K3 surfaces with maximal finite automorphism groups

(joint with C. Bonnafé)

1) Introduction

* In the 80's Nikulin started fundational work on auto of K3 surfaces
* Important Tool is lattice theory
* Nikulin classified: finite abelian groups acting symplectically on a K3 surface

There are 16 such groups where

are $\frac{\mathbb{Z}}{m\mathbb{Z}}$, $2 \le m \le 8$, $\left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^2$, $m = 3, 4$

$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}$; $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{6\mathbb{Z}}$, $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^t$, $t = 2, 3, 4$

* Work with A. Garbagnati (2008) (Morrison)
  compute the invariant lattices for the action of these groups on cohomology.

Def $g \curvearrowright X$, X K3 surface.

* g <u>acts symplectically</u>: $g^*\omega_X = \omega_X$
  $(H^{2,0}(X) = \mathbb{C}\cdot\omega_X$ holo. 2-form$)$

* otherwise we say that f
  acts non-sympl.

Mukai (1988) Gary (1996): classification
of all the groups finite acting
symplectically on K3 surfaces.
There are 11 maximal groups $\subset M_{23}$
It turns out that M_{20} is
the biggest one, $|M_{20}| = 960$
+ Several examples of K3 surfaces.

Aim of the talk:

Study finite groups $G \curvearrowright X$ K3 finite, $G \supsetneq M_{20}$, M_{20} acting symplectically. How big can be G? Can we describe the K3 surfaces?

Main theorem (Kondo (1988); Bonnafé–S. 2020; Brandhorst–Hashimoto 2020)

X K3, $G \curvearrowright X$ finite, $G \supsetneq M_{20}$ acting symplectically on X

Then:

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Main theorem (Kondo (1988); Bonnafé–S. 2020; Brandhorst–Hashimoto 2020)

X K3, $G \curvearrowright X$ finite, $G \supsetneq M_{20}$ acting symplectically on X

Then:

(1) $|G| \le 3840$

(2) If $|G| = 3840 \Rightarrow X = \mathrm{Km}(E_i \times E_i)$

E_i: $y^2 = x^3 + x$ ell. curve.

the K3 is unique and $G = G_{Ko}$ is unique too.

Kondo 1988

(3) $|G| < 3840 \Rightarrow |G| = 1920$ and there are exactly 2 cases (X_i, G_i) $i = 1, 2$.

Moreover the X_i are Kummer.

BS + BH

Remark

1. G any finite group, $G \curvearrowright X$ K3 $\Rightarrow |G| \le 3840$
2. " " $\Rightarrow$ the biggest groups are G_{Ko}, G_1, G_2

Results on Shioda question

(2008) K. Frantzen: who classifies extensions $M_{20} \times \mu_2$

(2020) BH: classifying all extensions of the 11 max Mukai groups

§ Facts on finite groups

Useful exact sequence, $G \curvearrowright X\ K3$, G finite.

$$1 \to G_0 \to G \xrightarrow{\alpha} \mathbb{C}^*$$
$$g \longmapsto \alpha(g)$$

$\alpha(g)$ def by $g^* \omega_X = \underline{\alpha(g)} \omega_X$

$G_0 = \{ g \in G \,/\, g^* \omega_X = \omega_X \}$

$\alpha(g) \in \mathbb{C}^*$

Thus precisely $Im(\alpha) < \mathbb{C}^*$

$\Rightarrow Im(\alpha) = \mu_m$ = cyclic groups of some m. root of unity

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \mu_m \longrightarrow 1$$

FACT $T_X = Pic(X)^\perp \cap H^2(X,\mathbb{Z})$ transv. lattice

$T_X \subset H^2(X,\mathbb{Z})^{G_0}$ = inv. lattice.

take $v \in T_X$ and consider

$\langle v, \omega_X \rangle = \langle g^* v, g^* \omega_X \rangle \quad g \in G_0$

$v - g^* v \in T_X \quad = \langle g^* v, \omega_X \rangle \Rightarrow v - g^* v \in Pic(X)$

sia $T_X \cap \mathrm{Pic}(X) = \{0\}$

$\Rightarrow v = g^* v \quad \forall v \in T_X$

$\Rightarrow T_X \subset H^2(X,\mathbb{Z})^{G_0}$

$\mathrm{Pic}(X) \supset (H^2(X,\mathbb{Z})^{G_0})^{\perp}$ 1

Sia $G_0 = M_{20}$.

<u>KONDO</u>

$$H^2(X,\mathbb{Z})^{M_{20}} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix} =: \mathbb{L}_{20}$$

$\Rightarrow \mathrm{rk}\,(H^2(X,\mathbb{Z})^{M_{20}})^{\perp} = 19 \Rightarrow \mathrm{rk}\,\mathrm{Pic}(X) = 20$
(X proj):

So $T_X = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ + conditions on a, b, c

$T_X \subset \Lambda_{24} = L'(2)$

and $T_X = \begin{pmatrix} 4a' & 2b' \\ 2b' & 4c' \end{pmatrix}$

$\Rightarrow$ X is Kummer.

Proposition G finite $G \curvearrowright X$ K3 $\Rightarrow |G| \leq 3840$

Idea of proof:

① first assume that $G_0 = M_{20}$.

$\mathrm{rk}\, T_X = 2$.

$$1 \to M_{20} \to G \to \mu_m \to 1$$

$\varphi(m) \mid \mathrm{rk}\, T_X = 2 \quad m \in \{1, 2, \cancel{3}, 4, \cancel{6}\}$.

$\parallel$

Euler totient function

G acts on $L_{20} = H^2(X, \mathbb{Z})^{G_0}$

G / M_{20}

$$|O(L_{20})| = 16$$

In any case $G \supset M_{20} \Rightarrow |G| \leq 3840$

② $G_0 \neq M_{20}$ one gets smaller orders.
(Xiao's list) ☒

We are left to study the two cases

$$1 \to M_{20} \to G \to \mu_4 \to 1 \quad \text{KONDO}$$

$$1 \to M_{20} \to G \circlearrowleft \tau \to \mu_2 \to 1 \quad \text{BS+BH}$$

§ Ingredients in the proof of main theorem (part 3)

Thm $M_{20} \circlearrowright X$ sympl.,
assume X admits $i \circlearrowright X$ s.t. $i^2 \in M_{20}$.
$(G = \langle M_{20}, i \rangle)$. Then we have 3 possibilities:

$\mathrm{Pic}(X)^G$	T_X	
$\langle 4 0 \rangle$	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	Kondo
$\langle 8 \rangle$	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$	
$\langle 4 \rangle$	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	

Proof We have an action of ι on $L_{20} = H^2(X,\mathbb{Z})^{G_0}$.

One can take $\underset{\mathrm{rk}\,2}{T_X} \oplus \underset{\langle 4m\rangle}{H^2(X,\mathbb{Z})^G} \xhookrightarrow[\text{index } 2]{} L_{20}$

$$T_X = \begin{pmatrix} 4a' & 2b' \\ 2b' & 4c' \end{pmatrix}.$$

$$4 = \left[\mathbb{Z}_{20} : T_X \oplus \langle 4m \rangle\right]^2 = \frac{16m(4a'c' - b'^2)}{160}.$$

one can show $b' = 2b''$ even.

so we get: $\quad m(a'c' - b''^2) = 10$

$m \in \{1, 2, 5, 10\}$

$4m \in \{4, 8, \cancel{20}, 40\}$

Existence of the K3 surfaces

* the case with $\langle 4\,0\rangle = ZL$, $T_X = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, described by Kondo

$$X = \mathrm{Km}(E_i \times E_i)$$

* the case with $ZL = \langle 4 \rangle$, $T_X = \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$

$$X = X_{Mu} = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 - 6(x_0^2 x_1^2 + \cdots + x_2^2 x_3^2) = 0\} \subset \mathbb{P}_3(\mathbb{C})$$

$$i : (x_0 : x_1 : x_2 : x_3) \longmapsto (x_0 : x_1 : x_2 : -x_3)$$

$|G_{Mu}| = 1920$, $G_{Mu} = PG_{29}$, G_{29} is a complex reflection groups (Shephard–Todd 1954)

* the cox with $\mathbb{Z}/\langle 8\rangle$, $T_X = \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$.

$$X = X_{BH} = \begin{cases} x_1^2 + x_4^2 - \phi x_5^2 + \phi x_6^2 = 0 \\ x_2^2 - \phi x_4^2 + x_5^2 - \phi x_6^2 = 0 \\ x_3^2 + \phi x_4^2 - \phi x_5^2 + x_6^2 = 0 \end{cases}$$

$\mathbb{P}^5$

$x_1 : \dots : x_6$

it answers a question of Brandhorst and Hashimoto.

$\phi = \frac{1+\sqrt{5}}{2}$ golden ratio.

Remark

- X_{KO}, X_{Mu}, X_{BH} are not iso
- G_{Mu}, G_{BH} are not iso.
- G_{Mu}, G_{BH} are not iso to subgroups of G_{KO}

* Singular projective model of Kondo:

Take Fermat quartic

$$F_4 : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

$$T_{F_4} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

Quotient by $j : (x_0 : x_1 : x_2 : x_3) \longrightarrow (x_0 : x_1 : -x_2 : -x_3)$

$$\begin{tikzcd}
\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \arrow[d] & F_4 \arrow[d] \\
X \arrow[r, "\text{min res.}"] & F_4/j
\end{tikzcd}$$

Ep B $\qquad F_4$ in some wps

$\mathbb{P}(2,1,2,2,2)$ $\qquad z_0 = x_0,\ z_1 = x_1,\ z_2 = x_2^2$

$z_3 = x_3^2,\ z_4 = x_2 x_3$

$$\begin{cases} z_0^4 + z_1^4 + z_2^2 + z_3^2 = 0 \\ z_4^2 = z_1 z_1 \end{cases}$$

4 A_1 sing. $z_0 = z_1 = 0$

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