Stochastic Euler-Poincaré reduction.

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Poitiers, 27 August 2015

joint work with

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Consider Euler equation for incompressible inviscid fluids

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\text{(E)} \quad \begin{cases} \frac{\partial u}{\partial t} & = -\nabla_u u - \nabla p \\ \text{div} u & = 0 \end{cases}
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for $u : [0, T] \times M \to TM$ ($u(t,x) \in T_x M$), with $M = \mathbb{T}^d$ or a compact Riemannian manifold.

[V.I. Arnold 1966] [D.G. Ebin, J.E. Marsden 1970] Integral curves $t \mapsto g(t)(x)$ of solutions to (E) are geodesics in the group of diffeomorphisms of M preserving the Lebesgue measure (integral curves satisfy $g(0)(x) = x$ and $\dot{g}(t)(x) = u(t, g(t)(x))$).

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if and only if u satisfies (E). This is the Euler-Poincaré reduction.

Aim: to establish a stochastic Euler-Poincaré reduction theorem in a general Lie group.

To apply it to volume preserving diffeomorphisms of a compact symmetric space.

Stochastic term will correspond for Euler equation to introducing viscosity.

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An $\mathbb{R}^n$-valued semimartingale ξ_t has a decomposition

$$\xi_t(\omega) = N_t(\omega) + A_t(\omega)$$

where (N_t) is a local martingale and (A_t) has finite variation.
If (N_t) is a martingale, then

$$\mathbb{E}[N_t|\mathscr{F}_s] = N_s, \quad t \geq s.$$

We are interested in semimartingales which furthermore satisfy

$$A_t(\omega) = \int_0^t a_s(\omega)\, ds.$$

Defining

$$\frac{D\xi_t}{dt} := \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\xi_{t+\varepsilon} - \xi_t}{\varepsilon} \middle| \mathscr{F}_t\right],$$

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Itô formula :

$$f(\xi_t) = f(\xi_0) + \int_0^t \langle df(\xi_s), dN_s\rangle + \int_0^t \langle df(\xi_s), dA_s\rangle + \frac{1}{2}\int_0^t \mathrm{Hess} f(d\xi_s \otimes d\xi_s).$$

From this we see that ξ_t is a local martingale if and only if for all $f \in C^2(\mathbb{R}^n)$,

$$f(\xi_t) - f(\xi_0) - \frac{1}{2}\int_0^t \mathrm{Hess} f(d\xi_s \otimes d\xi_s) \quad \text{is a real valued local martingale.}$$

This property becomes a definition for manifold-valued martingales. We say that ξ_t is a semimartingale if for all $f \in C^2(M)$ $f(\xi_t)$ is a real valued semimartingale. We say that the semimartingale ξ_t is a martingale if for all $f \in C^2(M)$

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Definition

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Let G be a Lie group with right invariant metric $\langle \cdot, \cdot \rangle$ and right invariant connection ∇.
Let $\mathscr{G} := T_e G$ be the Lie algebra of G.
Consider a countable family H_i, $i \geq 1$, of elements of $\mathscr{G}$, and $u \in C^1([0, T], \mathscr{G})$.
Consider the Stratonovich equation

$$\begin{aligned} dg_t &= \left(\sum_{i\geq 1} H_i \circ dW_t^i - \frac{1}{2} \nabla_{H_i} H_i \, dt + u(t) \, dt \right) \cdot g_t \\ g_0 &= e \end{aligned}$$

where the (W_t^i) are independent real valued Brownian motions. Itô formula

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This implies that $\frac{Dg_t}{dt} = u(t) g_t$.

Particular case

If (H_i) is an orthonormal basis, $\nabla_{H_i} H_i = 0$, ∇ is the Levi Civita connection associated to the metric and $u \equiv 0$, then g_t is a Brownian motion in G.

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On the space $\mathscr{S}(G)$ of G-valued semimartingales define

$$J(\xi) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\|\frac{D\xi}{dt}\right\|^2 dt\right].$$

Perturbation: for $v \in C^1([0,T],\mathscr{G})$ satisfying $v(0) = v(T) = 0$ and $\varepsilon > 0$, let $e_{\varepsilon,v}(\cdot) \in C^1([0,T],G)$ the flow generated by εv:

$$\begin{cases} \frac{d}{dt}e_{\varepsilon,v}(t) & = \varepsilon\dot{v}(t)\cdot e_{\varepsilon,v}(t) \\ e_{\varepsilon,v}(0) & = e \end{cases}$$

Definition

We say that $g \in \mathscr{S}(G)$ is a critical point of J if for all $v \in C^1([0,T],\mathscr{G})$ satisfying $v(0) = v(T) = 0$,

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} g_{\varepsilon,v} = 0 \quad \text{where} \quad g_{\varepsilon,v}(t) = e_{\varepsilon,v}(t)g(t).$$

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Theorem

g is a critical point of J if and only if

$$\frac{du(t)}{dt} = -\mathrm{ad}^*_{\tilde{u}(t)} u(t) - K(u(t))$$

with

$$\tilde{u}(t) = u(t) - \frac{1}{2}\sum_{i\geq 1} \nabla_{H_i} H_i, \quad \langle \mathrm{ad}^*_u v, w\rangle = \langle v, \mathrm{ad}_u v\rangle$$

and $K : \mathscr{G} \to \mathscr{G}$ satisfies

$$\langle K(u), v\rangle = -\left\langle u, \frac{1}{2}\sum_{i\geq 1} \nabla_{\mathrm{ad}_v H_i} H_i + \nabla_{H_i}\left(\mathrm{ad}_v(H_i)\right)\right\rangle$$

Remark 1

If for all $i \geq 1$, $H_i = 0$, or $\nabla_u v = 0$ for all $u, v \in \mathscr{G}$, then $K(u) = 0$ and we get the standard Euler-Poincaré equation.

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Proposition

If for all $i \geq 1$, $\nabla_{H_i} H_i = 0$ then

$$K(u) = -\frac{1}{2} \sum_{i \geq 1} \nabla_{H_i} \cdot \nabla_{H_i} u + R(u, H_i) H_i.$$

In particular if (H_i) is an o.n.b. of $\mathscr{G}$ then

$$K(u) = -\frac{1}{2} \Box u = -\frac{1}{2} \Delta u + \frac{1}{2} \mathrm{Ric}^\sharp u$$

Let

$$G_V^s = \{g : M \to M \quad \text{volume preserving bijection, such that} \quad g,\ g^{-1} \in H^s\}.$$

Assume $s > 1 + \frac{\dim M}{2}$. Then G_V^s is a C^∞ smooth manifold. Lie algebra

$$\mathscr{G}_V^s = T_e G_V^s = \{X : H^s(M, TM), \pi(X) = e, \operatorname{div}(X) = 0\}.$$

Notice that $\pi(X) = e$ means that X is a vector field on M: $X(x) \in T_x M$. On $\mathscr{G}_V^s$ consider the two scalar products

$$\langle X, Y\rangle^0 = \int_M \langle X(x), Y(x)\rangle \, dx$$

and

$$\langle X, Y\rangle^1 = \int_M \langle X(x), Y(x)\rangle \, dx + \int_M \langle \nabla X(x), \nabla Y(x)\rangle \, dx.$$

The Levi Civita connection on G_V^s is given by $\nabla_X^0 Y = P_e(\nabla_X Y)$ with ∇ the Levi Civita connection of $\langle \cdot, \cdot \rangle^0$ on G^s and P_e the orthogonal projection on $\mathscr{G}_V^s$:

$$H^s(TM) = \mathscr{G}_V^s \oplus dH^{s+1}(M).$$

One can find $(H_i)_{i\geq 1}$ such that for all $i \geq 1$, $\nabla_{H_i} H_i = 0$, $\operatorname{div}(H_i) = 0$, and

$$\sum_{i\geq 1} H_i^2 f = \nu \Delta f, \quad f \in C^2(M).$$

Let

$$G_V^s = \{g : M \to M \quad \text{volume preserving bijection, such that} \quad g,\ g^{-1} \in H^s\}.$$

Assume $s > 1 + \frac{\dim M}{2}$. Then G_V^s is a C^∞ smooth manifold. Lie algebra

$$\mathscr{G}_V^s = T_e G_V^s = \{X : H^s(M, TM), \pi(X) = e, \mathrm{div}(X) = 0\}.$$

Notice that $\pi(X) = e$ means that X is a vector field on M: $X(x) \in T_x M$. On $\mathscr{G}_V^s$ consider the two scalar products

$$\langle X, Y\rangle^0 = \int_M \langle X(x), Y(x)\rangle\, dx$$

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Corollary

(1) g is a critical point of $J^{\langle\cdot,\cdot\rangle^0}$ if and only if u solves Navier-Stokes equation

$$\begin{cases} \frac{\partial u}{\partial t} & = -\nabla_u u + \frac{\nu}{2}\Delta u - \nabla p \\ \mathrm{div} u & = 0 \end{cases}$$

(2) Assume $M = \mathbb{T}^2$ the 2-dimensional torus. Then g is a critical point of $J^{\langle\cdot,\cdot\rangle^1}$ if and only if u solves Camassa-Holm equation

$$\begin{cases} \frac{\partial u}{\partial t} & = -\nabla_u v - \sum_{j=1}^2 \nabla_{v_j} u_j + \frac{\nu}{2}\Delta v - \nabla p \\ v & = u - \Delta u \\ \mathrm{div} u & = 0 \end{cases}$$

From the proof, use Itô formula and compute in different situations $\mathrm{ad}_v^*(u)$ and $K(u)$. Notice that the term ∇p corresponds to the use of ∇ instead of ∇^0: the first system rewrites as

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