Quantifying repulsiveness of determinantal point processes

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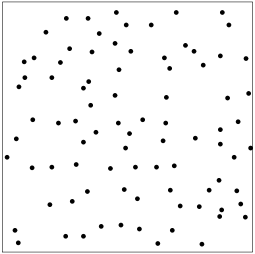
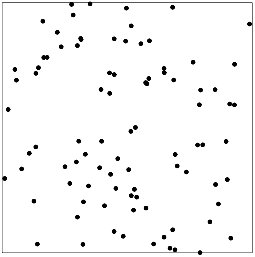
GDR GeoSto

Different situations

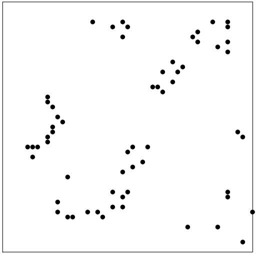
Inhibitive
(Norwegian spruce trees)



No interaction
(Poisson)

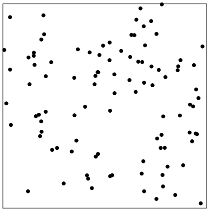


Clustering
(California redwood tree)

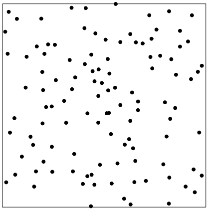


DPPs, a class of inhibitive point processes

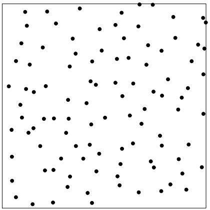
Determinantal point processes (DPPs) are adapted to inhibitive (or repulsive, or regular) point patterns.



Poisson



DPP with intermediate repulsion



DPP with stronger repulsion

Notation

- $\mathbf{X}$ is a spatial point process on $\mathbb{R}^d$.
- For any integer $n > 0$, $\rho^{(n)}$ denotes the n-th joint intensity (or product density function) of $\mathbf{X}$.
  Intuitively,
  $$\rho^{(n)}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$
  is the probability that for each $i = 1, \ldots, n$, $\mathbf{X}$ has a point in a region around x_i of volume dx_i.
- In particular $\rho = \rho^{(1)}$ is the intensity function.

Stationary DPPs

For a function C from $\mathbb{R}^d$ into $\mathbb{R}$, denote $[C](x_1, \ldots, x_n)$ the matrix $\big(C\,(x_i - x_j)\,\big)_{1\leq i,j\leq n}$.

For instance: $[C](x_1) = C(0)$, $\;[C](x_1, x_2) = \begin{pmatrix} C(0) & C(x_1 - x_2) \\ C(x_2 - x_1) & C(0) \end{pmatrix}$

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Definition

A point process $\mathbf{X}$ is a stationary determinantal point process with kernel C, denoted by $DPP(C)$, if for all $n \in \mathbb{N}$,

$$\rho^{(n)}(x_1,\dots,x_n) = \det[C](x_1,\dots,x_n).$$

- In particular, $\rho = C(0)$, $\rho^{(2)}(x_1,x_2) = C(0)^2 - C(x_1 - x_2)C(x_2 - x_1)$.

Existence and uniqueness

Theorem (Hough, Perez, Krishnapour, Virag, 09)

A function C defines at most one DPP.

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Theorem (Macchi, 75 and Lavancier, Møller, Rubak, 15)

A stationary determinantal point process with kernel C exists if C verifies the following assumptions,

- *C is real valued, symmetric and continuous,*
- $C \in L^2(\mathbb{R}^d)$,
- $0 \leq \mathcal{F}(C) \leq 1$.

We denote by $\mathcal{H}$ the set of functions that verify the last three assumptions.

- Up to a re-normalization, all continuous square-integrable covariance functions belong to $\mathcal{H}$.
- There already exist many functions of this kind in the literature.

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For instance, for $\nu > 0$ and $\alpha > 0$,

- the Whittle-Matérn functions

$$C(x) = \rho \frac{2^{1-\nu}}{\Gamma(\nu)} \left|\frac{x}{\alpha}\right| K_\nu \left(\left|\frac{x}{\alpha}\right|\right), \quad x \in \mathbb{R}^d,$$

  define DPPs with intensity ρ if and only if $\rho \leq \frac{\Gamma(\nu)}{\Gamma(\nu+d/2)(2\sqrt{\pi}\alpha)^d}$;
- the Generalized Cauchy functions

$$C(x) = \frac{\rho}{\left(1 + |x/\alpha|^2\right)^{\nu+d/2}}, \quad x \in \mathbb{R}^d,$$

  define DPPs with intensity ρ if and only if $\rho \leq \frac{\Gamma(\nu+d/2)}{\Gamma(\nu)(\sqrt{\pi}\alpha)^d}$.

First properties

DPPs have several advantages:

- By definition, their moments of any order exist and are known.
- They can be easily simulated (perfect simulation).
- On all bounded sets, the expression of the density is known.
- Parametric estimation is feasible:
  - by maximum likelihood,
  - by minimum contrast, for instance $\int (\widehat{K}(t) - K_\theta(t))^2 \, dt$.

For Gibbs point processes, none of these properties hold.

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How much DPPs are flexible (repulsive)?

The pair correlation function

The pair correlation function (pcf) is

$$g(x, y) = \frac{\rho^{(2)}(x, y)}{\rho(x)\rho(y)}, \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

- For a stationary and isotropic point process, $g(x, y) = g_0(r)$ where $r = |x - y|$;
- $g_0(r) < 1$ represents an inhibitive behaviour;
- $g_0(r) > 1$ represents a clustering behaviour.

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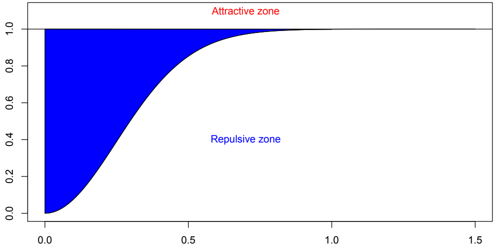
- $g_0(r) > 1$ represents a clustering behaviour.

For a stationary and isotropic DPP, $C(x) = C_0(|x|)$ and

$$g_0(r) = 1 - \left(\frac{C_0(r)}{C_0(0)}\right)^2 \leq 1$$

Repulsiveness



We consider only processes with given intensity $\rho > 0$. Let $\mathbf{X}$ and $\mathbf{Y}$ be two stationary DPPs with pcf $g_{\mathbf{X}}$ and $g_{\mathbf{Y}}$ respectively.

Global repulsiveness

$\mathbf{X}$ is more globally repulsive than $\mathbf{Y}$ if $g_{\mathbf{X}}$ has a larger "blue zone" than $g_{\mathbf{Y}}$, i.e. $\int_{\mathbb{R}^d}(1 - g_{\mathbf{X}}) \geq \int_{\mathbb{R}^d}(1 - g_{\mathbf{Y}})$.

The most globally repulsive DPPs in the global sense

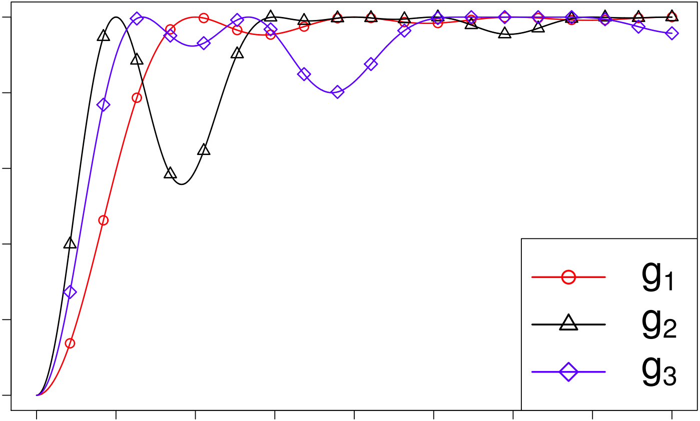
Theorem (Lavancier, Møller, Rubak, 15)

For a given $\rho > 0$, a DPP with kernel $C \in \mathcal{H}$ is the most globally repulsive if and only if there exists a set $A \subset \mathbb{R}^d$, with volume ρ, centred at 0 such that

$$\mathcal{F}(C)(x) = \mathbf{1}_A(x) \quad a.e.$$

Examples in dimension 1

- $C_1 = \mathcal{F}(\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]})$;
- $C_2 = \mathcal{F}(\mathbf{1}_{[-\frac{3}{4},-\frac{1}{4}]} + \mathbf{1}_{[\frac{1}{4},\frac{3}{4}]})$;
- $C_3 = \mathcal{F}(\mathbf{1}_{[-\frac{2}{3},-\frac{1}{3}]} + \mathbf{1}_{[-\frac{1}{6},\frac{1}{6}]} + \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]})$.

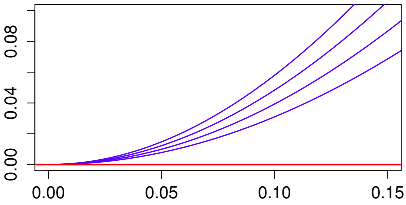


Repulsiveness: an other definition

Let $\mathbf{X}$ and $\mathbf{Y}$ be two stationary DPPs with pcf $g_{\mathbf{X}}$ and $g_{\mathbf{Y}}$ respectively.

Local repulsiveness

$\mathbf{X}$ is more locally repulsive than $\mathbf{Y}$ if $g_{\mathbf{X}}$ is more “flat“ at 0 than $g_{\mathbf{Y}}$, i.e. $g_{\mathbf{X}}(0) = g_{\mathbf{Y}}(0) = 0$, $\nabla g_{\mathbf{X}}(0) = \nabla g_{\mathbf{Y}}(0) = 0$ and $\Delta g_{\mathbf{X}}(0) \leq \Delta g_{\mathbf{Y}}(0)$.



Note that for a hardcore point process, $g(0) = \nabla g(0) = \Delta g(0) = 0$.

The most locally repulsive DPP

Theorem (Biscio, Lavancier, 15)

For a given $\rho > 0$, there exists an unique DPP with kernel C that is the most locally repulsive among all DPPs with kernels belonging to $\mathcal{H}$. This kernel is given by

$$C(x) = \frac{\sqrt{\rho\Gamma(\frac{d}{2}+1)}}{\pi^{d/4}} \frac{J_{\frac{d}{2}}\left(2\sqrt{\pi}\Gamma(\frac{d}{2}+1)^{\frac{1}{d}}\rho^{\frac{1}{d}}|x|\right)}{|x|^{\frac{d}{2}}}, \quad \forall x \in \mathbb{R}^d,$$

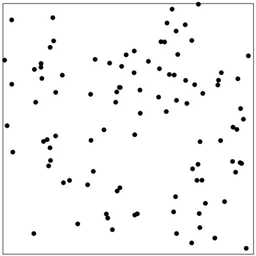
where $J_{\frac{d}{2}}$ denotes the Bessel function of the first kind. Moreover,

$$\mathcal{F}(C)(x) = \mathbf{1}_{B(0,\sqrt[d]{\rho/\omega_d})}, \quad \forall x \in \mathbb{R}^d,$$

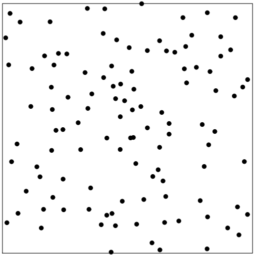
where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Summary

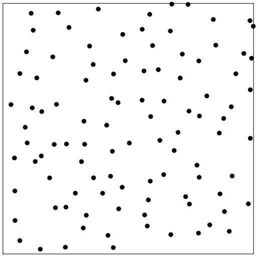
- There exists an unique DPP that is the most locally repulsive.
- Further, it is one of the most globally repulsive,
  - for $d = 1$, $C(x) = sinc(x) = \frac{\sin(\pi\rho|x|)}{\pi|x|}$,
  - for $d = 2$, $C(x) = \sqrt{\rho}\frac{J_1(2\sqrt{\pi\rho}|x|)}{\sqrt{\pi}|x|} := Jinc(x)$.
- The inhibition of DPPs is limited, hardcore inhibition is impossible.



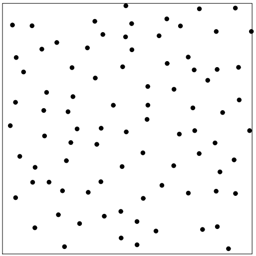
Poisson



DPP with intermediate repulsion



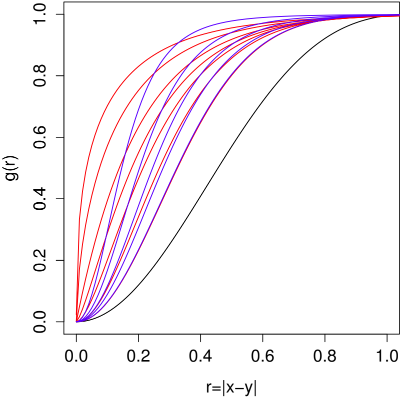
The most repulsive DPP



Hardcore

Pcfs for usual models of DPPs in dimension $d = 2$

- In blue, pcfs of DPPs with generalized Cauchy kernels, $\alpha = \alpha_{max}(\nu)$ and different values of ν.
- In red, pcfs of DPPs with Whittle-Matérn kernel, $\alpha = \alpha_{max}(\nu)$ and different values of ν.
- In black, the pcf of the most repulsive DPP, $DPP(Jinc)$.



Bessel type family

Bessel-type family

For $\sigma \geq 0$ and $0 \leq \alpha \leq \alpha_{max}(d, \rho, \sigma)$, the functions

$$C(x) = \rho 2^{\frac{\sigma+d}{2}} \Gamma\left(\frac{\sigma+d+2}{2}\right) \frac{J_{\frac{\sigma+d}{2}}\left(2|\frac{x}{\alpha}|\sqrt{\frac{\sigma+d}{2}}\right)}{\left(2|\frac{x}{\alpha}|\sqrt{\frac{\sigma+d}{2}}\right)^{\frac{\sigma+d}{2}}},$$

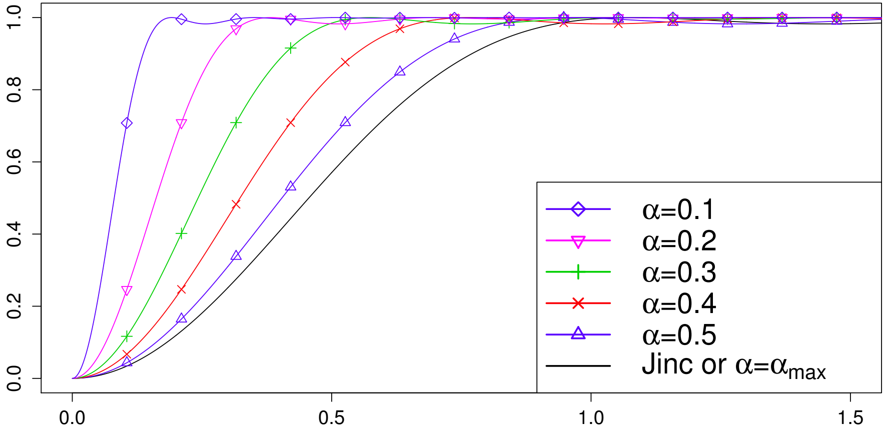
with Fourier transforms

$$\mathcal{F}\left(C\right)(x) = \frac{\rho(2\pi)^{\frac{d}{2}}\alpha^d\Gamma(\frac{\sigma+d+2}{2})}{(\sigma+d)^{\frac{d}{2}}\Gamma(\frac{\sigma+2}{2})}\left(1 - \frac{2(\pi|\alpha x|)^2}{\sigma+d}\right)_+^{\frac{\sigma}{2}}$$

define DPPs with intensity ρ.

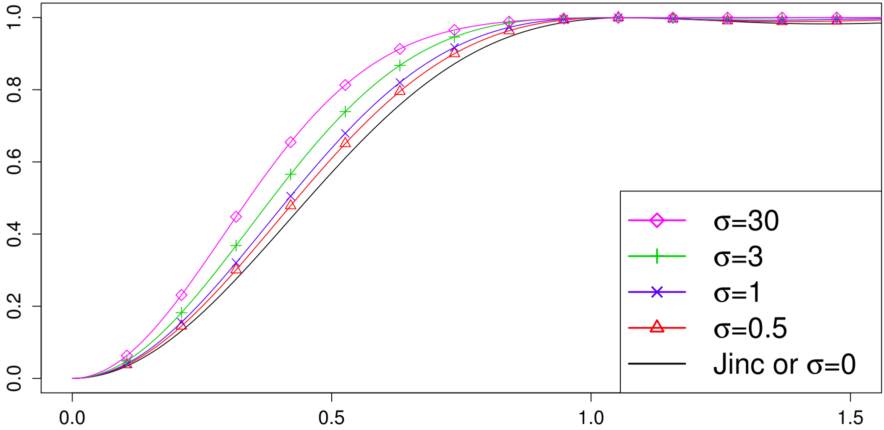
Pcfs of the Bessel type family

Associated pcfs in dimension $d = 2$ with $\rho = 1$, $\sigma = 0$ and different values of α.



Pcfs of the Bessel type family

Associated pcfs in dimension $d = 2$ with $\rho = 1$, $\alpha = \alpha_{\max}(d, \rho, \sigma)$ and different values of σ.



Laguerre-Gauss Family

The Laguerre polynomials are defined by

$$L_m^{\alpha}(x) = \sum_{k=0}^{m} \binom{m+\alpha}{m-k} \frac{(-x)^k}{k!}, \quad \forall x \in \mathbb{R}, \forall m \in \mathbb{N}, \alpha \in \mathbb{R}^+.$$

Laguerre-Gauss Family

For $m \in \mathbb{N}^*$ and $0 \leq \alpha \leq \alpha_{max}(d, \rho, m)$, the functions

$$C(x) = \rho \frac{\Gamma(m)\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(m+\frac{d}{2}\right)} L_{m-1}^{\frac{d}{2}}\left(\frac{1}{m}\left|\frac{x}{\alpha}\right|^2\right) e^{-\frac{1}{m}\left|\frac{x}{\alpha}\right|^2},$$

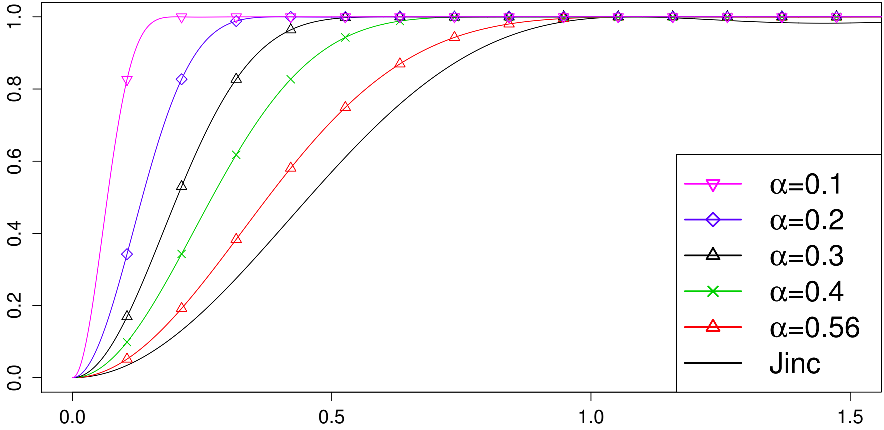
with Fourier transforms

$$\mathcal{F}\left(C\right)(x) = \rho \frac{\Gamma(m)\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(m+\frac{d}{2}\right)} \alpha^d \left(m\pi\right)^{\frac{d}{2}} e^{-m(\pi\alpha|x|)^2} \sum_{k=0}^{m-1} \frac{(\pi\sqrt{m}|\alpha x|)^{2k}}{k!}$$

define DPPs with intensity ρ.

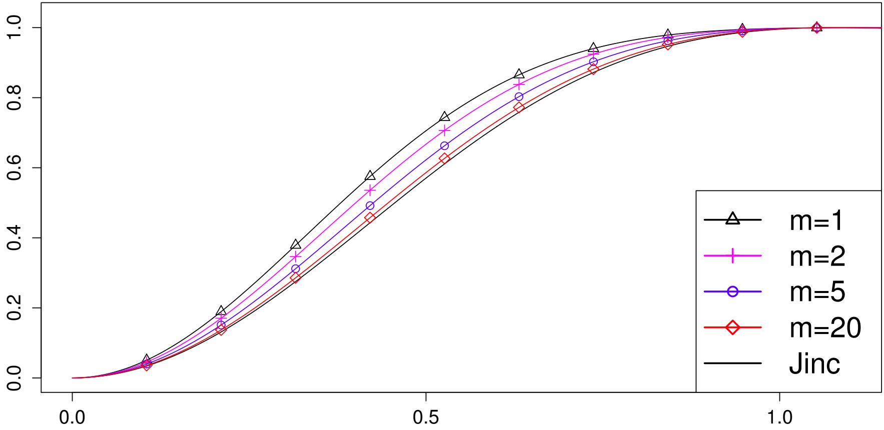
Pcf of the Laguerre-Gauss family

Associated pcfs in dimension $d = 2$ for $\rho = 1$, $m = 1$ and different values of α.



Pcf of the Laguerre-Gauss family

Associated pcfs in dimension $d = 2$ for $\rho = 1$, $\alpha = \alpha_{max}(d, \rho, m) \approx 0.56$ and different values of m.



Conclusion

- DPPs have some appealing properties: Their moments and density are known, inference is feasible.
- They are adapted to model a wide variety of repulsive points patterns,
- but they are not as flexible as Gibbs point processes, in particular they can not model hardcore phenomenon.
- There exist several parametric families that cover all the range of repulsiveness allowed by stationary DPPs.

- Christophe Ange Napoléon Biscio and Frédéric Lavancier.
  Quantifying repulsiveness of determinantal point processes.
  To appear in Bernoulli, 2015.
- John Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág.
  Zeros of Gaussian Analytic Functions and Determinantal Point Processes, volume 51 of *University Lecture Series*.
  American Mathematical Society, Providence, RI, 2009.
- Frédéric Lavancier, Jesper Møller, and Ege Rubak.
  Determinantal point process models and statistical inference.
  To appear in Journal of the Royal Statistical Society, series B, 2015.
- Odile Macchi.
  The coincidence approach to stochastic point processes.
  Advances in Applied Probability, 7:83–122, 1975.

Thank you for your attention.