Topological Phase Transitions

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Topological Phase Transitions

(Nature abhors complexity)



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and many others

What is a phase transition?

ISING MODEL

What is topology?

Starling murmuration, Rahat, Israel


euronews

The wonderful world of geotopology

SIMPLICIAL TOPOLOGY

Simplices, complexes,
cycles, numbers of simplices,
Betti numbers

$$\sum_k (-1)^k \#\{k\text{-dimensional simplices}\}$$

INTEGRAL GEOMETRY

Convexity, convex ring
kinematic formulae
Minkowski functionals

$$\mathcal{M}_k(M) = c_{dk} \int_{\mathrm{Graff}(d,d-k)} \chi(M \cap V)\, d\mu^d_{d-k}(V)$$

$$\chi$$

$$\sum_k (-1)^k \beta_k$$

$$\sum_k (-1)^k \#\{\text{critical points of index } k\}$$

$$\int_M \mathrm{Tr}(R^{m/2})\mathrm{Vol}_g$$

ALGEBRAIC TOPOLOGY

Homology, homotopy,
dimensions of groups,
Betti numbers, persistence

DIFFERENTIAL TOPOLOGY

Curvature, forms, Betti numbers,
Morse theory, integration,
Lipschitz-Killing curvatures

The Gaussian kinematic formula

A 1-slide course on Gaussian random fields

- On a topological space *M*

  (maybe a stratified Riemannian manifold)

- choose a set $\{\varphi_k\}$ of dense functions on *M*

  (e.g. Eigenfunctions of the Laplacian)

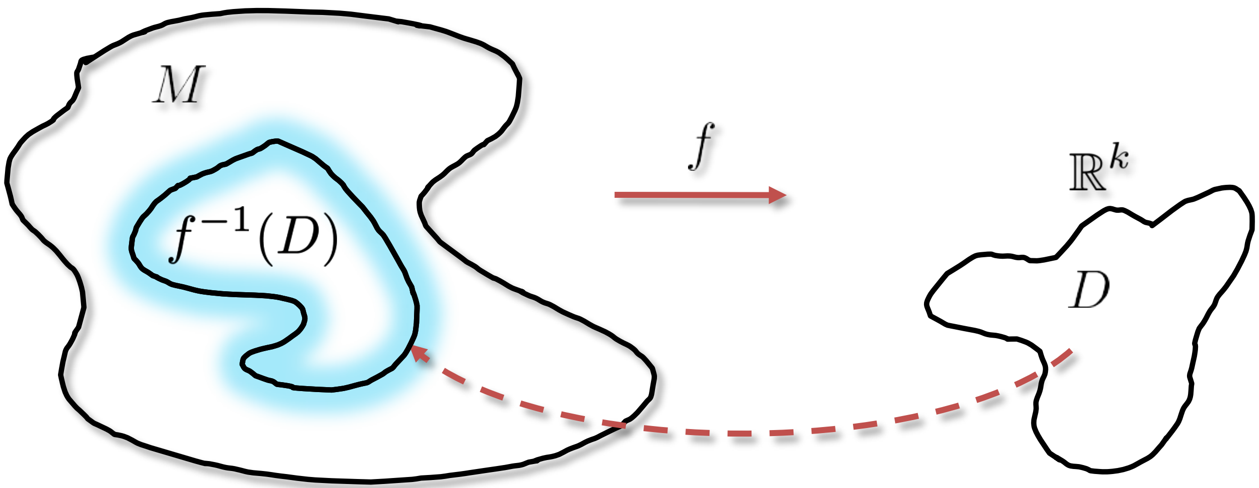
- Then take independent Gaussian random variables, $\xi_1, \xi_2, \ldots$

$$\mathcal{P}\left\{(\xi_{j_1}, \ldots, \xi_{j_k}) \ \in \ A\right\} \ = \ \frac{1}{(2\pi)^{k/2}} \int_A e^{-\|x\|^2/2} \, dx$$

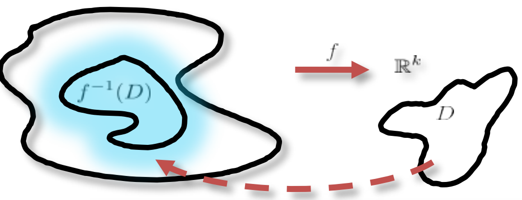
- and define a Gaussian random field:

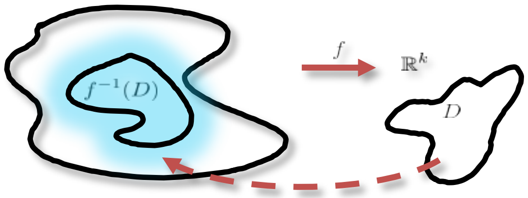
$$f(p) \ = \sum_j a_j \xi_j \varphi_j(p)$$

Excursion sets



The Gaussian Kinematic Formula





Theorem

Let $M \subset \mathbb{R}^N$ and $D \subset \mathbb{R}^d$ be nice stratified spaces. Let $f = (f_1, \ldots, f_d) : M \to \mathbb{R}^d$ be a d-dimensional Gaussian field, with *iid* components all having zero mean, unit variance and a nice covariance function. Then,

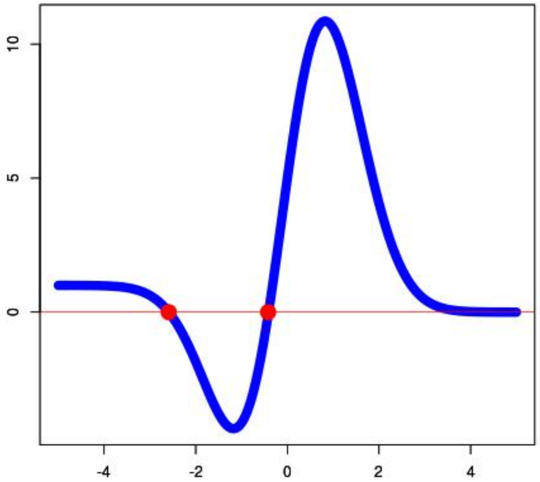
$$\mathbb{E}\left\{\chi(f^{-1}(D))\right\} = \sum_{j=0}^{\dim M} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j(D)$$

$$\mathbb{E}\left\{\mathcal{L}_i(f^{-1}(D))\right\} = \sum_{j=0}^{\dim M - i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M) \mathcal{M}_j(D)$$

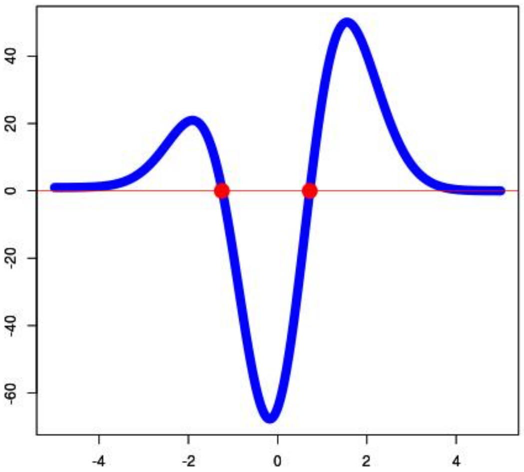
GKF - Examples

$$E\left\{\chi\left([0,T]^2 \cap f^{-1}[u,\infty)\right)\right\}$$

$$= \left[\frac{T^2\lambda}{(2\pi)^{3/2}}u + \frac{2T\lambda^{1/2}}{2\pi}\right]e^{-u^2/2} + \Psi(u)$$

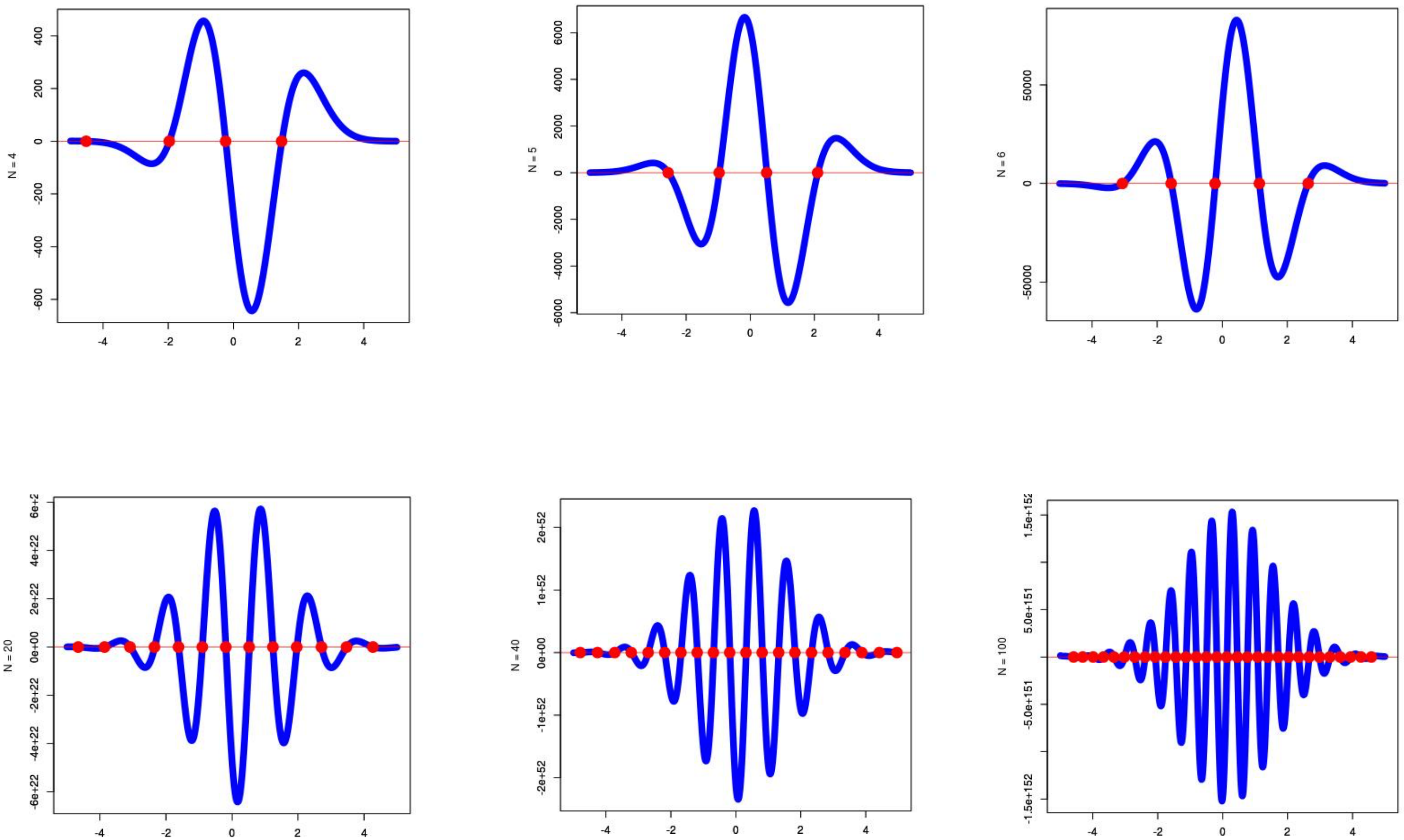


$d = 2$

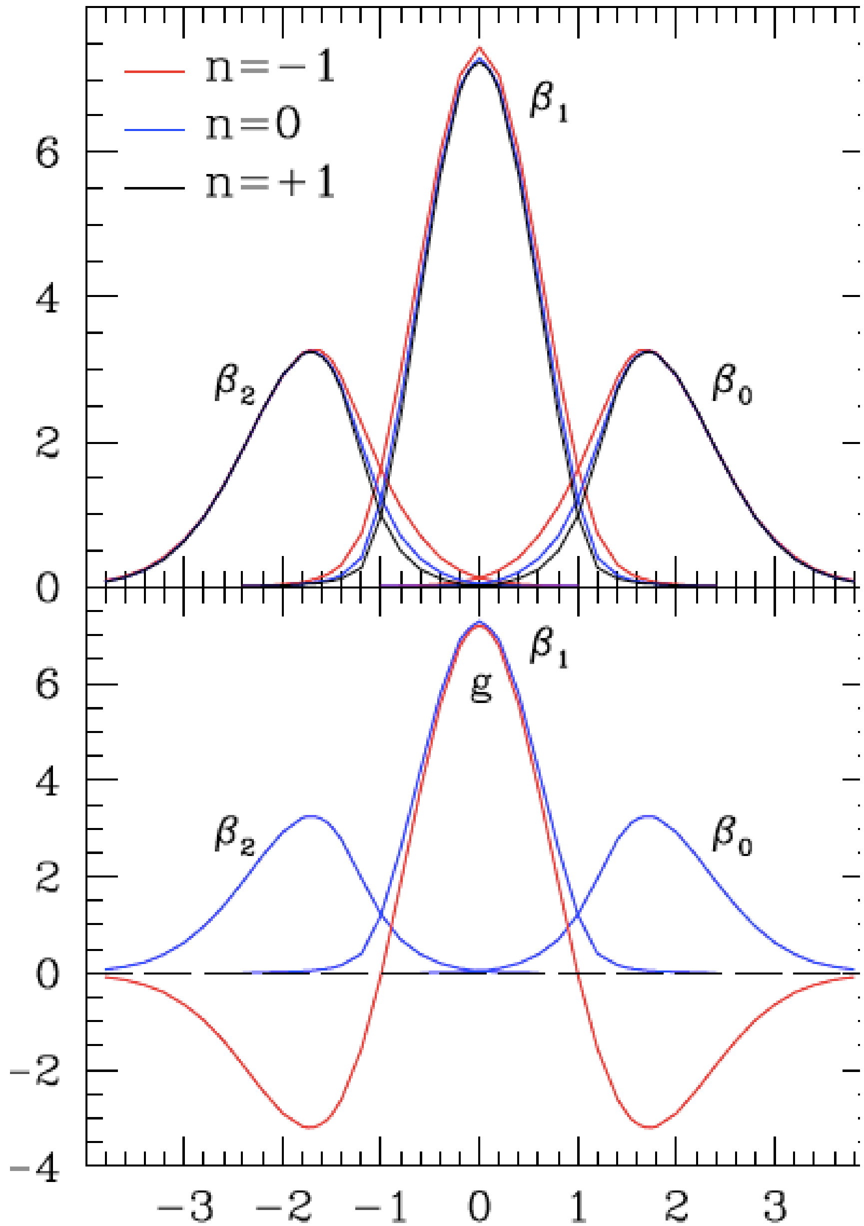


$d = 3$

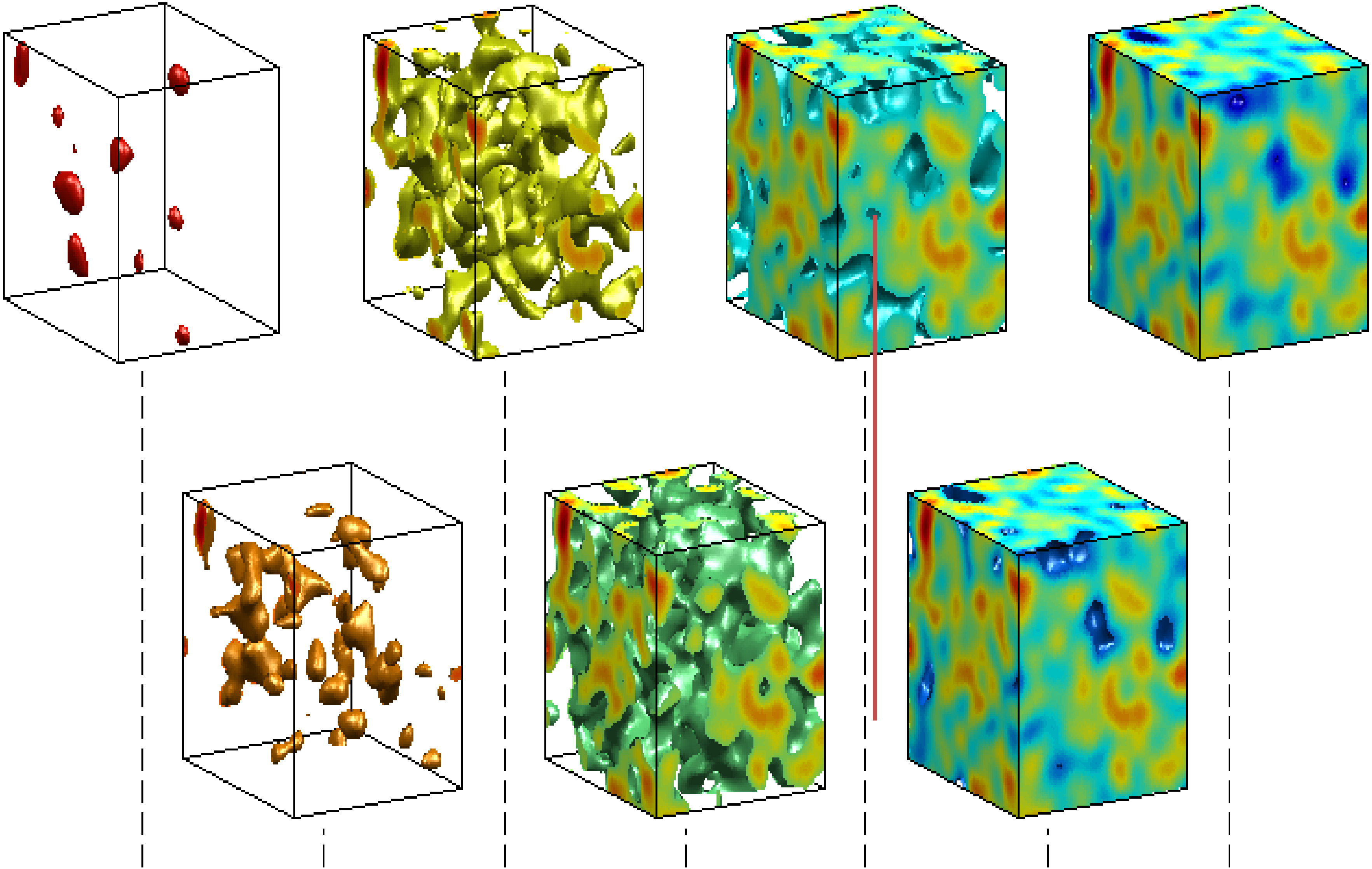
GKF and phase transitions



The separation of homologies



An example



A Gaussian topological (classical) phase transition

Spin glasses and Gaussian critical points

$$\sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p}\sigma_{i_1}\ldots\sigma_{i_p}$$

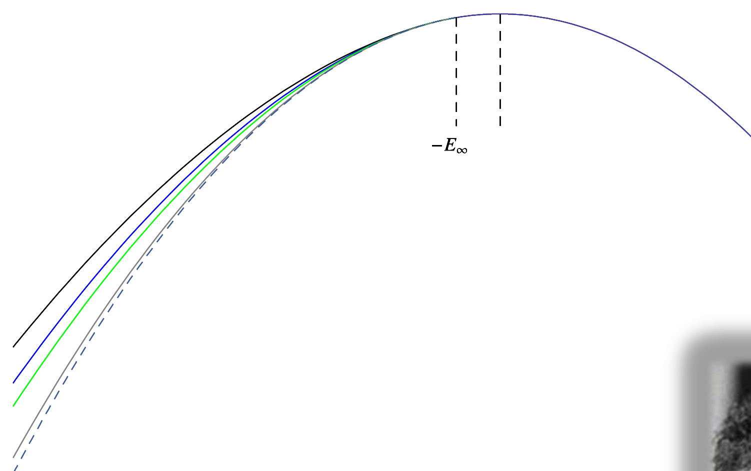
$$(\sigma_1,\ldots,\sigma_N)\in S^{N-1}(\sqrt{N})$$

$$\Big(\sum_{i=1}^{N}\sigma_i\sigma_i'\Big)^p$$

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\,\mathrm{Crt}_{N,k}(B)=\sup_{u\in B}\theta_{k,\nu}(u)$$

METHOD OF PROOF

- Rice formula for mean
- Exploitation of specific covariance
- Variance for concentration



The Euler characteristic heuristic

$$\mathbb{P}\left\{ \sup_M f \ \geq u \right\} \sim \ \mathbb{E}\{\chi(A_u(f, M))\}$$

$$\sim \ u^{\dim M - 1} e^{-u^2/2}$$

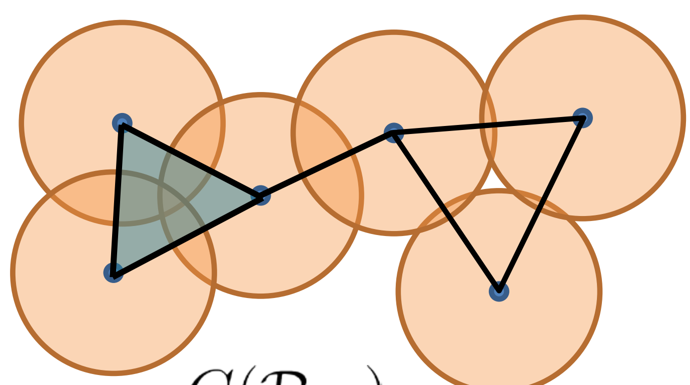
$$\liminf_{u \to \infty} u^{-2} \log |\mathbb{P} - \mathbb{E}| \ \geq \ \frac{1}{2} + \frac{1}{2\sigma^2(f)}$$

As well as a general theory of the critical points a Gaussian and related via Kac-Slepian models

Random simplicial complexes - Euler characteristics

The Čech Complex

- Take a set of vertices P (0-simplexes)
- Draw balls with radius ϵ
- Intersection of 2 balls → an edge (1-simplex)
- Intersection of 3 balls → a triangle (2-simplex)
- Intersection of n balls → a (n-1)-simplex



$C(\mathcal{P}, \epsilon)$

NERVE THEOREM

The Cech complex and union of balls have the same homology

E(EC) for Cech complexes on T^d

1: Points (n) chosen at random

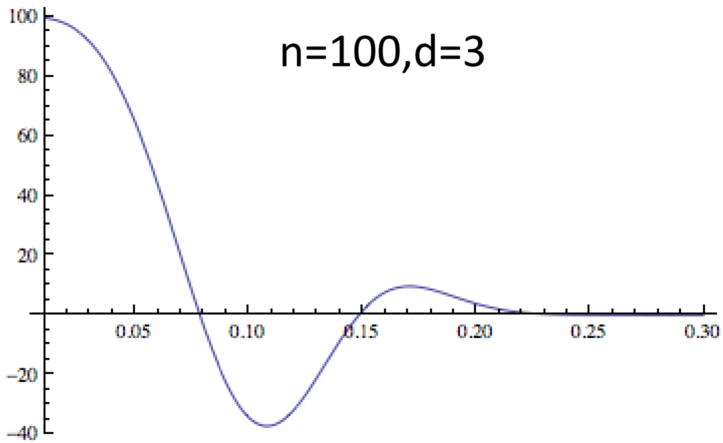
$$\mathbb{E}\{\chi\} = \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} k^d (2r)^{d(k-1)}$$

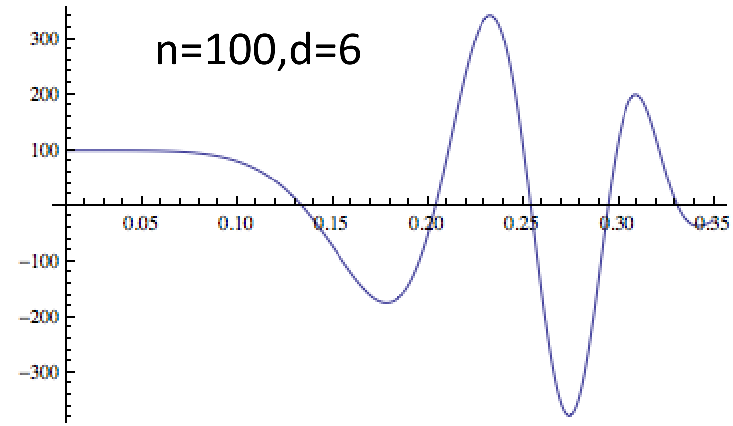
2: Poisson process

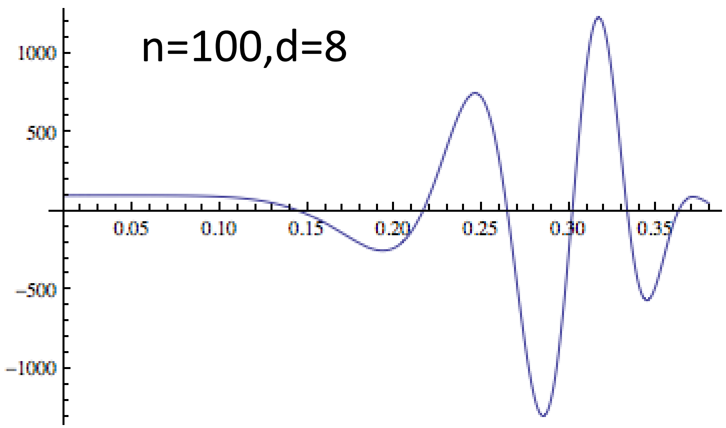
$$\mathbb{E}\{\chi\} = -(2r)^{-d} e^{-\lambda(2r)^d} \sum_{k=0}^{\infty} \frac{(-\lambda(2r)^d)^k k^d}{k!}$$

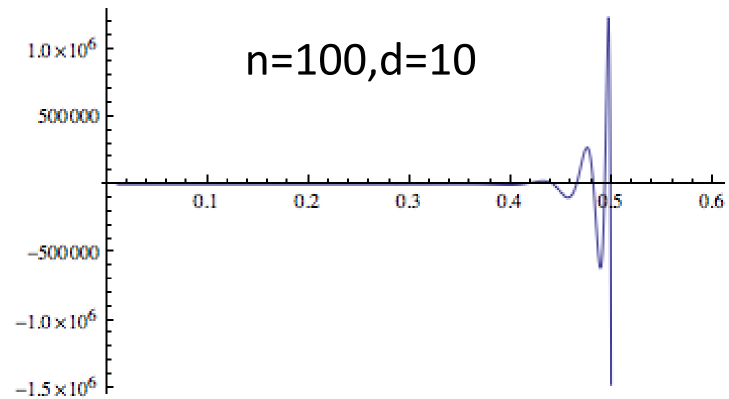
Decreusefond, Ferraz, Randriam, Vergne, by counting simplices

E(EC) for Cech complexes









Cech complex on 1000 points in $[0,1]^3$

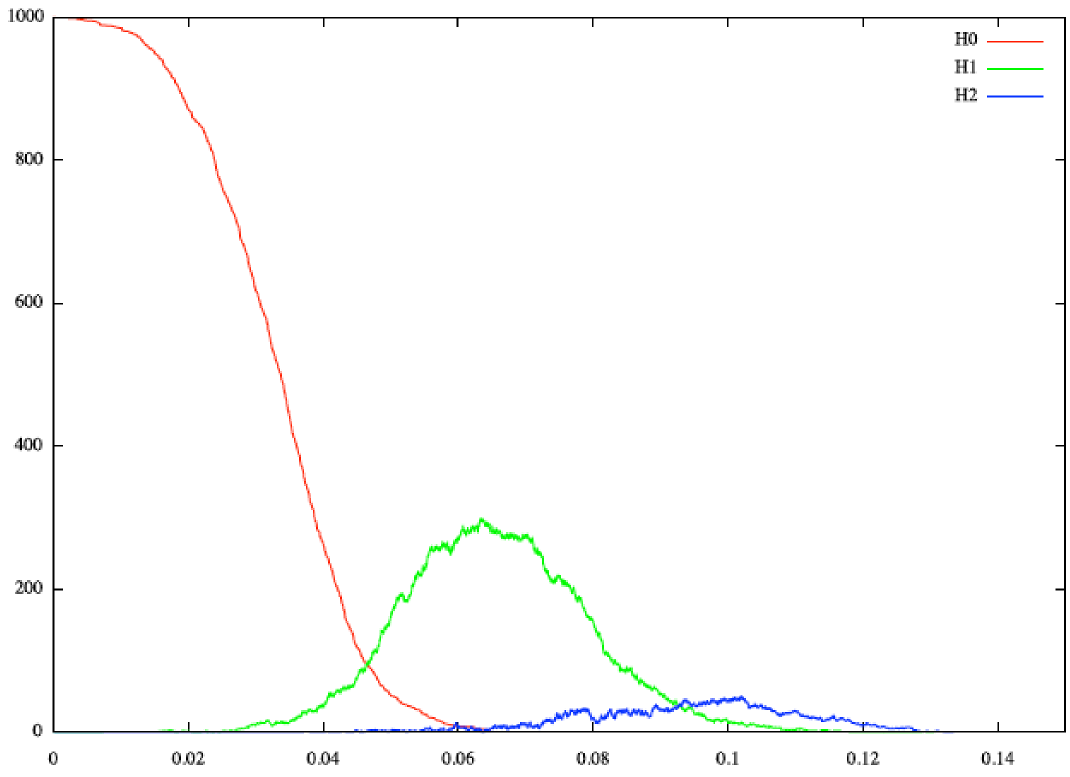
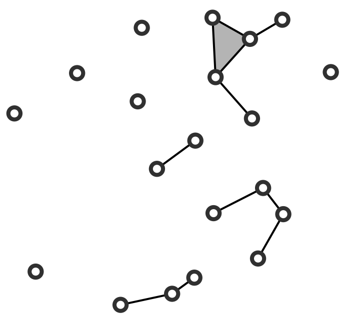
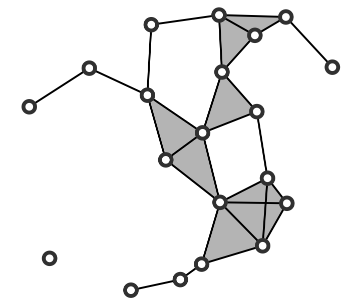


FIGURE 2. The Betti numbers of a random Čech complex on $n = 1000$ points versus the radius r. Computation and image courtesy of Dmitriy Morozov and his Dionysus library [39].

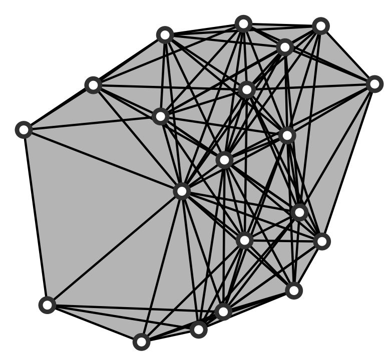
Phase transitions/persistence for complexes



Dust, or
subcritical phase



Thermodynamic,
or critical, or
percolation phase



Connectivity, or
supercritical phase

Theory: MK, EM, OB, YD, inter alia

Dust, subcritical

$r_n = o\left(n^{-1/d}\right)$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{N_k(r_n)\right\} = 0 \quad ; \quad \mathbb{E}\left\{N_k(r_n)\right\} = \theta(n^{k+1} r_n^{dk})$$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{\beta_k(r_n)\right\} = 0 \quad ; \quad \mathbb{E}\left\{\beta_k(r_n)\right\} = \theta(n^{k+2} r_n^{d(k+1)})$$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{\chi_n(r_n)\right\} = 1$$

Critical, percolative, thermodynamic

$r_n = cn^{-1/d}$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{N_k(r_n)\right\} = \frac{c^{dk}}{(k+1)!} C_k$$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{\beta_k(r_n)\right\} = const > 0$$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{\chi_n(r_n)\right\} = 1 + \sum_{k=1}^{d} \frac{(-c^d)^k}{(k+1)!} C_k$$

Dense, supercrictical

$r_n = \omega\left(n^{-1/d}\right)$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{N_k(r_n)\right\} = \lim_{n\to\infty} n^{-1}\mathbb{E}\left\{N_k\right\}$$

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\left\{\chi_n(r_n)\right\} = 0$$

Three phases of rigorous results (OB)

Recall: $N_{n,k}$ = # critical points p with $d_{\mathcal{X}_n}(p) \le r_n$

Theorem

- $nr_n^d \to 0 \quad \Rightarrow \quad (n^{k+1} r_n^{dk})^{-1} \mathbb{E}\{N_{k,n}\} \to \mu_k$
- $nr_n^d \to \lambda \in (0,\infty) \quad \Rightarrow \quad n^{-1} \mathbb{E}\{N_{k,n}\} \to \gamma_k(\lambda)$
- $nr_n^d \to \infty \quad \Rightarrow \quad n^{-1} \mathbb{E}\{N_{k,n}\} \to \gamma_k(\infty)$

$$\mu_k = \frac{1}{(k+1)!} \int_{\mathbb{R}^d} f(x)^{k+1} dx \int_{(\mathbb{R}^d)^k} h_1(0,\mathbf{y}) d\mathbf{y}$$

$$\gamma_k(\lambda) = \frac{\lambda^k}{(k+1)!} \int_{(\mathbb{R}^d)^{k+1}} f^{k+1}(x) h_1(0,\mathbf{y}) e^{-\lambda \omega_d R^d(0,\mathbf{y}) f(x)} dx d\mathbf{y}$$

$$\gamma_k(\infty) = \frac{1}{(k+1)!} \int_{(\mathbb{R}^d)^k} h_\infty(0,\mathbf{y}) e^{-\omega_d R^d(0,\mathbf{y})} dx d\mathbf{y}$$

$$h_\epsilon(0,\mathbf{y}) = \mathbb{1}\{C(0,\mathbf{y}) \in \mathrm{int}(\mathrm{conv}(0,\mathbf{y})) \text{ and } R(0,\mathbf{y}) \le \epsilon\}$$

Subcritical (dust) phase

$$\left(n^{k+1} r_n^{dk}\right)^{-1} \mathbb{E}\left\{N_{k,n}\right\} \to \mu_k$$

Theorem

If $n r_n^d \to 0$, and $k \geq 1$,

- $\left(n^{k+1} r_n^{dk}\right)^{-1} \mathrm{Var}(N_{k,n}) \to \mu_k$
- $n^{k+1} r_n^{dk} \to 0 \quad \Rightarrow \quad N_{k,n} \xrightarrow{L^2} 0$
- $n^{k+1} r_n^{dk} \to a \in (0, \infty) \quad \Rightarrow \quad N_{k,n} \xrightarrow{D} \text{Poisson}\left(a \mu_k\right)$
- $n^{k+1} r_n^{dk} \to \infty \quad \Rightarrow \quad \dfrac{N_{k,n} - \mathbb{E}\left\{N_{k,n}\right\}}{\sqrt{n^{k+1} r_n^{dk}}} \xrightarrow{D} \mathcal{N}(0, \mu_k)$

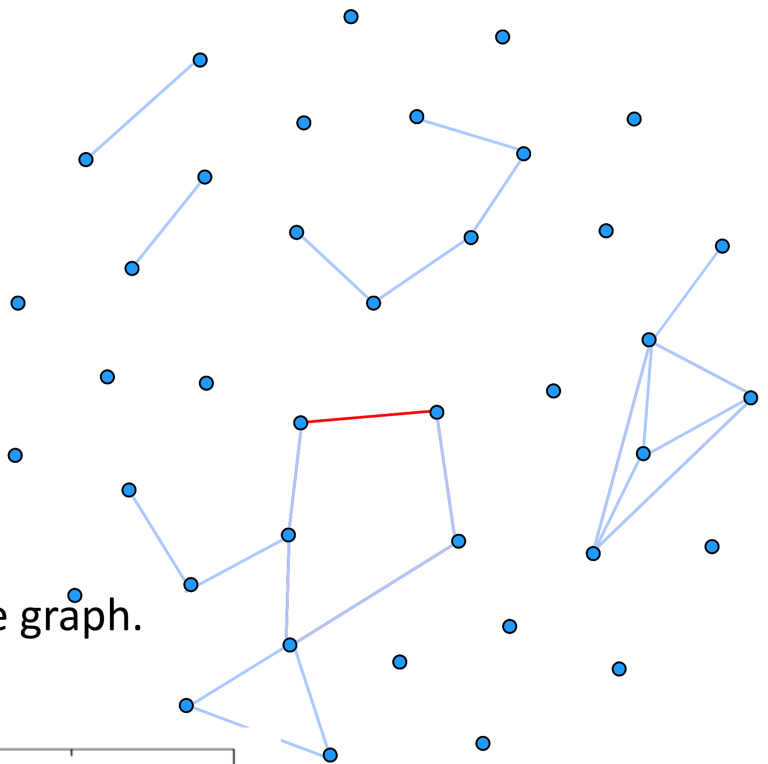
Erdos-Renyi graphs and complexes

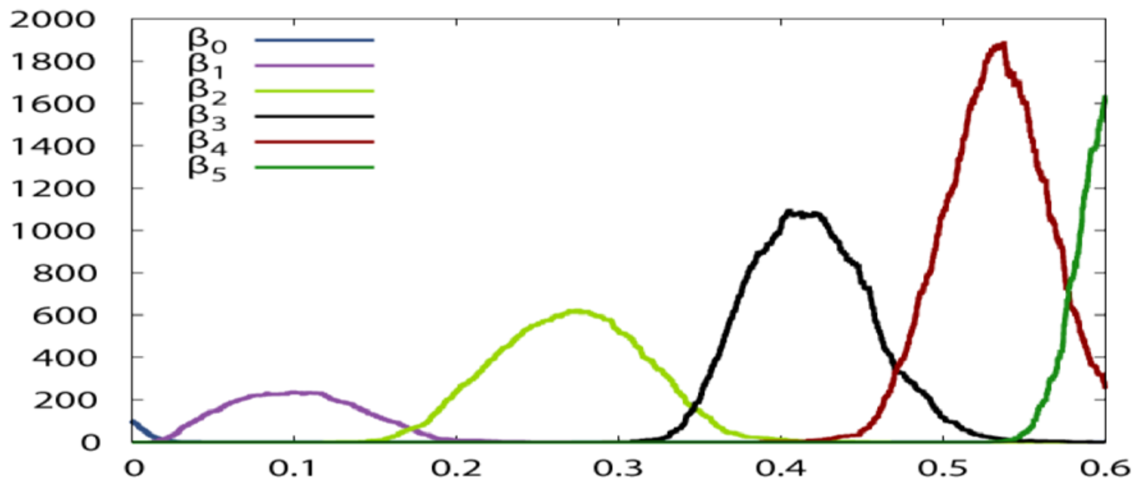
The graph is on n vertices where each edge appears with probability p, independently for each edge.

Traditionally, only the graph structure is studied. The literature is extensive and rich

This is a mean field model. Unrealistic, but mathematically tractable and often representative.

Increasing p increases the topological nature of the graph.





Linial-Meshulam model

- *X(d,n,p)*: A generalization of the Erdos-Renyi graph *G(n,p)*
- Start with a full *(d − 1)*-dimensional skeleton.
- For each d-dimensional face, independently and with probability p, decide whether to include it in *X(d,n,p)* or not.

$$\lim_{n\to\infty} Pr\left\{H_{k-1}(X(d,n,p))=0\right\} = \begin{cases} 0 & p=\frac{k\log n-\omega(n)}{n} \\ 1 & p=\frac{k\log n+\omega(n)}{n} \end{cases}$$

In the evolution of these complexes, with *d ≥ 2,* the first occurring cycle is, almost surely, either

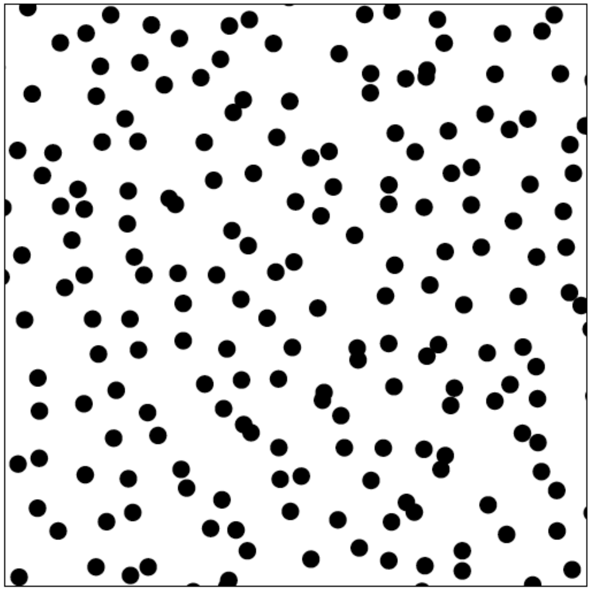
- The boundary of a *(d + 1)*-dimensional simplex, or
- A cycle that includes Ω*(n^d)* faces of dimension *d*.

Consequently, when *d ≥ 2,* different kinds of phase transitions occur.

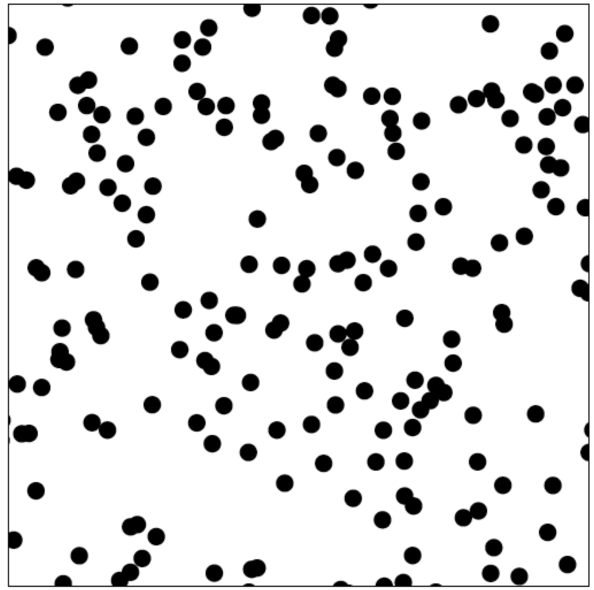
Bobrowski and Kahle. *Topology of random geometric complexes*.

Random simplicial complexes generated by point processes

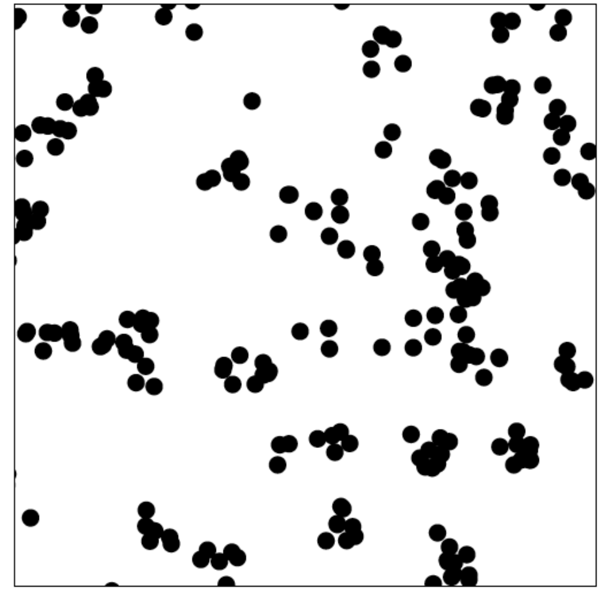
Different local structures



Simple perturbed lattice

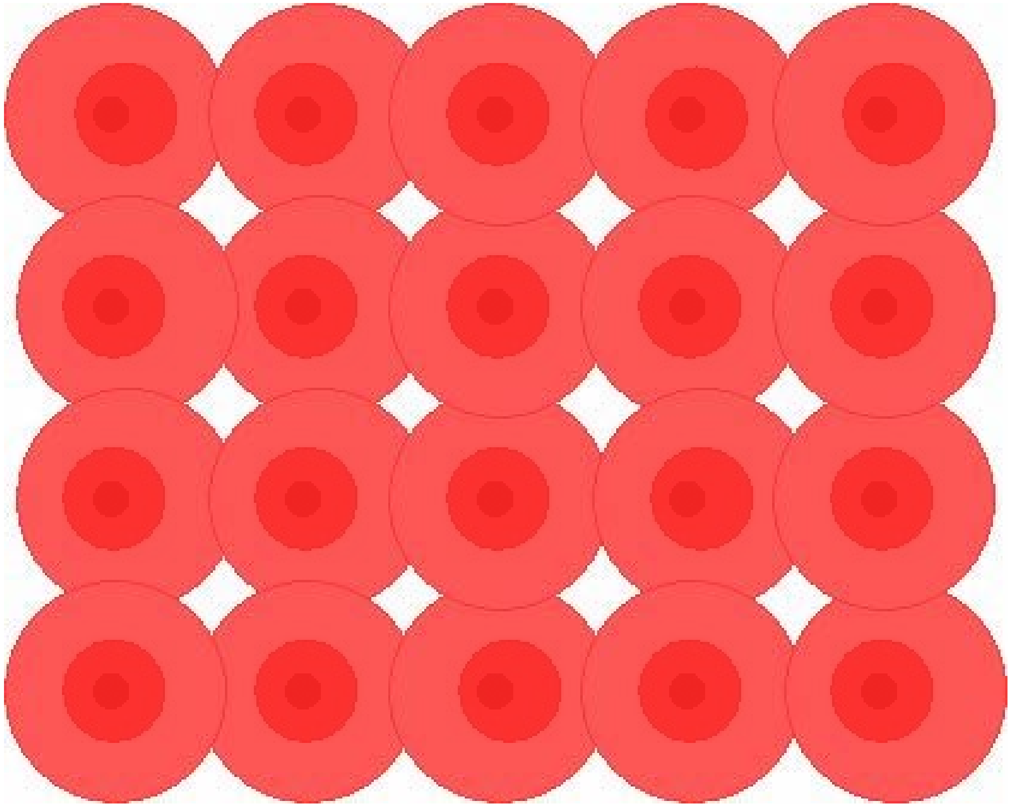


Poisson point process



Cox point process

(Un) Perturbed lattices



Structured point processes lead to a narrower range of topological behaviour

Models for clouds of interacting points

Poisson point process Φ:

$$\Phi(B_1), \ldots, \Phi(B_k) \qquad .$$

are independent for disjoint sets with distribution $\mathrm{Pois}(|B_i|)$.

Perturbed lattice:

$$\bigcup_{z \in \mathbb{Z}^d} \{z + U_z\}, \qquad U_z \sim Unif([0,1]^d) \qquad .$$

Ginibre point process: Eigenvalues of $N \times N$ matrix with i.i.d. standa
complex Gaussian entries as $N \to \infty$.

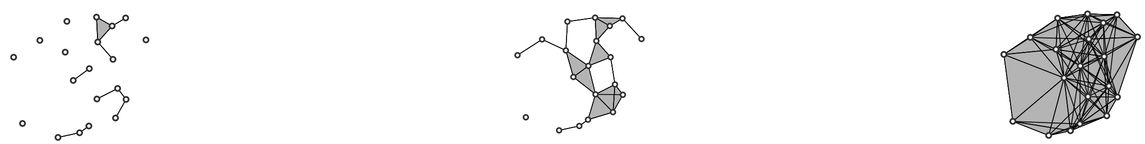
Zeros of Gaussian analytic functions:

$$f(z) = \sum_{n=0}^{\infty} \xi_n z^n / \sqrt{n!} \qquad .$$

Growth of Betti numbers (YD)

Betti numbers taken over the Cech complex with radii r_n over growing regions of the form

$$\left[\frac{-n^{1/d}}{2}, \frac{n^{1/d}}{2}\right]^d$$



	Sparse regime	Thermo–Perc	Connected
Radii r_n	$\to 0$	$\to (0, \infty)$	$\geq C(\log n)^{\frac{1}{d}}$
Poisson	$\sim nr_n^{d(k+1)}$	$\sim n$	0
Perturbed lattice	$o(nr_n^{d(k+1)})$	$\sim n$	$r_n \geq 2^d$
Ginibre	$\sim nr_n^{(k+1)(k+4)}$	$\sim n$	$\geq C(\log n)^{\frac{1}{4}}$
GAF	$\sim nr_n^{(k+1)(k+4)}$	$O(n)$	$\geq C(\log n)^{\frac{1}{4}}$

Two fluctuation theorems (DY, ES)

Theorem 4.5. *Let $1 \le k \le d-1$, $\mathcal{X}_n$ be a binomial point process, and assume that $nr_n^d \to r \in (0,\infty)$. Then, for any $a > \frac{1}{2}$ and $\varepsilon > 0$, for n large enough,*

$$\mathbb{P}\left\{\left|\beta_k(\mathcal{C}(\mathcal{X}_n, r_n) - \mathbb{E}\{\beta_k(\mathcal{C}(\mathcal{X}_n, r_n)\}\right| \ge \varepsilon n^a\right\} \;\le\; \frac{C}{\varepsilon} n^{2k+2-a} \exp(-n^\gamma),$$

where $\gamma = (2a-1)/4k$ and $C > 0$ is a constant depending only on a, r, k, d and the density f.

Theorem 4.7. *Let $\{B_n\}$ be a sequence of sets in $\mathbb{R}^d$ satisfying conditions (A)–(D) above, and let $\mathcal{P}$ and $\mathcal{U}_n$, $n \ge 1$, respectively, be the unit intensity Poisson process and the extended binomial point process described above. Take $k \in \{1,\dots,d-1\}$ and $r \in (0,\infty)$. Then there exists a constant $\sigma^2 > 0$ such that, as $n \to \infty$,*

$$n^{-1}\mathsf{VAR}(\beta_k(\mathcal{C}(\mathcal{P} \cap B_n, r)) \;\to\; \sigma^2,$$

and

$$n^{-1/2}\left(\beta_k(\mathcal{C}(\mathcal{P} \cap B_n, r)) - \mathbb{E}\{\beta_k(\mathcal{C}(\mathcal{P} \cap B_n, r))\}\right) \;\Rightarrow\; N(0,\sigma^2).$$

Furthermore, for $r \notin I_d(\mathcal{P})$, there exists a τ^2 with $0 < \tau^2 \le \sigma^2$ such that

$$n^{-1}\mathsf{VAR}(\beta_k(\mathcal{C}(\mathcal{U}_n, r)) \to \tau^2,$$

and

$$n^{-1/2}\left(\beta_k(\mathcal{C}(\mathcal{U}_n, r)) - \mathbb{E}\{\beta_k(\mathcal{C}(\mathcal{U}_n, r))\}\right) \;\Rightarrow\; N(0,\tau^2).$$

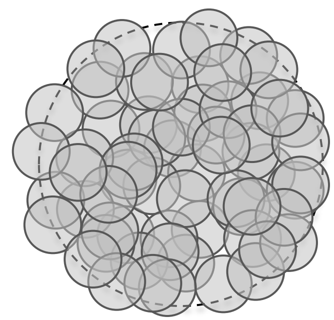
Crackle

(Boborowski, Owada, Weinberger, YT)

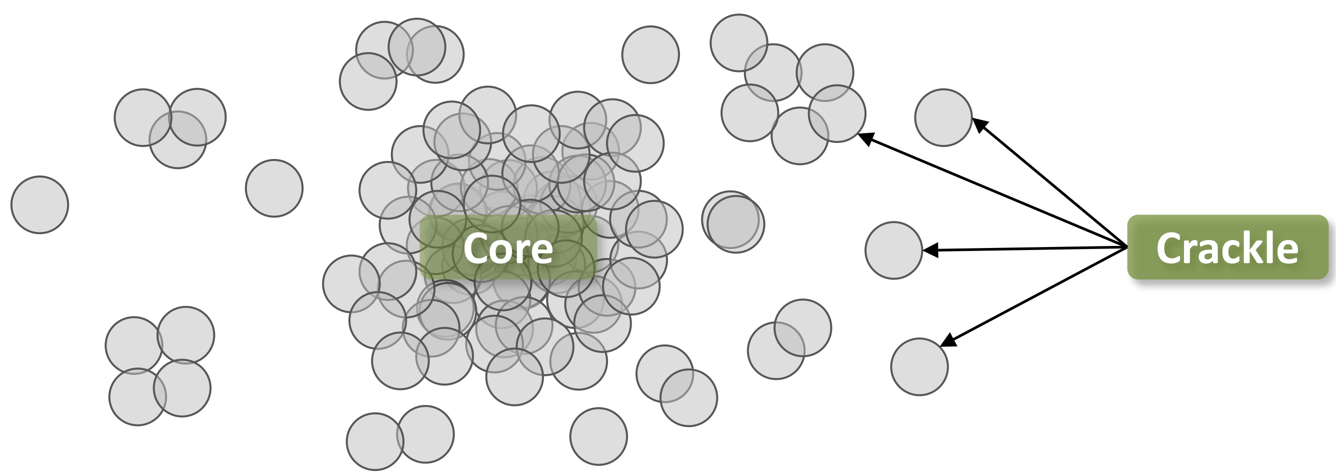
What is crackle, and why would one care?

Draw random balls with a <u>fixed</u> radius

Distributions with compact support on $\bullet\ \mathbb{R}^d$



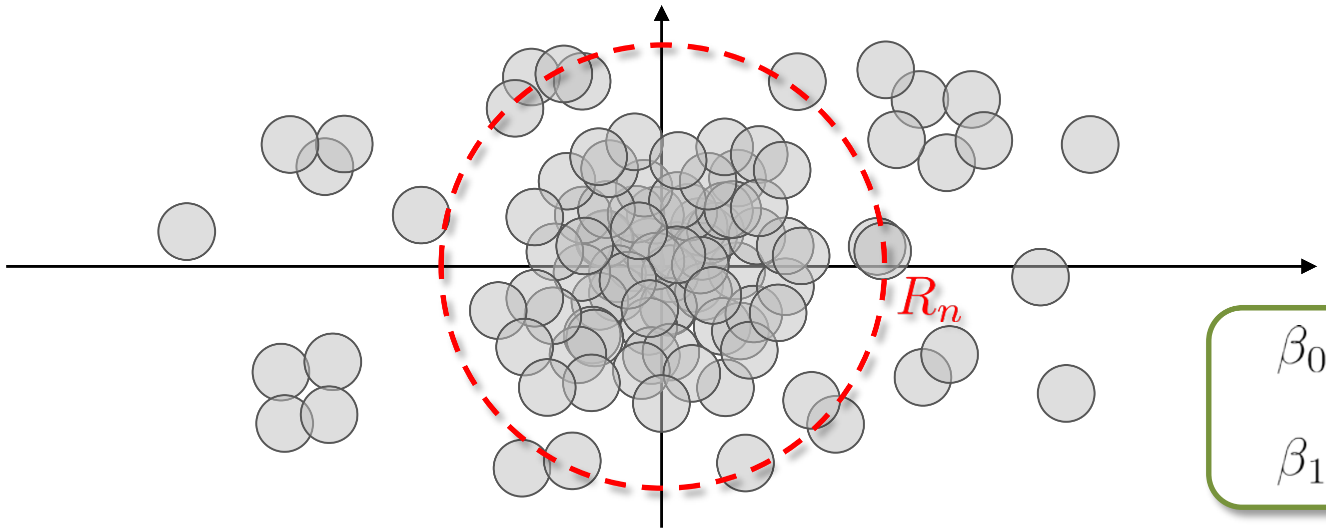
Distributions with unbounded support on $\bullet\ \mathbb{R}^d$



Crackle - Definitions

$$U_{n,R_n} = \bigcup_{X \in \mathcal{X}_n \cap (B_{R_n}(0))^c} B_1(X)$$

$$\beta_{k,n} \triangleq \beta_k(U_{n,R_n})$$



$\beta_{0,n} = 13$

$\beta_{1,n} = 2$

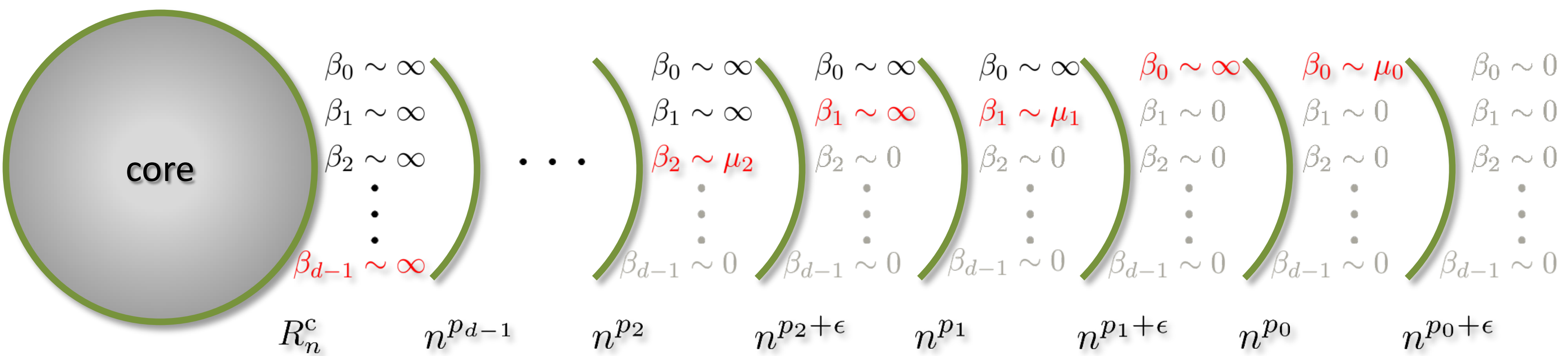
How Power-Law Noise Crackles

Theorem

If $f(x) = \frac{c}{1+\|x\|^{\alpha}}$, then for $0 \le k \le d-1$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left\{\beta_{k,n}\right\} = \begin{cases} 0 & R_n \ge n^{p_k+\epsilon}, \\ \mu_{\mathrm{p},k} & R_n = n^{p_k}, \\ \infty & R_n \le n^{p_k-\epsilon}, \end{cases}$$

where: $p_k \triangleq \dfrac{1}{\alpha - d/(k+2)} \quad \Rightarrow \quad p_{d-1} < \cdots < p_1 < p_0$



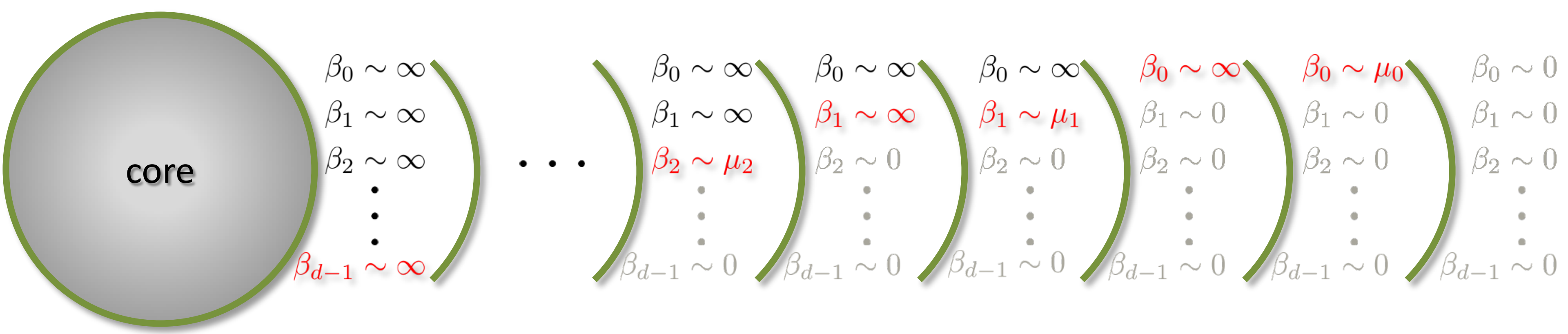
How Exponential Noise Crackles

Theorem

If $f(x) = ce^{-\|x\|}$, then for $0 \le k \le d-1$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{E}\{\beta_{k,n}\} = \begin{cases} 0 & R_n \ge \log n + (\delta_k + \epsilon)\log\log n, \\ \mu_{\mathrm{p},k} & R_n = \log n + \delta_k \log\log n, \\ \infty & R_n \le \log n + (\delta_k - \epsilon)\log\log n, \end{cases}$$

where: $\delta_k \triangleq \dfrac{d-1}{k+2} \quad \Rightarrow \quad \delta_{d-1} < \cdots < \delta_1 < \delta_0$



Gaussian Noise Does Not Crackle

Theorem

If $f = ce^{-\|x\|^2/2}$, for $\epsilon > 0$ set

$$R_n = R_n^\epsilon \triangleq \sqrt{2\log n + (d-2+\epsilon)\log\log n},$$

then for $0 \le k \le d-1$

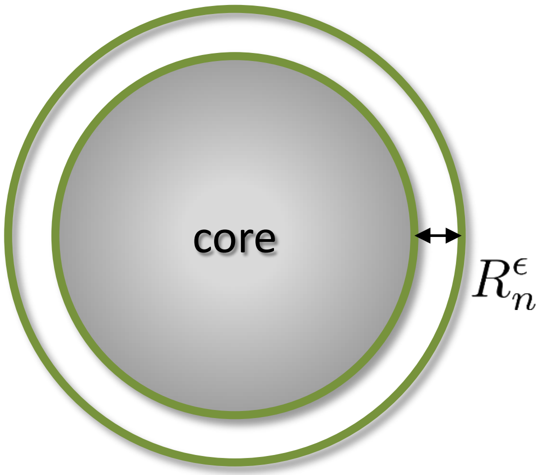
$$\lim_{n\to\infty} \mathbb{E}\{\beta_{k,n}\} = 0.$$

Recall:

$$R_n^{\mathrm{c}} = \sqrt{2\left(\log n - \log\log\log n - \delta_{\mathrm{g}} - \epsilon\right)}$$

So:

$$(R_n^\epsilon - R_n^{\mathrm{c}}) \to 0$$



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- Shmuel Weinberger
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THANK YOU

In collaboration with

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