FITTING SETS IN $\mathfrak{U}$-GROUPS

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Abstract. We consider different natural definitions for Fitting sets, and we prove the existence and conjugacy of the associated injectors in the largest possible classes of locally finite groups.

A *Fitting set* $\mathfrak{X}$ of a finite group G is a non-empty set of subgroups which is closed under taking normal products, taking subnormal subgroups and conjugation in G. This concept of Anderson generalized the work of Fischer, Gaschütz and Hartley about *Fitting classes* [2, p.535–537], where a *Fitting class* is a class of finite groups closed under taking normal products and taking subnormal subgroups.

The theory of Fitting sets concern the study of *injectors*: for a collection $\mathfrak{X}$ of finite groups, a subgroup V of a finite group G is an $\mathfrak{X}$-*injector* of G if $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A for every subnormal subgroup A. The basic theorem provides the existence and conjugacy of $\mathfrak{X}$-injectors in any finite soluble group G for each Fitting set $\mathfrak{X}$ [2, Theorem 2.9 p.539].

In this paper, we consider the natural generalizations of Fitting sets and injectors to locally finite groups. For each definition of a Fitting set, we prove the existence and conjugacy of associated injectors in a class of locally finite groups. It is shown in [3] that our classes of locally finite groups are the largest in which the results about injectors in finite soluble groups hold.

Note that, thanks to our new approach, we generalize in the last section the theorem of Hartley and Tomkinson about the existence and conjugacy of injectors in $\mathfrak{U}$-groups [6] from *Fitting classes* to Fitting sets.

1. Normal Fitting sets

In this section, we analyze the injectors associated to *normal Fitting sets*.

Notation 1.1. Let $\mathfrak{X}$ be a collection of groups. For each group G, we denote by $G_{\mathfrak{X}n}$ the join of its normal $\mathfrak{X}$-subgroups.

Definition 1.2. Let G be a locally finite group. A nonempty set $\mathfrak{X}$ of subgroups of G is a *normal Fitting set* of G if:
$(NF1)$ every normal subgroup of an $\mathfrak{X}$-group belongs to $\mathfrak{X}$;
$(NF2)$ when H is a subgroup of G then $H_{\mathfrak{X}n}$ is an $\mathfrak{X}$-subgroup of H;
$(NF3)$ every conjugate of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

Definition 1.3. Let G be a locally finite group. If $\mathfrak{X}$ is a set of subgroups of G, a subgroup V of G is an $\mathfrak{X}$-*n-injector* of G if, for every normal subgroup A of G, $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.

Date: November 11, 2013.
1991 *Mathematics Subject Classification.* Primary 20F17; Secondary 20D10.
Key words and phrases. locally finite groups, Fitting sets, Injectors.

The main result of this section is the following.

Theorem 1.4. *Let G be a* subsoluble $\mathfrak{U}$-group*, and $\mathfrak{X}$ a normal Fitting set of G. Then G has exactly one conjugacy class of $\mathfrak{X}$-n-injectors.*

It is likely that the class of subsoluble $\mathfrak{U}$-groups is the largest class of groups satisfying this result. A partial answer is obtained in [3].

We recall that a group is called a *Baer group* (resp. a *Gruenberg group*) if every cyclic subgroup is subnormal (resp. ascendant). A group G is said to be *subsoluble* (resp. an *SN^*-group*) if it possesses an ascending series with each term subnormal (resp. ascendant) in G and each factor abelian.

In any group G, there is a *Baer radical* $\beta(G)$ which is a Baer subgroup containing all the subnormal Baer subgroups of G, and a *Gruenberg radical* $\gamma(G)$ which is a Gruenberg subgroup containing all the ascendant Gruenberg subgroups of G. We refer to [7, §2.3] for more details.

As in [1, p.15], we denote by $\rho(G)$ the *Hirsch-Plotkin radical* of any group G, and $\rho_0(G) = 1$ and $\rho_{n+1}(G)/\rho_n(G) = \rho(G/\rho_n(G))$ for each integer n.

We recall that the class $\mathfrak{U}$ was introduced as below in [4], and that by an Hartley's Theorem [1, Theorem 4.4.7 p.163], the first condition is redundant.

Definition 1.5. The class $\mathfrak{U}$ is the largest subgroup-closed class of locally finite groups satisfying the conditions:

(U1) if $G \in \mathfrak{U}$ then $G = \rho_n(G)$ for an integer n;

(U2) if $G \in \mathfrak{U}$ and π is any set of primes, then the Sylow π-subgroups of G are conjugate in G.

Fact 1.6. [5, Theorem E] If G is a $\mathfrak{U}$-group, then $G/\rho(G)$ is (2-step soluble)-by-finite. In particular $G/\rho(G)$ is soluble. Moreover, $G/\rho(G)$ is hyperfinite.

We need to introduce *normal* Fitting sets* as the sets $\mathfrak{X}$ of subgroups of a locally finite group G satisfying the conditions $(NF1)$, $(NF3)$ and

$(NF2^*)$ *if H is a subgroup of G such that $H/H_{\mathfrak{X}n}$ is locally nilpotent, then $H_{\mathfrak{X}n}$ is a maximal $\mathfrak{X}$-subgroup of H.*

Any normal* Fitting set of a locally finite group G is a normal Fitting set too. By the result below, the converse is true in subsoluble $\mathfrak{U}$-groups.

Proposition 1.7. *If $\mathfrak{X}$ is a normal Fitting set of a subsoluble $\mathfrak{U}$-group G, then $\mathfrak{X}$ is a normal* Fitting set.*

Proof. Let H be a subgroup of G such that $H/H_{\mathfrak{X}n}$ is locally nilpotent. We define $\delta_0(H) = H_{\mathfrak{X}n}$, $\delta_{i+1}(H)/\delta_i(H) = \beta(H/\delta_i(H))$ for every ordinal i, and $\delta_\mu(H) = \cup_{i<\mu}\delta_i(H)$ for every limit ordinal μ. Since G is subsoluble, then H is subsoluble too, and there is an ordinal α such that $\delta_\alpha(H) = H$. Suppose toward a contradiction that $H_{\mathfrak{X}n}$ is not a maximal $\mathfrak{X}$-subgroup of H. Since it is an $\mathfrak{X}$-subgroup by $(NF2)$, there is a smallest ordinal γ such that $H_{\mathfrak{X}n}$ is strictly contained in an $\mathfrak{X}$-subgroup K of $\delta_\gamma(H)$. By $(NF1)$, we have $K \cap \delta_i(H) = H_{\mathfrak{X}n}$ for all $i < \gamma$, so there is an ordinal ν such that $\gamma = \nu + 1$. Now $K/H_{\mathfrak{X}n} \simeq K\delta_\nu(H)/\delta_\nu(H)$ is a Baer group. Hence, if g is an element of K, then $\langle g\rangle H_{\mathfrak{X}n}$ is subnormal in K, and $\langle g\rangle H_{\mathfrak{X}n}$ is an $\mathfrak{X}$-group by $(NF1)$. By the minimality of γ, it is a maximal $\mathfrak{X}$-subgroup of $\langle g\rangle\delta_i(H)$ for all $i < \gamma$.

For any element b of $\delta_\nu(H)$, the local nilpotence of $H/H_{\mathfrak{X}n}$ implies the nilpotence of $\langle g, b\rangle H_{\mathfrak{X}n}/H_{\mathfrak{X}n}$. Therefore $\langle g\rangle H_{\mathfrak{X}n}$ is a subnormal $\mathfrak{X}$-subgroup of $\langle g, b\rangle H_{\mathfrak{X}n}$. By the maximality of $\langle g\rangle H_{\mathfrak{X}n}$ in $\langle g\rangle\delta_\nu(H)$, we have $\langle g\rangle H_{\mathfrak{X}n} = (\langle g,b\rangle H_{\mathfrak{X}n})_{\mathfrak{X}n}$ and b normalizes $\langle g\rangle H_{\mathfrak{X}n}$, so $\langle g\rangle H_{\mathfrak{X}n} = (\langle g\rangle\delta_\nu(H))_{\mathfrak{X}n}$.

Since $\delta_\gamma(H)/\delta_\nu(H)$ is a Baer group, then $\langle g\rangle\delta_\nu(H)$ is subnormal in $\delta_\gamma(H)$. Hence $\langle g\rangle H_{\mathfrak{X}n}$ is a subnormal $\mathfrak{X}$-subgroup of $\delta_\gamma(H)$ and we have $\langle g\rangle H_{\mathfrak{X}n} \leq (\delta_\gamma(H))_{\mathfrak{X}n}$. As $\delta_\gamma(H)$ is a normal subgroup of H, we obtain $\langle g\rangle H_{\mathfrak{X}n} \leq H_{\mathfrak{X}n}$, contradicting the choice of g. This proves the proposition. □

For this section, we fix a normal* Fitting set $\mathfrak{F}$ of a fixed $\mathfrak{U}$-group G.

Lemma 1.8. *Let A a subgroup of G containing G'. If A possesses a maximal $\mathfrak{F}$-subgroup W, then every $\mathfrak{F}$-subgroup of G containing W is contained in a maximal $\mathfrak{F}$-subgroup of G.*

Proof. Let $(V_i)_{i<\alpha}$ be an increasing sequence of $\mathfrak{F}$-subgroups of G containing W, for an ordinal α. For each $i<\alpha$, since A is normal in G, $V_i \cap A$ is an $\mathfrak{F}$-subgroup of A containing W and we have $V_i \cap A = W$ by the maximality of W. As A contains G', then W contains V' where $V = \cup_{i<\alpha} V_i$. So V_i is normal in V for all $i<\alpha$. We obtain $V = V_{\mathfrak{F}n}$ and V is an $\mathfrak{F}$-group by $(NF2^*)$. □

A *Carter subgroup* of a group H is a locally nilpotent and self-serializing subgroup of H. The main result concerning these subgroups is the following:

Fact 1.9. [4, Theorem 5.4] Let H be a $\mathfrak{U}$-group, and A a normal subgroup of H. Then H has a unique conjugacy class of Carter subgroups. Moreover, if C is a Carter subgroup of H, then CA/A is a Carter subgroup of H/A and each Carter subgroup of H/A has this form.

Lemma 1.10. *Let A be a subgroup of G containing G'. If A has a maximal $\mathfrak{F}$-subgroup W, then the maximal $\mathfrak{F}$-subgroups of G containing W are conjugate.*

Proof. By Lemma 1.8, G has a maximal $\mathfrak{F}$-subgroup V containing W, and $W = V \cap A$ is normal in V. Let $U = N_G(W)$ and $N/W = N_{U/W}(V/W)$. We show that there exists a Carter subgroup C/W of U/W such that $V = C_{\mathfrak{F}n}$. As $[V,N]$ is contained in $V \cap G' \leq W$, N/W centralizes V/W. By Fact 1.9, N/W has a Carter subgroup C/W. Then C/W contains V/W, and V is a normal and maximal $\mathfrak{F}$-subgroup of C. In particular, we obtain $V = C_{\mathfrak{F}n}$ and V/W is normal in $N_{U/W}(C/W)$. Thus we have $N_{U/W}(C/W) = N_{N/W}(C/W) = C/W$ and C/W is a Carter subgroup of U/W.

Let V_1 and V_2 be two maximal $\mathfrak{F}$-subgroups of G containing W. Then there are two Carter subgroups C_1/W and C_2/W of U/W such that $V_1 = (C_1)_{\mathfrak{F}n}$ and $V_2 = (C_2)_{\mathfrak{F}n}$ respectively. The conjugacy of C_1 and C_2 is given by Fact 1.9, and the lemma is proved. □

For every group H, we denote by $H^{\mathrm{L}\mathfrak{N}}$ the intersection of all the normal subgroups K of H such that H/K is locally nilpotent.

We recall that a subgroup H of a group G is *pronormal* in G if H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$.

Lemma 1.11. *Let V be a pronormal subgroup of G such that $V \cap G^{\mathrm{L}\mathfrak{N}}$ is normal in G. Then $V/(V \cap G^{\mathrm{L}\mathfrak{N}})$ is a normal subgroup of a Carter subgroup of $G/(V \cap G^{\mathrm{L}\mathfrak{N}})$.*

Proof. We may assume $V \cap G^{\mathrm{L}\mathfrak{N}} = 1$. Let $G_0 = G$ and, for all $i \in \mathbb{N}$, $G_{i+1} = G_i^{\mathrm{L}\mathfrak{N}}$. We show that, for all $i \in \mathbb{N}$, VG_i/G_i is a normal subgroup of a Carter subgroup of G/G_i. It is true if $i = 0$. Assume that the result is true for $i \in \mathbb{N}$.

Then VG_i/G_i is a normal subgroup of a Carter subgroup C/G_i of G/G_i. By the pronormality of V in G, for every $c \in C$, there is $x \in G_i$ such that $V^x = V^c$, and we obtain $C = N_C(V)G_i$. By Fact 1.9, $N_C(V)$ has a Carter subgroup D_0, and as C/G_i is locally nilpotent, we have $C = D_0G_i$. Let D/G_{i+1} be a maximal locally nilpotent subgroup of C/G_{i+1} containing D_0G_{i+1}/G_{i+1}. Now D/G_{i+1} is a Carter subgroup of C/G_{i+1} [4, Theorem 5.12], and D/G_{i+1} is a Carter subgroup of G/G_{i+1} by Fact 1.9. But we have $(D_0V)^{\mathrm{L}\mathfrak{N}} \leq V \cap G^{\mathrm{L}\mathfrak{N}} = 1$, hence D_0V is locally nilpotent, and D_0 contains V by the definition of a Carter subgroup. Therefore, VG_{i+1}/G_{i+1} is contained in a Carter subgroup D/G_{i+1} of G/G_{i+1}. Since D/G_{i+1} is locally nilpotent, VG_{i+1}/G_{i+1} is a serial subgroup of D/G_{i+1}. As VG_{i+1}/G_{i+1} is a pronormal subgroup of D/G_{i+1}, it is a normal subgroup of D/G_{i+1}. This proves that VG_{i+1}/G_{i+1} is a normal subgroup of a Carter subgroup of G/G_{i+1} for each $i \in \mathbb{N}$. In particular, V is a normal subgroup of a Carter subgroup of G. □

Lemma 1.12. *Let A be a subgroup of G containing G', L a normal subgroup of G contaning $G^{\mathrm{L}\mathfrak{N}}$ and V a maximal $\mathfrak{F}$-subgroup of G. Suppose that, if B is a normal subgroup of A and if B_1 is a subgroup of B containing $V \cap B$, then $V \cap B$ is an $\mathfrak{F}$-n-injector of B_1. Then $V \cap L$ is a maximal $\mathfrak{F}$-subgroup of L.*

Proof. We show that V is pronormal in G. For every $g \in G$, V and V^g are two maximal $\mathfrak{F}$-subgroups of $\langle V, V^g\rangle$. By our hypothesis, for all $k \in \mathbb{N}$, $V \cap A^{(k)}$ is an $\mathfrak{F}$-n-injector of $\langle V, V^g\rangle \cap A^{(k)}$ and of $\langle V^{g^{-1}}, V\rangle \cap A^{(k)}$. So $V^g \cap A^{(k)}$ is an $\mathfrak{F}$-n-injector of $\langle V, V^g\rangle \cap A^{(k)}$. Since there exists $r \in \mathbb{N}$ such that $A^{(r)}$ is locally nilpotent (Fact 1.6), the unique $\mathfrak{F}$-n-injector of $\langle V, V^g\rangle \cap A^{(r)}$ is $(\langle V, V^g\rangle \cap A^{(r)})_{\mathfrak{F}n}$ by $(NF2^*)$, and we obtain $V \cap A^{(r)} = V^g \cap A^{(r)}$. Suppose that for an element $k \in \{1, ..., r\}$ there exists $\alpha \in A^{(k)}$ such that $V \cap A^{(k)} = V^{g\alpha} \cap A^{(k)}$. By hypothesis, $V \cap A^{(k)}$ is an $\mathfrak{F}$-n-injector of $\langle V, V^{g\alpha}\rangle \cap A^{(k)}$. Then $V \cap A^{(k-1)}$ and $V^{g\alpha} \cap A^{(k-1)}$ are conjugate in $A^{(k-1)}$ by Lemma 1.10. Thus $V \cap A$ and $V^g \cap A$ are conjugate in A. Now a last application of Lemma 1.10 provides the conjugacy of V and V^g in $\langle V, V^g\rangle$, as desired.

Let $N = N_G(V \cap L \cap A)$. Since $V \cap L \cap A$ is an $\mathfrak{F}$-n-injector of $L \cap A$, Lemma 1.8 shows that $V \cap L$ is contained in a maximal $\mathfrak{F}$-subgroup V_1 of L. We show that V_1 is pronormal in N. By hypothesis, $V_1 \cap A = V \cap L \cap A$ is an $\mathfrak{F}$-n-injector of $\langle V_1, V_1^g\rangle \cap A$ for all $g \in G$. So, for every $g \in N$, V_1 and V_1^g are two maximal $\mathfrak{F}$-subgroups of $\langle V_1, V_1^g\rangle \leq L$ containing the $\mathfrak{F}$-n-injector $V_1 \cap A = V_1^g \cap A$ of $\langle V_1, V_1^g\rangle \cap A$. By Lemma 1.10, we obtain conjugacy of V_1 and V_1^g in $\langle V_1, V_1^g\rangle$. Hence V_1 is pronormal in N.

Since $G^{\mathrm{L}\mathfrak{N}}$ is contained in A and L, $V \cap N^{\mathrm{L}\mathfrak{N}}$ and $V_1 \cap N^{\mathrm{L}\mathfrak{N}}$ are contained in $V \cap L \cap A = V_1 \cap A$. So $V \cap N^{\mathrm{L}\mathfrak{N}} = V_1 \cap N^{\mathrm{L}\mathfrak{N}}$ is a normal subgroup of N. By Lemma 1.11, $V/(V \cap N^{\mathrm{L}\mathfrak{N}})$ and $V_1/(V \cap N^{\mathrm{L}\mathfrak{N}})$ are normal subgroups of Carter subgroups $C/(V \cap N^{\mathrm{L}\mathfrak{N}})$ and $C_1/(V \cap N^{\mathrm{L}\mathfrak{N}})$ of $N/(V \cap N^{\mathrm{L}\mathfrak{N}})$ respectively, and Fact 1.9 gives $x \in N$ such that $C_1^x = C$. As V is a maximal $\mathfrak{F}$-subgroup of G and as $C_{\mathfrak{F}n} \in \mathfrak{F}$ contains V and V_1^x, we find $V_1^x \leq V$. But V_1^x is a maximal $\mathfrak{F}$-subgroup of L, so $V \cap L = V_1^x$ is a maximal $\mathfrak{F}$-subgroup of L. □

Lemma 1.13. *Let A be a subgroup of G containing G' and let V be a maximal $\mathfrak{F}$-subgroup of G. We suppose that, if B is a normal subgroup of G contained in A, then $V \cap B$ is an $\mathfrak{F}$-n-injector of B. Then $V \cap L$ is a maximal $\mathfrak{F}$-subgroup of L for every normal subgroup L of G such that G/L is soluble.*

Proof. We proceed by induction on the solubility class k of G/L. We may assume $k \geq 1$. Let $M = G^{(k-1)}L$. By induction, $V \cap M$ is a maximal $\mathfrak{F}$-subgroup of M. Since $V \cap M'$ is an $\mathfrak{F}$-n-injector of M' by our hypothesis, Lemma 1.8 says that $V \cap M'$ is contained in a maximal $\mathfrak{F}$-subgroup V_1 of L. Since V_1 is contained in a maximal $\mathfrak{F}$-subgroup V_2 of M (Lemma 1.8), we have two maximal $\mathfrak{F}$-subgroups $V \cap M$ and V_2 of M containing $V \cap M'$. Lemma 1.10 gives $g \in M$ such that $(V \cap M)^g = V_2$. As L is normal in G, we obtain $(V \cap L)^g = V_2 \cap L = V_1$, and $V \cap L$ is a maximal $\mathfrak{F}$-subgroup of L. □

The following proposition is an analogue of [2, Proposition 2.12].

Proposition 1.14. *Let $(A_i)_{i=0,\dots,n}$ ($n \in \mathbb{N}$) be a finite increasing series of subgroups of G such that A_0 is locally nilpotent and normal, G/A_0 is soluble, $A_n = G$ and, for every $i \in \{0, ..., n-1\}$, A_i is normal in A_{i+1} and A_{i+1}/A_i is abelian. Suppose that there is a subgroup V of G such that $V \cap A_0 = (A_0)_{\mathfrak{F}n}$ and such that $V \cap A_i$ is a maximal $\mathfrak{F}$-subgroup of A_i for every $i \in \{1, ..., n\}$. If B is a normal subgroup of G and if B_1 is a subgroup of B containing $V \cap B$, then $V \cap B$ is an $\mathfrak{F}$-n-injector of B_1.*

Proof. We proceed by induction on n. We may assume that G is not locally nilpotent and $n \geq 1$. First, we show that V is an $\mathfrak{F}$-n-injector of G. Let L be a normal subgroup of G. By induction, for $i \in \{0, \cdots, n-1\}$, if N is a normal subgroup of A_i, and if N_1 is a subgroup of N containing $V \cap N$, then $V \cap N$ is an $\mathfrak{F}$-n-injector of N_1. In particular, $V \cap \rho(G)L \cap A_i$ is a maximal $\mathfrak{F}$-subgroup of $\rho(G)L \cap A_i$, and we have $V \cap \rho(G)L \cap A_0 = \rho(G)L \cap (A_0)_{\mathfrak{F}n} = (\rho(G)L \cap A_0)_{\mathfrak{F}n}$. Moreover, by Lemma 1.13 and Fact 1.6, $V \cap \rho(G)L$ is a maximal $\mathfrak{F}$-subgroup of $\rho(G)L$. By induction, if E is a normal subgroup of $\rho(G)L \cap A_{n-1}$ and if E_1 is a subgroup of E containing $V \cap E$, then $V \cap E$ is an $\mathfrak{F}$-n-injector of E_1. By an application of Lemma 1.12 to $\rho(G)L$, $V \cap L$ is a maximal $\mathfrak{F}$-subgroup of L. Hence V is an $\mathfrak{F}$-n-injector of G.

Let B be a normal subgroup of G and let B_1 be a subgroup of B containing $V \cap B$. We show that $V \cap B$ is an $\mathfrak{F}$-n-injector of B_1. As V is an $\mathfrak{F}$-n-injector of G, $V \cap B \cap A_i = V \cap B_1 \cap A_i$ is a maximal $\mathfrak{F}$-subgroup of $B \cap A_i$ and of $B_1 \cap A_i$ for all $i \in \{0, ..., n\}$, and we have

$$V \cap B_1 \cap A_0 = V \cap B \cap A_0 = B \cap (A_0)_{\mathfrak{F}n} = (B_1 \cap A_0)_{\mathfrak{F}n}.$$

By induction, if D is a normal subgroup of $B_1 \cap A_{n-1}$ and if D_1 is a subgroup of D containing $V \cap D$, then $V \cap D$ is an $\mathfrak{F}$-n-injector of D_1. In particular, if D is a normal subgroup of B_1 contained in $B_1 \cap A_{n-1}$, then $V \cap D$ is an $\mathfrak{F}$-n-injector of D. Let L be a normal subgroup of B_1. Then, by Lemma 1.13 and Fact 1.6, $V \cap \rho(B_1)L \cap A_i$ is a maximal $\mathfrak{F}$-subgroup of $\rho(B_1)L \cap A_i$ for all $i \in \{0, ..., n\}$ and we have

$$V \cap \rho(B_1)L \cap A_0 = \rho(B_1)L \cap (A_0)_{\mathfrak{F}n} = (\rho(B_1)L \cap A_0)_{\mathfrak{F}n}.$$

By induction, if F is a normal subgroup of $\rho(B_1)L \cap A_{n-1}$ and if F_1 is a subgroup of F containing $V \cap F$, then $V \cap F$ is an $\mathfrak{F}$-n-injector of F_1. By an application of Lemma 1.12 to $\rho(B_1)L$, $V \cap L$ is a maximal $\mathfrak{F}$-subgroup of L. Hence $V \cap B$ is an $\mathfrak{F}$-n-injector of B_1. □

Corollary 1.15. *If G possesses an $\mathfrak{F}$-n-injector V, then $V \cap A$ is a maximal $\mathfrak{F}$-subgroup of A for every subnormal subgroup A of G.*

Proof. We may assume A normal in G. Then $V \cap A^{(k)}$ is a maximal $\mathfrak{F}$-subgroup of $A^{(k)}$ for every $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $A^{(n)}$ is locally nilpotent (n exists by Fact 1.6). Then we have $V \cap A^{(n)} = (A^{(n)})_{\mathfrak{F}n}$ by $(NF2^*)$, and Proposition 1.14 says that $V \cap A$ is an $\mathfrak{F}$-n-injector of A. □

Theorem 1.16. *G possesses a unique conjugacy class of $\mathfrak{F}$-n-injectors.*

Proof. We proceed by induction on the solubility class k of $G/\rho(G)$. By $(NF2^*)$ we may assume that G is not locally nilpotent, that is $k \geq 1$. By induction, G' has an $\mathfrak{F}$-n-injector W, and W is contained in a maximal $\mathfrak{F}$-subgroup V of G by Lemma 1.8. In particular, $V \cap G^{(i)} = W \cap G^{(i)}$ is a maximal $\mathfrak{F}$-subgroup of $G^{(i)}$ for every $i \in \{0, ..., k\}$, and we have $V \cap G^{(k)} = (G^{(k)})_{\mathfrak{F}n}$. Hence V is an $\mathfrak{F}$-n-injector of G by Proposition 1.14.

Let V_1 and V_2 be two $\mathfrak{F}$-n-injectors of G. Then $V_1 \cap G'$ and $V_2 \cap G'$ are $\mathfrak{F}$-n-injectors of G' (Corollary 1.15), and we can suppose $V_1 \cap G' = V_2 \cap G'$ by induction. Lemma 1.10 gives the result. □

Now we obtain Theorem 1.4 by Proposition 1.7 and Theorem 1.16.

Remark 1.17. If the word *normal* is replaced by *subnormal* in Definition 1.2 then clearly we do not obtain a new notion of Fitting set. Moreover, Corollary 1.15 shows that if we do the same thing in Definition 1.3, then *in the context of subsoluble* $\mathfrak{U}$*-groups*, we do not obtain a new notion of injector.

2. Ascendant Fitting sets

In this section we replace the word *normal* in §1 by *ascendant*.

Notation 2.1. Let $\mathfrak{X}$ be a collection of groups. For each group G, we denote by $G_{\mathfrak{X}a}$ the join of its ascendantl $\mathfrak{X}$-subgroups.

Definition 2.2. Let G be a locally finite group. A nonempty set $\mathfrak{X}$ of subgroups of G is an *ascendant Fitting set* of G if it satisfies $(NF3)$ and:
$(AF1)$ every ascendant subgroup of an $\mathfrak{X}$-group belongs to $\mathfrak{X}$;
$(AF2)$ when H is a subgroup of G then $H_{\mathfrak{X}a}$ is an $\mathfrak{X}$-subgroup of H.

Definition 2.3. Let G be a locally finite group. If $\mathfrak{X}$ is a set of subgroups of G, a subgroup V of G is an $\mathfrak{X}$*-a-injector* of G if, for every ascendant subgroup A of G, $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.

We will prove the following result.

Theorem 2.4. *Let G be both an SN^**-group *and a* $\mathfrak{U}$-group*, and $\mathfrak{X}$ an ascendant Fitting set. Then G has exactly one conjugacy class of $\mathfrak{X}$-a-injectors.*

It is likely that the class $SN^* \cap \mathfrak{U}$ is the largest class of groups satisfying this result. A partial answer is obtained in [3].

Similarly to §1, we define an *ascendant** *Fitting set* as a set $\mathfrak{X}$ of subgroups of a locally finite group G satisfying $(AF1)$, $(NF3)$ and

$(AF2^*)$ *if H is a subgroup of G such that $H/H_{\mathfrak{X}a}$ is locally nilpotent, then $H_{\mathfrak{X}a}$ is a maximal $\mathfrak{X}$-subgroup of H.*

We note that in any locally finite group G, an ascendant Fitting set is a normal Fitting set, and an ascendant* Fitting set is both a normal* Fitting set and an ascendant Fitting set.

Moreover, by a similar proof of Proposition 1.7, we show that if G is both an SN^**-group* and a $\mathfrak{U}$*-group*, then its an ascendant* Fitting sets are precisely its ascendant Fitting sets.

Theorem 2.5. *Let G be a $\mathfrak{U}$-group. For any ascendant* Fitting set $\mathfrak{F}$ of G, the $\mathfrak{F}$-n-injectors of G are precisely its $\mathfrak{F}$-a-injectors.*

Proof. Let A be an ascendant subgroup of G. Let $(A_i)_{i\leq\alpha}$ (α ordinal) be an ascending series from A to G. Let W be an $\mathfrak{F}$-n-injector of A (Theorem 1.16). We prove that there is an ascending series $(W_i)_{i\leq\alpha}$ such that $W_0 = W$ and W_i is an $\mathfrak{F}$-n-injector of A_i for every $i \leq \alpha$. Assume W_i has been constructed. Let U_{i+1} be an $\mathfrak{F}$-n-injector of A_{i+1}. Then $U_{i+1} \cap A_i$ is an $\mathfrak{F}$-n-injector of A_i by Corollary 1.15. By Theorem 1.16, there is $a \in A_i$ such that $(U_{i+1} \cap A_i)^a = W_i$. So U^a_{i+1} is an $\mathfrak{F}$-n-injector of A_{i+1} containing W_i, and we can take $W_{i+1} = U^a_{i+1}$. We must show that, if j is a limit ordinal, then $W_j = \cup_{i<j}W_i$ is an $\mathfrak{F}$-n-injector of A_j. We note that $W_j = (W_j)_{\mathfrak{F}a}$ is an $\mathfrak{F}$-subgroup of A_j by $(AF2^*)$. Let L be a normal subgroup of A_j and let K be an $\mathfrak{F}$-subgroup of L containing $W_j \cap L$. Since W_i is an $\mathfrak{F}$-n-injector of A_i for every $i < j$, $W_i \cap L$ is a maximal $\mathfrak{F}$-subgroup of $A_i \cap L$ for all $i < j$. Therefore, since $K \cap A_i$ is an $\mathfrak{F}$-subgroup of $A_i \cap L$ containing $W_i \cap L$ for all $i < j$ by $(AF1)$, we have $K \cap A_i = W_i \cap L$ for all $i < j$. Hence we obtain $K = W_j \cap L$ and $W_j \cap L$ is a maximal $\mathfrak{F}$-subgroup of L. So W_j is an $\mathfrak{F}$-n-injector of A_j.

Let V be an $\mathfrak{F}$-n-injectors of G. We show that $V \cap A$ is an $\mathfrak{F}$-n-injector of A. Otherwise there is a smallest ordinal β and an $\mathfrak{F}$-n-injector U of A_β such that $U \cap A$ is not an $\mathfrak{F}$-n-injector of A. If $\beta = \delta + 1$ for an ordinal δ, then $U \cap A_\delta$ is an $\mathfrak{F}$-n-injector of A_δ by Corollary 1.15, and $U \cap A = (U \cap A_\delta) \cap A$ is an $\mathfrak{F}$-n-injector of A by the minimality of β, contradicting the choice of β. So β is a limit ordinal. By Theorem 1.16, there exists $b \in A_\beta$ such that $U^b = W_\beta$. So there is $i < \beta$ such that $b \in A_i$ and we have $(U \cap A_i)^b = W_i$. Hence $U \cap A_i$ is an $\mathfrak{F}$-n-injector of A_i and $U \cap A = (U \cap A_i) \cap A$ is an $\mathfrak{F}$-n-injector of A by the minimality of β. Our result follows from this contradiction. ☐

Now Theorem 2.4 is obtained from Theorems 1.4 and 2.5.

3. Serial Fitting sets

In this section we replace the word *normal* in §1 by *ascendant*.

Notation 3.1. Let $\mathfrak{X}$ be a collection of groups. For each group G, we denote by $G_\mathfrak{X}$ the join of its serial $\mathfrak{X}$-subgroups.

Definition 3.2. Let G be a locally finite group. A nonempty set $\mathfrak{X}$ of subgroups of G is a *serial Fitting set* of G if it satisfies $(NF3)$ and:
$(SF1)$ every serial subgroup of an $\mathfrak{X}$-group belongs to $\mathfrak{X}$;
$(SF2)$ when H is a subgroup of G then $H_{\mathfrak{X}s}$ is an $\mathfrak{X}$-subgroup of H.

We note that in any locally finite group G, a serial Fitting set is an ascendant* Fitting set too.

Definition 3.3. Let G be a locally finite group. If $\mathfrak{X}$ is a set of subgroups of G, a subgroup V of G is an $\mathfrak{X}$*-injector* of G if, for every serial subgroup A of G, $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.

We will prove the following result.

Theorem 3.4. *Let G be a $\mathfrak{U}$-group, and $\mathfrak{X}$ a serial Fitting set of G. Then G has exactly one conjugacy class of $\mathfrak{X}$-injectors.*

It is proven in [3] *that the class of $\mathfrak{U}$-groups is the largest class of groups satisfying this result.* The difficulty of the proof in [3] is that without the conjugacy of Sylow subgroups, a Sylow subgroup might not be an injector.

From now on, G is a fixed $\mathfrak{U}$-group and $\mathfrak{F}$ is a serial Fitting set of G.

Lemma 3.5. *Let N be a normal subgroup of G such that G/N is a p-group for a prime p and $N_{\mathfrak{F}}$ is a maximal $\mathfrak{F}$-subgroup of N. If N contains a subgroup A of G and if V is an $\mathfrak{F}$-a-injector of G, then $V \cap A$ is an $\mathfrak{F}$-a-injector of A.*

Proof. By Theorems 1.16 and 2.5, A has an $\mathfrak{F}$-a-injector W. Then we have $W \cap N = N_{\mathfrak{F}}$, so $W/N_{\mathfrak{F}}$ is a p-group and there is a maximal p-subgroup $R/N_{\mathfrak{F}}$ of $G/N_{\mathfrak{F}}$ containing $W/N_{\mathfrak{F}}$. Likewise we have $V \cap N = N_{\mathfrak{F}}$ and there is a maximal p-subgroup $S/N_{\mathfrak{F}}$ of $G/N_{\mathfrak{F}}$ containing $V/N_{\mathfrak{F}}$. As G/N is a p-group, we have $G = RN = SN$ [4, Lemma 2.1 (ii)], and there is $g \in N$ such that $R^g = S$. But $S/N_{\mathfrak{F}}$ is locally nilpotent, so $S_{\mathfrak{F}}$ contains all the $\mathfrak{F}$-subgroups of S containing $N_{\mathfrak{F}}$. Hence $S_{\mathfrak{F}}$ contains V and W^g. By the maximality of V, we obtain $V = S_{\mathfrak{F}}$ and $W^g \leq V$. But W^g is an $\mathfrak{F}$-a-injector of A since $g \in N \leq A$. As A contains N and as G/N is locally nilpotent, A is a serial subgroup of G. Thus $V \cap A$ is an $\mathfrak{F}$-subgroup of A. Since $V \cap A$ contains an $\mathfrak{F}$-a-injector W^g of A, we obtain $V \cap A = W^g$ and the proof is complete. □

Lemma 3.6. *Assume that G has a normal p-subgroup M for a prime p and a serial subgroup A such that $G = MA$. If V is an $\mathfrak{F}$-a-injector of G, then $V \cap A$ is an $\mathfrak{F}$-a-injector of A.*

Proof. Let $P = \langle S : S \text{ is a } p'\text{-subgroup of } G\rangle$. We show that P is a subgroup of A. By the seriality of A, for each $x \in G \setminus A$, there exist U and V two subgroups of G containing A such that $V \trianglelefteq U$ and $x \in U \setminus V$. But $U = (M \cap U)V$, so U/V is a p-group and x is not a p'-element. This proves that each p'-subgroup of G is contained in A, hence P is a subgroup of A.

We show that $V \cap N_A(V \cap P)$ is an $\mathfrak{F}$-a-injector of $N_A(V \cap P)$. Let $G_1 = N_G(V \cap P)$ and $N = N_P(V \cap P)$. As P is normal in G, $V \cap P$ is an $\mathfrak{F}$-a-injector of P, so we have $N_{\mathfrak{F}} = V \cap P$, and $N_{\mathfrak{F}}$ is a maximal $\mathfrak{F}$-subgroup of P and of N. Since $N = G_1 \cap P$ contains all the p'-elements of G_1, G_1/N is a p-group. But, as A contains P, $N_A(V \cap P)$ is a subgroup of G_1 containing N and, since V is an $\mathfrak{F}$-a-injector of G_1 (Corollary 1.15 and Theorem 2.5), Lemma 3.5 says that $V \cap N_A(V \cap P)$ is an $\mathfrak{F}$-a-injector of $N_A(V \cap P)$.

By Theorems 1.16 and 2.5, A has an $\mathfrak{F}$-a-injector W. Then $W \cap P$ and $V \cap P$ are two $\mathfrak{F}$-a-injectors of P and there is $a \in P$ such that $(W \cap P)^a = V \cap P$ (Theorem 2.5). Now W^a is an $\mathfrak{F}$-a-injector of $N_A(V \cap P)$ by Corollary 1.15 and Theorem 2.5. Since $V \cap N_A(V \cap P)$ is an $\mathfrak{F}$-a-injector of $N_A(V \cap P)$, Theorem 2.5 says that W^a and $V \cap N_A(V \cap P)$ are conjugate in $N_A(V \cap P)$. Thus $V \cap N_A(V \cap P)$ is an $\mathfrak{F}$-a-injector of A contained in the $\mathfrak{F}$-subgroup $V \cap A$. Hence $V \cap N_A(V \cap P) = V \cap A$ and $V \cap A$ is as desired. □

Theorem 3.7. *The $\mathfrak{F}$-a-injectors of G are precisely its $\mathfrak{F}$-injectors.*

Proof. It is sufficient to prove that any $\mathfrak{F}$-a-injector V of G is an $\mathfrak{F}$-injector. We prove that, if A is a serial subgroup of G, then $V \cap A$ is an $\mathfrak{F}$-a-injector of A. Let

W be an $\mathfrak{F}$-a-injector of A. We show that W is contained in an $\mathfrak{F}$-a-injector U of $A\rho(G)$. Let $N_0 = 1$ and, for every $j \geq 1$, $N_j = \mathcal{O}_{\pi_j}(\rho(G))$ where $\pi_j = \{q : q$ is a the i^{th} prime for $i \leq j\}$. We construct an increasing sequence $(W_i)_{i\in\mathbb{N}}$ of $\mathfrak{F}$-subgroups of G such that $W_0 = W$ and, for every $i \in \mathbb{N}$, W_i is an $\mathfrak{F}$-a-injector of N_iA. Suppose W_i constructed for $i \in \mathbb{N}$. Let $W_{i+1,0}$ be an $\mathfrak{F}$-a-injector of $N_{i+1}A$. Let p be the $(i+1)^{th}$ prime. Then we have $N_{i+1} = N_i \times \mathcal{O}_p(\rho(G))$, and since N_iA is a serial subgroup of $N_{i+1}A$, then $W_{i+1,0} \cap N_iA$ is an $\mathfrak{F}$-a-injector of N_iA by Lemma 3.6. So there is $u \in N_iA$ such that $W_i = (W_{i+1,0} \cap N_iA)^u$ (Theorem 2.5). Now we may choose $W_{i+1} = W_{i+1,0}^u$, and $W_i = W_{i+1} \cap N_iA$ is serial in W_{i+1}. Let $U = \cup_{i\in\mathbb{N}} W_i$. Then W_i is a serial $\mathfrak{F}$-subgroup of U for each i, and $U = U_{\mathfrak{F}}$ is an $\mathfrak{F}$-group. Moreover we have $U \cap N_iA = W_i$ for every $i \in \mathbb{N}$. Let B be an ascendant subgroup of $A\rho(G)$ and U_1 be an $\mathfrak{F}$-subgroup of B containing $U \cap B$. For every $i \in \mathbb{N}$, as N_iA is a serial subgroup of $A\rho(G)$, $U_1 \cap N_iA$ is an $\mathfrak{F}$-group containing $W_i \cap B$. But $W_i \cap B$ is a maximal $\mathfrak{F}$-subgroup of $N_iA \cap B$ for every $i \in \mathbb{N}$ since W_i is an $\mathfrak{F}$-a-injector of N_iA. So we have $U_1 \cap N_iA = W_i \cap B$ for every $i \in \mathbb{N}$, and $U_1 = \cup_{i\in\mathbb{N}}(U_1 \cap N_iA) = \cup_{i\in\mathbb{N}}(W_i \cap B) = U \cap B$. Hence U is an $\mathfrak{F}$-a-injector of $A\rho(G)$ containing W.

It follows from Fact 1.6 that any serial subgroup of $G/\rho(G)$ is acendant [1, Lemma 7.2.11], so $A\rho(G)$ is ascendant in G, and $V \cap A\rho(G)$ is an $\mathfrak{F}$-a-injector of $A\rho(G)$. By Theorem 2.5, there is $g \in A\rho(G)$ such that $U^g = V \cap A\rho(G)$. But there is $k \in \mathbb{N}$ such that $g \in N_kA$ and we have $V \cap N_kA = (U \cap N_kA)^g = W_k^g$. So $V \cap N_kA$ is an $\mathfrak{F}$-a-injector of N_kA. Now, by successive applications of Lemma 3.6, $V \cap A$ is an $\mathfrak{F}$-a-injector of A. □

Now we obtain Theorem 3.4 from Theorems 1.16, 2.5 and 3.7.

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