FANO PLANE'S EMBEDDINGS ON COMPACT ORIENTABLE SURFACES

DOMENICO A. CATALANO AND CRISTINA SARTI

ABSTRACT. In this paper we study embeddings of the Fano plane as a bipartite graph. We classify all possible embeddings especially focusing on those with non-trivial automorphism group. We study them in terms of rotation systems, isomorphism classes and chirality. We construct quotients and show how to obtain information about face structure and genus of the covering embedding. As a by-product of the classification we determine the genus polynomial of the Fano plane.

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1. INTRODUCTION

Incidence structures, regarded as bipartite graphs, give rise to dessins d'enfants (oriented hypermaps) when they are embedded into orientable compact surfaces ([13],[12]). According to Belyĭ's Theorem any dessin d'enfant defines a complex structure on its underlying topological surface which becomes a Riemann surface with a model on the field of algebraic numbers $\overline{\mathbb{Q}}$ ([2]). This means that there exists a close relation between embeddings and Riemann surfaces (or algebraic curves).

Here we consider incidence structures describing finite projective spaces ([5], [10]). These spaces are of particular interest since their geometry is well-known and this is helpful to understand properties of the corresponding incidence structures and of their embeddings.

In this work we concentrate on the Fano plane as an example remarking that our results can be possibly generalised. We construct embeddings using rotation systems (see [14]) and analyze the action of Fano plane's automorphism group $PGL(3, 2)$ on them (Section 3 and 4). In this way, we achieve a classification of all the embeddings with information about isomorphism classes and automorphism groups. We determine under which conditions rotation systems generate embeddings with a non-trivial automorphism group. For these embeddings we construct quotients from which we derive combinatorial and topological properties of the covering embeddings (Section 5).

Finally, we give cell operations such that starting from particular embeddings with a non-trivial automorphism group one can construct all the others with the same automorphism group (Section 6).

We start with exhaustive preliminaries (Section 2) since, in this work, knowledge from different areas of mathematics is needed. Our aim is to provide readers having different backgrounds with the basic definitions and tools necessary to understand our approach. In this way, the paper becomes self-contained .

2. PRELIMINARIES

2.1. **Incidence structures.** An *incidence structure* is a triple of non-empty sets $\Gamma = (\mathfrak{p}, \mathfrak{B}, I)$ satisfying $\mathfrak{p} \cap \mathfrak{B} = \varnothing$ and $I \subseteq \mathfrak{p} \times \mathfrak{B}$. The elements of $\mathfrak{p}$ are called *points*, those of $\mathfrak{B}$ *blocks* and those of I *flags*.

A *homomorphism* from the incidence structure $\Gamma = (\mathfrak{p}, \mathfrak{B}, I)$ to the incidence structure $\overline{\Gamma} = (\overline{\mathfrak{p}}, \overline{\mathfrak{B}}, \overline{I})$ is a function $\phi : \mathfrak{p} \cup \mathfrak{B} \rightarrow \overline{\mathfrak{p}} \cup \overline{\mathfrak{B}}$ satisfying $\phi(\mathfrak{p}) \subseteq \overline{\mathfrak{p}}$, $\phi(\mathfrak{B}) \subseteq \overline{\mathfrak{B}}$ and $(\phi(p), \phi(b)) \in \overline{I}$ for any $(p, b) \in I$. A bijective homomorphism ϕ (from Γ to $\overline{\Gamma}$) for which ϕ^{-1} is a homomorphism (from $\overline{\Gamma}$ to Γ) is called an *isomorphism*. An isomorphism from Γ to Γ is called an *automorphism* of Γ. The set of automorphisms of an incidence structure $\Gamma = (\mathfrak{p}, \mathfrak{B}, I)$ with the composition of functions is obviously a (permutation) group (on $\mathfrak{p} \cup \mathfrak{B}$) called the *automorphism group* of Γ and denoted by $Aut(\Gamma)$.

An incidence structure $\Gamma = (\mathfrak{p}, \mathfrak{B}, I)$ is said to be *finite* if $\mathfrak{p} \cup \mathfrak{B}$ is a finite set. Writing pIb or bIp instead of $(p, b) \in I$, we say that Γ is *connected* if for any distinct $u, v \in \mathfrak{p} \cup \mathfrak{B}$ there is a (possible empty) sequence $u_1, ..., u_n$ in $\mathfrak{p} \cup \mathfrak{B}$ such that $uIu_1I \dots Iu_nIv$.

For our purposes, it is convenient to regard incidence structures $\Gamma = (\mathfrak{p}, \mathfrak{B}, I)$ as bipartite graphs with vertex set $V = \mathfrak{p} \cup \mathfrak{B}$ (partitioned by $\{\mathfrak{p}, \mathfrak{B}\}$) and edges corresponding to unordered pairs of elements of I. Therefore, in the following we denote an incidence structure by $\Gamma = (B, W, I)$ and call respectively *black vertices* and *white vertices* the elements of B and W (instead of points and blocks, see Fig. 1). Only finite and connected incidence structures are of our interest therefore this will be assumed from now on without mention.

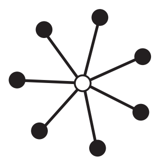


Figure 1: A block as a white vertex.

2.2. **Rotation systems.** A *rotation system* for the incidence structure $\Gamma = (B, W, I)$ is a function R assigning to any $v \in B \cup W$ a cyclic permutation $R_v = R(v)$ of the set $I(v) = \{u \in B \cup W : uIv\}$ of *neighbours of* v.

Any rotation system R for an incidence structure $\Gamma = (B, W, I)$ gives rise to a triangulation of an orientable compact and connected surface $\mathcal{S}$ in which Γ is embedded (as a bipartite graph). More precisely:

- Consider I as a set of disjoint plane rhombuses, where each rhombus $(b, w) \in I$ is the gluing of two plane triangles along an edge with vertices labeled b and w as in Fig. 2 (a).
- Fix now an orientation of the plane (say anti-clockwise as in Fig. 2) and
  - identify a side of (b, w) with a side of $(b, R_b(w))$ in such a way that $(b, R_b(w))$ is a positive rotation of (b, w) centered at vertex labeled b (see Fig. 2 (b)),
  - identify a side of (b, w) with a side of $(R_w(b), w)$ in such a way that $(R_w(b), w)$ is a positive rotation of (b, w) centered at vertex labeled w (see Fig. 2 (c)).

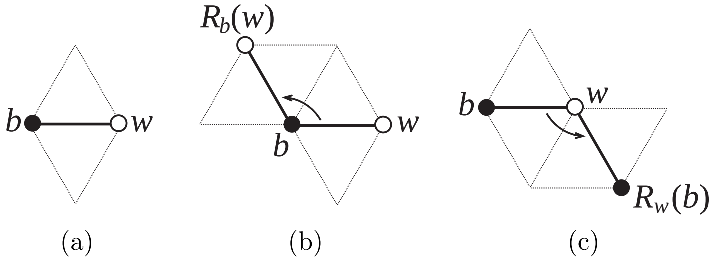


Figure 2: A rhombus and side identifications of two rhombuses.

After performing all possible side identifications one gets a triangulation of an orientable compact and connected surface $\mathcal{S}$ (with an orientation) in which Γ is embedded as a bipartite graph. Thus, we call the pair (Γ, R) an *embedding* of Γ by R into $\mathcal{S}$. Compactness of $\mathcal{S}$ is guaranteed by finiteness of Γ, while connectedness of $\mathcal{S}$ is a consequence of the connectedness of Γ. Orientability is a direct consequence of the construction and the orientation of $\mathcal{S}$ is given by the choice of an orientation of the plane. In particular, taking the mirror image of Fig. 2 is the same as taking R^{-1} as the rotation system for Γ, where $R^{-1}(v) = {R_v}^{-1}$. Therefore (Γ, R^{-1}) is called the *mirror image* of (Γ, R). By the given construction, according to Euler's formula, the genus g of $\mathcal{S}$ is

$$g = 1 - \frac{|B| + |W| + |F| - |I|}{2}, \tag{1}$$

where F is the set of unlabelled vertices of the triangulation called *faces* of (Γ, R). These are in one-to-one correspondence with the connected components of $\mathcal{S} \setminus \Gamma$, which are simply connected open sets of $\mathcal{S}$ by construction. Here $\mathcal{S} \setminus \Gamma$ denotes the topological space obtained by removing the embedded bipartite graph Γ from $\mathcal{S}$. We call $\mathcal{S}$ the *underlying surface* and g the *genus* of (Γ, R).

A homomorphism ϕ from the incidence structure $\Gamma = (B, W, I)$ with rotation system R to the incidence structure $\overline{\Gamma}$ with rotation system $\overline{R}$ is a *morphism* from (Γ, R) to $(\overline{\Gamma}, \overline{R})$ if it satisfies

$$\phi R_v = \overline{R}_{\phi(v)} \phi \qquad \text{for any } v \in B \cup W\,.$$

An isomorphism from Γ to $\overline{\Gamma}$ which is a morphism from (Γ, R) to $(\overline{\Gamma}, \overline{R})$ will be called an *isomorphism* from (Γ, R) to $(\overline{\Gamma}, \overline{R})$ as well as an automorphism of Γ which is a morphism from (Γ, R) to (Γ, R) will be called an *automorphism* of (Γ, R). The group of automorphisms of an embedding (Γ, R) will be denoted by $Aut(\Gamma, R)$. Remark that $Aut(\Gamma, R)$ is a subgroup of $Aut(\Gamma)$.

Let $\mathcal{R}(\Gamma)$ denote the set of all possible rotation systems for the incidence structure $\Gamma = (B, W, I)$. For $\phi \in Aut(\Gamma)$ we denote by R^ϕ the rotation system for Γ defined by

$$R^\phi_v = \phi R_{\phi^{-1}(v)} \phi^{-1}\,, \qquad v \in B \cup W\,. \tag{2}$$

Then we have an action of $Aut(\Gamma)$ on $\mathcal{R}(\Gamma)$

$$\mathcal{R}(\Gamma) \times Aut(\Gamma) \to \mathcal{R}(\Gamma),\ (R, \phi) \mapsto R^\phi$$

with the property that two embeddings (Γ, R_1) and (Γ, R_2) are isomorphic if and only if R_1 and R_2 are in the same orbit. Moreover, $Aut(\Gamma, R)$ is the stabilizer of $R \in \mathcal{R}(\Gamma)$ under this action. Hence

$$|Aut(\Gamma)| = |Aut(\Gamma, R)| \cdot |\{R^\phi : \phi \in Aut(\Gamma)\}| \tag{3}$$

for any $R \in \mathcal{R}(\Gamma)$, by the orbit-stabilizer theorem.

Proposition 2.2.1. *Let* $\phi \in Aut(\Gamma)$. *Then* $a^\phi = \phi a \phi^{-1} \in Aut(\Gamma, R^\phi)$ *for any* $a \in Aut(\Gamma, R)$ *and*

$$Aut(\Gamma, R) \to Aut(\Gamma, R^\phi),\ a \mapsto a^\phi$$

is a group isomorphism.

Proof. Let $\overline{R} = R^\phi$. Then

$$\begin{aligned}\overline{R}_v^{a^\phi} &= \phi a \phi^{-1} \overline{R}_{\phi a^{-1} \phi^{-1}(v)} \phi a^{-1} \phi^{-1} = \phi a R_{a^{-1}\phi^{-1}(v)} a^{-1} \phi^{-1} = \phi R^a_{\phi^{-1}(v)} \phi^{-1} \\ &= \phi R_{\phi^{-1}(v)} \phi^{-1} = R_v^\phi = \overline{R}_v\,,\end{aligned}$$

proving that $a^\phi \in Aut(\Gamma, R^\phi)$. It is now straightforward to conclude the statement. □

2.3. **Dessins d'enfants.** Another equivalent way of describing combinatorially embeddings of incidence structures is by means of dessins d'enfants.

A *dessin d'enfant* or (*oriented*) *hypermap* is a pair $\mathcal{D} = (\Omega, (x, y))$, where Ω is a finite non-empty set and x, y are permutations of Ω generating a transitive subgroup of the symmetric group on Ω, called the *monodromy group* of $\mathcal{D}$ and denoted by $Mon(\mathcal{D})$. Orbits of x are called *black vertices* (or *hypervertices*) of the dessin d'enfant $\mathcal{D}$, while orbits of y are called *white vertices* (or *hyperedges*) of $\mathcal{D}$. The orbits of xy are the *faces* (or *hyperfaces*) of $\mathcal{D}$. The *valency* of a black or of a white vertex, as well as the valency of a face, is given by its length as an orbit (of the action of x, y or xy, respectively).

Let $\mathcal{D} = (\Omega, (x, y))$ and $\overline{\mathcal{D}} = (\overline{\Omega}, (\overline{x}, \overline{y}))$ be dessins d'enfants. A *covering* (or *morphism*) from $\mathcal{D}$ to $\overline{\mathcal{D}}$ is a function $\phi : \Omega \to \overline{\Omega}$ satisfying

$$\overline{x}\,\phi = \phi\, x \qquad \text{and} \qquad \overline{y}\,\phi = \phi\, y\,.$$

By means of the transitivity of the monodromy groups of $\mathcal{D}$ and $\overline{\mathcal{D}}$, any covering from $\mathcal{D}$ to $\overline{\mathcal{D}}$ is uniquely determined by the image of an element and is onto. Therefore a one-to-one covering is called an *isomorphism.* An *automorphism* of $\mathcal{D}$ is then an isomorphism from $\mathcal{D}$ to $\mathcal{D}$. The set of automorphisms of $\mathcal{D} = (\Omega, (x, y))$ with the composition of functions is a group of permutations on Ω, called the *automorphism group* of $\mathcal{D}$ and denoted by $Aut(\mathcal{D})$. By definition, $Aut(\mathcal{D})$ is the centralizer of $Mon(\mathcal{D})$ in the symmetric group on Ω. Thus

$$|Aut(\mathcal{D})| \leq |\Omega| \leq |Mon(\mathcal{D})| \tag{4}$$

with $|Aut(\mathcal{D})| = |\Omega|$ if and only if $|\Omega| = |Mon(\mathcal{D})|$, in which case we say that $\mathcal{D}$ is a *regular* dessin d'enfant. The two inequalities in (4) are a consequence of the following

Proposition 2.3.1. *Let* $\mathcal{D} = (\Omega, (x, y))$ *be a dessin d'enfant. Then* $|Aut(\mathcal{D})|$ *divides* $|\Omega|$ *and* $|\Omega|$ *divides* $|Mon(\mathcal{D})|$.

Proof. The statement is a consequence of the transitivity of the action of $Mon(\mathcal{D})$ on Ω, which implies that $Aut(\mathcal{D})$, being the centralizer of $Mon(\mathcal{D})$ on the symmetric group on Ω, acts semiregularly on Ω ([6], Theorem 4.2A). Hence any orbit of $Aut(\mathcal{D})$ on Ω has $|Aut(\mathcal{D})|$ elements and therefore $|Aut(\mathcal{D})|$ divides $|\Omega|$. Stabilizers of transitive actions have all the same size, since they are conjugated and, as all elements are in the same orbit, the orbit-stabilizer theorem gives $|Mon(\mathcal{D})| = s|\Omega|$, where s is the size of a stabilizer of the action of $Mon(\mathcal{D})$ on Ω. □

Given an embedding (Γ, R) of the incidence structure $\Gamma = (B, W, I)$ one gets a dessin d'enfant $\mathcal{D}(\Gamma, R) = (I, (x, y))$ setting

$$x(b, w) = (b, R_b(w)) \qquad \text{and} \qquad y(b, w) = (R_w(b), w)\,, \tag{5}$$

for any $(b, w) \in I$. Connectedness of Γ guarantees transitivity of the monodromy group of $\mathcal{D}(\Gamma, R)$.

Let $\Gamma = (B, W, I)$ and $\overline{\Gamma} = (\overline{B}, \overline{W}, \overline{I})$ be incidence structures. If ϕ is a morphism from the embedding (Γ, R) to the embedding $(\overline{\Gamma}, \overline{R})$, then

$$I \to \overline{I},\ (b, w) \mapsto (\phi(b), \phi(w))$$

is a morphism from $\mathcal{D}(\Gamma, R)$ to $\mathcal{D}(\overline{\Gamma}, \overline{R})$.

Using the language of category theory, one can say that there is a faithful functor from the category of embeddings to the category of dessins d'enfants. In particular, this gives the following corollary to Proposition 2.3.1.

Corollary 2.3.2. *Let (Γ, R) be an embedding of the incidence structure $\Gamma = (B, W, I)$. Then $|Aut(\Gamma, R)|$ divides $|I|$.*

2.4. **Projective spaces.** We recall some known facts about finite projective spaces $\mathbb{P}^m(\mathbb{F}_n)$ (see [10], [5], [11]).

Let $\mathbb{F}_n^{m+1}$ be the vector space over the finite field $\mathbb{F}_n$ where $n = p^e$ is a prime power. Consider in $\mathbb{F}_n^{m+1} \setminus \{0\}$ the equivalence relation

$$y = tx \qquad \text{for some } t \in \mathbb{F}_n^* = \mathbb{F}_n \setminus \{0\}.$$

We define the projective space $\mathbb{P}^m(\mathbb{F}_n)$ of *dimension* m and *order* n as the set of equivalence classes $[x]$ and we call them *points* of $\mathbb{P}^m(\mathbb{F}_n)$. A *subspace of dimension* d is a subset S of $\mathbb{P}^m(\mathbb{F}_n)$ such that $\{0\} \cup \bigcup_{[x]\in S} [x]$ is a linear subspace of dimension $d + 1$. Hence points are subspaces of dimension 0. Subspaces of dimension $m - 1$ are called *hyperplanes*. Incidence between subspaces is given by set inclusion. Projective spaces satisfy the following *duality principle*: Any true statement on points and hyperplanes remains true interchanging the words "points" and "hyperplanes".

Permutations of $\mathbb{P}^m(\mathbb{F}_n)$ preserving incidence are called *collineations*. The set of all collineations of $\mathbb{P}^m(\mathbb{F}_n)$ with the composition of functions is a group, called the *collineation group* of $\mathbb{P}^m(\mathbb{F}_n)$ and denoted by $P\Gamma L(m + 1, n)$. By the fundamental theorem of projective geometry, any element of $P\Gamma L(m + 1, n)$ is induced by a semilinear transformation of the vector space $\mathbb{F}_n^{m+1}$ (here $m \geq 2$). A *semilinear transformation* of $\mathbb{F}_n^{m+1}$ is a permutation f of $\mathbb{F}_n^{m+1}$ for which there is $g \in Aut(\mathbb{F}_n)$ such that

$$f(a + b) = f(a) + f(b), \qquad f(sa) = g(s)f(a),$$

for any $a, b \in \mathbb{F}_n^{m+1}$ and any $s \in \mathbb{F}_n$. As $Aut(\mathbb{F}_n)$ is a cyclic group of order e generated by the *Frobenius automorphism*

$$\varphi : \mathbb{F}_n \to \mathbb{F}_n,\ s \mapsto s^p, \tag{6}$$

we have $g = \varphi^j$ for some $j \in \{0, \ldots, e - 1\}$.
The group $P\Gamma L(m + 1, n)$ contains as a normal subgroup the group $PGL(m + 1, n)$ of *linear transformations* of $\mathbb{P}^m(\mathbb{F}_n)$, induced by the linear transformations of $\mathbb{F}_n^{m+1}$. In case that $e = 1$ (that is $n = p$), the collineation group of $\mathbb{P}^m(\mathbb{F}_n)$ is equal to $PGL(m + 1, n)$. Among the linear transformations of $\mathbb{P}^m(\mathbb{F}_n)$ we find transformations permuting points in a single cycle (see [17]). A group Σ generated by such a transformation σ is called a *Singer group*. Any Σ is isomorphic to the cyclic group $\mathbb{Z}_\ell$, where $\ell = \frac{n^{m+1}-1}{n-1}$ is the size of $\mathbb{P}^m(\mathbb{F}_n)$. Its existence allows us to identify $\mathbb{P}^m(\mathbb{F}_n)$ with $\mathbb{Z}_\ell$, setting

$$\sigma : \mathbb{Z}_\ell \to \mathbb{Z}_\ell,\ j \mapsto j + 1. \tag{7}$$

The *Frobenius automorphism* φ of $\mathbb{F}_{n^{m+1}}$ acts then in the following way

$$\varphi : \mathbb{Z}_\ell \to \mathbb{Z}_\ell,\ j \mapsto pj. \tag{8}$$

The group Φ generated by φ is a cyclic group of order $e(m+1)$. It can be identified with the Galois group $Gal(\mathbb{F}_{n^{m+1}}/\mathbb{F}_p)$ cyclically permuting the elements of the field extension $\mathbb{F}_{n^{m+1}}$ of $\mathbb{F}_p$.

Combinatorially, $\mathbb{P}^m(\mathbb{F}_n)$ can be described by an incidence structure $\Gamma = (B, W, I)$, where B is the set of points, W is the set of hyperplanes of $\mathbb{P}^m(\mathbb{F}_n)$ and $(b, w) \in I$ if and only if b is incident with w. The incidence structure Γ is, thus, a *projective design* (denoted by $\mathbf{P}_{m-1}(m, n)$ in [5], see Chapter 2) with *parameters*

$$\ell = \frac{n^{m+1}-1}{n-1}, \qquad q = \frac{n^m-1}{n-1} \qquad \text{and} \qquad \lambda = \frac{n^{m-1}-1}{n-1}. \tag{9}$$

The integer ℓ is the number of points and, by duality, of hyperplanes of $\mathbb{P}^m(\mathbb{F}_n)$. The integer q is the number of points incident with a hyperplane and of hyperplanes incident with a point. The last parameter λ is the number of hyperplanes incident with any two distinct points or, dually, the number of points incident with any two distinct hyperplanes. The automorphism group of the projective design Γ is the collineation group $P\Gamma L(m+1, n)$ of the projective space $\mathbb{P}^m(\mathbb{F}_n)$ described above.

Projective designs are also called *symmetric balanced incomplete block designs* ([3], [9]) or *cyclic symmetric block designs* when they have a cyclic group of automorphisms like Γ above ([9], [1]). The interesting fact is that any cyclic symmetric block design can be constructed using difference sets (see next section).

2.5. **Difference sets.** In general, one can construct incidence structures having a group of automorphisms acting transitively on points and on blocks by quotient sets (see, for instance, [5] or [4]). In particular, since projective spaces have a cyclic sharply transitive (regular) group acting on points and on hyperplanes (blocks) they can be constructed by quotient sets of cyclic groups called difference sets.

Definition 2.5.1. A (v, k, λ)*-difference set* $D = \{d_1, ..., d_k\}$ is a collection of k (distinct) residues modulo v, such that for any residue $\alpha \not\equiv 0$ the congruence

$$d_i - d_j \equiv \alpha \mod v$$

has exactly λ solutions $(d_i, d_j) \in D \times D$.

The following lemmas are well-known and their proofs are straightforward.

Lemma 2.5.2. *For any (v, k, λ)-difference set holds $k(k-1) = \lambda(v-1)$.*

Lemma 2.5.3. *Let $D = \{d_1, ..., d_k\}$ be a (v, k, λ)-difference set and let s and t be residues modulo v. Then*

(i) *$D + s = \{d_1 + s, ..., d_k + s\}$ is a (v, k, λ)-difference set.*
(ii) *$tD = \{td_1, ..., td_k\}$ is a (v, k, λ)-difference set if and only if* $\gcd(t, v) = 1$.

Two (v, k, λ)-difference set D and $\overline{D}$ are called *equivalent*, if there are residues s and t modulo v with $\gcd(t, v) = 1$ such that $\overline{D} = tD + s$.

Given a (v, k, λ)-difference set D, one gets a projective design $\Gamma_D = (B, W, I_D)$, where B and W are two disjoint copies of $\mathbb{Z}_v = \mathbb{Z}/v\mathbb{Z}$, setting

$$I_D = \{(b, w) \in B \times W : b - w \in D\}.$$

In general, for v, k, λ as in Lemma 2.5.2, the existence of a (v, k, λ)-difference set cannot be guaranteed, but for ℓ, q, λ satisfying (9), with $n = p^e$ a prime power, (ℓ, q, λ)-difference sets exist. Moreover, any projective design Γ_D constructed with an (ℓ, q, λ)-difference set D can be regarded as the incidence structure of points and hyperplanes of a projective space ([17], see also [10] or[1]).

According to (7) and (8), we set

$$\sigma(b, w) = (b+1, w+1) \quad \text{and} \quad \varphi(b, w) = (pb, pw) \qquad \text{for any } (b, w) \in I_D. \tag{10}$$

Obviously $\sigma(b,w) \in I_D$ for any $(b,w) \in I_D$, but $\varphi(b,w) \in I_D$ if and only if $p(b-w) \in D$. Hence $\sigma \in Aut(\Gamma_D)$ for any D but $\varphi \in Aut(\Gamma_D)$ if and only if $D = pD$. We call this type of difference sets, fixed under the action of φ, *Frobenius difference sets*. For existence conditions and properties see [16]. We recall here only some facts:

- For any projective space there exists at least one Frobenius difference set.
- Under the action of φ elements of a Frobenius difference set D are subdivided into orbits of length a divisor of the order $e(m+1)$ of φ. If the elements of D all belong to the same orbit, then we can order the elements of D in such a way that the action of φ on them is a cyclic permutation.

The choice of a Frobenius difference set to construct designs Γ_D describing projective spaces is not necessary but in our work it turns out to be convenient for the study of Fano plane's embeddings.

In general, for an incidence structure $\Gamma_D = (B, W, I_D)$ constructed with a (v,k,λ)-difference set D, we regard automorphisms of Γ_D as pairs $\phi = (\phi_1, \phi_2)$ of permutations of $\mathbb{Z}_v$ such that $\phi_1(b) - \phi_2(w) \in D$ for any $(b,w) \in I_D$ (see (10) above). Then, setting

$$\phi_b(d) = \phi_1(b) - \phi_2(b-d) \qquad \text{and} \qquad \phi_w(d) = \phi_1(w+d) - \phi_2(w) \tag{11}$$

for any $b \in B$, $w \in W$ and $d \in D$, we get permutations ϕ_v of D depending on $v \in B \cup W$. We will now describe the action of an element $\phi \in Aut(\Gamma_D)$ on a rotation system R for Γ_D. For any $(b,w) \in I_D$ we set

$$R_b(w) = b - \rho_b(b-w) \qquad \text{and} \qquad R_w(b) = w + \varrho_w(b-w)\,, \tag{12}$$

where ρ_b and ϱ_w are cyclic permutations of D depending on the black vertex b and on the white vertex w. We write

$$R \cong \big((\rho_b),(\varrho_w)\big) = \big((\rho_0,\rho_1,\dots,\rho_{v-1}),(\varrho_0,\varrho_1,\dots,\varrho_{v-1})\big)$$

meaning (12).

Proposition 2.5.4. *For any $\phi = (\phi_1,\phi_2) \in Aut(\Gamma_D)$ and any rotation system $R \cong \big((\rho_b),(\varrho_w)\big)$ it holds true that $R^\phi \cong \big((\overline{\rho}_b),(\overline{\varrho}_w)\big)$ with*

$$\overline{\rho}_b = \rho_\beta^{\phi_\beta} = \phi_\beta \rho_\beta \phi_\beta^{-1} \qquad \textit{and} \qquad \overline{\varrho}_w = \varrho_\omega^{\phi_\omega} = \phi_\omega \varrho_\omega \phi_\omega^{-1}\,,$$

with $b = \phi_1(\beta)$ and $w = \phi_2(\omega)$, for any $\beta \in B$ and any $\omega \in W$.

Proof. Let $\phi = (\phi_1,\phi_2) \in Aut(\Gamma_D)$, $R \cong \big((\rho_b),(\varrho_w)\big)$ a rotation system for Γ_D and $(b,w) = (\phi_1(\beta),\phi_2(\omega))$ with $(\beta,\omega) \in I_D$. Then, according to (2),

$$\begin{aligned}\overline{\rho}_b(b-w) &= b - R_b^\phi(w) = \phi_1(\beta) - \phi_2 R_\beta(\omega)\\ &= \phi_1(\beta) - \phi_2(\beta - \rho_\beta(\beta-\omega)) = \phi_\beta\rho_\beta(\beta-\omega)\end{aligned}$$

and

$$\begin{aligned}\overline{\varrho}_w(b-w) &= R_w^\phi(b) - w = \phi_1 R_\omega(\beta) - \phi_2(\omega)\\ &= \phi_1(\omega + \varrho_\omega(\beta-\omega)) - \phi_2(\omega) = \phi_\omega\varrho_\omega(\beta-\omega)\,.\end{aligned}$$

Setting $d = b - w = \phi_1(\beta) - \phi_2(\omega) \in D$ we have $\beta = \phi_1^{-1}(\phi_2(\omega)+d)$ and $\omega = \phi_2^{-1}(\phi_1(\beta)-d)$. Hence

$$\begin{aligned}R_b^\phi(w) &= \phi_1(\beta) - \phi_\beta\rho_\beta(\beta - \phi_2^{-1}(\phi_1(\beta)-d))\\ &= \phi_1(\beta) - \phi_\beta\rho_\beta\phi_\beta^{-1}(d)\\ &= b - \phi_\beta\rho_\beta\phi_\beta^{-1}(b-w)\end{aligned}$$

and

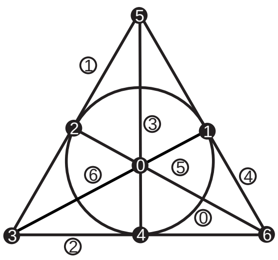
$$\begin{aligned} R^{\phi}_{w}(b) &= \phi_2(\omega) + \phi_\omega \varrho_\omega(\phi_1^{-1}(\phi_2(\omega) + d) - \omega) \\ &= \phi_2(\omega) + \phi_\omega \varrho_\omega \phi_\omega^{-1}(d) \\ &= w + \phi_\omega \varrho_\omega \phi_\omega^{-1}(b - w)\,. \end{aligned}$$

□

Proposition 2.5.4 gives the theoretical tool to describe all embeddings of a projective design arising from a difference set provided its automorphism group is known. In this paper we focus on the Fano plane as an example.

3. The collineation group of the Fano plane

The Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ is the smallest finite projective plane. According to Section 2.4 it has seven lines and seven points ($\ell = 7$). Each line is incident with three points and by duality each point is incident with three lines ($q = 3$). Through two points goes one line and two lines are incident in one point ($\lambda = 1$). According to Section 2.5 the Fano plane can be described by an incidence structure $\Gamma_D = (B, W, I_D)$, where B and W are disjoint copies of $\mathbb{Z}_7$ and D is a $(7, 3, 1)$-difference set. For our convenience we choose $D = \{1, 2, 4\}$ which is a Frobenius difference set, but we could choose any other $(7, 3, 1)$-difference set.



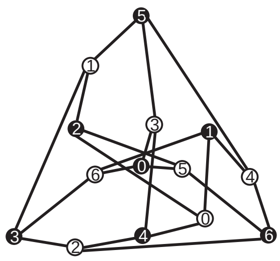


Figure 3: The Fano plane with points and lines numbered using the difference set $D = \{1, 2, 4\}$ and its representation as a bipartite graph.

Automorphisms of $\mathbb{P}^2(\mathbb{F}_2)$ are permutations of I_D that we write as pairs of permutations of $\mathbb{Z}_7$. According to Section 2.4 we have $Aut(\mathbb{P}^2(\mathbb{F}_2)) = PGL(3, 2)$. The following three permutations generate $PGL(3, 2)$:

$$\begin{aligned} &\sigma = (\sigma_1, \sigma_2) = \big((0123456), (0123456)\big)\,, \\ &\varphi = (\varphi_1, \varphi_2) = \big((124)(365), (124)(365)\big) \quad \text{and} \\ &\iota = (\iota_1, \iota_2) = \big((12)(36), (14)(56)\big)\,. \end{aligned} \tag{13}$$

The first generator σ is an element of order 7 cyclically permuting the set B of points and the set W of lines, i.e. it generates a Singer group. The second generator φ is the Frobenius automorphism acting on B and on W by multiplication with the prime $p = 2$ (compare with Section 2.4). The third generator ι is the mirroring of the Fano plane across the line passing through $\{5, 0, 4\}$.

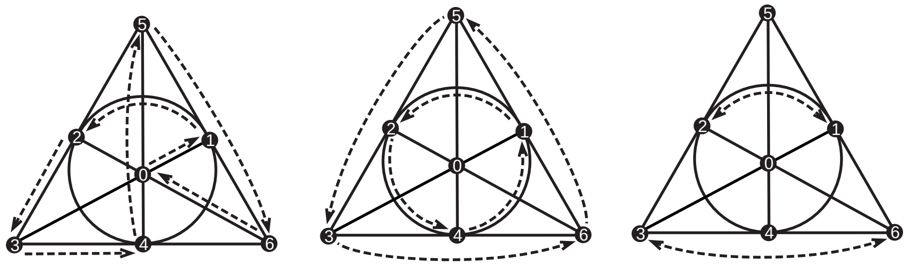


Figure 4: The collineations σ, φ and ι acting on black vertices.

With these generators we obtain the following presentation of $PGL(3,2)$:

$$P = \left\langle \sigma, \varphi, \iota \mid \sigma^7 = \varphi^3 = \iota^2 = (\varphi\iota)^2 = (\sigma\iota)^3 = 1,\ \varphi\sigma = \sigma^2\varphi \right\rangle .$$

From the group presentation P we see that

$$\langle \varphi, \iota \rangle = \langle \varphi \rangle \rtimes \langle \iota \rangle \quad \text{with} \quad \iota\varphi\iota^{-1} = \varphi^{-1} \qquad \text{and} \tag{14}$$

$$\langle \sigma, \varphi \rangle = \langle \sigma \rangle \rtimes \langle \varphi \rangle \quad \text{with} \quad \varphi\sigma\varphi^{-1} = \sigma^2 . \tag{15}$$

The groups $\langle \varphi, \iota \rangle$ and $\langle \sigma, \varphi \rangle$ are respectively the normalizers of $\Phi = \langle \varphi \rangle$ and of $\Sigma = \langle \sigma \rangle$ in $PGL(3,2)$ (see also [11]).

Remark 3.5. It follows from the presentation P, by means of (14) and (15), that $\varphi = (\sigma\iota)^3\varphi = \varphi^{-1}\sigma^2\iota\sigma^4\iota\sigma^2\iota$, which gives

$$\varphi = \iota\sigma^5\iota\sigma^3\iota\sigma^5 ,$$

that is $PGL(3,2) = \langle \sigma, \iota \rangle$. Hence φ is a redundant generator.

In view of Corollary 2.3.2 the automorphism group A of a Fano plane's embedding has size 1, 3, 7 or 21 since $|I_D| = 21$. By means of the presentation P, we can state:

- If $|A| = 3$, then A is a conjugate of Φ.
- If $|A| = 7$, then A is a conjugate of Σ.
- If $|A| = 21$, then A is a conjugate of $\Sigma \rtimes \Phi = \langle \sigma, \varphi \rangle$.

This is a consequence of the fact that $PGL(3,2)$ has only one conjugacy class of subgroups of order 3, 7 or 21. In the next section, we will see that no embedding of the Fano plane has automorphism group of order only 7.

4. Fano plane's embeddings

Let $\Gamma_D = (B, W, I_D)$ be the Fano plane constructed with the $(7,3,1)$-difference set $D = \{1,2,4\}$ we have chosen in the previous section. According to Section 2.2, any rotation system R for Γ_D gives rise to an embedding (Γ_D, R) of the Fano plane on an orientable compact and connected surface $\mathcal{S}$. We count $2^{14} = 16384$ possible rotation systems for Γ_D, since $|B \cup W| = 14$ and $|I_D(v)| = 3$, which gives two possible cyclic permutations R_v of $I_D(v)$ for any $v \in B \cup W$. According to Euler's Formula (1), these 16384 embeddings have following genera (and number of faces):

- genus 1 (7 faces);
- genus 2 (5 faces);
- genus 3 (3 faces);
- genus 4 (1 face).

Even if computationally possible, genus 0 (9 faces) is forbidden by geometry. In this case we would in fact have some faces of valency two and this is not possible since two lines incident with the same two points cannot be distinct. Alternatively, Kuratowsky's Theorem (see, for instance, [14]) also implies the non-existence of genus 0 embeddings of the Fano plane Γ_D since Γ_D, regarded as a bipartite graph, has a subdivision of $K_{3,3}$ as a subgraph.

From Proposition 2.5.4 we obtain the following corollaries.

Corollary 4.1. *For a rotation system* $R \cong \big((\rho_b),(\varrho_w)\big)$ *of the Fano plane* Γ_D *we have:*

$$R^\sigma \cong \big((\rho_{b-1}),(\varrho_{w-1})\big), \quad R^\varphi \cong \big((\rho_{4b}),(\varrho_{4w})\big) \quad \text{and}$$
$$R^\iota \cong \big((\rho_0^{-1},\rho_2,\rho_1,\rho_6^{-1},\rho_4,\rho_5^{-1},\rho_3^{-1}),(\varrho_0^{-1},\varrho_4^{-1},\varrho_2^{-1},\varrho_3,\varrho_1^{-1},\varrho_6,\varrho_5)\big)\,,$$

where σ, φ *and* ι *are the generators of* $PGL(3,2) = Aut(\Gamma_D)$ *given in* (13).

Proof. From (11) we have $\sigma_v = id_D$ and $\varphi_v = (124)$ for any $v \in B \cup W$. As $\rho_b, \varrho_w \in \{(124),(142)\}$, this implies $\rho_b^{\sigma_b} = \rho_b^{\varphi_b} = \rho_b$ for any $b \in B$ and similarly $\varrho_w^{\sigma_w} = \varrho_w^{\varphi_w} = \varrho_w$ for any $w \in W$. Hence

$$R^\sigma \cong \big((\rho_{\sigma_1^{-1}(b)}),(\varrho_{\sigma_2^{-1}(w)})\big) = \big((\rho_{b-1}),(\varrho_{w-1})\big) \quad \text{and}$$
$$R^\varphi \cong \big((\rho_{\varphi_1^{-1}(b)}),(\varrho_{\varphi_2^{-1}(w)})\big) = \big((\rho_{4b}),(\varrho_{4w})\big)$$

by Proposition 2.5.4. For the generator ι, from (11) we have that ι_β is an involution if and only if $\beta \in \{0,3,5,6\} \subset B$ and ι_ω is an involution if and only if $\omega \in \{0,1,2,4\} \subset W$. Hence

$$\rho_\beta^{\iota_\beta} = \begin{cases} \rho_\beta & \text{if } \beta \in \{1,2,4\} \\ \rho_\beta^{-1} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \varrho_\omega^{\iota_\omega} = \begin{cases} \varrho_\omega & \text{if } \omega \in \{3,5,6\} \\ \varrho_\omega^{-1} & \text{otherwise} \end{cases}$$

and Proposition 2.5.4 gives $R^\iota \cong \big((\overline{\rho}_b),(\overline{\varrho}_w)\big)$, with $\overline{\rho}_b = \rho_\beta^{\iota_\beta}$, $\overline{\varrho}_w = \varrho_\omega^{\iota_\omega}$ and $b = \iota_1(\beta)$, $w = \iota_2(\omega)$. ☐

Remark 4.2. It follows from Corollary 4.1 (or Proposition 2.5.4) that $R^\phi \neq R^{-1}$ for any $\phi \in PGL(3,2)$. Hence (Γ_D, R) and its mirror image (Γ_D, R^{-1}) cannot be isomorphic. In this case we say that (Γ_D, R) with its mirror image (Γ_D, R^{-1}) is a *chiral pair.*

Corollary 4.3. *If the Fano plane's embedding* (Γ_D, R) *has an automorphism of order* 7*, then* $|Aut(\Gamma_D, R)| = 21$ *and therefore* (Γ_D, R) *is a regular embedding (i.e.* $\mathcal{D}(\Gamma_D, R)$ *is a regular dessin). Up to isomorphism there are four regular embeddings of the Fano plane (two chiral pairs).*

Proof. Let $s \in Aut(\Gamma_D, R)$ be of order 7. A $\phi \in Aut(\Gamma_D) = PGL(3,2)$ exists such that $s^\phi = \sigma$ since there is only one conjugacy class of subgroups of order 7 in $PGL(3,2)$ (Remark 3.5). The embeddings (Γ_D, R) and (Γ_D, R^ϕ) are isomorphic and σ belongs to $Aut(\Gamma_D, R^\phi)$ by means of Proposition 2.2.1. Setting $\overline{R} = R^\phi \cong \big((\rho_b),(\varrho_w)\big)$, we have $\overline{R}^\sigma = \overline{R}$ and therefore, by means of Corollary 4.1,

$$\rho_b = \rho_0 \quad \text{and} \quad \varrho_w = \varrho_0 \quad \text{for any } b \in B \text{ and any } w \in W\,. \tag{16}$$

The equalities above imply that also $\varphi \in Aut(\Gamma_D, R^\phi)$, showing that $Aut(\Gamma_D, R^\phi) = \langle \sigma, \varphi \rangle$ since $Aut(\Gamma_D, R^\phi)$ cannot have more than $21 = |I_D| = |\langle \sigma, \varphi \rangle|$ elements. As there are four possible choices of $\big((\rho_b),(\varrho_w)\big)$ satisfying (16) and resulting into non-isomorphic embeddings (Corollary 4.1 or Proposition 2.5.4), we conclude that there are four non-isomorphic regular embeddings of the Fano plane. They are grouped into chiral pairs by means of Remark 4.2. ☐

Regular embeddings of the Fano plane are well-known. There is a chiral pair on the torus (see [12]) and a chiral pair on genus 3 (on the Klein quartic [18]). The chiral pair (Γ_D, R), (Γ_D, R^{-1}) with $R \cong \big((\rho_b),(\varrho_w)\big)$ and $\rho_b = \varrho_w = (124)$ is on the torus, while the pair with $\rho_b = (142)$ and $\varrho_w = (124)$ is on genus three. The last one corresponds to Wada dessins studied in [19], [15], [16].

Corollary 4.4. *Up to isomorphism there are* 28 *embeddings of the Fano plane with automorphism group of size* 3*.*

Proof. Let $f \in Aut(\Gamma_D, R)$ be of order 3. A $\phi \in Aut(\Gamma_D) = PGL(3,2)$ exists such that $f^\phi = \varphi$ since there is only one conjugacy class of elements of order 3 in $PGL(3,2)$ (Remark 3.5). As (Γ_D, R) and (Γ_D, R^ϕ) are isomorphic embeddings with $\varphi \in Aut(\Gamma_D, R^\phi)$ (Proposition 2.2.1), we can assume w.l.o.g. that $\varphi \in Aut(\Gamma_D, R)$. Then, setting $R \cong \big((\rho_b), (\varrho_w)\big)$, from $R^\varphi = R$ and Corollary 4.1 we have

$$\begin{aligned} &\rho_1 = \rho_2 = \rho_4\,, \quad \rho_3 = \rho_5 = \rho_6 \quad \text{and} \\ &\varrho_1 = \varrho_2 = \varrho_4\,, \quad \varrho_3 = \varrho_5 = \varrho_6\,. \end{aligned} \tag{17}$$

There are $2^3 \times 2^3 = 64$ such rotation systems. Consider that if R is one of those 64 rotation systems, then $R^\iota \neq R$ also belongs to these 64 rotation systems. As we have seen in Section 3, the normalizer of $\Phi = \langle\varphi\rangle$ is $\langle\varphi, \iota\rangle$ thus embeddings with rotation system R and embeddings with rotation system R^ι are isomorphic. It follows that we have 32 non-isomorphic embeddings with automorphism group of size at least 3. Four of them are the regular embeddings of Corollary 4.3. □

We can now state the following:

Theorem 4.5. *Up to isomorphism, there are 120 embeddings of the Fano plane:* 4 *of them are regular,* 28 *have (cyclic) automorphism group of size* 3 *and* 88 *have trivial automorphism group.*

Proof. It remains to prove that there are 88 embeddings of the Fano plane with trivial automorphism group. This follows from Section 2.2. Namely, from (3) we have that the 16384 possible rotation systems are grouped in the following way:

- 4 orbits of length 8 giving the regular embeddings;
- 28 orbits of length 56 giving the embeddings with automorphism group of size 3;
- x orbits of length 168 giving the embeddings with trivial automorphism group;

From $4 \times 8 + 28 \times 56 + x \times 168 = 16384$ we obtain $x = 88$. □

Given a rotation system R it is not straightforward to obtain the genus $g = g(R)$ of the embedding (Γ_D, R). One has to count the number of faces, that is, the number of orbits of the permutation xy on I_D, where $x(b,w) = (b, R_b(w))$and $y(b,w) = (R_w(b), w)$ (see Section 2.3 and Euler's formula (1)). Anyway, taking into account that

$$xy(b,w) = \big(R_w(b), R_{R_w(b)}(w)\big)\ ,$$

this task can be accomplished by a computer algebra system. Using GAP (see [7]), we get

- 2 embeddings of genus 1 resulting from 16 rotation systems;
- 10 embeddings of genus 2 resulting from 1008 rotation systems;
- 76 embeddings of genus 3 resulting from 10880 rotation systems;
- 32 embeddings of genus 4 resulting from 4480 rotation systems.

The *genus polynomial* of the Fano plane is then

$$\gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 = 16x + 1008x^2 + 10880x^3 + 4480x^4\,,$$

where γ_g is the number of rotation systems R with $g(R) = g$ (see [8]). The *genus distribution* $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is *log-concave* that is it satisfies $\gamma_{k-1}\gamma_{k+1} \leq \gamma_k^2$, $i = 2, 3$, according to a conjecture (stated in [8]) that this is true for any graph.

5. Quotients

As we have seen in Section 4 above we obtain 32 non-isomorphic embeddings of the Fano plane with non-trivial automorphism group. We now consider their quotients by a group of automorphisms of order three.

Proposition 5.1. *Let* (Γ_D, R) *be an embedding of the Fano plane* Γ_D *having a group of automorphisms* G *of order* 3*. Then, up to mirror image, either there is a covering of* (Γ_D, R) *onto one of the quotient embeddings in the sphere (see Fig. 5) or there is a covering onto one of the quotient embeddings in the torus (see Fig. 6).*

Proof. Let $\overline{B} = \{Gb : b \in B\}$, $\overline{W} = \{Gw : w \in W\}$ and $\overline{I} = \{G(b, w) : (b, w) \in I_D\}$, then $\overline{\Gamma} = (\overline{B}, \overline{W}, \overline{I})$ is an incidence structure and

$$\phi : B \cup W \to \overline{B} \cup \overline{W},\ v \mapsto Gv$$

is a covering from Γ_D to $\overline{\Gamma}$. As G is generated by a conjugate of the Frobenius automorphism φ of Γ_D, we have that $\overline{\Gamma}$ has

- one black vertex of valency one (branched with branch index three),
- one white vertex of valency one (branched with branch index three),
- two black vertices of valency three,
- two white vertices of valency three,
- seven incidences.

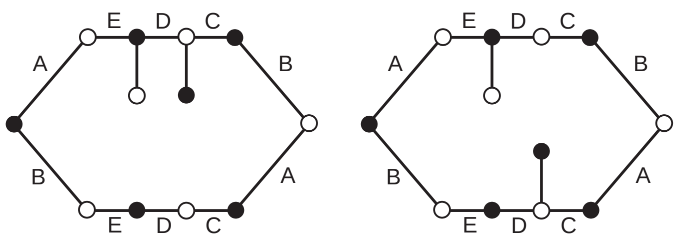
Applying Euler's formula (1) we obtain following relation between genus g and face number $|F|$:

$$2 - 2g = 3 + 3 - 7 + |F| \quad \Rightarrow \quad g = \frac{3 - |F|}{2}\ .$$

Hence either we have embeddings into the torus with $|F| = 1$ or we have embeddings into the sphere with $|F| = 3$. By construction it is now easy to see that, up to isomorphism and mirror image, $\overline{\Gamma}$ has two embeddings on the sphere (see Fig. 5) and two embeddings on the torus (see Fig. 6). □

(a) (b)

Figure 5: Quotients of the Fano plane embedded into the sphere.



(a) (b)

Figure 6: Quotients of the Fano plane embedded into the torus. Edges labeled with the same letter have to be identified.

From the constructed quotients we obtain information about the covering embeddings considering what follows:

- The group G fixes one black vertex b_0 and one white vertex w_0 cyclically permuting the edges incident with b_0 and the edges incident with w_0.
- The images of b_0 and w_0 by the covering are the black and the white vertex of valency one in a quotient (which are branched points of index 3).
- The resulting quotient does not depend on the choice of the rotations R_{b_0} and R_{w_0}.
- In a quotient, (the center of) a face f of valency ν can be ramified or not, e.g. preimages of f are either one face of valency 3ν or a set of three faces of valency ν.
- In a quotient, faces of valency one or two are always ramified since faces of such valency are not allowed in the covering embeddings.

Hence each of the four quotients has four covering embeddings, i.e.

- quotients with only one face of valency 7 (on the torus) have four covering embeddings with faces of valency $7, 7, 7$ or with one face of valency 21;
- quotients with three faces of valency $1, 2, 4$ have four covering embeddings with faces of valency $3, 4, 4, 4, 6$ or of valency $3, 6, 12$;
- quotients with faces of valency $1, 3, 3$ (on the sphere) have covering embeddings with faces of valency $3, 9, 9$ or $3, 3, 3, 3, 3, 3, 3$ or $3, 3, 3, 3, 9$.

In Appendix A, the above quotients and their covering embeddings are listed. They are constructed choosing different rotations around sets of black vertices $\{0\}, \{1, 2, 4\}, \{3, 6, 5\}$ and around sets of white vertices $\{0\}, \{1, 2, 4\}, \{3, 6, 5\}$ according to the equalities in (17) (see proof of Corollary 4.4).

6. Cell operations and unicellular embeddings

Let (Γ_D, R) with $R = \big((\rho_b), (\varrho_w)\big)$ be an embedding of the Fano plane whose automorphism group is of size equal to or larger than three. If the three faces incident with black (or white) vertex 0 are distinct with valency ν, then replacing the respective rotation ρ_0 (or ϱ_0) by ρ_0^{-1} (resp. ϱ_0^{-1}) we obtain a new embedding with a face of valency 3ν. This operation on rotation systems explains face valencies given in the Table of Appendix A. More precisely:

- In the first group of four embeddings, we obtain the unicellular ones (embeddings with one face) starting from the Wada embeddings with face valency sequence $(7, 7, 7)$.
- The second group of four embeddings contains the regular embedding on the torus with face valency sequence $(3, 3, 3, 3, 3, 3, 3)$. By construction the three faces incident with the black vertex zero are all different from the three faces incident with the white vertex zero, thus starting from $(3, 3, 3, 3, 3, 3, 3)$ we obtain first the embeddings with face valency sequence $(3, 3, 3, 3, 9)$ from which we obtain the embedding with face valency sequence $(3, 9, 9)$.
- For the third group of embeddings we obtain unicellular embeddings starting from $(7, 7, 7)$ in the same way as for the first group.
- In the fourth group of embeddings, the three faces of valency 4 of the embedding with face valency sequence $(3, 4, 4, 4, 6)$ are incident with black and white vertex zero by construction. This implies that starting from these embeddings we obtain the first two with face valency sequence $(3, 6, 12)$ from which we then obtain the third.

Remark 6.1. For any embedding (Γ, R) with $R \cong ((\rho_b), (\varrho_w))$ and genus g, if there is a black (or white) vertex v of odd valency r incident with r distinct faces of valency $\nu_1, \ldots, \nu_r$, replacing the rotation ρ_v (or ϱ_v) with ρ_v^{-1} (resp. ϱ_v^{-1}) we obtain a new embedding with a face of valency $\nu_1 + \cdots + \nu_r$ and genus $g + \frac{r-1}{2}$. If the valency r is even, then we obtain two faces whose valency sum is $\nu_1 + \cdots + \nu_r$ and the genus of the embedding is $g + \frac{r-2}{2}$ (see Fig. 7).

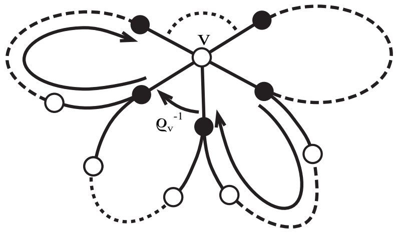


Figure 7: A cell operation.

Acknowledgments

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Appendix A. Embeddings of the Fano plane with non-trivial automorphism group

In the following table sixteen non-isomorphic embeddings of the Fano plane (Γ_D, R) are listed. For the construction we choose the Frobenius difference set $D = \{1, 2, 4\}$. The automorphism group of the embeddings is of order at least three and it contains the Frobenius automorphism. The remaining non-isomorphic embeddings with non-trivial automorphism group can be obtained mirroring the listed ones.

The first column of the table gives rotation systems such that the Frobenius automorphism is contained in $Aut(\Gamma_D, R)$ (see Section 4). The sign $+$ and $-$ denote respectively permutations (124) and (142) of the difference set D. The next two columns contain the resulting embeddings and their quotients with the corresponding face valencies. Embeddings are subdivided into four groups having the same quotient. The last column lists the numbers of the figures in which the quotients are sketched.

ρ_0 ϱ_0	ρ_1, ρ_2, ρ_4 $\varrho_1, \varrho_2, \varrho_4$	ρ_3, ρ_6, ρ_5 $\varrho_3, \varrho_6, \varrho_5$	(Γ_D, R)	$(\overline{\Gamma}, \overline{R})$	Fig.
$+$ $-$	$+$ $-$	$+$ $-$	$(7,7,7)$ regular	(7)	6 (a)
$-$ $+$	$+$ $-$	$+$ $-$	$(7,7,7)$	(7)	
$+$ $+$	$+$ $-$	$+$ $-$	(21)	(7)	
$-$ $-$	$+$ $-$	$+$ $-$	(21)	(7)	
$+$ $-$	$+$ $+$	$+$ $+$	$(3,3,3,3,9)$	$(1,3,3)$	7 (b)
$-$ $+$	$+$ $+$	$+$ $+$	$(3,3,3,3,9)$	$(1,3,3)$	
$+$ $+$	$+$ $+$	$+$ $+$	$(3,3,3,3,3,3,3)$ regular	$(1,3,3)$	
$-$ $-$	$+$ $+$	$+$ $+$	$(3,9,9)$	$(1,3,3)$	
$+$ $-$	$+$ $-$	$+$ $+$	$(7,7,7)$	(7)	6 (b)
$-$ $+$	$+$ $-$	$+$ $+$	$(7,7,7)$	(7)	
$+$ $+$	$+$ $-$	$+$ $+$	(21)	(7)	
$-$ $-$	$+$ $-$	$+$ $+$	(21)	(7)	
$+$ $-$	$+$ $+$	$+$ $-$	$(3,6,12)$	$(1,2,4)$	7 (a)
$-$ $+$	$+$ $+$	$+$ $-$	$(3,6,12)$	$(1,2,4)$	
$+$ $+$	$+$ $+$	$+$ $-$	$(3,6,12)$	$(1,2,4)$	
$-$ $-$	$+$ $+$	$+$ $-$	$(3,4,4,4,6)$	$(1,4,2)$	

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(D. Catalano) Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal

E-mail address, D. Catalano: `domenico@ua.pt`

(C. Sarti) Mathematisches Seminar, Universität Frankfurt, Anneliese-Hoevel-Str. 6, D-60435 Frankfurt a.M., Germany

E-mail address, C. Sarti: `crisarti@yahoo.de`