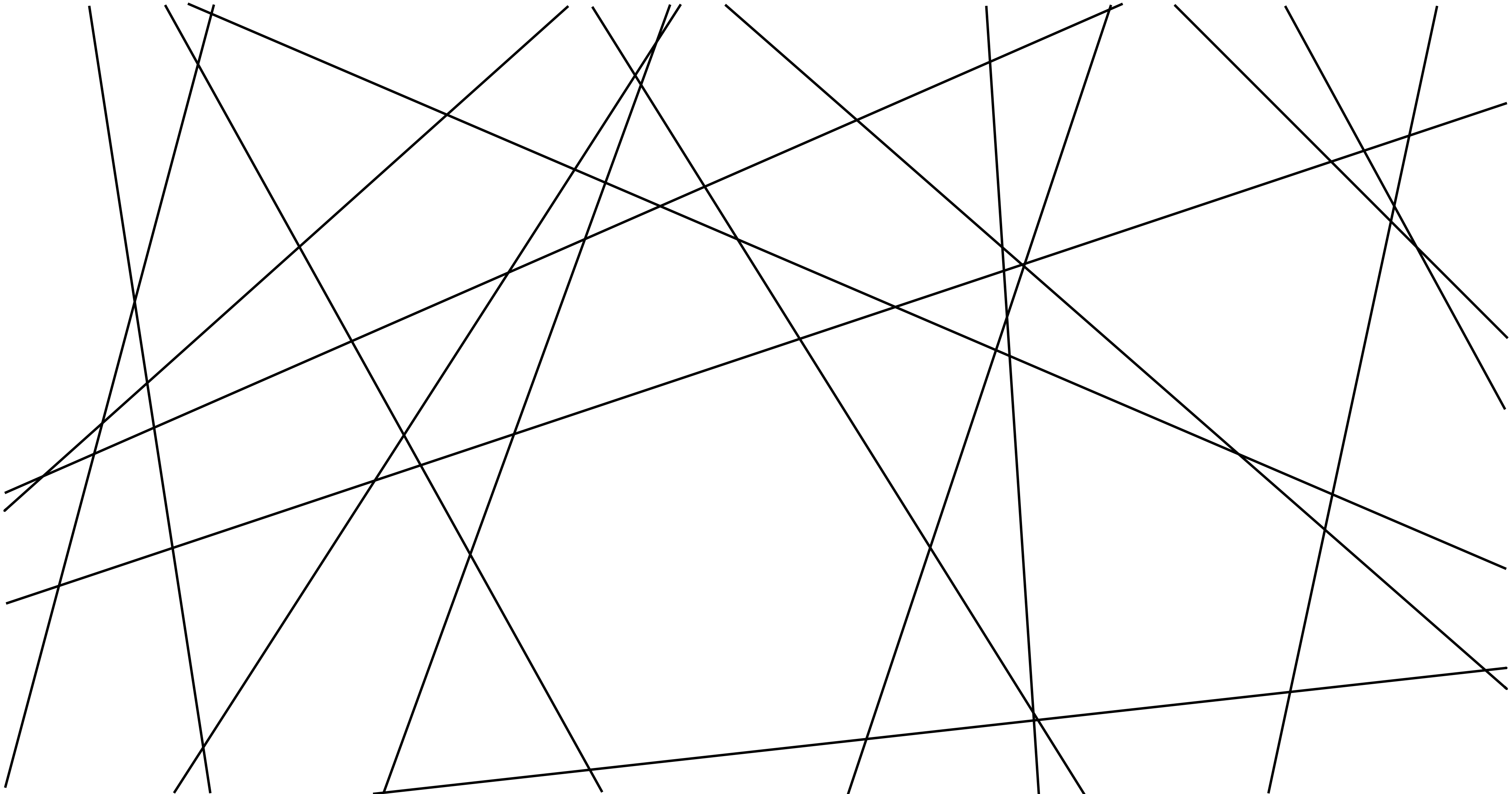

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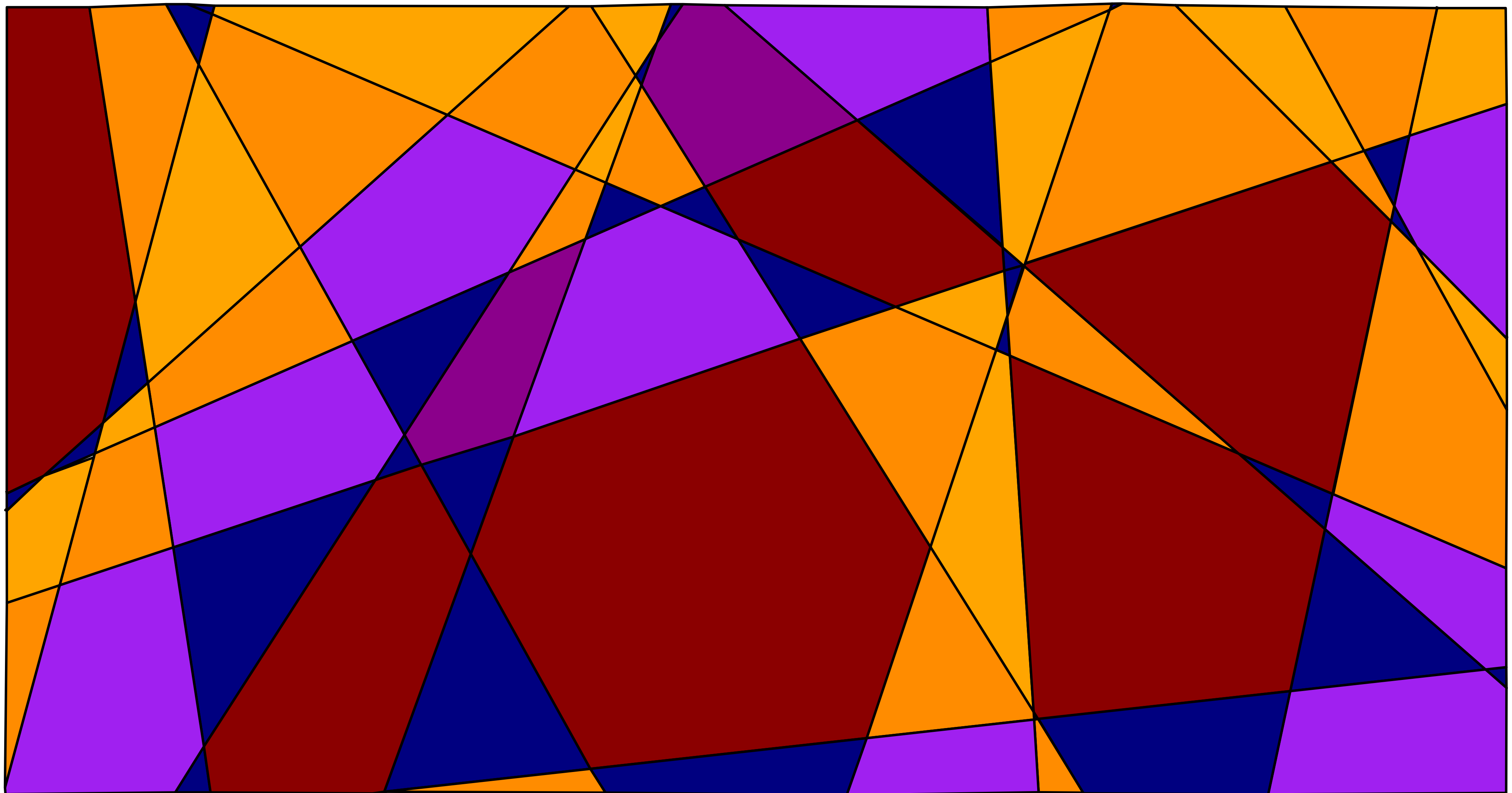
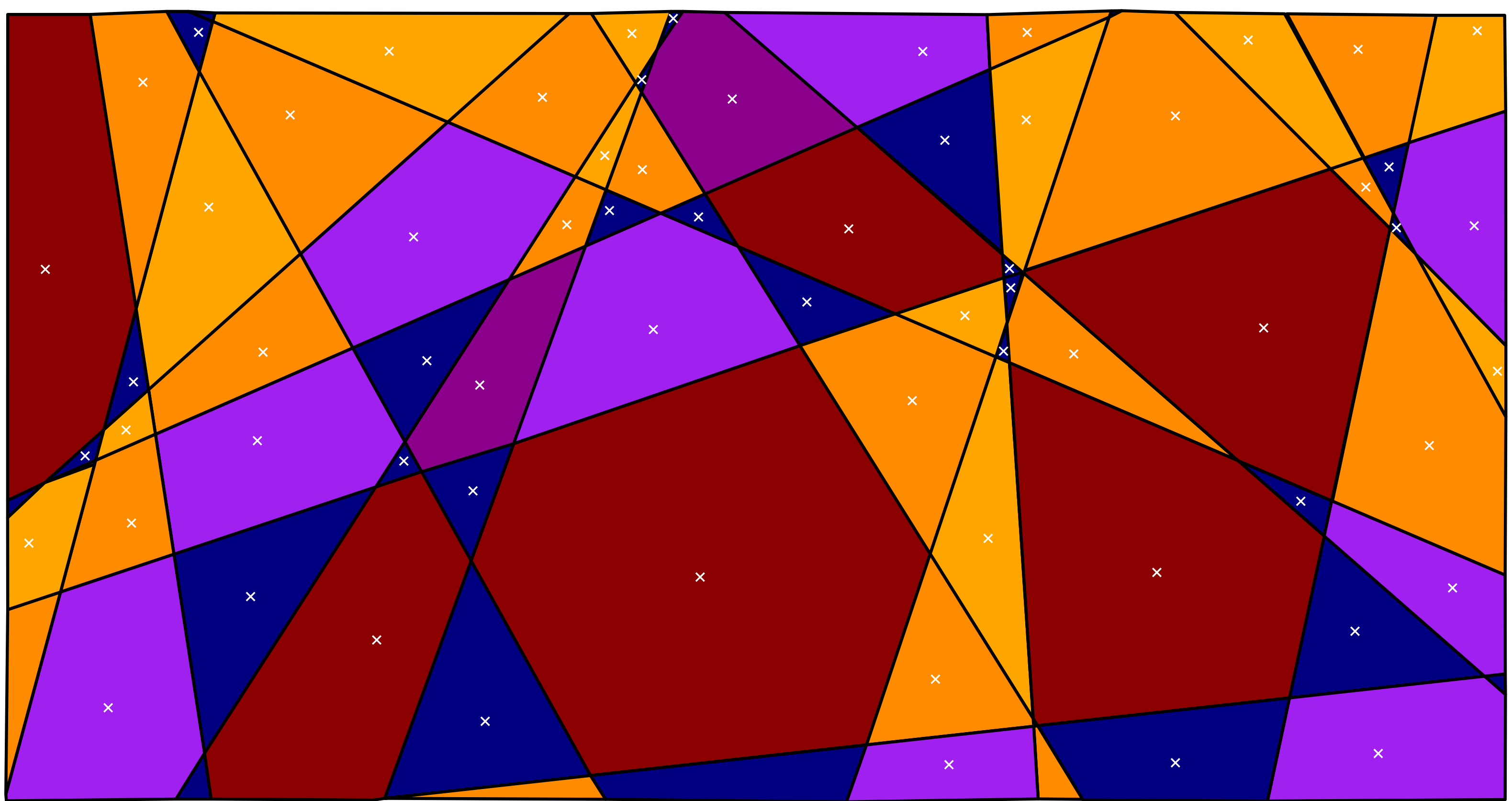


$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell



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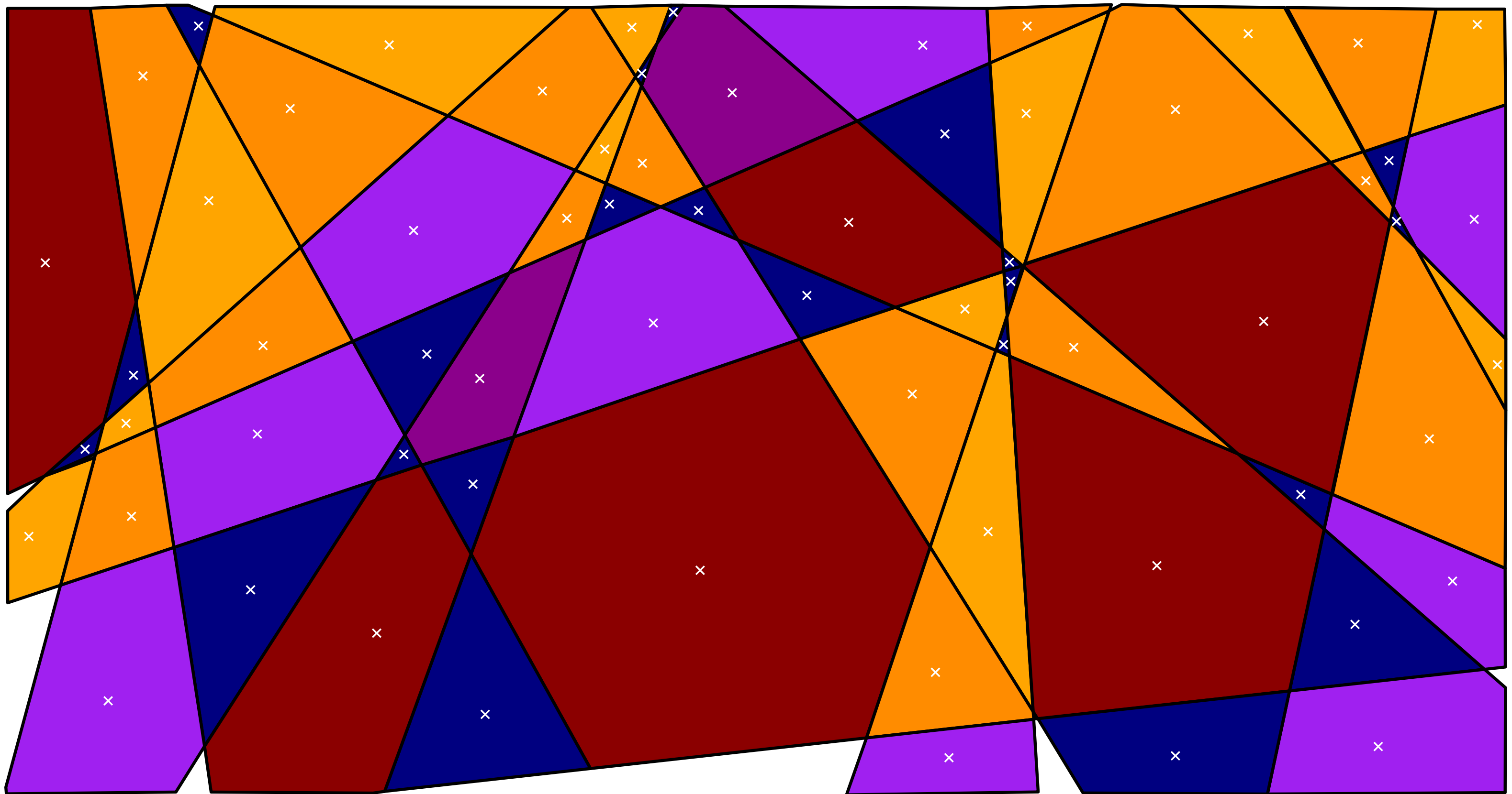

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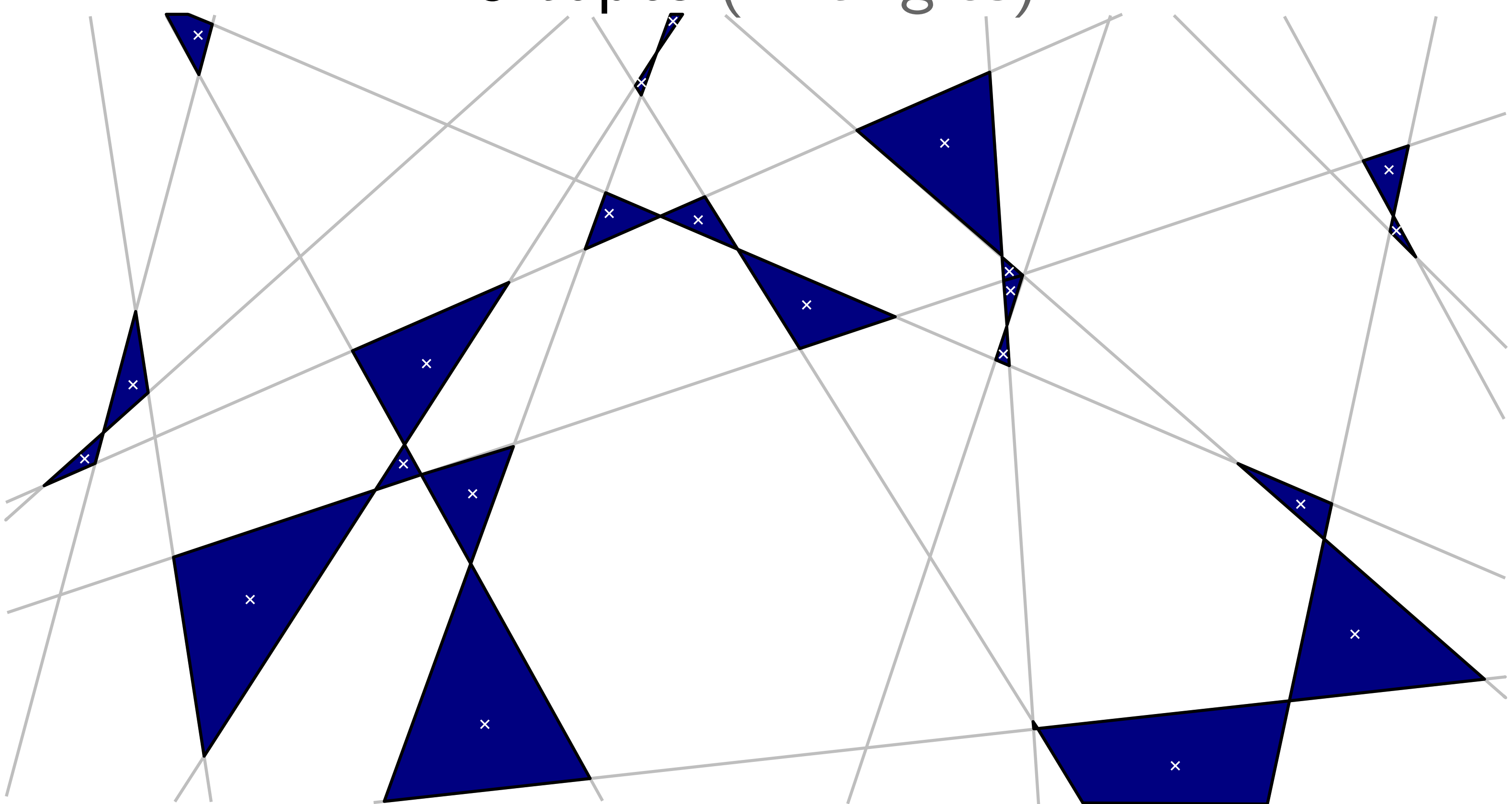
typical cell

23 3-topes

3-topes (Triangles)

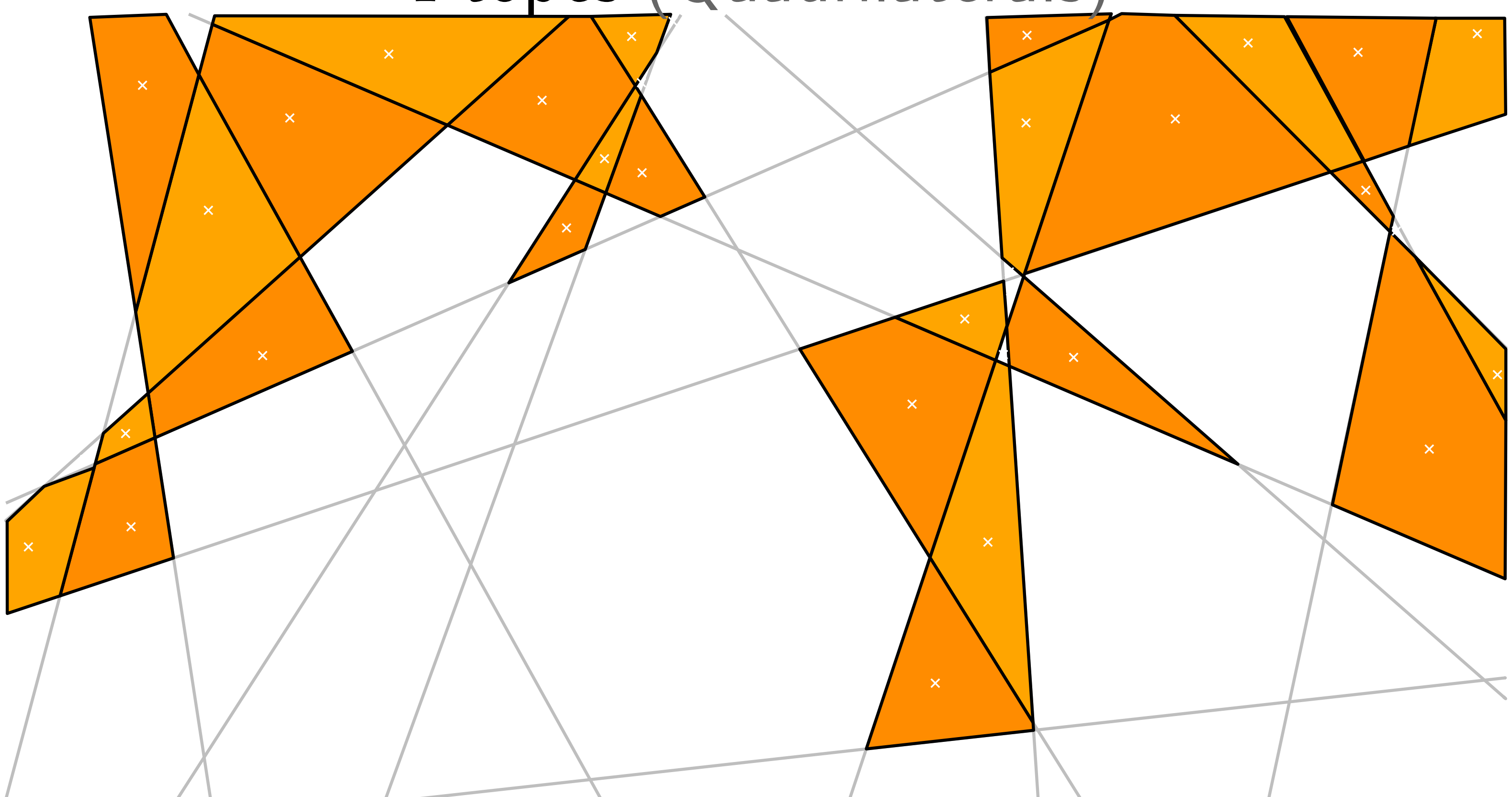


$\mathbb{P}(Z$ has n facets$)$

typical cell

23 3-topes
27 4-topes

4-topes (Quadrilaterals)

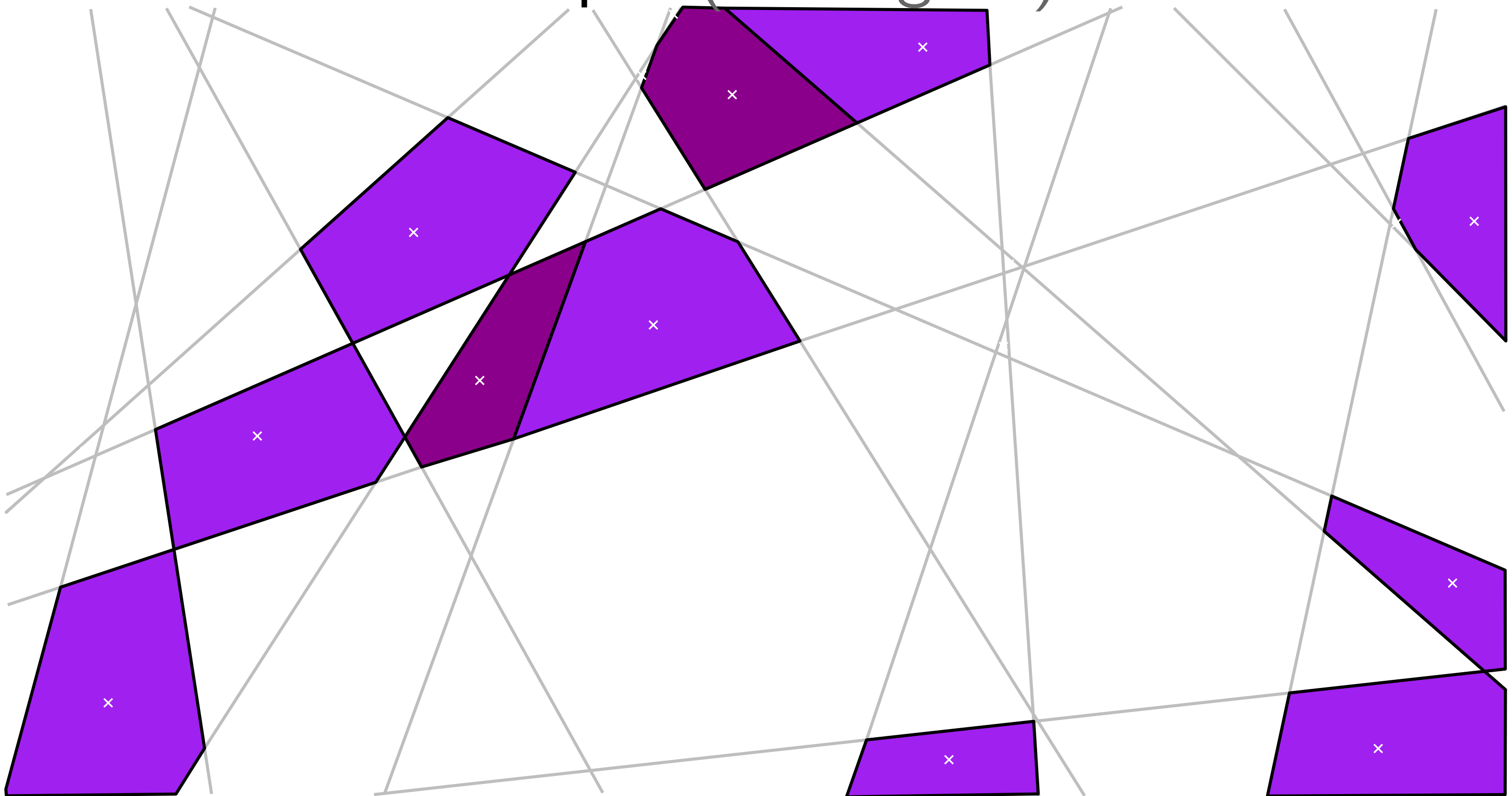


$\mathbb{P}(Z$ has n facets$)$

typical cell

23	3-topes
27	4-topes
11	5-topes

5-topes (Pentagons)

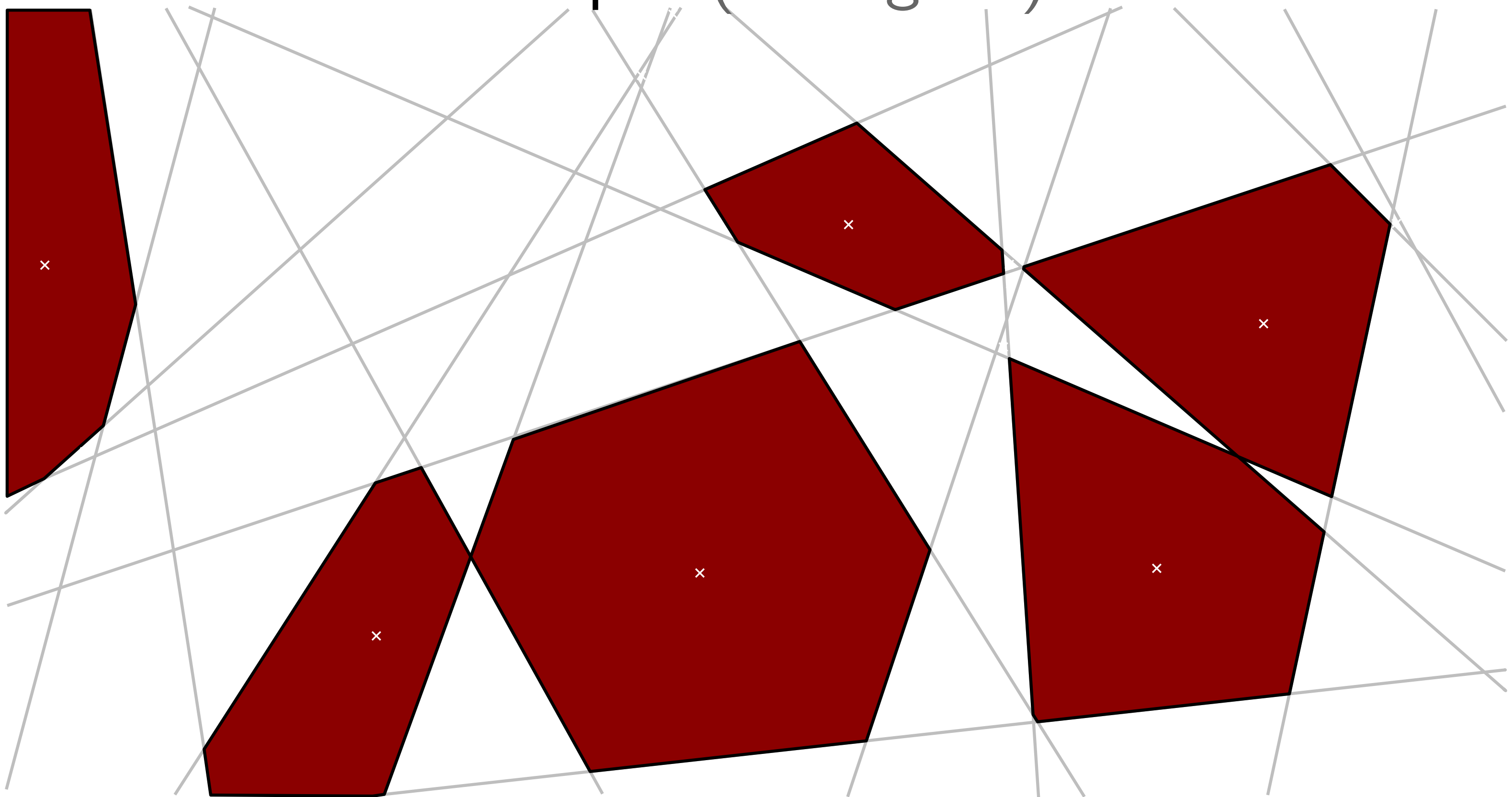


$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell

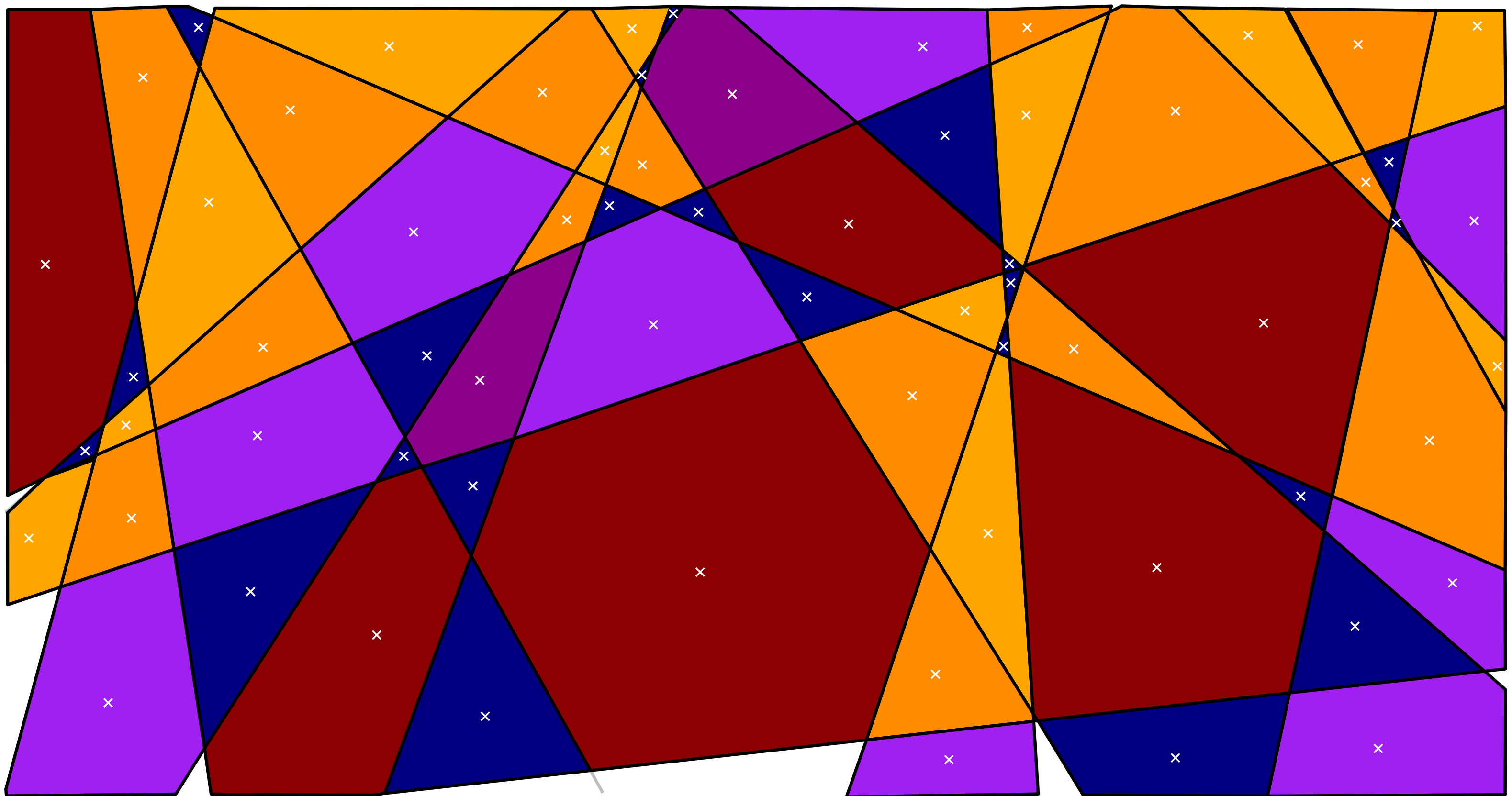
23	3-topes
27	4-topes
11	5-topes
6	6-topes

6-topes (Hexagons)



$\mathbb{P}(\underbrace{Z}_{\text{typical cell}} \text{ has } n \text{ facets})$

23	3-topes
27	4-topes
11	5-topes
6	6-topes
67	cells with center in the window



$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell

23	3-topes	$23/67 = 0.34\ldots$	$\simeq \mathbb{P}(Z \text{ has } 3 \text{ facets})$
27	4-topes	$27/67 = 0.40\ldots$	$\simeq \mathbb{P}(Z \text{ has } 4 \text{ facets})$
11	5-topes	$11/67 = 0.16\ldots$	$\simeq \mathbb{P}(Z \text{ has } 5 \text{ facets})$
6	6-topes	$6/67 = 0.09\ldots$	$\simeq \mathbb{P}(Z \text{ has } 6 \text{ facets})$

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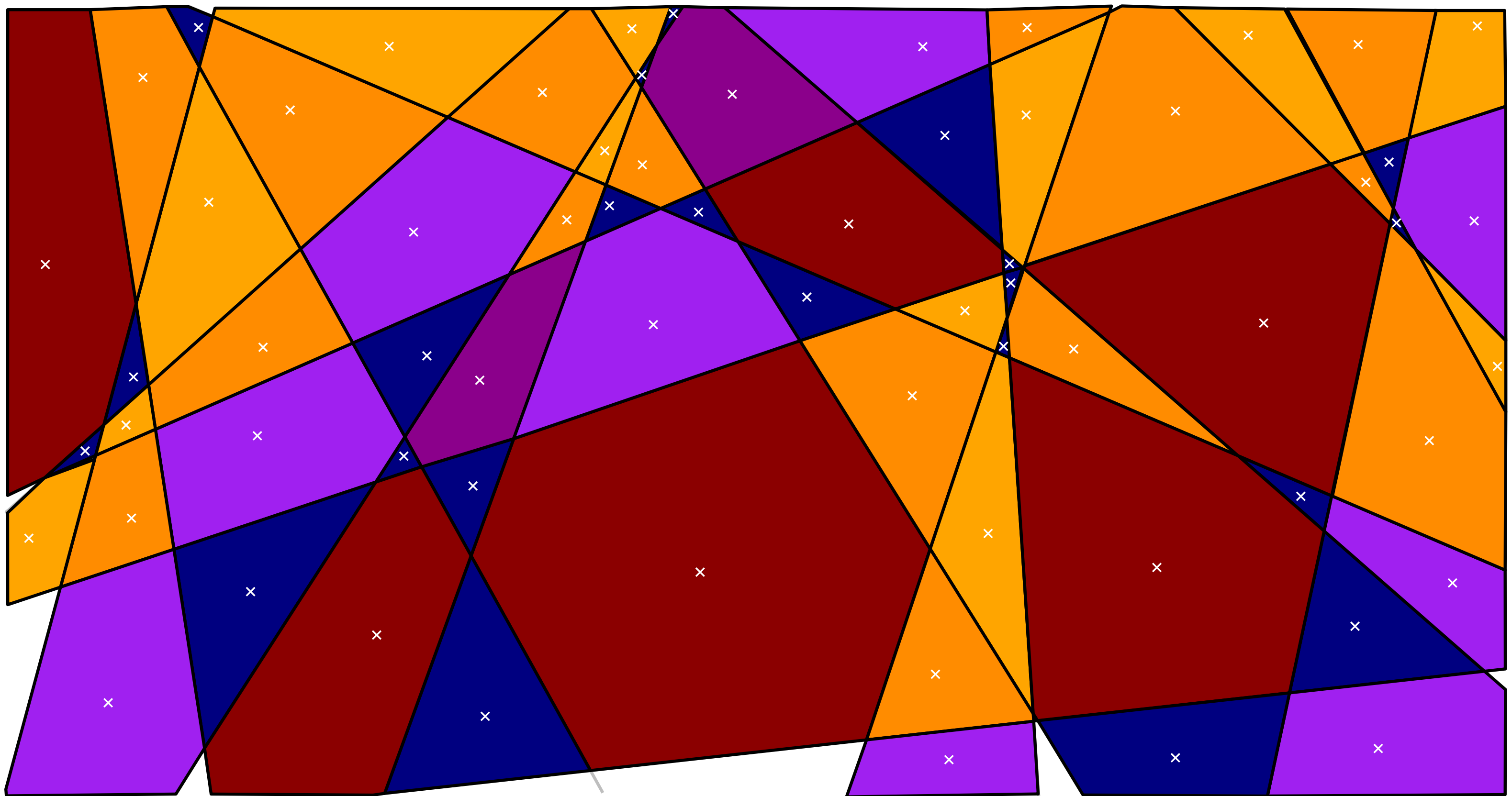

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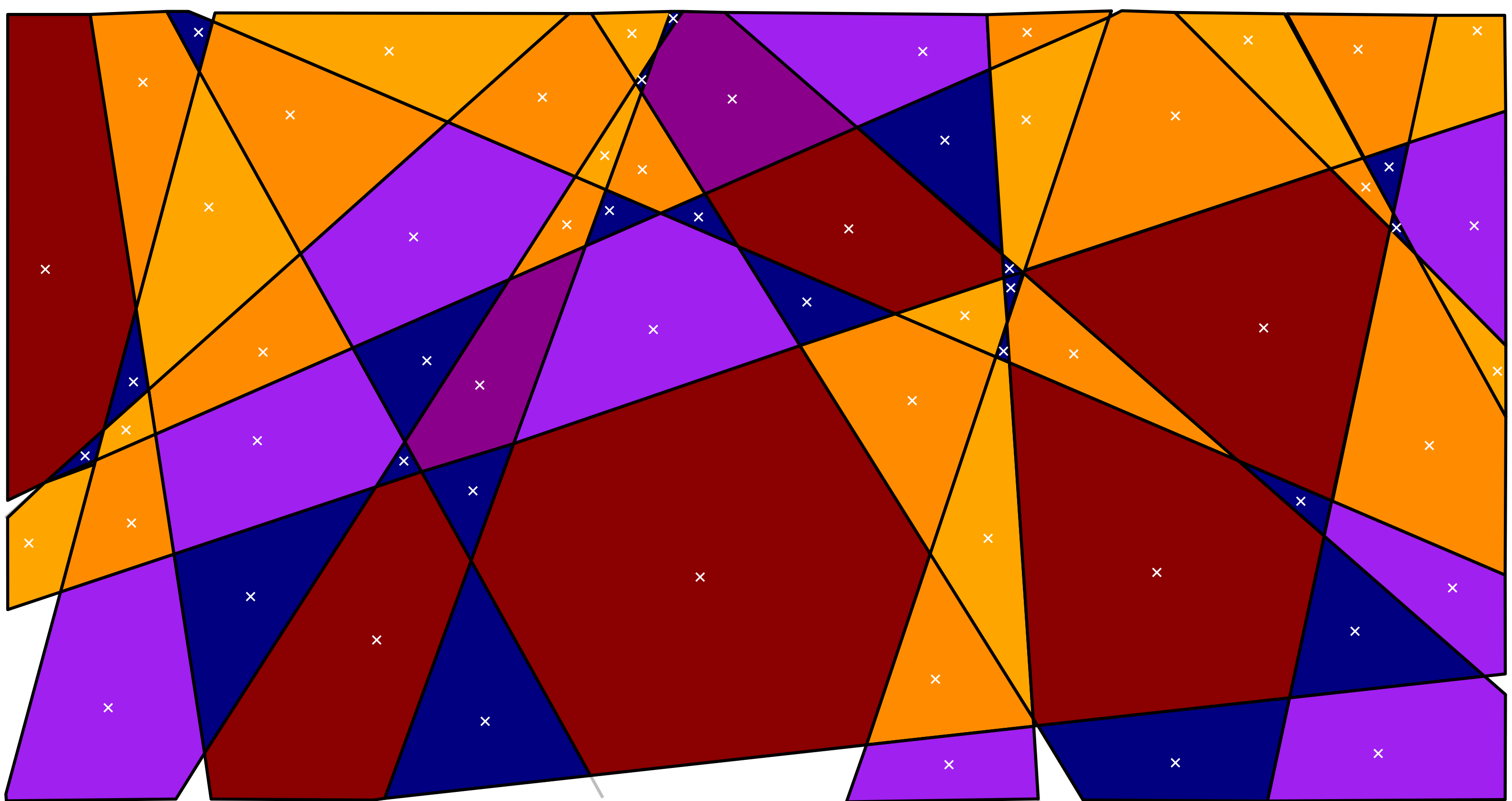
6	6-topes	$6/67 = 0.09...$	$\simeq \mathbb{P}(Z$ has 6 facets$)$
		$0/67 = 0$	$\simeq \mathbb{P}(Z$ has 7 or more facets$)$

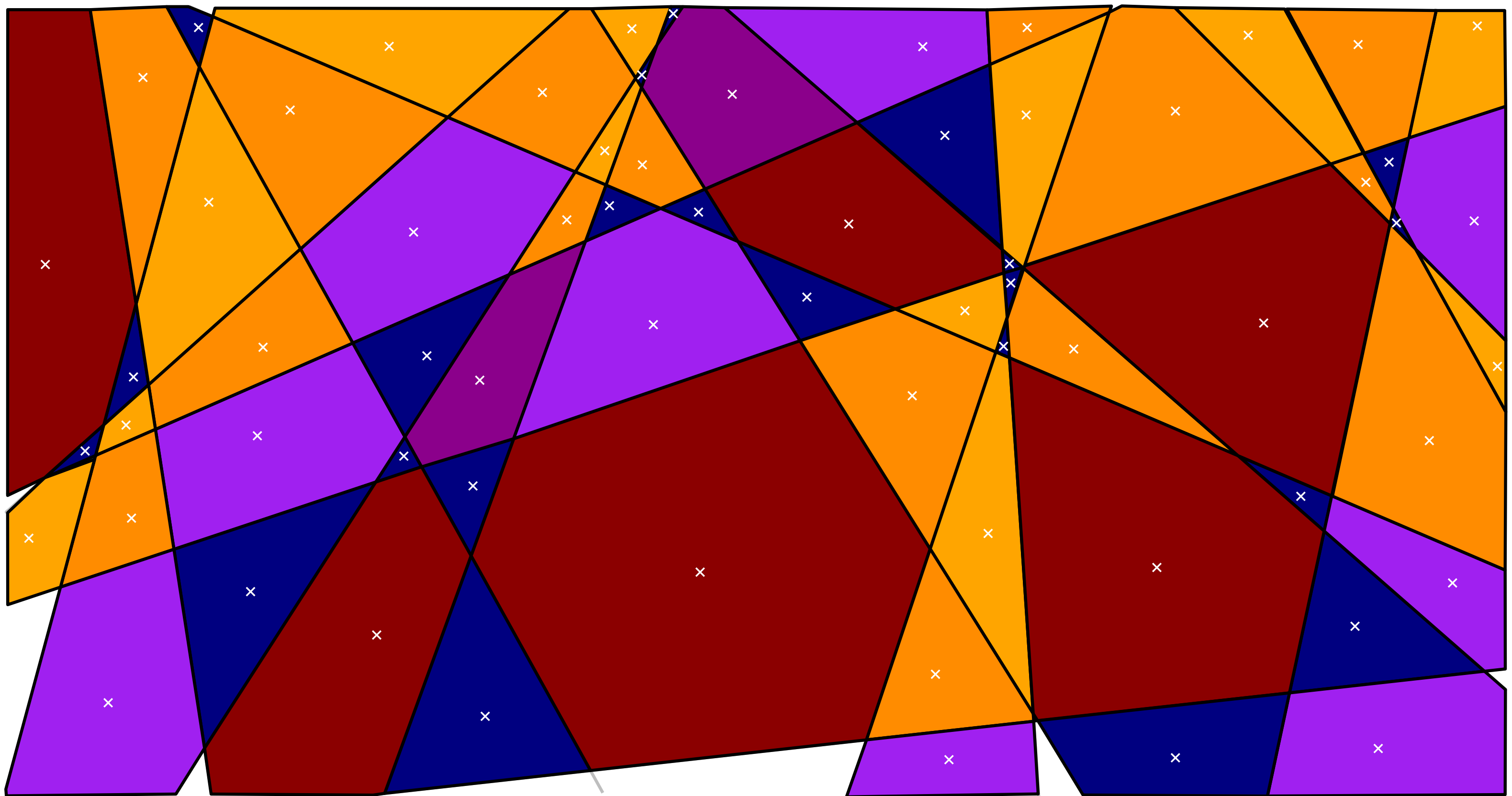


$\mathbb{P}(Z \text{ has } n \text{ facets})$

typical cell

$d = 2$, isotropy

$\mathbb{P}(Z \text{ has } 3 \text{ facets}) = 2 - \pi^2/6 = 0.36...$ [Miles 1964]



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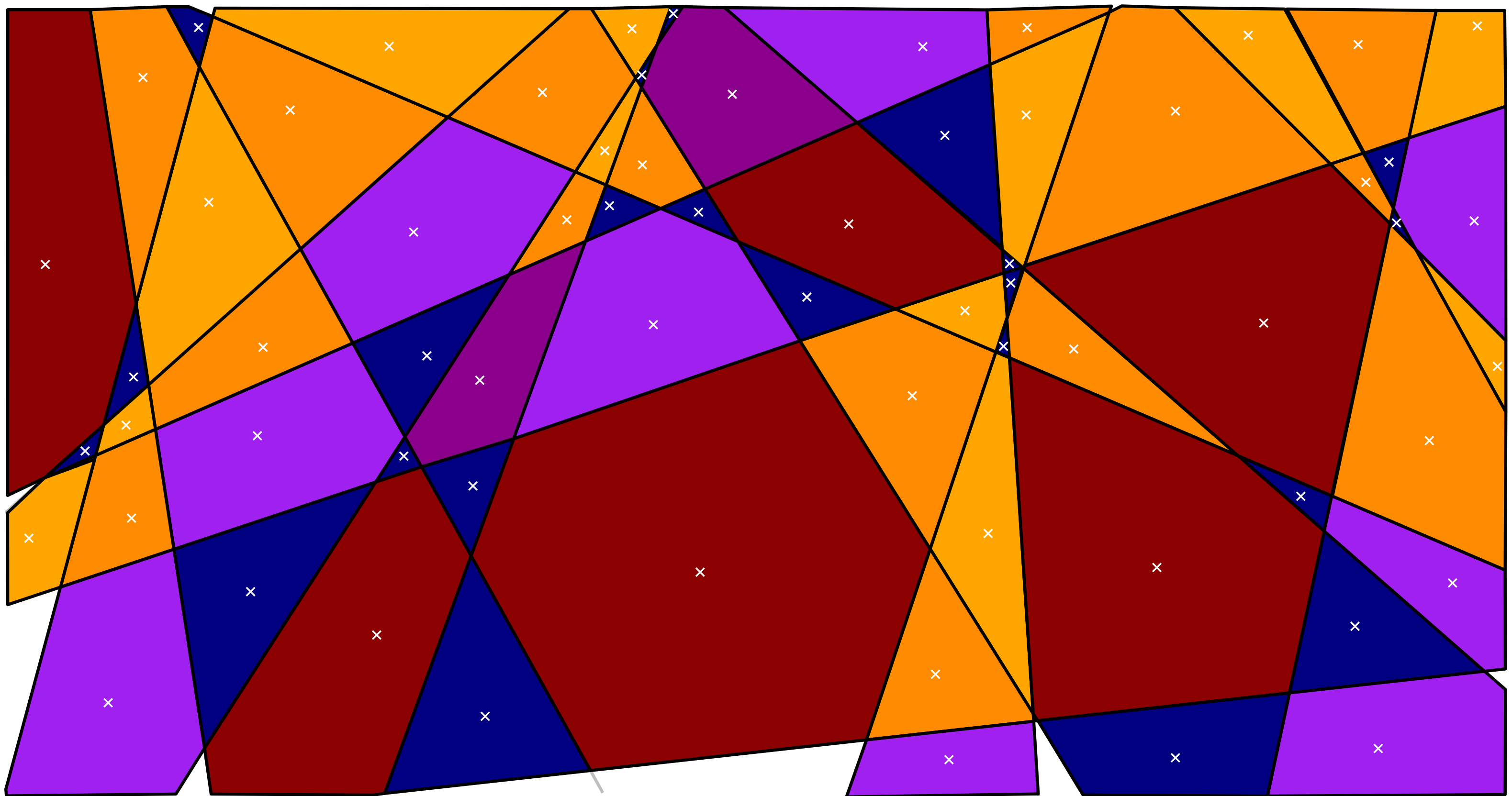
$\mathbb{P}(Z$ has 4 facets$) = \pi^2 \log 2 - \frac{1}{3} - \frac{7}{36}\pi^2 - \frac{7}{2}\zeta(3) = 0.38...$

[Tanner 1983]



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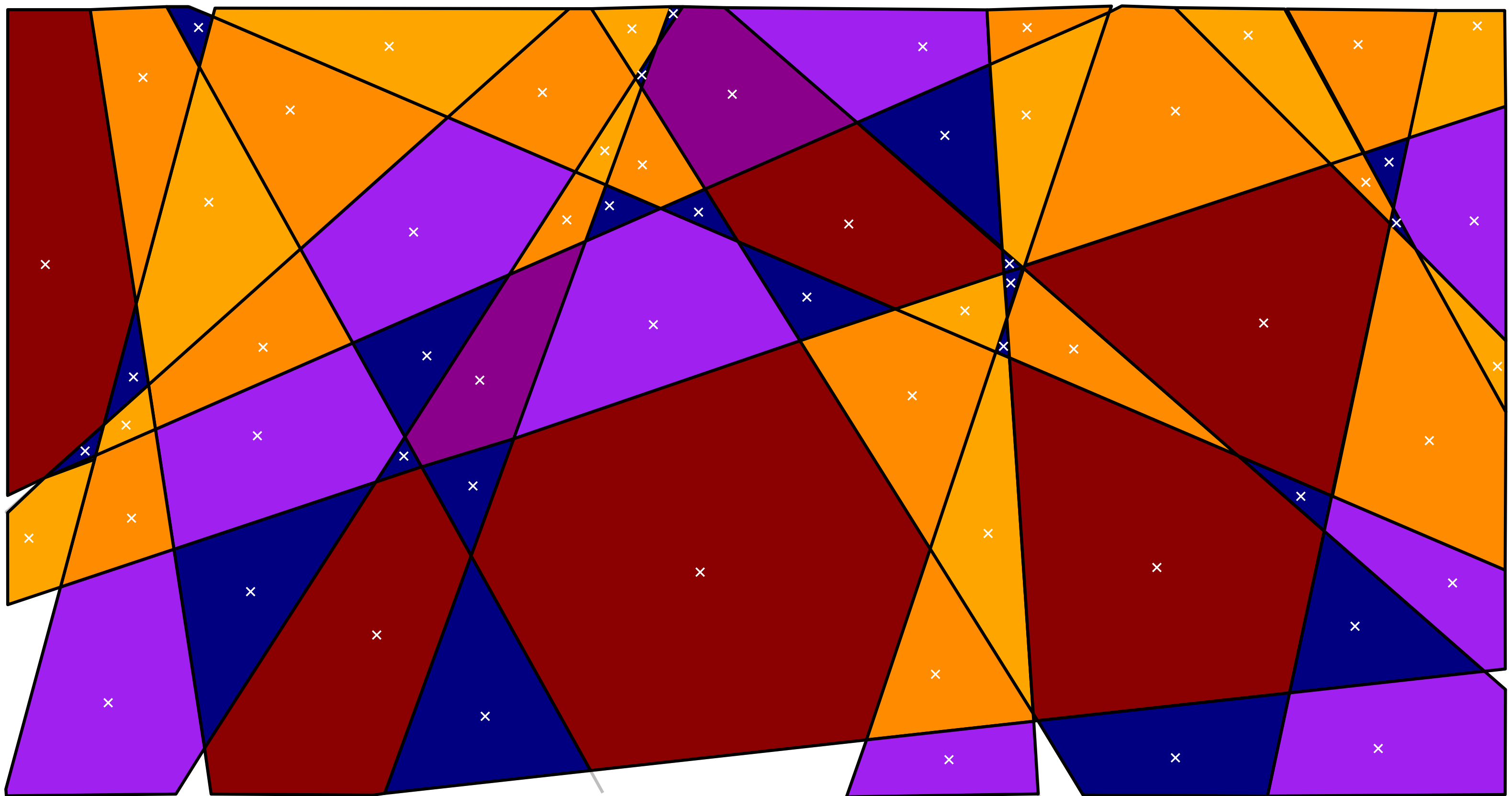
[Tanner 1983]

Approximation by Monte Carlo simulation for $n = 5, ..., 12$

[Crain and Miles 1976] [George 1987] [Michel and Paroux 2007]



Goal

$$\mathbb{P}(Z \text{ has } n \text{ facets})?$$

when $n \to \infty$

typical cell

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In a specific case it is already known:

Theorem [Calka and Hilhorst 2008]

In the 2-dimensional isotropic case we have that

$$\mathbb{P}(Z \text{ has } n \text{ facets}) \sim \alpha \boldsymbol{\beta^n n^{-2n}} n^{-3/2} \text{ when } n \to \infty$$

with $\alpha = 2/(3\pi^{5/2})$ and $\beta = \pi^2 e^2$

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We generalize this to any dimension and nice directional distribution:

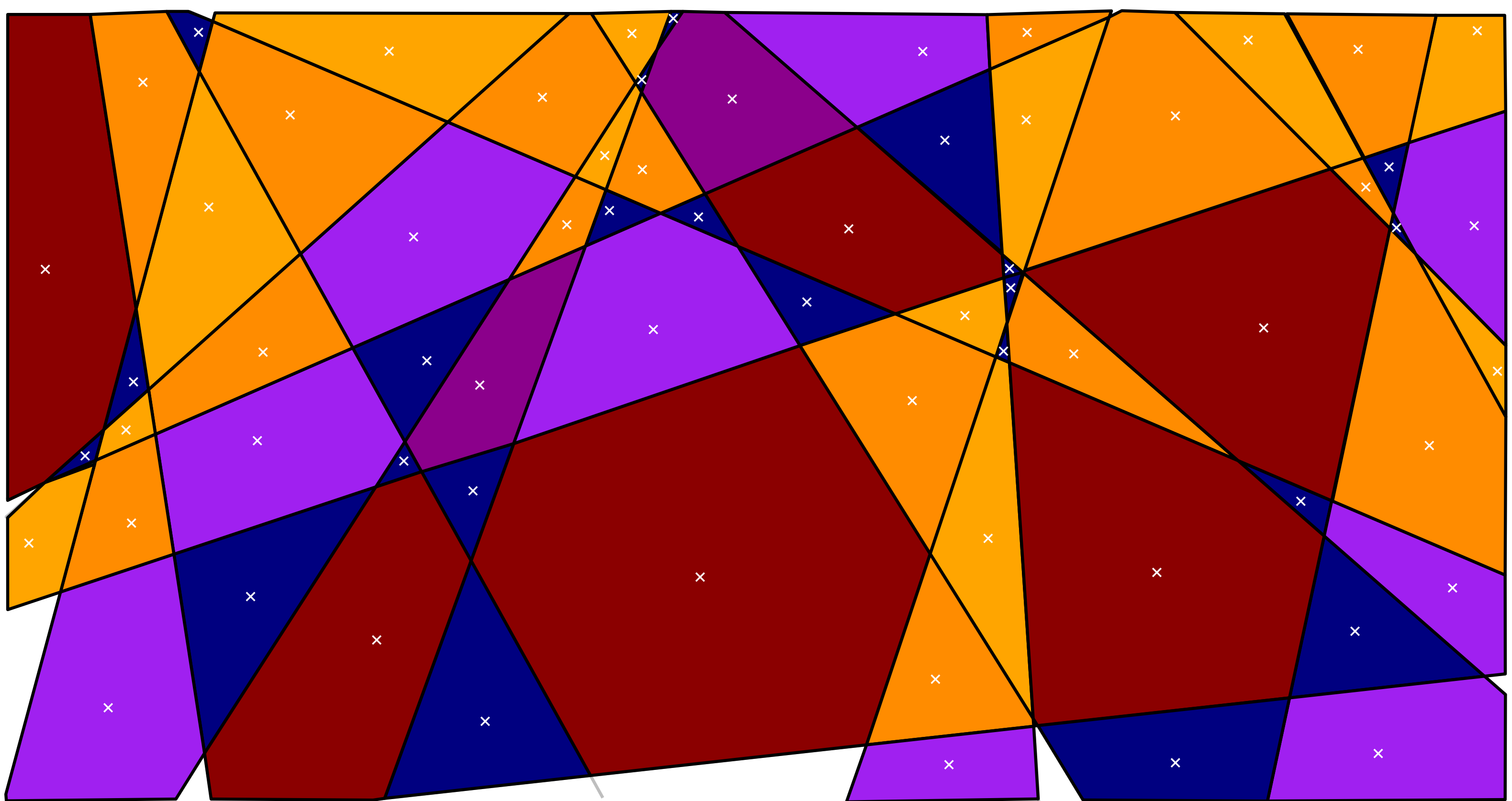
Main Theorem

There exist constants c_1 and c_2 depending on d and φ such that for n big enough we have

$$c_1^n n^{-2n/(d-1)} < \mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

$$\mathbb{P}(\overset{\text{typical cell}}{Z} \text{ has } n \text{ facets})$$

$$\mathbb{P}(Z \text{ has } n \text{ facets}) = \gamma^{-1}\mathbb{E}\,\underbrace{X(\mathcal{P}_{n,[0,1]^d})}_{\text{number of } n\text{-topes of } X \text{ with center in } [0,1]^d}$$



$$\mathbb{P}(Z \text{ has } n \text{ facets})$$

typical cell

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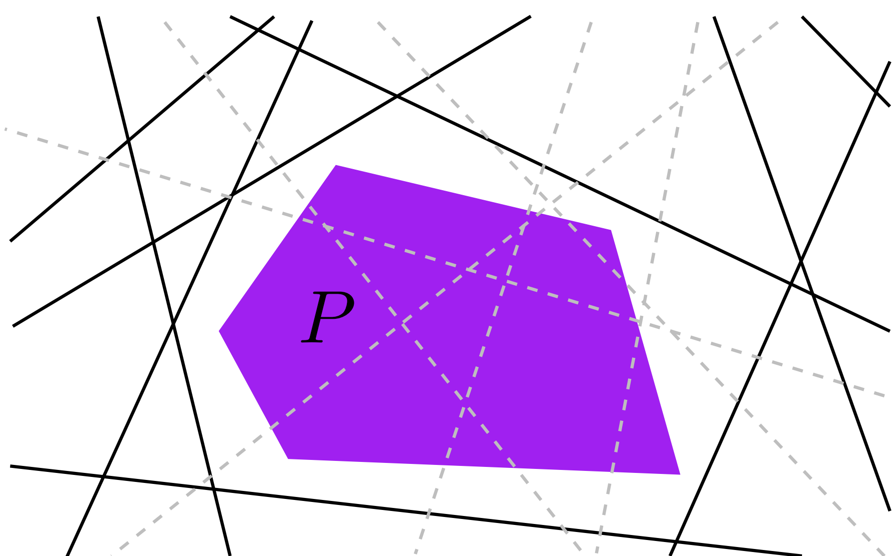
$$\boxed{\gamma\mathbb{P}(Z \text{ has } n \text{ facets}) = \int_{\mathcal{P}_n} \mathbf{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \exp\left(-\Phi\left(P\right)\right) \Theta_n(\mathrm{d}P)}$$



Θ_n measure on $\mathcal{P}_n$

$$\Theta_n(t \cdot + x) = t^n \Theta_n(\cdot)$$

$$\Phi : \mathcal{P} \to (0, \infty)$$

$$\mathbb{P}(\eta \cap P = \emptyset) = \exp(-\Phi(P))$$

$$\Phi(tP + x) = t\Phi(P)$$

Complementary Theorem

Complementary Theorem (Miles 1971)

If we condition the typical cell Z to have n facets, then

- $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent
- $\Phi(Z)$ is Gamma distributed with parameter $n - d$

[Møller and Zuyev 1996]

[Møller 1999]

[Cowan 2006]

[Baumstark and Last 2009]

$\mathbb{P}(Z$ has n facets$)$ = **simple** n-fold integral

$\gamma\mathbb{P}(Z$ has n facets$) = \mathbb{E}X(\mathcal{P}_{n,[0,1]^d})$

$$= \int_{\mathcal{P}^1_{n,\mathfrak{c}}} \int_{(0,\infty)} \int_{[0,1]^d} e^{-t} t^{n-d-1} \, \mathrm{d}c \, \mathrm{d}t \, \Theta^1_{n,\mathfrak{c}}(\mathrm{d}P)$$

center

size

shape

$$= (n-d-1)!\Theta^1_{n,\mathfrak{c}}(\mathcal{P}^1_{n,\mathfrak{c}})$$

$$= \frac{(n-d-1)!}{n!} \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

Lower Bound

$$\gamma \frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

$$= \int \cdots \int_{\mathcal{H}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \, \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

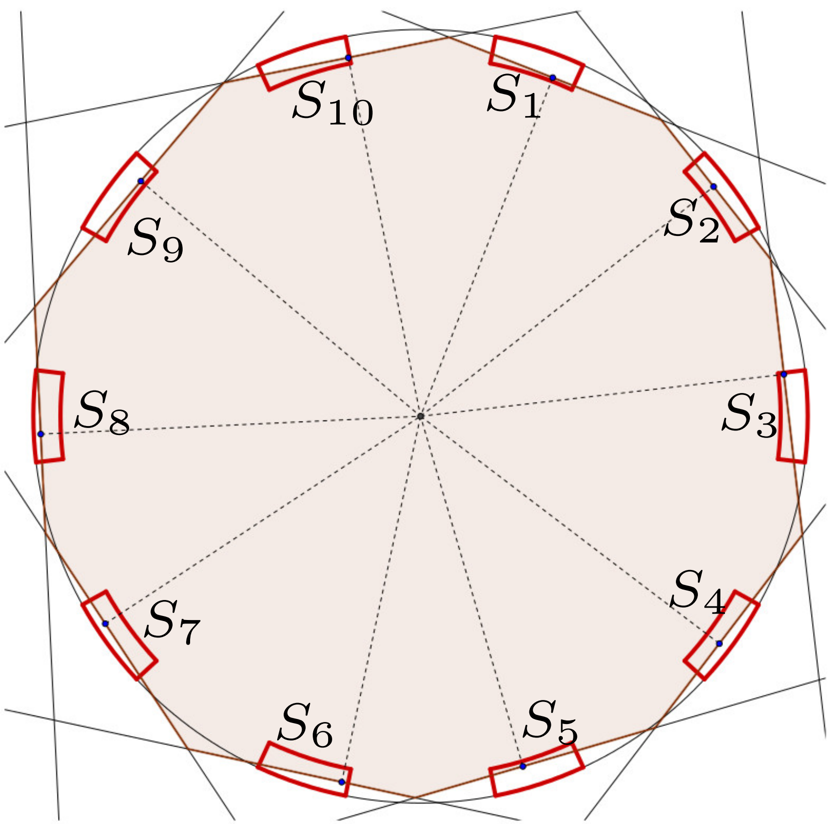
$$> n! \int \cdots \int_{\mathcal{H}^n} \mathbb{1}(H_1 \in S_1) \cdots \mathbb{1}(H_n \in S_n) \, \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

$$= n! \Theta(S_1)^n$$

$$> n! \left(c n^{-(d-1)/(d+1)}\right)^n$$

$$\sim c^n n^{-2n/(d-1)}$$

Stirling

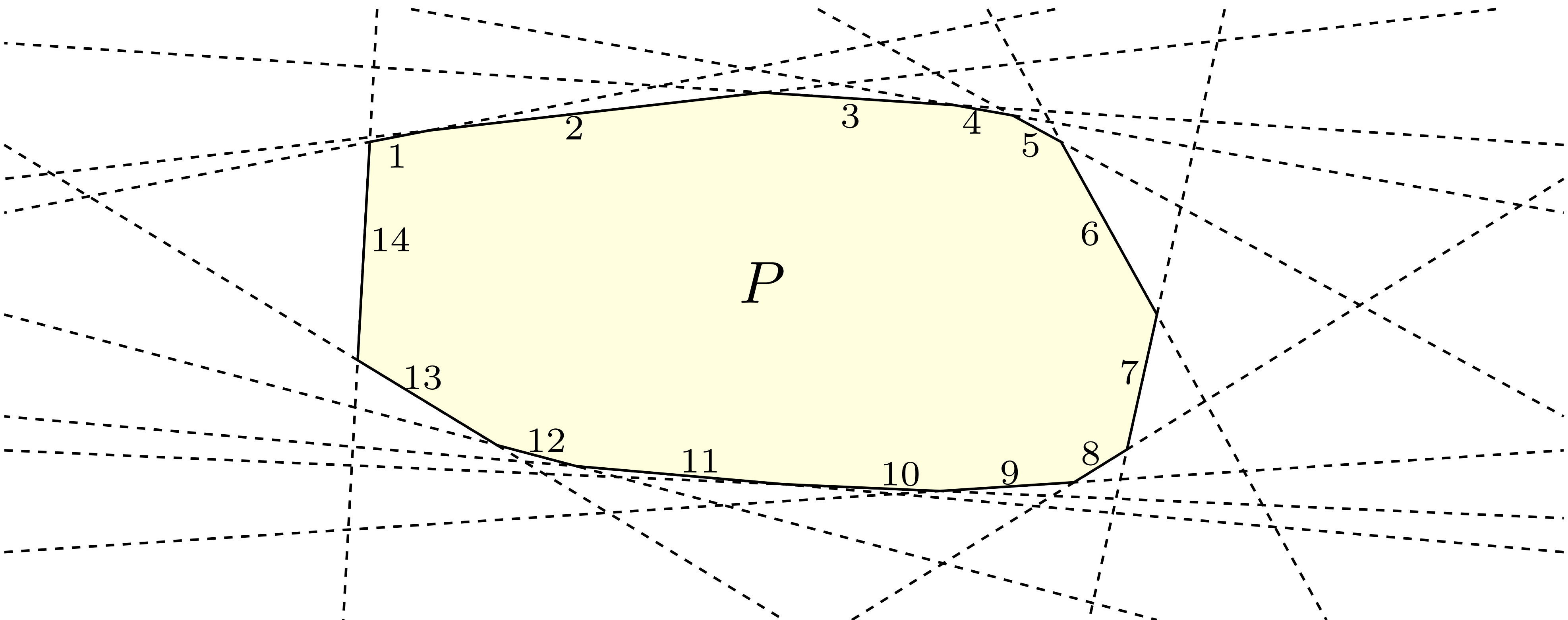


Theorem: Lower bound

There exists a constant c_1 depending on d and φ such that for n big enough we have

$$\mathbb{P}(Z \text{ has } n \text{ facets}) > c_1^n n^{-2n/(d-1)}$$

Approximation by Deleting One Facet



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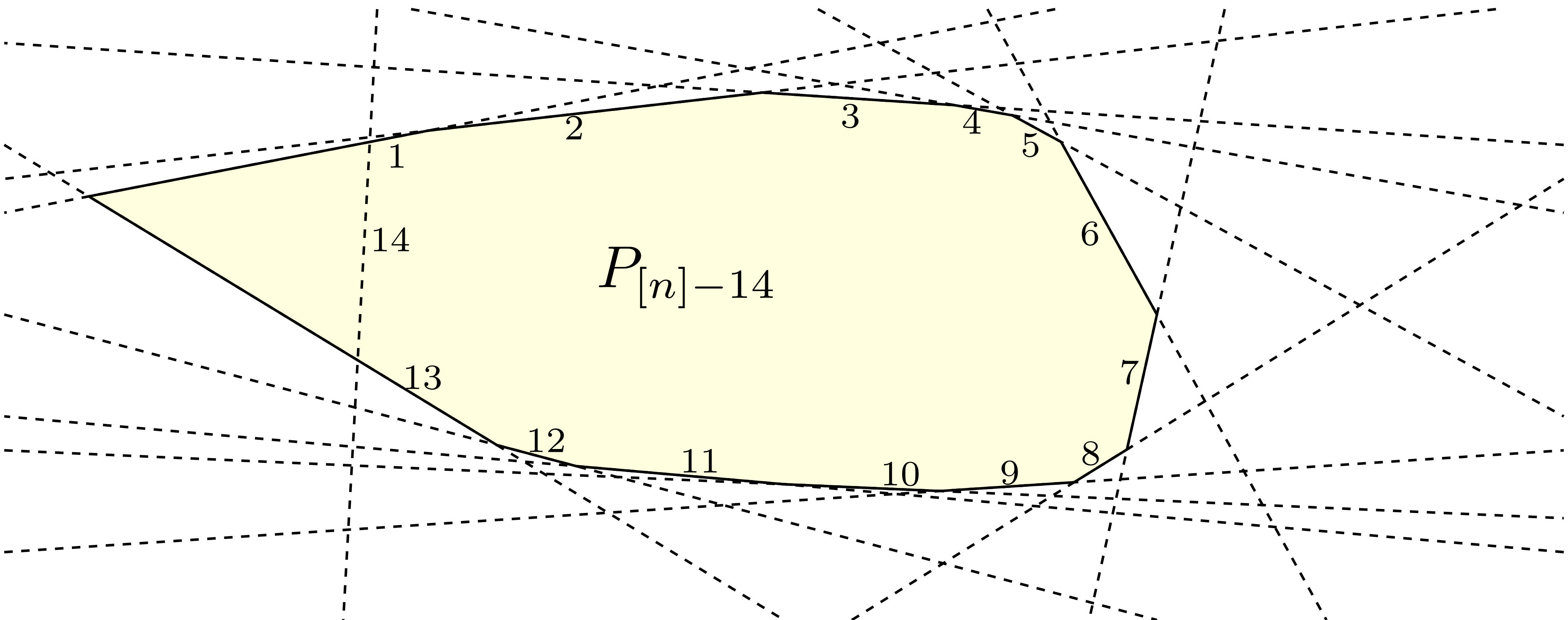

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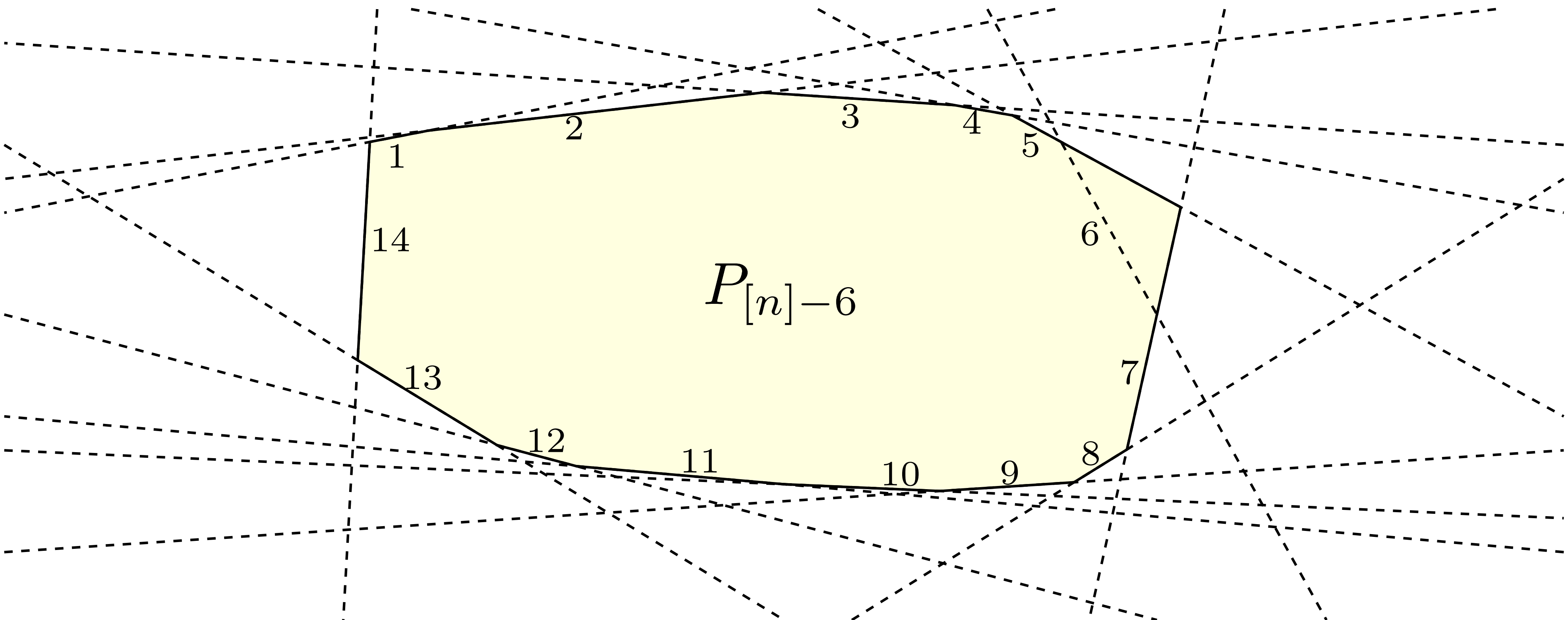

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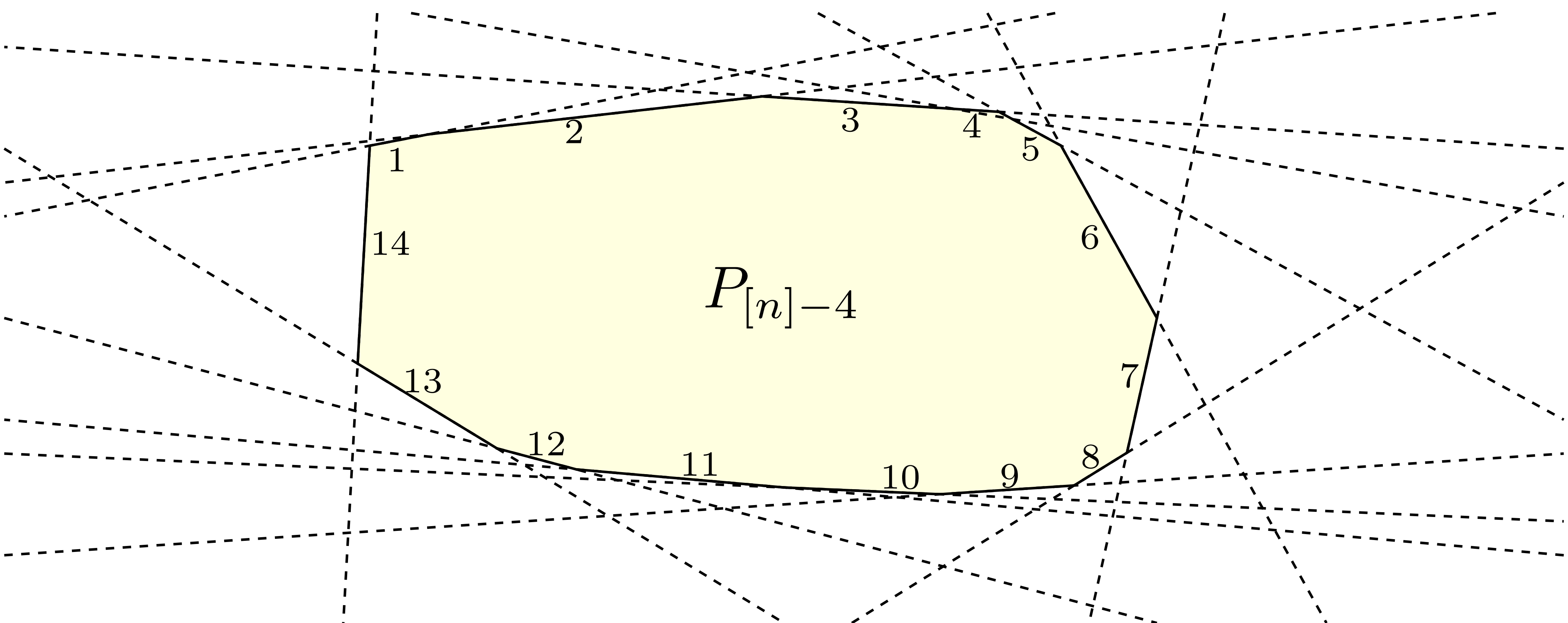
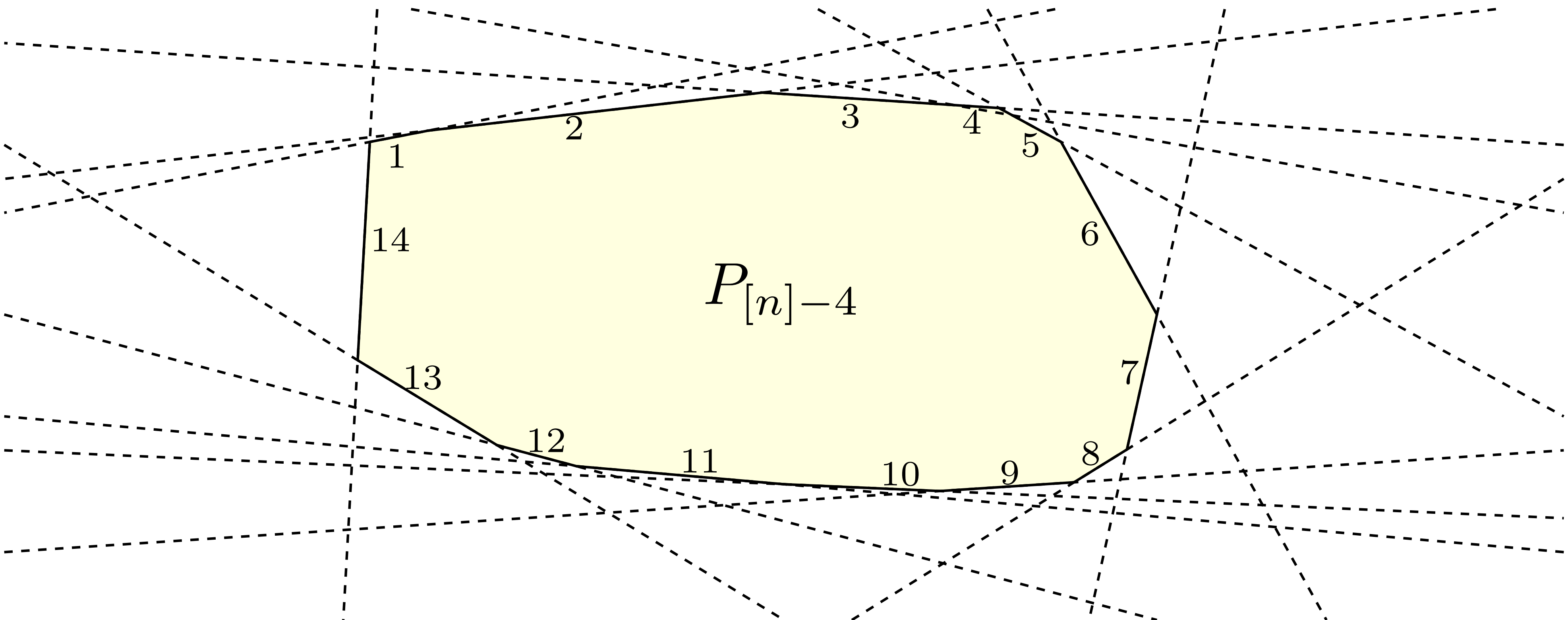


Approximation by Deleting One Facet

There exists a constant c_0 such that:

Theorem

Let $P = \cap_{i=1}^{n} H_i^-$ be a simple polytope with n big enough.
There exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have
$d_H\left(P, P_{[n]-j}\right) < c_0 n^{-2/(d-1)} \Phi\left(P\right)$
and
$\Phi\left(P_{[n]-j}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right) \Phi(P).$



Upper Bound

$$\gamma \frac{n!}{(n-d-1)!} \mathbb{P}(Z \text{ has } n \text{ facets})$$
$$= \int \cdots \int_{\mathcal{H}^n} \sum_{\boldsymbol{\epsilon} \in \{\pm 1\}^n} \mathbb{1}\left(\mathfrak{c}(P) \in [0,1]^d\right) \mathbb{1}\left(\Phi(P) < 1\right) \mathbb{1}\left(P \in \mathcal{P}_n\right) \Theta(\mathrm{d}H_n) \cdots \Theta(\mathrm{d}H_1)$$

Upper Bound

$$\frac{\gamma}{4}\frac{n!}{(n-d-1)!}\mathbb{P}(Z \text{ has } n \text{ facets})$$

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$$\Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$

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$$< \int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\epsilon \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,2]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < \exp\left(c_0 n^{-1-2/(d-1)}\right)\right)$$
$$\mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$
$$\left(\int_{\mathcal{H}} \sum_{\epsilon_n = \pm 1} \mathbb{1}\left(d_H\left(P, P_{[n-1]}\right) < c_0 n^{-2/(d-1)}\right) \Theta(\mathrm{d}H_n)\right) \Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$
$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)}$$
$$\int \cdots \int_{\mathcal{H}^{n-1}} \sum_{\epsilon \in \{\pm 1\}^{n-1}} \mathbb{1}\left(\mathfrak{c}(P_{[n-1]}) \in [0,1]^d\right) \mathbb{1}\left(\Phi\left(P_{[n-1]}\right) < 1\right) \mathbb{1}\left(P_{[n-1]} \in \mathcal{P}_{n-1}\right)$$
$$\Theta(\mathrm{d}H_{n-1}) \cdots \Theta(\mathrm{d}H_1)$$
$$< 2^d \exp\left(c_0 n^{-1-2/(d-1)}\right)^{n-d-1} c_0 n^{-2/(d-1)} \gamma \frac{(n-1)!}{(n-d-2)!} \mathbb{P}(Z \text{ has } n-1 \text{ facets})$$

Theorem: Upper bound

There exists a constant c_2 depending on d and φ such that for n big enough we have
$$\mathbb{P}(Z \text{ has } n \text{ facets}) < c_2^n n^{-2n/(d-1)}$$

THANK YOU!