The main object of my talk are Kummer surfaces, that are a beautiful example of K3 surfaces.

Recall the Enriques – Castelnuovo – classification [1910] of projective surfaces:

One uses the Kodaira dimension:

S proj surface, K_S = canonical divisor, recall that Ω^2_S = v.b. of holo 2-forms

one can associate K_S s.t. $\Omega^2_S = \mathcal{O}_S(K_S)$

$+ (\Omega^2_S)^{\otimes n} = \mathcal{O}_S(nK_S)$

$|K_S| = \{$ linear system of effective divisors lin. equiv. to $K_S \} = \mathbb{P}(H^0(\mathcal{O}_S(K_S)))$

$$\varphi_{|nK_S|} : S \dashrightarrow (\mathbb{P}^N)^\vee = \mathbb{P}(H^0(\mathcal{O}_S(nK_S)))^\vee$$
$$x \longmapsto (s_0(x) : \ldots : s_N(x))$$

with $s_0, \ldots, s_N$ basis of $H^0(\mathcal{O}_S(nK_S))$

Def.

$$\max_n \dim \overline{\mathrm{Im}\,\varphi_{|nK_S|}(S)} =: \kappa(S) \text{ Kodaira dimension of } S.$$

if $|nK_S| = \emptyset \ \forall n$

$\bullet\ \kappa(S) \in \{-\infty, 0, 1, 2\}$

$\boxed{\kappa(S) = -\infty}$: Ruled surfaces i.e. $\exists$ birational map $X \dashrightarrow C \times \mathbb{P}^1$, C smooth curve

(in part. if $C \cong \mathbb{P}^1 \Rightarrow X$ birat. $\mathbb{P}^1 \times \mathbb{P}^1$ and X is $\boxed{\text{rational}}$ i.e. birational to $\mathbb{P}^2$.

$\boxed{\kappa(S) = 0}$ S is birat. to a pt and the case

Abelian: $\mathbb{C}^2 / \Lambda$, Λ = rank 4 lattice

K3 surfaces

Enriques surfaces = quot of K3 by fixed pt free invol.

Bielliptic = quotients of $E \times E'$, E, E' elliptic; cell classif. by Bagnera–De F. 1907

Most easy example of K3 surface:

$$\mathbb{P}^3(\mathbb{C}) \supset \underset{\overset{\shortparallel}{X_4}}{\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}} \quad (\forall \overset{\text{any}}{\text{smooth}} \text{ quartic surface in } \mathbb{P}_3)$$

Using adjunction formula one can show: H = hyperplane section in $\mathbb{P}^3$.

$$\underline{\underline{K_{X_4}}} = (K_{\mathbb{P}_3} + X_4)_{|X_4} = (-4H + 4H)_{|X_4} = \underline{\underline{0}}$$

Moreover one can show by using exact sequence of sheaves:

$$0 \to \mathcal{O}_{\mathbb{P}_3}(-4H) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{X_4} \to 0$$

that $\underline{\underline{h^1(\mathcal{O}_{X_4}) = 0}}$.

These properties characterise K3 surfaces:

<u>Definition (K3 surface)</u> A K3 surface is a compact complex smooth surface S s.t. $\underset{\text{Trivial canonical class}}{K_S \sim 0}$ and $q(S) = 0$

$$\overset{\shortparallel}{h^1(\mathcal{O}_S)}$$

<u>Examples</u>:
- smooth quartic surfaces in $\mathbb{P}_3$
- c.i., double plane, ... $(4) \subset \mathbb{P}^3$, $(2,4) \subset \mathbb{P}^4$, $(2,2,2) \subset \mathbb{P}^5$, $X \xrightarrow{2:1} \underset{\underset{C_6}{\cup}}{\mathbb{P}^2}$
- <u>Kummer surfaces</u>

<u>Kummer surfaces</u> let T be ~~an abelian surface~~ cplx Torus (i.e. not necessarily projective) $T = \frac{\mathbb{C}^2}{\Lambda}$, $\underset{\text{a rank 4 lattice}}{\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4}$, $e_i \in \mathbb{C}^2$

Consider the involution $i: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $(x,y) \longmapsto (-x,-y)$ induce $i \circlearrowright A$ which has 16 fixed pts

i.e. the pts with $2(x,y) \in \Lambda$, called also 2-torsion pts. We have a commutative diagram:

$E_1, \dots, E_{16}$ excep. div.

$p_1, \dots, p_{16}$ the 16 fixed pts

$$\begin{tikzcd}
\tilde{\iota} \circlearrowright \tilde{A} \arrow[r, "\beta", "\text{blow-up}"'] \arrow[d, "\tilde{\pi}"', "2:1"] \arrow[dr, phantom, "\circlearrowright"] & A \circlearrowleft \iota \arrow[d, "\pi", "2:1"'] \\
\dfrac{\tilde{A}}{\langle \tilde{\iota} \rangle} = \tilde{Z} \arrow[r, "\tilde{\beta}", "\text{min. res}"'] & \dfrac{A}{\langle \iota \rangle} =: Z \ni
\end{tikzcd}$$

16 A_1 singularities at $\pi(p_i) = f_i$, $i = 1, \dots, 16$.

$D_1, \dots, D_{16}$ (-2)-curves.

Rem: 1) β is the blow-up at the 16 fixed-pts
$\beta^{-1}(p_i) = E_i \quad i = 1, \dots, 16$

2) the action of ι on A induces an action $\tilde{\iota} \circlearrowright \tilde{A}$ that fixes pointwise E_i (direct comp. using blow-up)

3) $\tilde{Z} = \dfrac{\tilde{A}}{\langle \tilde{\iota} \rangle}$ is smooth ($\tilde{\iota}$ is a reflection in E_i $\Rightarrow$ quotient is smooth, thm of Chevalley)

4) $\tilde{\beta}$ is the minimal res. of sing. on $\dfrac{A}{\langle \iota \rangle}$, $\tilde{\beta}^{-1}(f_i) = D_i$
$D_i^2 = -2$, $D_i \cong \mathbb{P}_1$

5) $\tilde{\pi}$ is the double cover of $\tilde{Z}$ which ramifies on $D_1 + \dots + D_{16}$
$\Rightarrow$ we have a cyclic cov. $\left(\dfrac{D_1 + \dots + D_{16}}{2} \in NS(\tilde{Z}) \text{ \& } \tilde{\pi}^*(D_i) = 2E_i \right)$

By using $\tilde{\pi}$: $-2 = \overbrace{2E_i \cdot E_i}^{-1} = \tilde{\pi}^*(D_i) \cdot E_i \underset{\substack{\uparrow \\ \text{proj. formula}}}{=} D_i \cdot \tilde{\pi}_*(E_i) = D_i^2$

Definition: $\tilde{Z}$ is called Kummer surface, we have $\tilde{Z} := \mathrm{Km}(A)$

By using pull back & push-down of 2 form on $\dfrac{\mathbb{C}^2}{\Lambda}$, that can be written as $dx \wedge dy$ with (x, y) coord on $\mathbb{C}^2$ one shows $K_{\tilde{Z}} \sim 0$

We show $q(\tilde{Z}) = 0$.

Noether formula:

$$\underset{\shortparallel}{\chi(\mathcal{O}_{\tilde{Z}})} = \frac{1}{12}\left(\overset{\overset{0}{\shortparallel}}{K_{\tilde{Z}}}^2 + \chi_{top}(\tilde{Z})\right) = \frac{\chi_{top}(\tilde{Z})}{12}$$

$$h^0(\mathcal{O}_{\tilde{Z}}) - h^1(\mathcal{O}_{\tilde{Z}}) + \underset{\underset{h^0(\Omega^2_{\tilde{Z}})}{\shortparallel}}{h^2(\mathcal{O}_{\tilde{Z}})} = 2 - h^1(\mathcal{O}_{\tilde{Z}}) \qquad (*)$$

Serre duality

τ gen by holo 2-form nowhere 0.

We compare $\chi_{top}(\tilde{Z})$:

$$\chi_{Top}(\tilde{A}) = 2\left(\chi_{top}(\tilde{Z}) - 16\,\chi_{top}(\mathbb{P}^1)\right) + 16\,\chi_{top}(\mathbb{P}_1)$$

$$= 2\,\chi_{top}(\tilde{Z}) - 16\cdot 2 = 2\,\chi_{Top}(\tilde{Z}) - 32$$

On the other hand $\overset{1}{b_0} - \overset{4}{b_1} + \overset{6}{b_2} - \overset{4}{b_3} + \overset{1}{b_4} = 0$

$$\chi_{Top}(\tilde{A}) = \left(\overset{\shortparallel}{\chi_{Top}}(A) - 16\,\chi_{Top}(pt)\right) + 16\,\chi_{Top}(\mathbb{P}^1)$$

$$= 0 - 16 + 2\cdot 16 = 16$$

$$\Rightarrow \quad \chi_{Top}(\tilde{Z}) = \frac{\overset{16}{\chi_{Top}(\tilde{A})} + 32}{2} = \frac{48}{2} = 24$$

Hence in $(*)$

$$2 - h^1(\mathcal{O}_{\tilde{Z}}) = 2 \Rightarrow \boxed{h^1(\mathcal{O}_{\tilde{Z}}) = 0.}$$

In part $\rho(NS(\mathrm{Km}(A))) \geq 16 + 1 = 17$ $\quad NS(\mathrm{Km}(A))$ is $\boxed{\text{Néron-Severi group.}}$

exponential sequence of analytic sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_S \xrightarrow{e^{2\pi i f}} \mathcal{O}_S^* \to 0.$$

0 (in K3)

$$\cdots \dashrightarrow \overset{\overset{0}{\shortparallel}}{H^1(\mathcal{O}_S)} \to H^1(\mathcal{O}_S^*) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \to \overset{\mathbb{Z}}{H^2(\mathcal{O}_S)} \to \cdots$$

$$L \longmapsto c_1(L)$$

$\mathrm{Im}(c_1) = NS(S)$ $\quad$ i.e. $NS(S) = \underset{\mathrm{Pic}^0(S)}{\underline{\mathrm{Pic}(S)}}$ $\quad \Rightarrow NS(S) = \mathrm{Pic}(S)$

$\mathrm{rk}\, NS(S) = \mathrm{rk}\,\mathrm{Pic}(S) =: \rho(S)$ Picard number of

"Nikulin configurations on Kummer surfaces" (joint work with X. Roulleau) (1)

§1 Introduction

Main object of the talk are Kummer surfaces.

Recall Take A = abelian surface (or tori), $i \circlearrowright A$ by $a \mapsto -a$. It has 16 fixed pts on A & we have a commutative diagram.

$$\begin{tikzcd} \tilde{A} \arrow[r, "\text{blow-up these 16 pts}"] \arrow[d, "2:1", "\tilde{\pi}"'] & A \quad p_1, \dots, p_{16} \arrow[d, "2:1", "\pi"'] \\ X \arrow[r, "\text{min res.}"'] & A/\langle i \rangle \end{tikzcd}$$

double cover ramified on 16 (-2)-curves.

16 A_1 singularities.

Def ① $X := Km(A)$ is a Kummer surface, it contains 16 disjoint smooth rational curves ($\rho(S)$ [illegible]).

Rmk $X = Km(A)$ is a K3 surface (cpt. cplx. $K_X \sim 0$ & $h^0(\Omega^1_S) = 0$)

Nikulin If X is a K3 surface containing 16 disjoint (-2)-curves $\Rightarrow X = Km(A)$ for some abelian surface A.

Def ②

1) We call a Kummer structure on X an abelian surface A s.t. $X = Km(A)$

2) We call Nikulin configuration a set of 16 disjoint (-2)-curves on a K3 surface

By the result of Nikulin we have a bijection

$$\{\text{Kummer str.}\}/\text{iso} \xleftrightarrow{1:1} \{\text{Nikulin configurations}\}/\text{iso}.$$

Question of Shioda 1977 Is it possible to have several non iso Kummer structures on a Kummer surface? i.e. $\exists A, B$ ab. sfces s.t. $A \not\cong B$ but $Km(A) \cong Km(B)$?

(2)

$\boxed{\text{Shioda - Mitani}}$: No if $\rho(Km(A)) = 16$ $NS(Km(A)) \cong \Lambda_0$
(here $\rho(A) = 4$).

+ if A p.p.a.s. (i.e. $A = J(C)$, $g(C) = 2$)
with $\rho(A) = 1$ i.e. $\rho(Km(A)) = 17$ (see later...)

$\boxed{\text{Gritsenko - Hulek 1998}}$: Yes it is possible, take A a generic (i.e. $\rho(A)=1$) $(1,t)$-polarized abelian surface, $t > 1$ and $A^* = $ dual abelian surface ($\mathrm{Pic}(A)$)
then $A \not\cong A^*$ (are only isogenous and isom. for $t = 1$)
but $Km(A) \cong Km(A^*)$ (here $\rho(Km(A)) = 17$)

$\boxed{\text{Study of certain moduli spaces of ab. surfaces that are } (1,t)\text{-pol.}}$

$\boxed{\text{Hosono - Lian - Oguiso - Yau 2003}}$: $\#\dfrac{\text{Kummer str.}}{\text{iso}}$ is finite and $\forall N \in \mathbb{N}^*$
see

they construct a Kummer surface of Picard rank 18 with at least N Kummer structures (uses lattice theory, but not explicit example).

$\boxed{\text{Orlov 2003}}$ If $rk(NS(Km(A))) = 17 \Rightarrow \#\dfrac{\text{Kummer str.}}{\text{iso}}$ is 2^ν
where $\nu = \#\{$prime divisors of $\frac{1}{2}M^2\}$ where M is a polarisation (i.e. an (ample) divisor) on A.
big + nef

<u>Ex</u> if $A = J(C) = $ Jacobian of a genus-2 curve $\Rightarrow M^2 = 2$ ($NS(A) = \mathbb{Z}M$)
so that $\nu = 0 \Rightarrow 2^\nu = 1 \Rightarrow$ only 1 $\dfrac{\text{Kummer str.}}{\text{iso}}$

<u>Remarks</u>
(1) If $rk(NS(Km(A))) = 17 \Rightarrow NS(Km(A)) \supset \mathbb{Z}L \oplus K$, $L^2 = 2M^2$
finite index (in fact $= 2$)

$K = $ Kummer lattice $=$ smallest primitive sublattice of $H^2(Km(A), \mathbb{Z})$ that contains the 16 curves (-2), $\det K = 2^6$
$rk\, K = 16$ isomorphic to $U^3 \oplus E_8^{\oplus 1}$ ($\forall$ K3 surface) $U(?)$; ...
rep of

One can show that Π^2 does not depend on the choice of the 16 curves (-2), recall that $\dfrac{\text{Kummer str.}}{\text{iso}} \xleftrightarrow{1:1} \dfrac{\text{Nik. config.}}{\text{iso.}}$ (3)

(2) One can ask: A, B ab. spaces s.t. $Km(A) \cong Km(B) \Rightarrow A$ & B isogenous i.e. $A \xrightarrow{\pi} B$ π surjective with finite kernel (i.e. hom.-morph.)? $\underline{\underline{\text{Yes.}}}$ (Gaulleau - S. easy conseq. of a result of P. Stell...)

$\underline{\underline{\text{Ex.}}}$ if A is $(1,t)$-pol. $\Rightarrow \exists\, A \xrightarrow{\pi} A^*$ isogeny $Ker(\pi) = \dfrac{\mathbb{Z}}{t\mathbb{Z}}$

$\boxed{\text{Aim of the talk}}$ Construct explicitly (i.e. with geometry) non-isomorphic Kummer structures on $Km(A)$ in case that $\rho(Km(A)) = 17$ (We give the set of disjoint (-2)-curves by using lattice theory).

§2 A basic example. Recall first the following $\underline{\text{classical construction}}$

Let C be a genus 2 curve $J(C) = \dfrac{H^0(\Omega^1_C)^{\vee}}{H_1(C)}$ (2 dim / 4-dim) the Jacobian

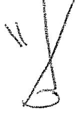
suppose it is a ppas with polarisation (i.e. bip + upf div.) Π, so that on $Km(J(C))$ we have a pol. L, with $L^2 = 4$ and $\Pi^2 = 2$

$$NS(Km(J(C))) \supset \mathbb{Z}L \oplus K$$

(index 2; K Kummer lattice), let $X := Km(J(C))$

We have a map: $\varphi_L : X \longrightarrow \mathbb{P}^3$ birat. onto image

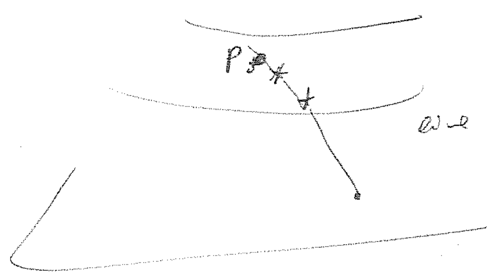
$\underline{\text{Thm}}$ $\varphi_L(X)$ contains 16 nodes locally $\cong$



Projecting from one node p one gets:

$$X \overset{2:1}{- - - \to} \mathbb{P}^2$$

$\downarrow$ double cover of $\mathbb{P}^2$



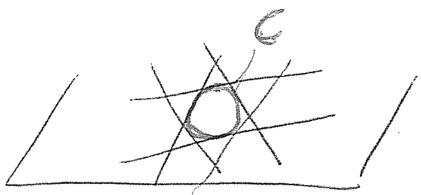
one through p cuts in 2 pts more.

Let $Y = Bl_p(X)$.

And A_1 be the excep. curve that goes to p.

Then we extend the cover to:

$$\begin{array}{ccc} & i \circlearrowright & \\ \varphi: & Y & \xrightarrow[|L-A_1|]{2:1} \mathbb{P}^2 \\ A_1, A_1' & \cup & \\ & A_1 & \end{array}$$



It is well known that Y is the double cover of $\mathbb{P}^2$ ramified on 6 lines tangent to a common conic C & Y is singular at the $\binom{6}{2}=15$ int pts of the 6 lines.

· $\varphi^{-1}(C) = A_1 \cup A_1'$, the conic splits completely & $A_1 \cap A_1' = 6$ pts coming from the 6 tang pts of C with the 6 lines.

· φ is given in fact by the linear system $|L - A_1|$

· With φ we have a covering involution i exchanging 2 by 2 the pts in the fiber (generally) and $Y/i = \mathbb{P}^2$, and i exchanges A_1 & A_1'

· If $\widetilde{X} = Bl(Y)$ in the other 15 nodes $\Rightarrow$ $\widetilde{X}$ contains 2 Nikulin configurations:

$$\begin{array}{l} A_1, \boxed{A_2, \dots, A_{16}} \\ A_1', \boxed{A_2, \dots, A_{16}} \end{array}$$

same ~~excep.~~ curves, are the excep. curves coming from the nodes.

Properties: $i \in \mathrm{Aut}(Y)$ and exchanges $A_1 \leftrightarrow A_1'$, fixes $A_2, .., A_{16}$

So that the two Nikulin config. $A_1, .., A_{16}$ & $A_1', A_2, .., A_{16}$ are equivalent. In fact in this case $\frac{1}{2} n^2 = 1 \Rightarrow r = 0 \Rightarrow 2^0 = 1 \xRightarrow{\text{order}} 1$ $\frac{\text{Kum str.}}{\text{iso.}}$

Question: ① Can one generalise this construction? Yes

② If yes, does there exist an involution as i? No

§3 Main results

Let (A,M) be ~~a~~ ~~...~~ pol. ab. surface s.t.

$$M^2 = k(k+1),\ k \text{ integer s.t. } \boxed{k \geq 1}$$

If A is general projective $\Rightarrow NS(A) = \mathbb{Z}M \quad (\rho(A)=1)$

Let $X = \mathrm{Km}(A)$ and L be the big + nef divisor on X coming from M

i.e. $L^2 = 2M^2 = 2k(k+1)$ and let

$$\begin{tikzcd} & A \arrow[d] & M \\ \mathrm{Km}(A) \arrow[r] \arrow[ur, "2:1"] & A/\iota & \\ L & & \end{tikzcd}$$

$C = A_1 + \dots + A_{16}$ the sum of the 16 (-2)-curves on X.

then $L \in \langle A_1, \dots, A_{16}\rangle^{\perp}$ & $NS(X) \supset \mathbb{Z}L \oplus K$, $\rho(X) = 17$.

<u>theorem 1</u> (RS) Let $j \in \{1, \dots, 16\}$ then $\exists$ a (-2)-curve $\boxed{k \geq 1}$

A_j' on $\mathrm{Km}(A)$ s.t. $A_j \cdot A_j' = 4k+2$

and $C_j = A_j' + \sum_{i \neq j} A_i$ is a <u>Nikulin configuration</u>

where $A_j' = 2L - (2k+1)A_j \in NS(\mathrm{Km}(A))$ &

$\exists \quad L_j' = (2k+1)L - 2k(k+1)A_j$ has $(L_j')^2 = L^2$ is big + nef also

and L_j' generates the orthogonal complement of

$$\{A_j', A_i\ i \neq j\}_{i=1,\dots,16} \text{ in } NS(\mathrm{Km}(A))$$

Rmk For $k=1 \Rightarrow$ the gen case of $A = J(C)$, $\rho(C) = 2$ where $A_1 \cdot A_1' = 6$.

But for $k > 1$ we have:

Theorem 2 I. Let $k>1 \Rightarrow \exists$ auto of X s.t.

(RS)

$$e = \sum_{i\geq 1}^{16} A_i \longrightarrow e_j = A_j' + \sum_{i\neq j} A_i$$

So we get two non iso Nikulin configuration.

Remark (2) If $k=2 \Rightarrow \Pi^2 = 2\cdot 3 = 6 \Rightarrow \gamma = \#$ pro... of $\frac{1}{2}\Pi^2 = 3 = 2 \Rightarrow 2$ Nik. conf

$\Rightarrow$ our result is optimal! If $e \leftrightarrow A$ then $\underline{e_1 \leftrightarrow A^+}$, $A \not\cong A^+$ but iso $Km(A) \cong Km(A^+)$

(1) One can fix $\boxed{j\geq 1}$ in fact $G = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4 \subset Km(A)$ (Symplectically) — " — stop

and $f \in G$ is induced by transl. by a pt of order 2 on $A \Rightarrow$ the curves

$A_1, .., A_{16}$ are permuted.

3) $\exists$ similar involutions i of case $k=1$, but it is only isometry of $NS(X)$ for $k>1$

and it does not extend to an invol. on $H^2(X, \mathbb{Z})$.

One has several of such isometries, composing 2 of them one gets an

∞-order auto of X.

(4) Work in progress w. X. Roulleau: ext. the theorem to all pol. $\Pi^2 = 2t$

if possible ...

An interesting map: fix $j=1$ so we have the 2-conf.

$\underline{k>1}$ $\boxed{A_1, .., A_{16}}$ $\Rightarrow$ $A_1 A_1' = 4k+2$ on $\boxed{A_1', A_2, .., A_{16}}$

We have a map: birat. onto img. $\varphi : X \xrightarrow{|L-kA_1|} Y \subset \mathbb{P}^{k+1}$

15 nodes at $\varphi(A_2), .., \varphi(A_{16})$

and: $(L-kA_1)\cdot A_1 = 2k = (L-kA_1)\cdot A_1'$ images are rational curves of degree $2k$.

& $A_1 \cdot A_1' = 4k+2$ (with mult.)

One computes also $A_1 + A_1' = 2(L-kA_1) \Rightarrow A_1, A_1'$ are cut by a quadric.

(for $k=1$ one gets $X \xrightarrow{2:1} \mathbb{P}^2$) (for $k=2$ $\varphi : X \longrightarrow Y \subset \mathbb{P}^3$ quartic with 15 nodes